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An explicit determination of the Petrov type N space-times on which the conformally invariant scalar wave equation satisfies Huygens' principle

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ABSTRACT. — It is shown that the conformally invariant scalar wave equation on a general Petrov type-N space-time satisfies Huygens' principle if and only if the space-time is conformally related to an exact plane wave space-time. Some related intermediate results on the validity of Huygens' principle for Maxwell's equations and for Weyl's equation are also given.

RÉSUMÉ. — On démontre que l'équation invariante conforme des ondes scalaires sur un espace-temps général de type N de Petrov satisfait au principe d'Huygens si et seulement si l'espace-temps est conforme à l'espace-temps des ondes planes. On donne aussi quelques résultats partiels de nature analogue sur la validité du principe d'Huygens pour les équations de Maxwell et pour l'équation de Weyl.

1. INTRODUCTION

We consider the general second order linear, homogeneous, hyperbolic partial differential equation for an unknown function u of n independent variables. Such an equation may be expressed in coordinate invariant form as

$$F[u] := g^{ab}u_{;ab} + A^{a}u_{,a} + Cu = 0, \tag{1.1}$$

where g^{ab} are the contravariant components of the metric tensor on the underlying n-dimensional Lorentzian space V_n, and « , » and « ; » denote respectively the partial derivative and the covariant derivative with respect to the Levi-Civita connection. The coefficients g^{ab}, A^a, and C and the space V_n are assumed to be of class C^∞.

Cauchy's problem for the equation (1.1) is the problem of determining a solution which assumes given values of u and its normal derivative on a given space-like $(n - 1)$ dimensional submanifold S. These values are called Cauchy data. Local existence and uniqueness of the solution has been proved by Hadamard [*15*] who introduced the concept of the fundamental solution. Alternate solutions have been presented by Mathison [*20*], Sobolev [*32*], Bruhat [*3*], Douglis [*8*], and Friedlander [*10*]. The global theory of Cauchy's problem has been developed by Leray [*19*]. The considerations of the present paper will be purely local.

Of particular importance in Cauchy's problem is the domain of dependence of the solution. In this regard Hadamard has shown that for any $x_0 \in V_n$ sufficiently close to S, $u(x_0)$ depends only on the Cauchy data in the interior of the intersection of the past null conoid $C^-(x_0)$ with S and on the intersection itself. The equation (1.1) is said to satisfy *Huygens' principle* (in the strict sense) if and only if for every Cauchy problem and for every $x_0 \in V_n$, the solution depends only on the data in an arbitrarily small neighbourhood of $S \cap C^-(x_0)$. Such an equation is called a *Huygens' differential equation.* Hadamard posed the problem, as yet unsolved, of determining up to equivalence all the Huygens' differential equations. He showed that in order that (1.1) satisfy Huygens' principle it is necessary that $n \geq 4$, and be even. Furthermore he established that a necessary and sufficient condition for a Huygens' differential equation is the vanishing of the logarithmic term in the elementary solution. He then suggested that one should attempt to show that every such an equation is equivalent to one of the ordinary wave equations that may be obtained from (1.1) by setting $g^{ab} = \text{diag}(1, -1, \ldots, -1)$, $A^a = C = 0$, and $n = 2m$, $m = 2, 3, \ldots$, which satisfy Huygens' principle [*6*]. This is often referred to as « Hadamard's conjecture » in the literature. We recall that two equations of the form (1.1) are *equivalent* if and only if one may be transformed into the other by any of the following *trivial transformations* which preserve the Huygens' character of the equation:

a) a general coordinate transformation;

b) multiplication of the equation (1.1) by the function $\exp(-2\phi(x))$, which induces a conformal transformation of the metric:

$$\tilde{g}_{ab} = e^{2\phi} g_{ab}\,; \tag{1.2}$$

c) replacement of the unknown function by λu, where $\lambda(x)$ is a non-vanishing function.

Hadamard's conjecture has been proved in the case $n = 4$, g^{ab} constant by Mathisson [*21*], Hadamard [*16*], and Asgeirson [*1*]. However, it has been disproved in general by Stellmacher [*33*] [*34*] who gave counter-examples for $n = 6, 8, \ldots$ and Günther [*12*] who provided a family of counter examples in the physically interesting case $n = 4$, with $A^a = C = 0$, based on the *plane wave space-time* [*29*] whose metric may be written as [*9*]

$$ds^2 = 2dv\,[du + (D(v)z^2 + \overline{D}(v)\bar{z}^2 + e(v)z\bar{z})dv] - 2dzd\bar{z}, \tag{1.3}$$

where D and e are arbitrary functions. These are the only known Huygens' differential equations not equivalent to the ordinary 4-dimensional wave equation.

In their attempts to solve Hadamard's problem workers [*17*] [*11*] [*22*] [*35*] [*24*] have derived the following series of necessary conditions that must be satisfied by the coefficients of a Huygens' differential equation:

I $$C = \frac{1}{2} A^k{}_{;k} + \frac{1}{4} A_k A^k + \frac{1}{6} R, \tag{1.4}$$

II $$H_{ab;}{}^{b} = 0, \tag{1.5}$$

III $$S_{abk;}{}^{k} - \frac{1}{2} C^k{}_{ab}{}^{l} L_{kl} = -5\left(H_{ka}H_b{}^k - \frac{1}{4} g_{ab}H_{kl}H^{kl}\right), \tag{1.6}$$

IV $$TS(S_{abk}H^k{}_c + C^k{}_{ab}{}^{l}H_{ck;l}) = 0, \tag{1.7}$$

V $$\begin{aligned} TS(&3C_{kabl;m}C^k{}_{cd}{}^{l}{}_{;}{}^{m} + 8C^k{}_{ab}{}^{l}{}_{;c}S_{kld} + 40S_{ab}{}^kS_{cdk} \\ &- 8C^k{}_{ab}{}^{l}S_{klc;d} - 24C^k{}_{ab}{}^{l}S_{cdk;l} + 4C^k{}_{ab}{}^{l}C_l{}^m{}_{ck}L_{dm} \\ &+ 12C^k{}_{ab}{}^{l}C^m{}_{cdl}L_{km} \\ &+ 12H_{ka;bc}H^k{}_d - 16H_{ka;b}H^k{}_{c;d} - 84H^k{}_aC_{kbcl}H^l{}_d - 18H_{ka}H^k{}_bL_{cd}) = 0. \end{aligned} \tag{1.8}$$

In the above conditions

$$H_{ab} := A_{[a,b]}\,, \tag{1.9}$$

$$L_{ab} := -R_{ab} + \frac{1}{6} R g_{ab}\,, \tag{1.10}$$

$$S_{abc} := L_{a[b;c]}\,, \tag{1.11}$$

$$C_{abcd} := R_{abcd} - 2g_{[a[d}L_{b]c]}\,, \tag{1.12}$$

where R_{abcd} denotes the Riemann curvature tensor corresponding to the metric g_{ab}, $R_{ab} := g^{cd}R_{cabd}$ the Ricci tensor and $R := g^{ab}R_{ab}$ the curvature scalar, our sign conventions being the same as those in [*24*]. The notation TS( ) denotes the trace-free symmetric part of the enclosed tensor [*22*]. It is important to note that the above necessary conditions are invariant under the trivial transformations.

The Conditions I and III were used by Mathisson to prove Hadamard's conjecture in the case g_{ab} constant. One of us [*22*] used Conditions I, III, and V to show that the Eq. (1.1) on a space-time conformal to a space-time satisfying $R_{ab} = 0$ (empty space-time) is a Huygens' equation only if it is equivalent to the wave equation on the plane wave space-time with metric (1.3). This result combined with that of Günther previously mentioned solves Hadamard's problem in this case. However, the Conditions I to V are not sufficient to characterize the Huygens' differential equations in the general case. This conclusion follows from the fact that the conformally invariant wave equation

$$g^{ab}u_{;ab} + \frac{1}{6}Ru = 0, \tag{1.13}$$

on the generalized plane wave space-time of McLenaghan and Leroy [*23*] with metric

$$\begin{aligned} ds^2 = {} & 2dv\,[du + (a(v)(z+\bar{z})u + D(v)z^2 + \bar{D}(v)\bar{z}^2 + e(v)z\bar{z} + F(v)z + \bar{F}(v)\bar{z})dv\,] \\ & -2(dz + a(v)z^2dv)(d\bar{z} + a(v)\bar{z}^2dv), \end{aligned} \tag{1.14}$$

where a, D, e, and F are arbitrary functions, satisfies Conditions III and V [*25*], but does not satisfy the following additional necessary condition for (1.13) to be a Huygens' differential equation, derived by Rinke and Wünsch [*27*], unless $a(v) = 0$ [*40*] [*5*], when (1.14) modulo a coordinate transformation reduces to the plane wave metric (1.3):

$$\begin{aligned} \text{VII}\quad & \mathrm{TS}\big[3C^k{}_{ab;}{}^{lm}{}_c C_{kdel;mf} + C^k{}_{ab}{}^l{}_{;cd}(10S_{kle;f} + 6S_{efk;l}) \\ & + 64S_{abk;c}S_{de}{}^k{}_{;f} - C^k{}_{ab}{}^l(3C^m{}_{cdk;ef}L_{lm} + 5C_{kcdl;me}L^m{}_f \\ & + 7C^m{}_{cdk;le}L_{mf} + 13S_{klc;d}L_{ef} + 12S_{cdk;l}L_{ef} + 71S_{cdk;e}L_{lf}) \\ & -10C^k{}_{ab}{}^l{}_{;c}(S_{kld;ef} + 3S_{dek;lf}) - 20S_{abk;cd}S_{ef}{}^k \\ & + 50S_{abk}S_{cd}{}^kL_{ef} + 5C^k{}_{ab}{}^l{}_{;c}(2C^m{}_{kld;e}L_{mf} + 3C^m{}_{dek;l}L_{mf} \\ & + S_{kld}L_{ef} + 3C_{kde}{}^m{}_{;f}L_{lm} + 15S_{dek}L_{lf}) + 10C^k{}_{ab}{}^l(C_{kcd}{}^m{}_{;e}L_{(lm;f)} \\ & + S_{cdk}L_{(le;f)} - \frac{1}{12}R_{;c}C_{kdel;f}\Big) - 4C^k{}_{ab}{}^l(2C_k{}^{mn}{}_cC_{lnmd;ef} \\ & - 10C^m{}_{cd}{}^nC_{kefl;mn} + 20C_{lcd}{}^mS_{kme;f}) - 20C_k{}^{mn}{}_aC_{lmnb}C^k{}_{cd}{}^l{}_{;ef} \\ & + 4C^k{}_{ab}{}^l(7C_k{}^{mn}{}_{c;d}C_{lmnd}L_{ef} - 10C_{kefl}C^m{}_{cd}{}^nL_{mn}) \\ & - 5C^k{}_{ab}{}^l(3C_k{}^{mn}{}_{c;d}C_{lmne;f} + 54C_{lcd}{}^m{}_{;e}S_{kmf} + 74C_{lcd}{}^m{}_{;k}S_{efm} \end{aligned}$$

$$- \frac{76}{3} C_{ckl}{}^{m}{}_{;d}S_{efm} - \frac{404}{3} S_{cdk}S_{efl} \Big) + 30 C_{k}{}^{mn}{}_{a} C^{k}{}_{bc}{}^{l}{}_{;d} C_{lefm;n}$$

$$+ 25 C^{k}{}_{ab}{}^{l} C_{lcd}{}^{m} L_{km} L_{ef} + \frac{1}{6} C^{k}{}_{ab}{}^{l} C_{kcdl} (87 L^{m}{}_{e} L_{mf} + 19 R L_{ef}) \Big] = 0 . \quad (1.15)$$

It should be noted that Rinke and Wünsch used Conditions III and VII to show that the conformally invariant wave equation on the plane fronted wave with parallel rays space-time with metric

$$ds^2 = 2dv(du + m(v, z, \bar{z})dv) - 2dzd\bar{z},$$

where m is an arbitrary function, is a Huygens' differential equation only if there exists a system of coordinates for which

$$m(v, z, \bar{z}) = D(v)z^2 + \bar{D}(v)\bar{z}^2 + e(v)z\bar{z},$$

in which case the metric reduces to the plane wave metric (1.3).

The results described above suggest that the Conditions III, V, and VII may be sufficient to show that *every space-time on which the conformally invariant equation* (1.13) *satisfies Huygens' principle, is conformally related to the plane wave space-time* (1.3) *or is conformally flat.* A plan of attack for proving this conjecture, suggested by the authors [*4*], is to consider separately each of the five possible Petrov types [*7*] [*28*] of the Weyl conformal curvature tensor C_{abcd} of space-time. This is a natural approach since Petrov type is invariant under a general conformal transformation (1.2). The conjecture is indeed true for Petrov type N, as has been stated without proof by the authors in [*5*].

The main purpose of the present paper is to provide a detailed proof of this result. Our analysis enables us to obtain « en passant » some related intermediate results on the validity of Huygens' principle for Maxwell's equations for a 2-form ω on V_4 which may be written as

$$d\omega = 0, \qquad \delta\omega = 0, \quad (1.16)$$

where d and δ denote the exterior derivative and co-derivative respectively, and for Weyl's neutrino equation for a 2-spinor ϕ_A which may be written as

$$\nabla_{\dot{A}}{}^{B}\phi_B = 0, \quad (1.17)$$

where $\nabla_{\dot{A}}{}^{B}$ denotes the convariant derivative on spinors. It has been shown by Künzle [*18*] that Huygens' principle is satisfied for Maxwells' equations on the plane wave space-time [*31*] according to a criterion given by Günther [*13*], and that the only empty space-time with this property has metric (1.3) with $e = 0$ [*14*]. Wünsch [*38*] [*39*] has shown that Weyl's equation also satisfies Huygens' principle on the plane wave space-time with a converse analogous to that for Maxwell's equations when $R_{ab} = 0$. Necessary conditions for the validity of Huygens' principle for the equations (1.16) and (1.17) have also been derived [*36*] [*39*]. The first of these is the vanishing of the Bach tensor [*2*], C_{ab}, which is defined to be the tensor on the left hand side of Eq. (1.6). The next level of conditions are similar to Condition V in the case of the conformally invariant scalar wave equation (1.13) but with different numerical coefficients. These conditions can be combined into the following form valid for all three of the equations considered:

$$\text{III}' \qquad C_{ab} := S_{abk;}{}^{k} - \frac{1}{2} C^{k}{}_{ab}{}^{l} L_{kl} = 0 , \tag{1.18}$$

$$\begin{aligned} \text{V}' \quad \text{TS}[k_1 C^{k}{}_{ab}{}^{l}{}_{;}{}^{m} C_{kcdl;m} &+ 2k_2 C^{k}{}_{ab}{}^{l}{}_{;c} S_{kld} + 2(8k_1 - k_2) S_{ab}{}^{k} S_{cdk} \\ &- 2k_2 C^{k}{}_{ab}{}^{l} S_{klc;d} - 8k_1 C^{k}{}_{ab}{}^{l} S_{cdk;l} + k_2 C^{k}{}_{ab}{}^{l} C_{l}{}^{m}{}_{ck} L_{dm} \\ &+ 4k_1 C^{k}{}_{ab}{}^{l} C^{m}{}_{cdl} L_{km}] = 0 , \end{aligned} \tag{1.19}$$

where the values of k_1 and k_2 in each of the three cases are given in the following table:

TABLE 1

Equation	k_1	k_2
Scalar	3	4
Maxwell	5	16
Weyl	8	13

It will be shown in the sequel that the general solution of Eqs. (1.18) and (1.19) for Petrov type N are conformally related to special cases of the complex recurrent space-times of McLenaghan and Leroy [*23*]. In the case of the conformally invariant scalar wave equation (1.13), imposition of the additional necessary condition VII yields the solution of Hadamard's problem already stated. However, in the case of Maxwell's and Weyl's equations the corresponding problems remain open. It seems that additional conditions analogous to Condition VII are required in order to settle Hadamard's problem for these equations.

The plan of the remainder of the paper is as follows. In Section 2 the main theorems are stated and a description of the method of proof given. The formalisms used are briefly described in Section 3 and the details of the proofs given in Sections 4 and 5.

2. HYPOTHESES AND STATEMENT OF RESULTS

We assume that the space-time V_4 is Petrov type N. This is equivalent to the existence of a necessarily null vector field l such that the non-vanishing Weyl conformal curvature tensor satisfies the equation [7]

$$C_{abcd}l^d = 0, \qquad (2.1)$$

at each point. We note that the plane wave space-time with metric (1.3) is Petrov type N with $l = \partial/\partial u$.

The main results of this paper are contained in the following theorems.

THEOREM 1. — *For every Petrov type* N *space-time for which the conformally invariant scalar wave equation (1.13), or Maxwell's equations (1.16), or Weyl's equation (1.17) satisfies Huygens' principle there exists a coordinate system* $(u, v, z, \bar{z})$ *and a function* ϕ *such that the metric of* V_4 *has the form*

$$ds^2 = e^{-2\phi(u,v,z,\bar{z})}\left\{ 2dv\left[du + \left(\frac{1}{2}(p_z(v, z) + \bar{p}_{\bar{z}}(v, \bar{z}))u + m(v, z, \bar{z})\right)dv\right] - 2(dz + p(v, z)dv)(d\bar{z} + \bar{p}(v, \bar{z})dv) \right\}, \qquad (2.2)$$

where the functions p and m satisfy

$$p(v, z) = p_2(v)z^2 + p_1(v)z + p_0(v), \qquad (2.3)$$

$$m(v, z, \bar{z}) = \bar{z}G(v, z) + z\bar{G}(v, \bar{z}) + H(v, z) + \bar{H}(v, \bar{z}). \qquad (2.4)$$

The functions G *and* H *appearing in (2.4) are either given explicitly by*

$$G(v, z) = g_1(v)z + g_0(v), \qquad H(v, z) = h_2(v)z^2 \qquad (2.5)$$

where g_0, $\bar{g}_1 = g_1$, *and* $h_2 \neq 0$ *are arbitrary functions in which case there exists coordinates in which the function* p_2 *of (2.3) is real and* p_0 *and* p_1 *are zero, or they satisfy the following differential equations*

$$G_{zz}(v, z) = a_2(v)[a_1(v)z]^{1/a_1(v)}, \qquad (2.6)$$

$$H_{zz}(v, z) = [a_2(v)b_1(v)/(1 + a_1(v))][a_1(v)z]^{1/a_1(v)}z, \qquad (2.7)$$

where a_2 *is an arbitrary non-vanishing function and* a_1 *and* b_1 *must satisfy*

$$a_1(v) = -1/5, \qquad |b_1(v)|^2 = (2/25)(k_2/k_1 - 1) \qquad (2.8)$$

or

$$(17 - 2k_2/k_1)|a_1(v)|^2 + 4(a_1(v) + \bar{a}_1(v)) + 1 = 0, \qquad b_1 = 0. \quad (2.9)$$

In the latter case the functions p_i, $i = 0, 1, 2$, *of (2.3) are arbitrary and the values of the parameters* k_1 *and* k_2 *are those of Table 1.*

When $k_1 = 3$, $k_2 = 4$, which yields the scalar case, Theorem 1 reduces to the proposition stated without proof in [*4*]. When $\phi = 0$, the metric (2.2) is a special case of the type N complex recurrent metric of McLenaghan and Leroy. The values of G and H given by (2.5), yield the generalized plane wave metric [*23*], which is already known to satisfy the necessary conditions III′ and V′ [*25*]. When $p_2 = 0$, there exists coordinates in which the metric has the form (1.3) of the plane wave space-time. The metrics defined by Eqs. (2.2) to (2.4) and (2.6) to (2.9) are apparently new solutions of these conditions [*4*] [*5*].

THEOREM 2. — *Every Petrov type* N *space-time on which the conformally invariant scalar wave equation (1.13) satisfies Huygens' principle is conformally related to a plane-wave space-time, that is there exists a system of coordinates* $(u, v, z, \bar{z})$ *and a function* ϕ *such that the metric of* V_4 *has the form*

$$ds^2 = e^{-2\phi(u,v,z,\bar{z})}\{2dv\,[du + (D(v)z^2 + \bar{D}(v)\bar{z}^2 + e(v)z\bar{z})dv] - 2dzd\bar{z}\}, \quad (2.10)$$

where D *and* e *are arbitrary functions.*

Conversely [*12*] *the conformally invariant wave equation* (1.13) *on any* V_4 *which is conformally related to a plane wave space-time satisfies Huygens' principle.* Theorem 2 and this result may be combined in

THEOREM 3. — *The conformally invariant wave equation (1.13) on any Petrov type* N *space-time satisfies Huygens' principle if and only if the space-time is conformally related to a plane wave space-time for which there exists a coordinate system in which the metric has the form (1.3).*

Theorem 3 is equivalent to the theorem stated without proof in [*5*]. Theorem 2 follows from Theorem 1 in the case $k_1 = 3$, $k_2 = 4$, when the necessary Condition VII is imposed.

3. FORMALISMS

We employ the two-component spinor formalism of Penrose [*28*] [*30*] and the spin coefficient formalism of Newman and Penrose (NP) [*26*] whose conventions we follow. In the spinor formalism tensors and spinors are related by the complex connection quantities $\sigma_a{}^{A\dot{A}}$ ($a = 1, \ldots, 4$; $A=0,1$) which are Hermitian in the spinor indices $A\dot{A}$ and satisfy the conditions

$$\sigma_a{}^{A\dot{A}}\sigma^a{}_{B\dot{B}} = \delta^A_B\delta^{\dot{A}}_{\dot{B}}. \quad (3.1)$$

In this equation spinor indices have been lowered by the skew symmetric spinors ε_{AB} and $\varepsilon_{\dot{A}\dot{B}}$ defined by $\varepsilon_{01} = \varepsilon_{\dot{0}\dot{1}} = 1$, according to the convention

$$\xi_A = \xi^B\varepsilon_{BA}. \quad (3.2)$$

Spinor indices are raised by the respective inverses of these spinors denoted by ε^{AB} and $\varepsilon^{\dot{A}\dot{B}}$. The spinor equivalents of the Weyl conformal curvature tensor (1.12) and the tensor L_{ab} defined by (1.10) are given respectively by

$$C_{abcd}\sigma^a{}_{A\dot{A}}\sigma^b{}_{B\dot{B}}\sigma^c{}_{C\dot{C}}\sigma^d{}_{D\dot{D}} = \Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{D}\dot{C}} + \overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}\varepsilon_{AB}\varepsilon_{DC}, \qquad (3.3)$$

$$L_{ab}\sigma^a{}_{A\dot{A}}\sigma^b{}_{B\dot{B}} = 2(\Phi_{AB\dot{A}\dot{B}} - \Lambda\varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}}), \qquad (3.4)$$

where Ψ_{ABCD}, called the Weyl spinor, is symmetric on its four indices, where $\Phi_{AB\dot{A}\dot{B}}$, called the trace free Ricci spinor, is a Hermitian spinor symmetric on the indices AB and $\dot{A}\dot{B}$, and where

$$\Lambda = (1/24)R. \qquad (3.5)$$

The covariant derivative of a spinor ξ_A is defined by

$$\xi_{A;a} = \xi_{A,a} - \xi_B\Gamma^B{}_{Aa}, \qquad (3.6)$$

where $\Gamma^B{}_{Aa}$ denotes the spinor affine connection. This connection is determined by the requirements that the covariant derivative is real, linear, obeys Leibnitz's rule and satisfies

$$\sigma_a{}^{A\dot{A}}{}_{;b} = \varepsilon_{AB;b} = 0. \qquad (3.7)$$

It is useful to introduce a basis $\{ o_A, \iota_A \}$ for the space of valence one spinors satisfying

$$o_A\iota^A = 1. \qquad (3.8)$$

These spinors may be used to define a spinor dyad $\zeta_a{}^A$ by

$$\zeta_0{}^A = o^A, \qquad \zeta_1{}^A = \iota^A. \qquad (3.9)$$

Associated to the spinor dyad is a null tetrad $(l, n, m, \overline{m})$ defined by

$$l^a = \sigma^a{}_{A\dot{A}}o^A\bar{o}^{\dot{A}}, \qquad n^a = \sigma^a{}_{A\dot{A}}\iota^A\bar{\iota}^{\dot{A}}, \qquad m^a = \sigma^a{}_{A\dot{A}}o^A\bar{\iota}^{\dot{A}}, \qquad (3.10)$$

whose only non-zero inner products are

$$l_an^a = - m_a\overline{m}^a = 1. \qquad (3.11)$$

The metric tensor may be expressed in terms of the null tetrad by

$$g_{ab} = 2l_{(a}n_{b)} - 2m_{(a}\overline{m}_{b)}. \qquad (3.12)$$

The NP spin coefficients associated to the dyad $\zeta_a{}^A$ or equivalently to the corresponding null tetrad are defined by

$$\begin{aligned} o_{A;B\dot{B}} = {} & \gamma o_Ao_B\bar{o}_{\dot{B}} - \alpha o_Ao_B\bar{\iota}_{\dot{B}} - \beta o_A\iota_B\bar{o}_{\dot{B}} + \varepsilon o_A\iota_B\bar{\iota}_{\dot{B}} \\ & - \tau\iota_Ao_B\bar{o}_{\dot{B}} + \rho\iota_Ao_B\bar{\iota}_{\dot{B}} + \sigma\iota_A\iota_B\bar{o}_{\dot{B}} - \kappa\iota_A\iota_B\bar{\iota}_{\dot{B}}, \qquad (3.13) \\ \iota_{A;B\dot{B}} = {} & \nu o_Ao_B\bar{o}_{\dot{B}} - \lambda o_Ao_B\bar{\iota}_{\dot{B}} - \mu o_A\iota_B\bar{o}_{\dot{B}} + \pi o_A\iota_B\bar{\iota}_{\dot{B}} \\ & - \gamma\iota_Ao_B\bar{o}_{\dot{B}} + \alpha\iota_Ao_B\bar{\iota}_{\dot{B}} + \beta\iota_A\iota_B\bar{o}_{\dot{B}} - \varepsilon\iota_A\iota_B\bar{\iota}_{\dot{B}}. \qquad (3.14) \end{aligned}$$

The NP components of the Weyl tensor and trace free Ricci tensor are defined respectively by

$$\left.\begin{array}{ll} \Psi_0 := \Psi_{ABCD}o^{ABCD}, & \Psi_1 := \Psi_{ABCD}o^{ABC}\iota^{D}, \\ \Psi_2 := \Psi_{ABCD}o^{AB}\iota^{CD}, & \Psi_3 := \Psi_{ABCD}o^{A}\iota^{BCD}, \\ \Psi_4 := \Psi_{ABCD}\iota^{ABCD}, & \end{array}\right\} \tag{3.15}$$

$$\left.\begin{array}{ll} \Phi_{00} := \Phi_{AB\dot{A}\dot{B}}o^{AB}\bar{o}^{\dot{A}\dot{B}}, & \Phi_{01} := \Phi_{AB\dot{A}\dot{B}}o^{AB}\bar{o}^{\dot{A}}\bar{\iota}^{\dot{B}}, \\ \Phi_{02} := \Phi_{AB\dot{A}\dot{B}}o^{AB}\bar{\iota}^{\dot{A}\dot{B}}, & \Phi_{11} := \Phi_{AB\dot{A}\dot{B}}o^{A}\iota^{B}\bar{o}^{\dot{A}}\bar{\iota}^{\dot{B}}, \\ \Phi_{12} := \Phi_{AB\dot{A}\dot{B}}o^{A}\iota^{B}\bar{\iota}^{\dot{A}}\bar{\iota}^{\dot{B}}, & \Phi_{22} := \Phi_{AB\dot{A}\dot{B}}\iota^{AB}\iota^{\dot{A}\dot{B}}, \end{array}\right\} \tag{3.16}$$

where the notation $o_{A_1 \dots A_P} := o_{A_1} \dots o_{A_P}$, etc. has been used. The NP differential operators are defined by

$$\mathrm{D} := l^a \frac{\partial}{\partial x^a}, \qquad \Delta := n^a \frac{\partial}{\partial x^a}, \qquad \delta := m^a \frac{\partial}{\partial x^a}, \qquad \bar{\delta} := \bar{m}^a \frac{\partial}{\partial x^a}. \tag{3.17}$$

The equations relating the curvature components and the spin coefficients and the commutation relations satisfied by the differential operators (3.17) may be found in NP.

The null tetrad preserving the direction of l is determined up to the subgroup G_4 of proper orthochronous Lorentz transformations $L_+^\uparrow$ defined by

$$\left.\begin{array}{l} l' = e^a l, \quad m' = e^{ib}(m + \bar{q}l), \\ n' = e^{-a}(n + qm + \bar{q}\bar{m} + q\bar{q}l), \end{array}\right\} \tag{3.18}$$

where the functions a and b are real valued and q is complex valued. The corresponding transformation of the spinor dyad $\{ o, \iota \}$ is given by

$$\left.\begin{array}{l} o' = e^{w/2}o, \\ \iota' = e^{-w/2}(\iota + qo), \end{array}\right\} \tag{3.19}$$

where $w = a + ib$. These transformations induce transformations of the spin coefficients and curvature components which will be used later.

Finally by (3.12) the conformal transformation (1.2) is induced by the following transformation of the null tetrad:

$$\tilde{l}_a = l_a, \qquad \tilde{n}_a = e^{2\phi}n_a, \qquad \tilde{m}_a = e^{\phi}m_a. \tag{3.20}$$

Some of the more useful transformations of the spin coefficients induced by (3.20) are

$$\left.\begin{array}{ll} \tilde{\kappa} = e^{-3\phi}\kappa, & \tilde{\rho} = e^{-2\phi}(\rho - \mathrm{D}\phi), \\ \tilde{\sigma} = e^{-2\phi}\sigma, & \tilde{\tau} = e^{-\phi}(\tau - \delta\phi). \end{array}\right\} \tag{3.21}$$

4. PROOF OF THEOREM 1

We begin by expressing Conditions III′ and V′ in spinor form. By contracting Eqs. (1.18) and (1.19) with the appropriate number of σ's and noting that the spinor equivalent of a trace free symmetric tensor is a Hermitian spinor symmetric in its dotted and undotted indices [22] we obtain with the help of Eqs. (3.3) and (3.4)

$$\text{III}'\text{s} \quad \Psi_{ABKL;}{}^{K}{}_{\dot{A}}{}^{L}{}_{\dot{B}} + \overline{\Psi}_{\dot{A}\dot{B}\dot{K}\dot{L};}{}^{\dot{K}}{}_{A}{}^{\dot{L}}{}_{B} + \Psi_{AB}{}^{KL}\Phi_{KL\dot{A}\dot{B}} + \overline{\Psi}_{\dot{A}\dot{B}}{}^{\dot{K}\dot{L}}\Phi_{\dot{K}\dot{L}AB} = 0, \quad (4.1)$$

$$\begin{aligned}\text{V}'\text{s} \quad & k_1\Psi_{ABCD;K\dot{K}}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D};}{}^{\dot{K}K} + k_2\Psi^{K}{}_{(ABC;D)(\dot{A}}\overline{\Psi}_{\dot{B}\dot{C}\dot{D})\dot{L};}{}^{\dot{L}}{}_{K} \\ & + k_2\overline{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C};\dot{D})(A}\Psi_{BCD)L;}{}^{L}{}_{\dot{K}} - 2(8k_1 - k_2)\Psi_{(ABC|K|;}{}^{K}{}_{(\dot{A}}\overline{\Psi}_{\dot{B}\dot{C}\dot{D})\dot{K};}{}^{\dot{K}}{}_{D)} \\ & - k_2\Psi^{K}{}_{(ABC}\overline{\Psi}_{(\dot{A}\dot{B}\dot{C}|\dot{L}|;}{}^{\dot{L}}{}_{|K|D)\dot{D})} - k_2\overline{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C}}\Psi_{(ABC|L|;}{}^{L}{}_{|\dot{K}|D)\dot{D})} \\ & + 4k_1\Psi^{K}{}_{(ABC}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}|\dot{L}|;}{}^{\dot{L}}{}_{D)K\dot{D})} + 4k_1\overline{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C}}\Psi_{(ABC|L|;}{}^{L}{}_{\dot{D})D)\dot{K}} \\ & + 2(k_2 - 4k_1)\Psi^{K}{}_{(ABC}\Phi_{D)K\dot{K}(\dot{A}}\overline{\Psi}_{\dot{B}\dot{C}\dot{D})}{}^{\dot{K}} - 2(4k_1 + k_2)\Lambda\Psi_{ABCD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} = 0. \quad (4.2)\end{aligned}$$

We next make the hypothesis that space-time is of Petrov type N. The condition for this is Eq. (2.1) which in spinor form is equivalent to the existence of a principal null spinor o_A, such that the Weyl spinor has the form

$$\Psi_{ABCD} = \Psi o_A o_B o_C o_D, \quad (4.3)$$

where $\Psi := \Psi_4$.

We select o_A to be the first spinor in a spinor dyad which implies by (3.15) that

$$\Psi_i = 0, \quad (4.4)$$

for all $i = 0, \ldots, 3$. In view of the transformation (3.19) it is possible to choose a spinor dyad satisfying (4.4) in which

$$\Psi = 1. \quad (4.5)$$

For the present we shall assume that this choice of dyad has been made which implies that

$$\Psi_{ABCD} = o_A o_B o_C o_D. \quad (4.6)$$

We proceed by substituting for Ψ_{ABCD} in Eqs. (4.1) and (4.2) from (4.6). The covariant derivatives of o_A and ι_A that appear are eliminated using Eqs. (3.13) and (3.14) respectively. The dyad form of the resulting equations is obtained by contracting them with appropriate products of o^A and ι^A. In view of the conformal invariance of Conditions III′ and V′ [24] [37] or equivalently of III′s and V′s it follows that each dyad equation must be individually invariant under the conformal transformation (3.20).

The first contraction to consider is $o^{ABC}\iota^{D}\bar{o}^{\dot{A}}\bar{\iota}^{\dot{B}\dot{C}\dot{D}}$ with Condition V′s which yields the condition

$$(k_2 - 4k_1)\kappa^2 = 0. \tag{4.7}$$

This implies

$$\kappa = 0, \tag{4.8}$$

since from Table 1 $k_2 \neq 4k_1$. The condition (4.8), which is invariant under the transformations (3.19) and (3.20), implies that the principal null congruence of C_{abcd} defined by the principal null vector field

$$l^a = \sigma^a{}_{A\dot{A}} o^A \bar{o}^{\dot{A}},$$

is *geodesic.*

Before proceeding with the derivation of further dyad equations from III's and V's we exploit the conformal invariance of the problem by using the transformation (3.21) for ρ to set (dropping tildes)

$$\bar{\rho} = -\rho. \tag{4.13}$$

We note that in order to obtain (4.13) the function ϕ must satisfy the linear inhomogeneous partial differential equation

$$D\phi = \frac{1}{2}(\rho + \bar{\rho}), \tag{4.14}$$

which always has a solution.

The next contractions to consider are $o^{AB}\iota^{CD}\bar{\iota}^{\dot{A}\dot{B}\dot{C}\dot{D}}$ and $o^{A}\iota^{BCD}\bar{o}^{\dot{A}}\bar{\iota}^{\dot{B}\dot{C}\dot{D}}$ with V′s and $o^{AB}\bar{\iota}^{\dot{A}\dot{B}}$ with III′s which yield respectively

$$2k_1 D\sigma + \sigma[(12k_1 + k_2)\rho + 2(k_1 - k_2)\varepsilon + 2(17k_1 - k_2)\bar{\varepsilon}] = 0, \tag{4.15}$$

$$(k_2 - 4k_1)[2D(\varepsilon + \bar{\varepsilon}) - 7\rho^2 + 12\rho(\varepsilon - \bar{\varepsilon}) + 6\varepsilon^2 + 6\bar{\varepsilon}^2 + \Phi_{00}] + (15k_2 - 92k_1)\sigma\bar{\sigma} - 4(28k_1 + k_2)\varepsilon\bar{\varepsilon} = 0, \tag{4.16}$$

$$D(4\varepsilon - \rho) - 4\varepsilon\bar{\varepsilon} - 7\varepsilon\rho + \rho^2 - \sigma\bar{\sigma} + \bar{\varepsilon}\rho + 12\varepsilon^2 + 2\sigma^2 + \Phi_{00} = 0. \tag{4.17}$$

Analysis of these equations requires the NP Eqs. (4.2*a*) and (4.2*b*) which in view of (4.4), (4.8) and (4.13) reduce to

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \rho(\varepsilon + \bar{\varepsilon}) + \Phi_{00}, \tag{4.18}$$

$$D\sigma = \sigma(3\varepsilon - \bar{\varepsilon}). \tag{4.19}$$

By eliminating $D\sigma$ between Eqs. (4.15) and (4.19) we obtain

$$\sigma[(12k_1 + k_2)\rho + 2(4k_1 - k_2)\varepsilon + 2(16k_1 - k_2)\bar{\varepsilon}] = 0. \tag{4.20}$$

If we assume

$$\sigma \neq 0 \tag{4.21}$$

Eq. (4.20) implies

$$(12k_1 + k_2)\rho + 2(4k_1 - k_2)\varepsilon + 2(16k_1 - k_2)\bar{\varepsilon} = 0. \tag{4.22}$$

By adding this equation to its complex conjugate we have

$$(10k_1 - k_2)(\varepsilon + \bar{\varepsilon}) = 0\,, \tag{4.23}$$

which implies

$$\bar{\varepsilon} = -\varepsilon\,, \tag{4.24}$$

since $k_2 \neq 10k_1$ in Table 1. Solving (4.22) we obtain

$$\varepsilon = (12k_1 + k_2)\rho/(24k_1)\,. \tag{4.25}$$

We next add Eq. (4.18) to its complex conjugate to obtain

$$\Phi_{00} = -\rho^2 - \sigma\bar{\sigma}\,. \tag{4.26}$$

Finally we substitute for ε and Φ_{00} in (4.16) from (4.25) and (4.26) respectively obtaining

$$k_2(240k_1^2 + 64k_1k_2 + k_2^2)\rho^2 + 72k_1^2(7k_2 - 44k_1)\sigma\bar{\sigma} = 0\,. \tag{4.27}$$

We now observe that the coefficient of $\sigma\bar{\sigma}$ in (4.27) is negative in each of the three cases since from Table 1, $7k_2 < 44k_1$. It thus follows from (4.27) that

$$\sigma = 0\,, \tag{4.28}$$

which contradicts the inequality (4.21). We are hence led to conclude that

$$\sigma = 0\,, \tag{4.29}$$

a condition that remains invariant under the transformations (3.18) and (3.20). The condition (4.29) implies that the principal null congruence of C_{abcd} defined by the vector field l^a is *shear free* [*30*].

We proceed with our analysis by using Eqs. (4.17) to eliminate the term $D(\varepsilon + \bar{\varepsilon})$ from Eq. (4.16). The resulting equation has the form

$$(4k_1 - k_2)[\rho^2 + 2\rho(\bar{\varepsilon} - \varepsilon)] - 16k_1\varepsilon\bar{\varepsilon} = 0\,. \tag{4.30}$$

By completing squares we obtain

$$(4k_1 - k_2)(\rho + \bar{\varepsilon} - \varepsilon)^2 - (4k_1 - k_2)(\varepsilon + \bar{\varepsilon})^2 - 4k_2\varepsilon\bar{\varepsilon} = 0\,. \tag{4.31}$$

It follows from this equation that

$$\rho = 0\,, \tag{4.32}$$

$$\varepsilon = 0\,, \tag{4.33}$$

since by Table 1, $4k_1 > k_2$ for each of our three cases. We note that the condition (4.32) is invariant under the transformation (3.18) but obviously not under the conformal transformation (3.20). However, it is clear from the transformations (3.21) that the condition

$$\rho = \bar{\rho}\,, \tag{4.34}$$

is invariant under (3.20) and that this is the form that (4.32) takes in an arbitrary conformal gauge. On the other hand the condition (4.33) is not invariant under (3.18) since the induced transformation law for ε is

$$\varepsilon' = e^{a}\left(\varepsilon + \frac{1}{2}\,\mathrm{D}w\right). \tag{4.35}$$

From the transformation formula

$$\mathrm{D}'\Psi' + 4\varepsilon'\Psi' = e^{-a-2ib}(\mathrm{D}\Psi + 4\varepsilon\Psi), \tag{4.36}$$

and the choice (4.5) it follows that the form of the condition (4.33) invariant under (3.18) is

$$\mathrm{D}\Psi + 4\varepsilon\Psi = 0. \tag{4.37}$$

We also note that this condition is also invariant under the conformal transformation (3.20).

An important consequence of the conditions (4.8), (4.29), (4.32) and (4.33) is the vanishing of some of the trace free Ricci tensor components. Indeed from NP Eqs. (4.2*a*) and (4.2*k*) we find that

$$\Phi_{00} = \Phi_{01} = 0. \tag{4.38}$$

Further exploitation of conformal invariance is now possible. From the transformations (3.21) we may set

$$\tilde{\tau} = 0, \tag{4.39}$$

while preserving $\tilde{\rho} = 0$, To achieve this, ϕ must satisfy the following system of first order linear partial differential equations

$$\mathrm{D}\phi = 0, \qquad \delta\phi = \tau, \qquad \bar{\delta}\phi = \bar{\tau}. \tag{4.40}$$

In order to establish that this system has a solution we must show that the integrability conditions for (4.40) are satisfied. The relevant commutation relations are given by NP Eqs. (4.4) which read

$$[\delta, \mathrm{D}] = (\bar{\alpha} + \beta - \bar{\pi})\mathrm{D}, \tag{4.41}$$

$$[\bar{\delta}, \mathrm{D}] = (\alpha + \bar{\beta} - \pi)\mathrm{D}, \tag{4.42}$$

$$[\bar{\delta}, \ \delta] = (\bar{\mu} - \mu)\mathrm{D} + (\alpha - \bar{\beta})\delta + (\beta - \bar{\alpha})\bar{\delta}. \tag{4.43}$$

To verify that the first of these relations is satisfied by ϕ we note that by Eqs. (4.38) and (4.40) and NP Eq. (4.2*c*) the left hand side is given by

$$[\delta, \mathrm{D}]\phi = \delta\mathrm{D}\phi - \mathrm{D}\delta\phi = -\,\mathrm{D}\tau = 0, \tag{4.44}$$

while the right hand side becomes

$$(\bar{\alpha} + \beta - \bar{\pi})\mathrm{D}\phi = 0. \tag{4.45}$$

The verification of (4.42) is identical. Turning to (4.43) we note that on the one hand by (4.38), (4.40) and NP Eq. (4.2q)

$$[\bar{\delta}, \delta]\phi = \bar{\delta}\delta\phi - \delta\bar{\delta}\phi = \bar{\delta}\tau - \delta\bar{\tau}$$
$$= \tau(\alpha - \bar{\beta}) - \bar{\tau}(\bar{\alpha} - \beta), \tag{4.46}$$

while on the other hand

$$(\bar{\mu} - \mu)D\phi + (\alpha - \bar{\beta})\delta\phi + (\beta - \bar{\alpha})\bar{\delta}\phi = (\alpha - \bar{\beta})\tau + (\beta - \bar{\alpha})\bar{\tau}. \tag{4.47}$$

Comparison of (4.46) and (4.47) shows that the commutation relation (4.43) is satisfied. We thus conclude that the system (4.40) has a solution and hence that (4.39) holds. We shall assume in the sequel, that we are using a conformal gauge in which (4.39) is true and consequently drop the tildes from all the transformed NP quantities. We note that by Eqs. (4.8), (4.29), and (4.32) the condition $\tau = 0$, is invariant under the tetrad transformation (3.18), but clearly not under the conformal transformation (3.20).

The results obtained at this point may be summarized as follows: *Conditions* III's *and* V's *imply that with respect to any null tetrad* $(l, n, m, \bar{m})$ *in which* l *is a principal null vector of the type* N *Weyl tensor there exists a conformal transformation* ϕ *in which*

$$\kappa = \sigma = \rho = \tau = 0, \tag{4.48}$$
$$D\Psi + 4\varepsilon\Psi = 0, \tag{4.49}$$
$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, \tag{4.50}$$
$$\Phi_{00} = \Phi_{01} = \Phi_{02} = \Lambda = 0. \tag{4.51}$$

The form of the above conditions is dependent on the fact that the tetrad vector l is a principal null vector of the type N Weyl tensor. It is advantageous for what follows to find a spinorial or tensorial form for these conditions. To this end we take the covariant derivative of (4.3) obtaining

$$\Psi_{ABCD;E\dot{E}} = \Psi_{;E\dot{E}}o_A o_B o_C o_D + 4\Psi o_{(A}o_B o_C o_{D);E\dot{E}}. \tag{4.52}$$

We then substitute for $o_{D;E\dot{E}}$ using (3.13) and (4.48) to get

$$\Psi_{ABCD;E\dot{E}} = \Psi_{ABCD}K_{E\dot{E}}, \tag{4.53}$$

where

$$K_{E\dot{E}} = \Psi^{-1}\Psi_{;E\dot{E}} + 4\gamma o_E\bar{o}_{\dot{E}} - 4\alpha o_E\bar{\iota}_{\dot{E}} - 4\beta\iota_E\bar{o}_{\dot{E}} + 4\varepsilon\iota_E\bar{\iota}_{\dot{E}}. \tag{4.54}$$

The spinor equation (4.53) is equivalent to the tensor equation

$$\overset{+}{C}_{abcd;e} = \overset{+}{C}_{abcd}K_e, \tag{4.55}$$

where

$$\overset{+}{C}_{abcd} = \frac{1}{2}(C_{abcd} - i^*C_{abcd}), \tag{4.56}$$
$$K_e = \sigma_e{}^{E\dot{E}}K_{E\dot{E}}. \tag{4.57}$$

The tensor $^*C_{abcd}$ in (4.56) is the dual of C_{abcd} defined by

$$^*C_{abcd} = \frac{1}{2}\,\varepsilon_{abef}C^{ef}{}_{ab}\,, \tag{4.58}$$

where ε_{abcd} is the Levi-Civita tensor.

We now observe that (4.55) is the defining equation of the *complex recurrent* space-times [23]. We are thus able to restate our result in the following tensorial form: *Any Petrov type* N *space-time satisfying Conditions* III′ *and* V′ *is conformally related to a complex recurrent space-time.* The required conformal factor, which is identical with that appearing in Eq. (2.2) of Theorem 1, is given by $e^{-2\phi}$, where ϕ is the function satisfying Eqs. (4.14) and (4.40), whose existence has just been established. We thus have obtained a generalization of a similar result which holds in the vacuum case (see Theorem 1 of [22]). However, the similarity with the vacuum case ends here since the vacuum complex recurrent space-times are necessarily plane wave while the Petrov type N complex recurrent space-times includes a much wider class of metrics [23].

We proceed with the proof by using the fact that the Weyl spinor satisfies (4.53) to extract the remaining information from Conditions III′s and V′s. We drop the assumption (4.5) and work in a general spinor dyad with o_A a principal null spinor of Ψ_{ABCD}. It follows from (4.3), (4.49), (4.53), and (4.54) that

$$\Psi_{ABCD;E\dot{E}} = o_{ABCD}(\Theta_1 o_E \bar{o}_{\dot{E}} + \Theta_2 o_E \bar{\iota}_{\dot{E}} + \Theta_3 \iota_E \bar{o}_{\dot{E}})\,, \tag{4.59}$$

where

$$\Theta_1 := \Delta\Psi + 4\gamma\Psi\,, \tag{4.60}$$

$$\Theta_2 := -\,(\bar{\delta}\Psi + 4\alpha\Psi)\,, \tag{4.61}$$

$$\Theta_3 := -\,(\delta\Psi + 4\beta\Psi)\,. \tag{4.62}$$

For future use we need the transformation law for Θ_3 induced by (3.19) which is

$$\Theta_3' = e^{-2a-ib}\Theta_3\,. \tag{4.63}$$

By contracting (4.59) with ε^{DC} we obtain

$$\Psi_{ABCL;}{}^{L}{}_{\dot{E}} = \Theta_3 o_{ABC}\bar{o}_{\dot{E}}\,. \tag{4.64}$$

Covariant differentiation of this equation yields

$$\Psi_{ABCL;}{}^{L}{}_{\dot{E}F\dot{F}} = o_{ABC}\bar{o}_{\dot{E}}(\Sigma_8 o_F \bar{o}_{\dot{F}} + \Sigma_9 o_F \bar{\iota}_{\dot{F}} + \Sigma_{10}\iota_F \bar{o}_{\dot{F}})\,, \tag{4.65}$$

where

$$\Sigma_8 := \Delta\Theta_3 + (3\gamma + \bar{\gamma})\Theta_3\,, \tag{4.66}$$

$$\Sigma_9 := -\,[\bar{\delta}\Theta_3 + (3\alpha + \bar{\beta})\Theta_3]\,, \tag{4.67}$$

$$\Sigma_{10} := -\,[\delta\Theta_3 + (3\beta + \bar{\alpha})\Theta_3]\,, \tag{4.68}$$

The function

$$\Sigma_{11} := D\Theta_3 + (3\varepsilon + \bar{\varepsilon})\Theta_3\,, \tag{4.69}$$

that one might have expected to appear in (4.65) as the coefficient of $\iota_F \bar{\iota}_{\dot{F}}$, is identically zero. This follows from the transformation law

$$\Sigma'_{11} = e^{-w}\Sigma_{11}, \tag{4.70}$$

induced by (3.19), and the fact that in a tetrad for which (4.5) holds

$$\Sigma_{11} = -4D\beta = 0, \tag{4.71}$$

by Eqs. (4.33), (4.62) and NP Eq. (4.2*e*). We also need the formula

$$\Phi_{AB\dot{A}\dot{B}} = \Phi_{22} o_{AB} \bar{o}_{\dot{A}\dot{B}} - 2\Phi_{21} o_{AB} \bar{o}_{(\dot{A}} \bar{\iota}_{\dot{B})} - 2\Phi_{12} o_{(A} \iota_{B)} \bar{o}_{\dot{A}\dot{B}} + 4\Phi_{11} o_{(A}\iota_{B)} \bar{o}_{(\dot{A}} \bar{\iota}_{\dot{B})}, \tag{4.72}$$

which follows from (4.51).

In view of the Eqs. (4.59), (4.65) and (4.72) it is an easy matter to show that the only remaining information in Conditions III's and V's is expressed by the following equations:

$$\Sigma_{10} + \bar{\Sigma}_{10} = 0, \tag{4.73}$$

$$4k_1\bar{\Psi}\Sigma_9 + 4k_1\Psi\bar{\Sigma}_9 + k_1\Theta_2\bar{\Theta}_2 + (17k_1 - 2k_2)\Theta_3\bar{\Theta}_3 + 2(4k_1 - k_2)\Psi\bar{\Psi}\Phi_{11} = 0. \tag{4.74}$$

In order to solve these equations we need a canonical coordinate system for the type N complex recurrent space-times. Such a coordinate system $(u, v, z, \bar{z})$ has been provided by McLenaghan and Leroy with respect to which the metric of any type N space-time satisfying (4.55) has the form

$$\begin{aligned} ds^2 = {} & 2dv\,[du + (ek^2(v)u^2 + l(v, z, \bar{z})u + m(v, z, \bar{z}))dv] \\ & - 2k^{-2}(v)(1 + ez\bar{z})^{-2}(dz + p(v, z)dv)(d\bar{z} + \bar{p}(v, \bar{z})dv), \end{aligned} \tag{4.75}$$

where

$$e = -1, 0, 1, \tag{4.76}$$

$$k^2(v) = 1 + e^2 K^2(v), \tag{4.77}$$

$$l(v, z, \bar{z}) = \frac{1}{2}(p_z(v, z) + \bar{p}_{\bar{z}}(v, \bar{z})) - e(1 + ez\bar{z})^{-1}(\bar{z}p(v, z) + z\bar{p}(v, \bar{z})). \tag{4.78}$$

A canonical null tetrad for (4.75) in which l^a is a principal null vector of the type N Weyl tensor is defined by the following choice of NP operators [*23*]:

$$\begin{aligned} & D = \partial/\partial u, \quad \delta = -k(v)(1 + ez\bar{z})\partial/\partial\bar{z} \\ & \Delta = \partial/\partial v - p(v, z)\partial/\partial z - \bar{p}(v, \bar{z})\partial/\partial\bar{z} - (ek^2(v)u^2 + l(v, z, \bar{z})u + m(v, z, \bar{z}))\partial/\partial u. \end{aligned} \tag{4.79}$$

The non-vanishing NP spin coefficients and curvature components corresponding to this tetrad are

$$\alpha(v, \bar{z}) = -\frac{1}{2} ek(v)\bar{z}, \qquad \beta(v, z) = \frac{1}{2} ek(v)z, \tag{4.80}$$

$$\gamma(v, z, \bar{z}) = ek^2(v)u - \frac{1}{2}p_z(v, z) + \frac{1}{2}e(1+ez\bar{z})^{-1}(\bar{z}p(v, z) + z\bar{p}(v, \bar{z})), \tag{4.81}$$

$$\nu(v, z, \bar{z}) = -k(v)(1 + ez\bar{z})(l_z(v, z, \bar{z})u + m_z(v, z, \bar{z})), \tag{4.82}$$

$$\mu(v, z, \bar{z}) = -(k^{-1}(v)k'(v) + l(v, z, \bar{z})), \tag{4.83}$$

$$\Psi(v, z, \bar{z}) = k^2(v)(1+ez\bar{z})\left[\frac{1}{2}u(1+ez\bar{z})p_{zzz}(v, z) + (1+ez\bar{z})m_{zz}(v, z, \bar{z}) + 2e\bar{z}m_z(v, z, \bar{z})\right], \tag{4.84}$$

$$\Phi_{11}(v) = ek^2(v), \quad \Phi_{21}(v, z, \bar{z}) = -k(v)(1 + ez\bar{z})l_z(v, z, \bar{z}), \tag{4.85}$$

$$\begin{aligned}\Phi_{22}(v, z, \bar{z}) = {} & (k^{-1}k')'(v) - p(v, z)l_z(v, z, \bar{z}) - \bar{p}(v, \bar{z})l_{\bar{z}}(v, z, \bar{z}) + l_v(v, z, \bar{z}) \\ & + k^2(v)(1+ez\bar{z})^2 m_{z\bar{z}}(v, z, \bar{z}) + 2ek(v)k'(v)u \\ & - k^{-1}(v)k'(v)(k^{-1}(v)k'(v) + l(v, z, \bar{z})).\end{aligned} \tag{4.86}$$

We begin this stage of the proof by imposing the condition (4.49). On account of (4.79) and (4.84) this condition may be expressed as

$$p_{zzz}(v, z) = 0. \tag{4.87}$$

The general solution of this equation is given by

$$p(v, z) = p_2(v)z^2 + p_1(v)z + p_0(v), \tag{4.88}$$

where p_i, $i = 0, 1, 2$, are arbitrary functions. This establishes the form (2.3) of Theorem 1.

We next examine the condition (4.73). In view of (4.62) and (4.68) we may write

$$\Sigma_{10} = \delta^2\Psi + (7\beta + \bar{\alpha})\delta\Psi + 4\Psi[\delta\beta + \beta(\bar{\alpha} + 3\beta)]. \tag{4.89}$$

Substitution for δ, α, and β in the above from (4.79) and (4.80) yields

$$\Sigma_{10} = k^2[(1 + ez\bar{z})^2\Psi_{\bar{z}\bar{z}} - 2ez(1 + ez\bar{z})\Psi_{\bar{z}} + 2e^2z^2\Psi]. \tag{4.90}$$

Finally by eliminating Ψ using (4.84) we obtain on noting (4.87)

$$\Sigma_{10} = k^4(1+ez\bar{z})^3[(1+ez\bar{z})m_{zz\bar{z}\bar{z}} + 2e(zm_{zz\bar{z}} + \bar{z}m_{z\bar{z}\bar{z}}) + 4em_{z\bar{z}}]. \tag{4.91}$$

From this equation we observe that

$$\bar{\Sigma}_{10} = \Sigma_{10}. \tag{4.92}$$

It follows that the condition (4.73) may be expressed as

$$\Sigma_{10} = 0, \tag{4.93}$$

an equation which is invariant under a general tetrad transformation (3.18) and conformal transformation (3.20). By (4.90) the above condition may be expressed as

$$(1 + ez\bar{z})^2\Psi_{\bar{z}\bar{z}} - 2ez(1 + ez\bar{z})\Psi_{\bar{z}} + 2e^2z^2\Psi = 0. \qquad (4.94)$$

The general solution of this equation is given by

$$\Psi(v, z, \bar{z}) = (1 + ez\bar{z})(\bar{z}\mathrm{A}(v, z) + \mathrm{B}(v, z)), \qquad (4.95)$$

where A and B are arbitrary functions. An alternate form of the solution to (4.93) in the case $e = 0$, is given by

$$m(v, z, \bar{z}) = \bar{z}\mathrm{G}(v, z) + z\overline{\mathrm{G}}(v, \bar{z}) + \mathrm{H}(v, z) + \overline{\mathrm{H}}(v, \bar{z}) \qquad (4.96)$$

where G and H are arbitrary functions. This result, which is the form (2.4) of Theorem 1, is obtained using (4.91). When $e = 0$, the function Ψ given by (4.84) has the form

$$\Psi(v, z, \bar{z}) = m_{zz}(v, z, \bar{z}). \qquad (4.97)$$

Comparison of (4.95) and (4.97) yields on account of (4.94)

$$\mathrm{G}_{zz}(v, z) = \mathrm{A}(v, z), \qquad \mathrm{H}_{zz}(v, z) = \mathrm{B}(v, z). \qquad (4.98)$$

The last equation to be solved is (4.74). In view of Eqs. (4.61), (4.62), (4.67), (4.79), and (4.95) this equation has the form

$$\begin{aligned}&(1+ez\bar{z})^2\,[4k_1(z\bar{\mathrm{A}}+\bar{\mathrm{B}})\mathrm{A}'+4k_1(\bar{z}\mathrm{A}+\mathrm{B})\bar{\mathrm{A}}'+k_1(\bar{z}\mathrm{A}'+\mathrm{B}')(z\bar{\mathrm{A}}'+\bar{\mathrm{B}}')\\&\quad+(17k_1-2k_2)\mathrm{A}\bar{\mathrm{A}}]+e(1+ez\bar{z})[(2k_2-5k_1)\bar{z}(z\bar{\mathrm{A}}+\bar{\mathrm{B}})\mathrm{A}+(2k_2-5k_1)z(\bar{z}\mathrm{A}+\mathrm{B})\bar{\mathrm{A}}\\&\quad-k_1z(z\bar{\mathrm{A}}+\bar{\mathrm{B}})(\bar{z}\mathrm{A}'+\mathrm{B}')-k_1\bar{z}(\bar{z}\mathrm{A}+\mathrm{B})(z\bar{\mathrm{A}}'+\bar{\mathrm{B}}')]\\&\quad-2e\,[k_2+e(k_2-k_1)z\bar{z}\,](\bar{z}\mathrm{A}+\mathrm{B})(z\bar{\mathrm{A}}+\bar{\mathrm{B}})=0,\end{aligned} \qquad (4.99)$$

where «$'$» denotes the partial derivative with respect to z or $\bar{z}$. In order to solve (4.99) it is convenient to distinguish the cases $e = 0$ and $e \neq 0$.

CASE $e = 0$. — The Eq. (4.99) reduces to

$$4(z\bar{\mathrm{A}}+\bar{\mathrm{B}})\mathrm{A}'+4(\bar{z}\mathrm{A}+\mathrm{B})\bar{\mathrm{A}}'+(\bar{z}\mathrm{A}'+\mathrm{B}')(z\bar{\mathrm{A}}'+\bar{\mathrm{B}}')+(17-2k_2/k_1)\mathrm{A}\bar{\mathrm{A}}=0. \qquad (4.100)$$

If

$$\mathrm{A}' = 0, \qquad (4.101)$$

the above equation reduces to

$$\mathrm{B}'\bar{\mathrm{B}}' + (17 - 2k_2/k_1)\mathrm{A}\bar{\mathrm{A}} = 0, \qquad (4.102)$$

which implies

$$\mathrm{A} = 0, \qquad (4.102)$$

$$\mathrm{B}' = 0, \qquad (4.103)$$

since from Table 1, $2k_2/k_1 < 17$, for each of the three cases. It thus follows from (4.98) that the functions G and H have the form

$$G(v, z) = g_1(v)z + g_0(v), \tag{4.104}$$

$$H(v, z) = h_2(v)z^2 + h_1(v)z + h_0(v), \tag{4.105}$$

where g_i, $i = 0, 1$, and $h_2 \neq 0$, h_i, $i = 0, 1$, are arbitrary functions. By (4.96) and the above

$$m(v, z, \bar{z}) = D(v)z^2 + \bar{D}(v)\bar{z}^2 + e(v)z\bar{z} + F(v)z + \bar{F}(v)\bar{z} + g(v), \tag{4.106}$$

where

$$D(v) := h_2(v), \tag{4.107}$$

$$e(v) := g_1(v) + \bar{g}_1(v), \tag{4.108}$$

$$F(v) := \bar{g}_0(v) + h_1(v), \tag{4.109}$$

$$g(v) := h_0(v) + \bar{h}_0(v). \tag{4.110}$$

By means of coordinate transformations which preserve the form of the metric defined by Eqs. (4.75) to (4.78) and the form of the functions p and m given by (4.88) and (4.106) respectively it is possible to set [23]

$$p_0 = p_1 = 0, \qquad \bar{p}_2 = p_2, \tag{4.111}$$

and

$$g = 0. \tag{4.112}$$

The resulting form of m may be obtained from (4.107) to (4.110) without loss of generality by setting

$$h_0 = h_1 = 0, \tag{4.113}$$

$$\bar{g}_1 = g_1. \tag{4.114}$$

The above results justify the form (2.5) *of Theorem 1, which as already noted, yields the generalized plane wave metric.*

We now assume

$$A' \neq 0, \tag{4.115}$$

Dividing Eq. (4.100) by $A'\bar{A}'$ we obtain

$$4z(\bar{A}/\bar{A}') + 4(\bar{B}/\bar{A}') + 4\bar{z}(A/A') + 4(B/A') + z\bar{z} + \bar{z}(\bar{B}'/\bar{A}') + z(B'/A')$$
$$+ (B'/A')(\bar{B}'/\bar{A}') + (17 - 2k_2/k_1)(A/A')(\bar{A}/\bar{A}') = 0. \tag{4.116}$$

Taking $\partial^4/\partial z^2\partial\bar{z}^2$ of this equation yields

$$(B'/A')''(\bar{B}'/\bar{A}')'' + (17 - 2k_2/k_1)(A/A')''(\bar{A}/\bar{A}')'' = 0. \tag{4.117}$$

This equation implies

$$(A/A')'' = 0, \tag{4.118}$$

$$(B'/A')'' = 0, \tag{4.119}$$

since, as previously noted, $2k_2/k_1 < 17$, in each case. The general solutions of (4.118) and (4.119) are

$$(A/A')(v, z) = a_1(v)z + a_0(v), \qquad (4.120)$$

$$(B'/A')(v, z) = b_1(v)z + b_0(v), \qquad (4.121)$$

where a_0, a_1, b_0, and b_1 are functions to be determined. In particular they must satisfy the following equation

$$(17-2k_2/k_1)\,|\,a_1(v)\,|^2+4(a_1(v)+\bar{a}_1(v))+|\,b_1(v)\,|^2+1=0, \qquad (4.122)$$

which is obtained by substituting from (4.120) and (4.121) into the equation which results from taking $\partial^2/\partial z\partial\bar{z}$ of Eq. (4.116).

In order to simplify the solution of (4.120) and (4.121) for A and B we recall that the complex recurrent metric defined by (4.75) to (4.78) is form invariant when $e = 0$, under the coordinate transformation

$$u = \hat{u}, \qquad v = \hat{v}, \qquad z = \hat{z} + k(\hat{v}), \qquad (4.123)$$

where k is arbitrary [23]. This transformation preserves the form (4.88) of p, and induces the following transformation of A and B:

$$\hat{A}(\hat{v}, \hat{z}) = A(\hat{v}, \hat{z} + k(\hat{v})), \qquad (4.124)$$

$$\hat{B}(\hat{v}, \hat{z}) = B(\hat{v}, \hat{z} + k(\hat{v})) + \bar{k}(\hat{v})A(\hat{v}, \hat{z}+k(\hat{v})), \qquad (4.125)$$

It follows from these equations that Eq. (4.100) is invariant under the transformation (4.123). We are thus able to conclude that the functions a_0 and a_1 of (4.120) transform as

$$\hat{a}_1(\hat{v}) = a_1(\hat{v}), \qquad (4.126)$$

$$\hat{a}_0(\hat{v}) = a_0(\hat{v}) + a_1(\hat{v})k(\hat{v}). \qquad (4.127)$$

Since by (4.122), $a_1 \neq 0$, it follows from the above that we may choose k such that $\hat{a}_0 = 0$. Dropping the hats from the transformed quantities we may write (4.120) as

$$(A/A')(v, z) = a_1(v)z. \qquad (4.128)$$

The general solution of this equation is

$$A(v, z) = a_2(v)(a_1(v)z)^{1/a_1(v)}, \qquad (4.129)$$

where $a_2 \neq 0$, is an arbitrary function. Integrating by parts we find that the general solution of (4.121) has the form

$$B(v, z) = a_2(v)(1+a_1(v))^{-1}[b_1(v)z+b_0(v)(1+a_1(v))](a_1(v)z)^{1/a_1(v)}+b_2(v), \qquad (4.130)$$

where b_2 is a function to be determined. By substituting for A and B from (4.129) and (4.130) into (4.100) we obtain an equation of the form

$$b_2(v)a_1^{-1}(v)(a_1(v)z)^{1-1/a_1(v)} + P(v, z, \bar{z}) = 0, \qquad (4.131)$$

where P is a polynomial of degree less than three in z and in $\bar{z}$. Now it follows from (4.122) that $1-1/a_1(v) \neq 0$, 1, or 2. Thus by taking $\frac{\partial^3}{\partial z^3}$ of (4.131) we obtain

$$b_2(v)a_2^{-1}(v)(a_1(v)-1)(a_1(v)+1)[a_1(v)z]^{-2-1/a_1(v)}=0, \qquad (4.132)$$

from which it follows that

$$b_2 = 0, \qquad (4.133)$$

since $a_1(v) = 1$, or -1, by (4.122). By now equating the coefficients of the polynomial P to zero we find the additional conditions

$$b_1(v)(5a_1(v) + 1) = 0, \qquad (4.134)$$

$$b_0(v)(4a_1(v)+1) + \bar{b}_0(v)b_1(v) = 0, \qquad (4.135)$$

$$|b_0(v)|^2 = 0. \qquad (4.136)$$

The last of the above equations implies

$$b_0 = 0, \qquad (4.137)$$

while the first equation implies

$$b_1 = 0, \qquad (4.138)$$

or

$$a_1(v) = -1/5. \qquad (4.139)$$

If (4.138) holds the solutions for A and B are given by (4.129) and

$$B = 0,$$

where the function a_1, according to (4.122), must satisfy

$$(17 - 2k_2/k_1)|a_1(v)|^2 + 4(a_1(v) + \bar{a}_1(v)) + 1 = 0, \qquad (4.140)$$

which by Table 1 has solutions for each of the three cases. The above forms of A and B combined with (4.98) establishes Eqs. (2.6), (2.7) and (2.9) of Theorem 1.

If (4.139) is satisfied the functions A and B are given by

$$A(v, z) = a_2(v)(-z/5)^{-5}, \qquad (4.141)$$

$$B(v, z) = (5/4)a_2(v)b_1(v)(-z/5)^{-5}z, \qquad (4.142)$$

where the function b_1, by (4.122), must satisfy

$$|b_1(v)|^2 = (2/25)(k_2/k_1 - 1), \qquad (4.143)$$

which has solutions for each of the three cases since by Table 1, $k_2/k_1 > 1$. *The above forms for* A *and* B *in conjunction with* (4.98) *justify Eqs.* (2.6) (2.7), *and* (2.8) *of Theorem 1.*

CASE $e \neq 0$. — *We will show that Eq.* (4.99) *has no solution in this case.* If

$$A = 0, \qquad (4.144)$$

Eq. (4.99) reduces to

$$(1 + ez\bar{z})^2 B'\bar{B}' - ez(1 + ez\bar{z})\bar{B}B' - e\bar{z}(1 + ez\bar{z})B\bar{B}' - 2e[k_2/k_1 + e(k_2/k_1 - 1)z\bar{z}]B\bar{B} = 0. \quad (4.145)$$

If $B' = 0$, the above equation implies $B = 0$, which by (4.95) implies $\Psi = 0$, which is impossible. Thus we must have

$$B' \neq 0. \quad (4.146)$$

Dividing Eq. (4.145) by $B'\bar{B}'$ we have

$$(1 + ez\bar{z})^2 - ez(1 + ez\bar{z})(\bar{B}/\bar{B}') - e\bar{z}(1 + ez\bar{z})(B/B') - 2e[k_2 k_1 + e(k_2/k_1 - 1)z\bar{z}](B/B')(\bar{B}/\bar{B}') = 0 \quad (4.147)$$

Taking $\partial^6/\partial z^3 \partial\bar{z}^3$ of the above we obtain

$$|(B/B')'''|^2 + e(1 - k_1/k_2)|(zB/B')'''|^2 = 0. \quad (4.147\,a)$$

When $e = 1$, this equation implies, since $k_1/k_2 < 1$

$$(B/B')''' = 0, \quad (4.148)$$

$$(zB/B')''' = 0. \quad (4.149)$$

The general solution of these equations is given by

$$(B/B')(v, z) = d(v)z^2 + f(v)z + g(v), \quad (4.150)$$

$$z(B/B')(v, z) = h(v)z^2 + j(v)z + l(v), \quad (4.151)$$

where d, f, g, h, j, and l are functions of integration. For consistency we must have

$$d = l = 0, \qquad h = f, \qquad j = g. \quad (4.152)$$

It follows that

$$(B/B')(v, z) = f(v)z + g(v). \quad (4.153)$$

We proceed by substituting for B/B', from the above in (4.147), which yields

$$(1 + ez\bar{z})^2 - e\bar{z}(1 + ez\bar{z})(fz + g) - ez(1 + ez\bar{z})(\bar{f}\bar{z} + \bar{g}) - 2e(k_2/k_1)[1 + e(1 - k_1/k_2)z\bar{z}](fz + g)(\bar{f}\bar{z} + \bar{g}) = 0. \quad (4.154)$$

The vanishing of the constant term in the above equation implies

$$2e(k_2/k_1)g\bar{g} = 1, \quad (4.155)$$

from which it follows (when $e = 1$) that

$$g \neq 0. \quad (4.156)$$

It thus follows that the vanishing of the coefficients of z and $z^2\bar{z}$ in (4.154) implies

$$f = -k_1/(2k_2), \quad (4.157)$$

$$f = -k_1/(2(k_2 - k_1)). \quad (4.158)$$

These equations imply $k_1 = 0$, which by Table 1 is impossible. We thus conclude that Eq. (4.147) has no solutions when $e = 1$. For future use we note that Eq. (4.155) implies that Eq. (4.147) has no solution of the form (4.153) when $e = -1$.

We now consider the case $e = -1$, when (4.147) may be written as

$$(B/B')'''(\overline{B}/\overline{B}')''' - (1 - k_1/k_2)(zB/B')'''(\overline{z}\overline{B}/\overline{B}')''' = 0 . \quad (4.159)$$

We may assume

$$(B/B')''' \neq 0 , \quad (4.160)$$

which by (4.159) is equivalent to

$$(zB/B')''' \neq 0 , \quad (4.161)$$

since the case $(B/B')''' = 0$, which is equivalent to $(zB/B')''' = 0$, has already been shown to be impossible. In view of this remark we may separate the variables in (4.159) yielding

$$(B/B')'''/(zB/B')''' = (1 - k_1/k_2)(\overline{z}\overline{B}/\overline{B}')'''/(\overline{B}/\overline{B}')''' = k , \quad (4.162)$$

where k is a separation function. For consistency it follows that k must satisfy

$$| k(v) |^2 = 1 - k_1/k_2 , \quad (4.163)$$

which has solutions since $k_1 < k_2$ by Table 1. The general solution of (4.162) is given by

$$(B/B')(v, z) = (d(v)z^2 + f(v)z + g(v))/(1 - k(v)z) , \quad (4.164)$$

where d, f, and g are functions of integration. When this form of B/B' is substituted into (4.147) a polynomial equation in z and $\overline{z}$ is obtained after multiplication by $(1-kz)(1-\overline{k}\overline{z})$. The vanishing of the term independent of z and $\overline{z}$ in this equation yields

$$2(k_2/k_1) | g(v) |^2 + 1 = 0 , \quad (4.165)$$

which has no solutions. We thus conclude that Eq. (4.147) has no solution when $e = -1$. Combining the results just proven for $e = 1$ and -1, *we conclude that Eq.* (4.99) *has no solutions when* $A = 0$.

We now consider the possibility

$$A \neq 0 . \quad (4.166)$$

Division of Eq. (4.99) by $A\overline{A}$ yields

$$(1+ez\overline{z})^2 [4(z+\overline{P})N+4(\overline{z}+P)\overline{N}+(\overline{z}N+Q)(z\overline{N}+\overline{Q})+(17-2k_2/k_1)]$$
$$+e(1+ez\overline{z})[(2k_2/k_1-5)(\overline{z}(z+\overline{P})+z(\overline{z}+P))-z(z+\overline{P})(\overline{z}N+Q)$$
$$-\overline{z}(\overline{z}+P)(z\overline{N}+\overline{Q})]-2e(k_2/k_1)[1+e(1-k_1/k_2)z\overline{z}](\overline{z}+P)(z+\overline{P})=0, \quad (4.167)$$

where

$$N := A'/A, \qquad P := B/A, \qquad Q = B'/A. \tag{4.168}$$

By a process of successive division and differentiation, similar but more elaborate than that used in the case $A = 0$, the details of which are given in the Appendix, it is possible after a finite number of steps to separate the variables in Eq. (4.167). Integrating back the required number of times it is possible to show *that the functions* N, P, *and* Q *are rational functions of* z. We may thus write

$$N = N_1/K, \qquad P = P_1/K, \qquad Q = Q_1/K, \tag{4.169}$$

where K, N_1, P_1, and Q_1 are polynomials in z whose coefficients are functions of v and K is the common denominator. Substituting for N, P, and Q in (4.167) from (4.169) and multiplying the resulting equation by $K\overline{K}$ we obtain an equation which may be written as

$$2e(k_2/k_1)[1+e(1-k_1/k_2)z\bar{z}](\bar{z}K+P_1)(z\overline{K}+\overline{P}_1)=(1+ez\bar{z})S. \tag{4.170}$$

where S is a polynomial in z and $\bar{z}$. It follows from this equation that $1+ez\bar{z}$ must be a factor of the polynomial $\bar{z}K+P_1$, since $1+ez\bar{z}$ does not divide the polynomial $1+e(1-k_1/k_2)z\bar{z}$, for any of the values of k_1 and k_2 permitted by Table 1. Thus we must have

$$P_1(z) + \bar{z}K(z) = (1 + ez\bar{z})T(z), \tag{4.171}$$

where T is some polynomial in z. It follows from this equation that

$$K(z) = ezT(z), \tag{4.172}$$

$$T(z) = P_1(z). \tag{4.173}$$

Thus we have

$$K(z) = ezP_1(z), \tag{4.174}$$

which by (4.169) implies that

$$P(z) = 1/(ez). \tag{4.175}$$

Employing the definitions (4.168) we find that

$$A(z) = ezB(z). \tag{4.176}$$

We now use (4.176) to eliminate A from Eq. (4.99). The resulting equation, when simplified, has the following form:

$$4e(1+ez\bar{z})(z\overline{B}B'+\bar{z}B\overline{B}')+(1+ez\bar{z})^2B'\overline{B}'+2e[(4-k_2/k_1)+8ez\bar{z}]B\overline{B} = 0. \tag{4.177}$$

We proceed by dividing the above equation by $B\overline{B}$, since $B \neq 0$. This yields

$$\begin{aligned}4ezB'/B+4e\bar{z}\overline{B}'/\overline{B}+4e^2\bar{z}z^2B'/B+4e^2z\bar{z}^2\overline{B}'/\overline{B}\\ +(B'/B)(\overline{B}'/\overline{B})+2e(zB'/B)(\bar{z}\overline{B}'/\overline{B})+e^2(z^2B'/B)(\bar{z}^2\overline{B}'/\overline{B})\\ +2e(4-k_2/k_1)+16e^2z\bar{z}=0.\end{aligned} \tag{4.178}$$

Taking $\partial^4/\partial z^2\partial\bar{z}^2$ of this equation yields

$$(B'/B)''(\bar{B}'/\bar{B})''+2e(zB'/B)''(\bar{z}\bar{B}'/\bar{B})''+e^2(z^2B'/B)''(\bar{z}^2\bar{B}'/\bar{B})''=0. \quad (4.179)$$

If $e = 1$, the above equation implies

$$(B'/B)'' = 0,$$

$$(zB'/B)'' = 0, \quad (4.180)$$

$$(z^2B'/B)'' = 0. \quad (4.181)$$

The general solution of these equations is

$$(B'/B)(v, z) = c(v)z + d(v), \quad (4.182)$$

$$z(B'/B)(v, z) = f(v)z + g(v), \quad (4.183)$$

$$z^2(B'/B)(v, z) = h(v)z + j(v), \quad (4.184)$$

where c, d, f, g, h, and j are functions of integration. For consistency we must have

$$c = d = f = g = h = j = 0, \quad (4.185)$$

from which it follows that

$$B'/B = 0. \quad (4.186)$$

It follows that Eq. (4.178) reduces to

$$4 - k_2/k_1 + 8ez\bar{z} = 0, \quad (4.187)$$

which is impossible. We conclude that Eq. (4.177) has no solutions when $e = 1$.

We now consider the possibility $e = -1$. We first note that

$$(zB'/B)'' = 0, \quad (4.188)$$

and Eq. (4.179) imply that

$$(B'/B)'' = (z^2B'/B)'' = 0, \quad (4.189)$$

which is the case just considered. Thus we assume

$$(zB'/B)'' \neq 0. \quad (4.190)$$

We proceed by dividing both sides of Eq. (4.179) by $(zB'/B)''$ $(\bar{z}\bar{B}'/\bar{B})''$ to obtain

$$[(B'/B)''/(zB'/B)''][(\bar{B}'/\bar{B})''/(\bar{z}\bar{B}'/\bar{B})''] - 2$$
$$[(z^2B'/B)''/(zB'/B)''][(\bar{z}^2\bar{B}'/\bar{B})''/(\bar{z}\bar{B}'/\bar{B})''] = 0. \quad (4.191)$$

Taking the derivative $\partial^2/\partial z\partial\bar{z}$ of this equation gives

$$[(B'/B)''/(zB'/B)'']'[(\bar{B}'/\bar{B})''/(\bar{z}\bar{B}'/\bar{B})'']'$$
$$+ [(z^2B'/B)''/(zB'/B)'']'[(\bar{z}^2\bar{B}'/\bar{B})''/(\bar{z}\bar{B}'/\bar{B})'']' = 0, \quad (4.192)$$

from which it follows that

$$[(B'/B)''/(zB'/B)'']' = 0, \tag{4.193}$$

$$[(z^2B'/B)''/(zB'/B)'']' = 0. \tag{4.194}$$

Integrating these equations we obtain

$$(B'/B)''(v, z) = a(v)(zB'/B)''(v, z), \tag{4.195}$$

$$(z^2B'/B)''(v, z) = b(v)(zB'/B)''(v, z), \tag{4.196}$$

where a and b are functions of integration which by (4.191) must satisfy

$$| a(v) |^2 + | b(v) |^2 = 2. \tag{4.197}$$

Further integration of Eqs. (4.195) and (4.196) yields

$$(1 - a(v)z)(B'/B)(v, z) = c(v)z + d(v), \tag{4.198}$$

$$z(z - b(v))(B'/B)(v, z) = f(v)z + g(v), \tag{4.199}$$

where c, d, f, and g are additional functions of integration. Consistency of the above equations requires

$$c = g = 0, \qquad d = -af, \qquad f = -bd, \tag{4.200}$$

from which it follows that

$$d(1 - ab) = 0. \tag{4.201}$$

If $d = 0$, Eq. (4.200) implies $f = 0$, from which it follows by (4.198) that $B'/B = 0$, which violates (4.190). If $d \neq 0$, Eq. (4.201) implies

$$b = 1/a, \tag{4.202}$$

which in turn implies by (4.197) that

$$| a(v) |^2 = | b(v) |^2 = 1. \tag{4.203}$$

We thus have from (4.198)

$$(B'/B)(v, z) = d(v)/(1 - a(v)z). \tag{4.204}$$

We now substitute for B'/B in (4.178) from the above and multiply the resulting equation by $| 1 - a(v)z |^2$. The vanishing of the term independent of z and $\bar{z}$ and of the coefficient of $z^2\bar{z}$ in this polynomial equation yields

$$d(v)\bar{d}(v) = 2(4 - k_2/k_1), \tag{4.205}$$

$$d(v) = 4a(v). \tag{4.206}$$

These equations and (4.203) imply that

$$k_2 = -4k_1,$$

which by Table 1 is impossible. We thus conclude that the differential equation (4.177) has no solution when $e = -1$.

Combining the results for $e = 1$ and -1, *we conclude that the differential equation* (4.99) *has no solutions when* $A \neq 0$. Recalling the analogous result for $A = 0$, we have thus proven the claim that the differential equation (4.99) has no solutions when $e \neq 0$, for the values of k_1 and k_2 permitted by Table 1. This completes the proof of Theorem 1.

5. PROOF OF THEOREM 2

The proof consists of the application of Condition VII to the general solutions of Conditions III′ and V′ given in Theorem 1. We continue to use the spinor formalism and spin coefficient formalism described in Section 3. Exploiting the conformal invariance of the problem we employ the tetrad (4.79) and the corresponding spinor dyad. We also use the special properties of the above mentioned solutions in order to simplify the calculations as much as possible. We recall that with respect to the tetrad (4.79) the non-vanishing spin coefficients and curvature components given by Eqs. (4.80) to (4.86), when $e = 0$, reduce to

$$\gamma(v, z, \bar{z}) = -\frac{1}{2} p_z(v, z), \tag{5.1}$$

$$\nu(v, z, \bar{z}) = -l_z(v, z, \bar{z})u - \frac{1}{2} m_z(v, z, \bar{z}). \tag{5.2}$$

$$\mu(v, z, \bar{z}) = -l(v, z, \bar{z}), \tag{5.3}$$

$$\Psi(v, z, \bar{z}) = \bar{z}A(v, z) + B(v, z) \tag{5.4}$$

$$\Phi_{21}(v, z, \bar{z}) = l_z(v, z, \bar{z}), \tag{5.5}$$

$$\Phi_{22}(v, z, \bar{z}) = m_{z\bar{z}}(v, z, \bar{z}) - p(v, z) l_z(v, z, \bar{z}) - \bar{p}(v, \bar{z}) l_{\bar{z}}(v, z, \bar{z}) + l_v(v, z, \bar{z}), \tag{5.6}$$

where

$$l(v, z, \bar{z}) = \frac{1}{2}(p_z(v, z) + \bar{p}_{\bar{z}}(v, \bar{z})), \tag{5.7}$$

and where p and m are given by (2.3) and (2.4) respectively. The NP operators have the form

$$D = \partial/\partial u, \qquad \delta = -\partial/\partial\bar{z}, \qquad \Delta = \partial/\partial v - p(v, z)\partial/\partial z - \bar{p}(v, \bar{z})\partial/\partial\bar{z} - (l(v, z, \bar{z}) + m(v, z, \bar{z}))\partial/\partial u. \tag{5.8}$$

We require the covariant derivatives of the Weyl spinor up to third order. Differentiation of (4.59) and the use of (3.13) and (3.14) yields

$$\Psi_{ABCD;E\dot{E}F\dot{F}} = o_{ABCD}(\Sigma_1 o_{EF}\bar{o}_{\dot{E}\dot{F}} + \Sigma_2 o_{EF}\bar{o}_{\dot{E}}\bar{l}_{\dot{F}} + \Sigma_3 o_E l_F \bar{o}_{\dot{E}\dot{F}} \\ \Sigma_5 o_{EF}\bar{l}_{\dot{E}}\bar{o}_{\dot{F}} + \Sigma_6 o_{EF}\bar{l}_{\dot{E}\dot{F}} \\ + \Sigma_7 o_E l_F \bar{l}_{\dot{E}}\bar{o}_{\dot{F}} + \Sigma_8 l_E o_F \bar{o}_{\dot{E}\dot{F}} + \Sigma_9 l_E o_F \bar{o}_{\dot{E}}\bar{l}_{\dot{F}}), \tag{5.9}$$

where

$$\Sigma_1 := \Delta\Theta_1 + (5\gamma + \bar{\gamma})\Theta_1 + \bar{\nu}\Theta_2 + \nu\Theta_3, \tag{5.10}$$
$$\Sigma_2 := -(\bar{\delta}\Theta_1 + \bar{\mu}\Theta_2), \tag{5.11}$$
$$\Sigma_3 := -(\delta\Theta_1 + \mu\Theta_3), \tag{5.12}$$
$$\Sigma_5 := \Delta\Theta_2 + (5\gamma - \bar{\gamma})\Theta_2, \tag{5.13}$$
$$\Sigma_6 := -(\bar{\delta}\Theta_2 + (5\alpha - \bar{\beta})\Theta_2), \tag{5.14}$$
$$\Sigma_7 := -(\delta\Theta_2 + (5\beta - \bar{\alpha})\Theta_2), \tag{5.15}$$
$$\Sigma_8 := \Delta\Theta_3 + (3\gamma + \bar{\gamma})\Theta_3, \tag{5.16}$$
$$\Sigma_9 := -(\bar{\delta}\Theta_3 + (3\alpha + \bar{\beta})\Theta_3). \tag{5.17}$$

By a similar method we obtain by differentiating (4.65)

$$\Psi_{ABCK;}{}^{K}{}_{\dot{E}\dot{F}\dot{F}\dot{G}\dot{G}} = o_{ABCF}\bar{o}_{\dot{E}}(\Upsilon_1 o_G\bar{o}_{\dot{F}\dot{G}} + \Upsilon_2 o_G\bar{o}_{\dot{F}}\bar{l}_{\dot{G}} + \Upsilon_3 l_G\bar{o}_{\dot{F}\dot{G}}$$
$$+ \Upsilon_5 o_G\bar{l}_{\dot{F}}\bar{o}_{\dot{G}} + \Upsilon_6 o_G\bar{l}_{\dot{F}\dot{G}} + \Upsilon_7 l_G\bar{l}_{\dot{F}}\bar{o}_{\dot{G}}), \tag{5.18}$$

where

$$\Upsilon_1 := \Delta\Sigma_8 + 2(2\gamma + \bar{\gamma})\Sigma_8 + \bar{\nu}\Sigma_9, \tag{5.19}$$
$$\Upsilon_2 := -(\bar{\delta}\Sigma_8 + \bar{\mu}\Sigma_9), \tag{5.20}$$
$$\Upsilon_3 := -\delta\Sigma_8, \tag{5.21}$$
$$\Upsilon_5 := \Delta\Sigma_9 + 4\gamma\Sigma_9, \tag{5.22}$$
$$\Upsilon_6 := -\bar{\delta}\Sigma_9, \tag{5.23}$$
$$\Upsilon_7 := -\delta\Sigma_9. \tag{5.24}$$

We also require the covariant derivative of the trace free Ricci spinor. From (4.72), (3.13), and (3.14) we obtain

$$\Phi_{AB\dot{A}\dot{B};C\dot{C}} = \Gamma_1 o_{ABC}\bar{o}_{\dot{A}\dot{B}\dot{C}} + \Gamma_2 o_{ABC}\bar{o}_{\dot{A}\dot{B}}\bar{l}_{\dot{C}} + \bar{\Gamma}_2 o_{AB}l_C\bar{o}_{\dot{A}\dot{B}\dot{C}}$$
$$+ \Gamma_4 o_{ABC}\bar{o}_{(\dot{A}}\bar{l}_{\dot{B})}\bar{o}_{\dot{C}} + \Gamma_5 o_{ABC}\bar{o}_{(\dot{A}}\bar{l}_{\dot{B})}\bar{l}_{\dot{C}} + \Gamma_6 o_{AB}l_C\bar{o}_{(\dot{A}}\bar{l}_{\dot{B})}\bar{o}_{\dot{C}}$$
$$+ \bar{\Gamma}_4 o_{(A}l_{B)}o_C\bar{o}_{\dot{A}\dot{B}\dot{C}} + \bar{\Gamma}_6 o_{(A}l_{B)}o_C\bar{o}_{\dot{A}\dot{B}}\bar{l}_{\dot{C}} + \bar{\Gamma}_5 o_{(A}l_{B)}l_C\bar{o}_{\dot{A}\dot{B}\dot{C}}, \tag{5.25}$$

where

$$\Gamma_1 := \Delta\Phi_{22} + 2(\gamma + \bar{\gamma})\Phi_{22} - 2\bar{\nu}\Phi_{21} - 2\nu\Phi_{12}, \tag{5.26}$$
$$\Gamma_2 := -(\bar{\delta}\Phi_{22} - 2\bar{\mu}\Phi_{21}), \tag{5.27}$$
$$\Gamma_4 := -2(\Delta\Phi_{21} + 2\gamma\Phi_{21}), \tag{5.28}$$
$$\Gamma_5 := 2\bar{\delta}\Phi_{21}, \tag{5.29}$$
$$\Gamma_6 := 2\delta\Phi_{21}. \tag{5.30}$$

We proceed by expressing Condition VII in spinor form. We use the same procedure as that employed to derive Conditions III′s and V′s.

The sought after condition, where the terms which vanish identically as a result of Eqs. (5.9) (5.18) and (5.25) are omitted, has the following form:

VII s

$$
\begin{aligned}
& 3\Psi_{(ABCD;K\dot{K}E(\dot{A}}\overline{\Psi}_{\dot{B}\dot{C}\dot{D}\dot{E};}{}^{\dot{K}K}{}_{\dot{F})F)} + 6\Psi_{K(ABC;D(\dot{A}E\dot{B}}\overline{\Psi}_{\dot{C}\dot{D}\dot{E}|\dot{K}|;}{}^{\dot{K}}{}_{F)\dot{F})}{}^{K} \\
& - 64\Psi_{(ABC|K|;}{}^{K}{}_{\dot{A}D\dot{B}}\overline{\Psi}_{\dot{C}\dot{D}\dot{E}|\dot{K}|;}{}^{\dot{K}}{}_{E\dot{F})F)} - 10\Psi_{(ABCD}\overline{\Psi}_{(\dot{A}\dot{B}\dot{C}\dot{D};}{}^{\dot{K}K}{}_{\dot{E}E}\Phi_{F)K\dot{F})\dot{K}} \\
& - 14\Psi_{K(ABC}\overline{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C};\dot{D}}{}^{K}{}_{\dot{E}D}\Phi_{EF)\dot{F})\dot{K}} - 24\Psi_{K(ABC}\overline{\Psi}_{(\dot{A}\dot{B}\dot{C}|\dot{K}|;}{}^{\dot{K}}{}_{D\dot{D}}{}^{K}\Phi_{EF)\dot{E}\dot{F})} \\
& - 142\Psi_{K(ABC}\overline{\Psi}_{(\dot{A}\dot{B}\dot{C}|\dot{K}|;}{}^{\dot{K}}{}_{D\dot{D}E}\Phi_{F)}{}^{K}{}_{\dot{E}\dot{F})} - 30\Psi_{K(ABC;D(\dot{A}}\Psi_{\dot{B}\dot{C}\dot{D}|\dot{K}|;}{}^{\dot{K}}{}_{E\dot{E}}{}^{K}{}_{\dot{F})F)} \\
& + 20\Psi_{(ABC|K|;}{}^{K}{}_{(\dot{A}}\overline{\Psi}_{\dot{B}\dot{C}\dot{D}|\dot{K}|;}{}^{\dot{K}}{}_{D\dot{E}E\dot{F})F)} - 100\Psi_{(ABC|K|;}{}^{K}{}_{(\dot{A}}\overline{\Psi}_{\dot{B}\dot{C}\dot{D}|\dot{K}|;}{}^{\dot{K}}{}_{D}\Phi_{EF)\dot{E}\dot{F}} \\
& + 30\Psi_{K(ABC;D(\dot{A}}\Psi_{\dot{B}\dot{C}\dot{D}}{}^{\dot{K}}{}_{;\dot{E}}{}^{K}\Phi_{EF)\dot{F}\dot{K}} + 150\Psi_{K(ABC;D(\dot{A}}\overline{\Psi}_{\dot{B}\dot{C}\dot{D}|\dot{K}|;}{}^{\dot{K}}{}_{E}\Phi_{F)}{}^{K}{}_{\dot{E}\dot{F})} \\
& - \frac{20}{3}\Psi_{K(ABC}\overline{\Psi}_{\dot{K}(\dot{A}\dot{B}\dot{C};\dot{D}D}\Phi_{EF)\dot{E}}{}^{\dot{K}}{}_{;}{}^{K}{}_{\dot{F})} - \frac{20}{3}\Psi_{K(ABC}\overline{\Psi}_{\dot{K}(\dot{A}\dot{B}\dot{C};\dot{D}D}\Phi_{E}{}^{K}{}_{\dot{E}\dot{F});F)}{}^{\dot{K}} \\
& + \frac{40}{3}\Psi_{K(ABC}\overline{\Psi}_{(\dot{A}\dot{B}\dot{C}|\dot{K}|;}{}^{\dot{K}}{}_{D}\Phi_{E}{}^{K}{}_{\dot{D}\dot{E};F)\dot{F}} + \frac{20}{3}\Psi_{K(ABC}\overline{\Psi}_{(\dot{A}\dot{B}\dot{C}|\dot{K}|;}{}^{\dot{K}}{}_{D}\Phi_{EF)\dot{D}\dot{E};}{}^{K}{}_{\dot{F})} \\
& + 40\Psi_{(ABCD}\Psi^{K}{}_{EF)}{}^{L}\overline{\Psi}_{(\dot{A}\dot{B}\dot{C}\dot{D};\dot{E}K\dot{L})L} + 40\Psi_{(ABCD}\overline{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}}{}^{\dot{L}}\overline{\Psi}_{\dot{C}\dot{D}\dot{E}\dot{F});\dot{K}E\dot{L}F)} \\
& + 370\Psi_{K(ABC}\overline{\Psi}_{(\dot{A}\dot{B}\dot{C}\dot{D};\dot{E}}{}^{K}\Psi_{DEF)L;}{}^{L}{}_{F)} + \frac{2020}{3}\Psi_{(ABCD}\overline{\Psi}_{(\dot{A}\dot{B}\dot{C}|\dot{K}|;}{}^{\dot{K}}{}_{E}\overline{\Psi}_{\dot{D}\dot{E}\dot{F}|\dot{L}|}{}^{\dot{L}}{}_{F)} \\
& + 58\Psi_{(ABCD}\overline{\Psi}_{(\dot{A}\dot{B}\dot{C}\dot{D}}\Phi^{K}{}_{E}{}^{\dot{K}}{}_{\dot{E}}\Phi_{F)K\dot{F})\dot{K}} + \text{complex conjugate} = 0. \qquad (5.31)
\end{aligned}
$$

The next step is to substitute for the Weyl spinor, the trace free Ricci spinor and their covariant derivatives in the above equation from (4.3), (4.72), (5.9), (5.18) and (5.25). Equating to zero the coefficients of the basis element $o_{(ABCD}\iota_{EF)}\bar{o}_{\dot{A}\dot{B}\dot{C}\dot{D}\dot{E}\dot{F}}$ in the resulting equation we obtain

$$3\overline{\Sigma}_6(\Sigma_7 - \Sigma_9) + 15\overline{\Theta}_2\Upsilon_7 + 5\Theta_3\overline{\Upsilon}_6 = 0. \qquad (5.32)$$

We now apply this condition to the metric defined by Eqs. (2.2) to (2.4) and (2.6) to (2.9) of Theorem 1. By the definitions (4.61), (4.62), (5.14), (5.15), (5.17), (5.23) and Eqs. (5.1) to (5.7) we obtain the following intermediate results:

$$\Theta_2 = \Psi_z, \qquad (5.33)$$

$$\Theta_3 = \Psi_{\bar{z}} = \mathrm{A}, \qquad (5.34)$$

$$\Sigma_6 = \Theta_{2,z} = \Psi_{zz}, \qquad (5.35)$$

$$\Sigma_7 = \Theta_{2,z} = \Psi_{z\bar{z}} = \mathrm{A}_z, \qquad (5.36)$$

$$\Sigma_9 = \Theta_{3,z} = \Psi_{\bar{z}z} = \mathrm{A}_z, \qquad (5.37)$$

$$\Upsilon_6 = \Sigma_{9,z} = \mathrm{A}_{zz}, \qquad (5.38)$$

$$\Upsilon_7 = \Sigma_{9,z} = 0. \qquad (5.39)$$

In view of the above results the complex conjugate of Eq. (5.32) reduces to

$$\overline{\mathrm{A}}\mathrm{A}_{zz} = 0. \qquad (5.40)$$

Since $\mathrm{A} \neq 0$, this implies

$$\mathrm{A}_{zz} = 0. \qquad (5.41)$$

This in turn implies by (2.6) and (4.98) that

$$a_2(v)(1 - a_1(v))[a_1(v)z]^{1/a_1(v)-2} = 0. \qquad (5.42)$$

This equation is inconsistent with the requirements that $a_2 \neq 0$, and that a_1 satisfy (2.8) or (2.9). We thus conclude that the metric of Theorem 1 defined by the Eqs. (2.2) to (2.4) and (2.6) to (2.9) does not satisfy Condition VIII. It follows that *the conformally invariant wave equation* (1.13) *on a space-time with metric given by Eqs.* (2.2) *to* (2.4) *and* (2.6) *to* (2.9) *of Theorem 1 does not satisfy Huygens' principle* [*5*].

We now turn to the metric defined by Eqs. (2.2) (2.3) (2.4) and (2.5) of Theorem 1. In this case we have by (4.98)

$$A = 0, \qquad B(v, z) = 2h_2(v). \qquad (5.43)$$

It follows from the above and Eqs. (5.33) and (5.34) that

$$\Theta_2 = \Theta_3 = 0. \qquad (5.44)$$

Thus the condition (5.32) reduces to an identity. The equations (5.44) in fact imply that the condition (5.31) reduces to

$$|\Psi|^2 |\Phi_{12}|^2 o_{ABCDEF}\bar{o}_{\dot{A}\dot{B}\dot{C}\dot{D}\dot{E}\dot{F}} = 0, \qquad (5.45)$$

where the left hand side arises from the last term on the left hand side of (5.31). The condition (5.45) implies

$$|\Psi|^2 |\Phi_{12}|^2 = 0, \qquad (5.46)$$

which in turn implies

$$\Phi_{21} = 0, \qquad (5.47)$$

since $\Psi = 2h_2 \neq 0$. Now by (2.3), (5.5) and (5.7)

$$\Phi_{21}(v, z, \bar{z}) = p_2(v). \qquad (5.48)$$

Thus (5.47) implies

$$p_2 = 0. \qquad (5.49)$$

This establishes the fact that *the generalized plane wave metric does not satisfy Condition* VII *if* a $:= p_2 \neq 0$ [*5*] [*40*]. On the other hand if (5.49) holds there exists a coordinate system in which the metric of Theorem 1 defined by Eqs. (2.2) (2.3) (2.4) and (2.5) has the form (2.10) of the metric of Theorem 2. The required coordinate transformation may be found in Ref. [*23*]. This completes the proof of Theorem 2.

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APPENDIX

In this Appendix it is shown that the functions N, P and Q defined by (4.168) and satisfying Eq. (4.167) are necessarily rational functions in z. There are numerous special cases to consider. However, we shall only give the details for the most general case since the method and the result are the same for the special cases. We proceed by taking the derivative $\partial^6/\partial z^3 \partial \bar{z}^3$ of (4.167) which yields

$$\begin{aligned} 12\mathrm{C}\overline{\mathrm{G}} &+ 21e\mathrm{D}\overline{\mathrm{H}} + 9e^2\mathrm{E}\overline{\mathrm{I}} + 12\overline{\mathrm{C}}\mathrm{G} + 21e\overline{\mathrm{D}}\mathrm{H} + 9e^2\overline{\mathrm{E}}\mathrm{I} \\ &+ 3\mathrm{D}\overline{\mathrm{D}} + 6e\mathrm{E}\overline{\mathrm{E}} + 9e^2\mathrm{F}\overline{\mathrm{F}} + 3\mathrm{C}\overline{\mathrm{K}} + 6e\mathrm{D}\overline{\mathrm{L}} + 3e^2\mathrm{E}\overline{\mathrm{M}} \\ &+ 3\overline{\mathrm{C}}\mathrm{K} + 6e\overline{\mathrm{D}}\mathrm{L} + 3e^2\overline{\mathrm{E}}\mathrm{M} + 3\mathrm{J}\overline{\mathrm{J}} + 6e\mathrm{K}\overline{\mathrm{K}} + 3e^2\mathrm{L}\overline{\mathrm{L}} \\ &- 3e\overline{\mathrm{G}}\mathrm{K} - 3e^2\overline{\mathrm{H}}\mathrm{L} - 3e\mathrm{G}\overline{\mathrm{K}} - 3e^2\mathrm{H}\overline{\mathrm{L}} - 8ea\mathrm{G}\overline{\mathrm{G}} - 2e^2b\mathrm{H}\overline{\mathrm{H}} = 0, \end{aligned} \quad (\mathrm{A}.1)$$

where

$$a := (3k_2)/(4k_1), \qquad b := 3(k_2/k_1 - 1), \quad (\mathrm{A}.2)$$

$$\left.\begin{aligned} \mathrm{C} &:= \mathrm{N}''', \quad \mathrm{D} := (z\mathrm{N})''', \quad \mathrm{E} := (z^2\mathrm{N})''', \quad \mathrm{F} := (z^3\mathrm{N})''', \\ \mathrm{G} &:= \mathrm{P}''', \quad \mathrm{H} := (z\mathrm{P})''', \quad \mathrm{I} := (z^2\mathrm{P})''', \\ \mathrm{J} &:= \mathrm{Q}''', \quad \mathrm{K} := (z\mathrm{Q})''', \quad \mathrm{L} := (z^2\mathrm{Q})''', \quad \mathrm{M} := (z^3\mathrm{Q})'''. \end{aligned}\right\} \quad (\mathrm{A}.3)$$

Assuming

$$\mathrm{D}\overline{\mathrm{G}} \neq 0, \quad (\mathrm{A}.4)$$

we divide Eq. (A.1) by $\mathrm{D}\overline{\mathrm{G}}$ and differentiate the resulting equation with respect to z and then with respect to $\bar{z}$ obtaining

$$\begin{aligned} 9e^2\left(\frac{\mathrm{E}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{I}}}{\overline{\mathrm{G}}}\right)' &+ 12\left(\frac{\mathrm{G}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{C}}}{\overline{\mathrm{G}}}\right)' + 21e\left(\frac{\mathrm{H}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{D}}}{\overline{\mathrm{G}}}\right)' + 9e^2\left(\frac{\mathrm{I}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{E}}}{\overline{\mathrm{G}}}\right)' + 6e\left(\frac{\mathrm{E}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{E}}}{\overline{\mathrm{G}}}\right)' \\ &+ 3e^2\left(\frac{\mathrm{F}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{F}}}{\overline{\mathrm{G}}}\right)' + 3\left(\frac{\mathrm{C}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{K}}}{\overline{\mathrm{G}}}\right)' + 3e^2\left(\frac{\mathrm{E}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{M}}}{\overline{\mathrm{G}}}\right)' + 3\left(\frac{\mathrm{K}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{C}}}{\overline{\mathrm{G}}}\right)' + 6e\left(\frac{\mathrm{L}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{D}}}{\overline{\mathrm{G}}}\right)' \\ &+ 3e^2\left(\frac{\mathrm{M}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{E}}}{\overline{\mathrm{G}}}\right)' + 3\left(\frac{\mathrm{J}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{J}}}{\overline{\mathrm{G}}}\right)' + 6e\left(\frac{\mathrm{K}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{K}}}{\overline{\mathrm{G}}}\right)' + 3e^2\left(\frac{\mathrm{L}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{L}}}{\overline{\mathrm{G}}}\right)' - 3e^2\left(\frac{\mathrm{L}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{H}}}{\overline{\mathrm{G}}}\right)' \\ &- 3e\left(\frac{\mathrm{G}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{K}}}{\overline{\mathrm{G}}}\right)' - 3e^2\left(\frac{\mathrm{H}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{L}}}{\overline{\mathrm{G}}}\right)' - 2e^2b\left(\frac{\mathrm{H}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{H}}}{\overline{\mathrm{G}}}\right)' = 0. \end{aligned} \quad (\mathrm{A}.5)$$

We now divide (A.5) by $(\mathrm{E}/\mathrm{D})'(\overline{\mathrm{K}}/\overline{\mathrm{G}})'$ assuming

$$\left(\frac{\mathrm{E}}{\mathrm{D}}\right)'\left(\frac{\overline{\mathrm{K}}}{\overline{\mathrm{G}}}\right)' \neq 0, \quad (\mathrm{A}.6)$$

and differentiate $\partial^2/\partial z \partial \bar{z}$ to obtain

$$\begin{aligned} 12\mathrm{P}\overline{\mathrm{W}} &+ 21e\mathrm{Q}\overline{\mathrm{X}} + 9e^2\mathrm{R}\overline{\mathrm{Y}} + 3e^2\mathrm{N}\overline{\mathrm{Z}} + 3\mathrm{T}\overline{\mathrm{W}} + 6e\mathrm{U}\overline{\mathrm{X}} \\ &+ 3e^2\mathrm{V}\overline{\mathrm{Y}} + 3\mathrm{S}\overline{\Lambda} + 3e^2\mathrm{U}\overline{\Pi} - 3e^2\mathrm{U}\overline{\Theta} - 3e^2\mathrm{Q}\overline{\Pi} - 2e^2b\mathrm{Q}\overline{\Theta} = 0, \end{aligned} \quad (\mathrm{A}.7)$$

where

$$
\left.
\begin{array}{llll}
N := \left[\dfrac{\left(\frac{F}{D}\right)'}{\left(\frac{E}{D}\right)'}\right]', &
P := \left[\dfrac{\left(\frac{G}{D}\right)'}{\left(\frac{E}{D}\right)'}\right]', &
Q := \left[\dfrac{\left(\frac{H}{D}\right)'}{\left(\frac{E}{D}\right)'}\right]', &
R := \left[\dfrac{\left(\frac{I}{D}\right)'}{\left(\frac{E}{D}\right)'}\right]', \\[2ex]
S := \left[\dfrac{\left(\frac{J}{D}\right)'}{\left(\frac{E}{D}\right)'}\right]', &
T := \left[\dfrac{\left(\frac{K}{D}\right)'}{\left(\frac{E}{D}\right)'}\right]', &
U := \left[\dfrac{\left(\frac{L}{D}\right)'}{\left(\frac{E}{D}\right)'}\right]', &
V := \left[\dfrac{\left(\frac{M}{D}\right)'}{\left(\frac{E}{D}\right)'}\right]', \\[2ex]
W := \left[\dfrac{\left(\frac{C}{G}\right)'}{\left(\frac{K}{G}\right)'}\right]', &
X := \left[\dfrac{\left(\frac{D}{G}\right)'}{\left(\frac{K}{G}\right)'}\right]', &
Y := \left[\dfrac{\left(\frac{E}{G}\right)'}{\left(\frac{K}{G}\right)'}\right]', &
Z := \left[\dfrac{\left(\frac{F}{G}\right)'}{\left(\frac{K}{G}\right)'}\right]', \\[2ex]
\Theta := \left[\dfrac{\left(\frac{H}{G}\right)'}{\left(\frac{K}{G}\right)'}\right]', &
\Lambda := \left[\dfrac{\left(\frac{J}{G}\right)'}{\left(\frac{K}{G}\right)'}\right]', &
\Pi := \left[\dfrac{\left(\frac{L}{G}\right)'}{\left(\frac{K}{G}\right)'}\right]'. &
\end{array}
\right\} \qquad (A.8)
$$

Next we divide (A.7) by $Q\overline{W}$ assuming

$$Q\overline{W} \neq 0, \qquad (A.9)$$

and take $\partial^2/\partial z\partial\bar{z}$ to find

$$
\begin{aligned}
9e^2\left(\frac{R}{Q}\right)'\left(\frac{\overline{Y}}{\overline{W}}\right)' + 3e^2\left(\frac{N}{Q}\right)'\left(\frac{\overline{Z}}{\overline{W}}\right)' + 6e\left(\frac{U}{Q}\right)'\left(\frac{\overline{X}}{\overline{W}}\right)' + 3e^2\left(\frac{V}{Q}\right)'\left(\frac{\overline{Y}}{\overline{W}}\right)' \\
+ 3\left(\frac{S}{Q}\right)'\left(\frac{\overline{\Lambda}}{\overline{W}}\right)' + 3e^2\left(\frac{U}{Q}\right)'\left(\frac{\overline{\Pi}}{\overline{W}}\right)' - 3e^2\left(\frac{U}{Q}\right)'\left(\frac{\overline{\Theta}}{\overline{W}}\right)' = 0. \qquad (A.10)
\end{aligned}
$$

We finally obtain a separable equation by dividing (A.10) by $\left(\frac{U}{Q}\right)'\left(\frac{\overline{Y}}{\overline{W}}\right)'$, assuming

$$\left(\frac{U}{Q}\right)'\left(\frac{\overline{Y}}{\overline{W}}\right)' \neq 0, \qquad (A.11)$$

and by taking the derivative $\partial^2/\partial z\partial\bar{z}$. The resulting equation may be written as

$$
e^2\left[\dfrac{\left(\frac{N}{Q}\right)'}{\left(\frac{U}{Q}\right)'}\right]'\left[\dfrac{\left(\frac{\overline{Z}}{\overline{W}}\right)'}{\left(\frac{\overline{Y}}{\overline{W}}\right)'}\right]' + \left[\dfrac{\left(\frac{S}{Q}\right)'}{\left(\frac{U}{Q}\right)'}\right]'\left[\dfrac{\left(\frac{\overline{\Lambda}}{\overline{W}}\right)'}{\left(\frac{\overline{Y}}{\overline{W}}\right)'}\right]' = 0. \qquad (A.12)
$$

Assuming

$$
\left[\dfrac{\left(\frac{S}{Q}\right)'}{\left(\frac{U}{Q}\right)'}\right]'\left[\dfrac{\left(\frac{\overline{Z}}{\overline{W}}\right)'}{\left(\frac{\overline{Y}}{\overline{W}}\right)'}\right]' \neq 0, \qquad (A.13)
$$

we find that (A.12) implies

$$e^2\left[\frac{\left(\frac{N}{Q}\right)'}{\left(\frac{U}{Q}\right)'}\right]' = a_1\left[\frac{\left(\frac{S}{Q}\right)'}{\left(\frac{U}{Q}\right)'}\right]', \tag{A.14}$$

$$\left[\frac{\left(\frac{\bar{\Lambda}}{\bar{W}}\right)'}{\left(\frac{\bar{Y}}{\bar{W}}\right)'}\right]' = -a_1\left[\frac{\left(\frac{\bar{Z}}{\bar{W}}\right)'}{\left(\frac{\bar{Y}}{\bar{W}}\right)'}\right]', \tag{A.15}$$

where a_1 is some (separation) function only of v. Integrating (A.14) and (A.15) with respect to z we obtain

$$e^2\left(\frac{N}{Q}\right)' = a_1\left(\frac{S}{Q}\right)' + a_2\left(\frac{U}{Q}\right)', \tag{A.16}$$

$$\left(\frac{\bar{\Lambda}}{\bar{W}}\right)' = -a_1\left(\frac{\bar{Z}}{\bar{W}}\right)' + a_3\left(\frac{\bar{Y}}{\bar{W}}\right)', \tag{A.17}$$

where a_2 and a_3 are functions only of v. We proceed backwards by substituting from (A.16) and (A.17) into (A.10). On account of the cancellation of some terms the resulting equation separates and it follows by (A.11) that we have

$$9e^2\left(\frac{R}{Q}\right)' + 3e^2\left(\frac{V}{Q}\right)' + 3a_3\left(\frac{S}{Q}\right)' = b_1\left(\frac{U}{Q}\right)', \tag{A.18}$$

$$3a_2\left(\frac{\bar{Z}}{\bar{W}}\right)' + 6e\left(\frac{\bar{X}}{\bar{W}}\right)' + 3e^2\left(\frac{\bar{\Pi}}{\bar{W}}\right)' - 3e^2\left(\frac{\bar{\Theta}}{\bar{W}}\right)' = -b_1\left(\frac{\bar{Y}}{\bar{W}}\right)', \tag{A.19}$$

where b_1 is another separation function. The Eqs. (A.18) (A.19) (A.16) and (A.17) may now be integrated with respect to z and $\bar{z}$ yielding

$$9e^2R + 3e^2V + 3a_3S = b_1U + b_2Q, \tag{A.20}$$

$$3a_2\bar{Z} + 6e\bar{X} + 3e^2\bar{\Pi} - 3e^2\bar{\Theta} = -b_1\bar{Y} + b_3\bar{W}, \tag{A.21}$$

$$e^2N = a_1S + a_2U + a_4Q, \tag{A.22}$$

$$\bar{\Lambda} = -a_1\bar{Z} + a_3\bar{Y} + a_5\bar{W}, \tag{A.23}$$

where a_4, a_5, b_2 and b_3 are further functions of integration. We now proceed a further step backwards by substituting for $9e^2R$ and $3e^2\bar{\Pi}$ from (A.20) and (A.21) respectively into (A.7). Due to appropriate cancellations the resulting equation separates yielding two additional equations, which we may integrate with respect to z and $\bar{z}$ respectively. We repeat the process just described two more times to separate the Eq. (A.5) and finally (A.1). The whole procedure yields ten separated equations which may be written as follows:

$$12C + d_3E - (b_7 - c_5)H - 8eaG + 3a_9J - 3eK + b_7L = f_1D, \tag{A.24}$$

$$c_4\bar{C} + 3\bar{D} + b_6\bar{E} + 3a_8\bar{F} + 21e\bar{H} + (2eC_4 + d_2)\bar{K} + 6e\bar{L} = -f_1\bar{G}, \tag{A.25}$$

$$3C - 27eG + (2eb_3 - b_5 + 2ec_1 + c_3)H + 3(a_7 - 2ea_5)J + (b_5 - 2eb_3)L = d_1E + d_2D, \tag{A.26}$$

$$c_2\bar{C} + (6e + b_4)\bar{E} + 3a_6\bar{F} + 9e^2\bar{I} + 2ec_2\bar{K} + 3e^2M = -d_1K + d_3\bar{G}, \tag{A.27}$$

$$12G + 3K + 3a_5J + b_3L = (b_3 + c_1)H + c_2E + c_4D, \tag{A.28}$$

$$27e\bar{D} + (b_1 + b_2)\bar{E} + 3(a_2 + a_4)\bar{F} - e^2(2b + 3)\bar{H} = -c_1\bar{C} + c_3\bar{K} + c_5\bar{G}, \tag{A.29}$$

$$9e^2I + 3e^2M + 3a_3J = b_1L + b_2H + b_4E + b_6D, \tag{A.30}$$

$$6e\overline{D}+3a_2\overline{F}+3e^2\overline{L}-3e^2\overline{H}=-b_1\overline{E}+b_3\overline{C}+b_5\overline{K}+b_7\overline{G}, \quad (A.31)$$

$$e^2F=a_1J+a_2L+a_4H+a_6E+a_8D, \quad (A.32)$$

$$\overline{J}=-a_1\overline{F}+a_3\overline{E}+a_5\overline{C}+a_7\overline{K}+a_9\overline{G}. \quad (A.33)$$

The next step is to replace in the above equations the functions C, ..., M by their definitions given by (A.3) and integrate the resulting equations three times with respect to z after taking complex conjugates of the Eqs (A.25) (A.27) (A.29) (A.31) and (A.33). The equations obtained may be written (in a different order) as follows:

$$h_2P = g_1N - h_1Q - \Gamma_1, \quad (A.34)$$

$$h_4P = -h_3N + g_2Q - \Gamma_2, \quad (A.35)$$

$$g_3P = h_5N + g_4Q + \Gamma_3, \quad (A.36)$$

$$g_7P = g_5N - g_6Q - \Gamma_4, \quad (A.37)$$

$$g_8P = h_6N - g_9Q + \Gamma_5, \quad (A.38)$$

$$g_{11}P = g_{10}N - h_7Q - \Gamma_6, \quad (A.39)$$

$$g_{13}P = -g_{12}N - h_8Q + \Gamma_7, \quad (A.40)$$

$$g_{14}P = -g_{16}N - g_{15}Q + \Gamma_8, \quad (A.41)$$

$$g_{18}P = g_{17}N + g_{19}Q - \Gamma_9, \quad (A.42)$$

$$g_{21}P = -g_{20}N - g_{22}Q + \Gamma_{10}, \quad (A.43)$$

where

$$g_1(z) := z^3 - a_6z^2 - a_8z, \qquad h_1(z) := a_2z^2 + a_1, \quad (A.44)$$

$$g_2(z) := -r_7z + 1, \qquad h_2(z) := a_4z, \quad (A.45)$$

$$g_3(z) := 9z^2 - b_2z, \qquad h_3(z) := -r_1z^3 + r_3z^2 + r_5, \quad (A.46)$$

$$g_4(z) := -3z^3 + b_1z^2 - 3a_3, \qquad h_4(z) := r_9, \quad (A.47)$$

$$g_5(z) := 3r_2z^3 + s_1z^2 + 6ez - s_3, \qquad h_5(z) := b_4z^2 + b_6z, \quad (A.48)$$

$$g_6(z) := -3z^2 + s_5z, \qquad h_6(z) := c_2z^2 + c_4z, \quad (A.49)$$

$$g_7(z) := 3z + s_7, \qquad h_7(z) := t_3z, \quad (A.50)$$

$$g_8(z) := -(b_3 + c_1)z + 12, h_8(z) := (b_5 - 2eb_3)z^2 + 3(a_7 - 2ea_5), \quad (A.51)$$

$$g_9(z) := b_3z^2 + 3z + 3a_5, \quad (A.52)$$

$$g_{10}(z) := 3(r_2 + r_4)z^3 + (s_1 + s_2)z + 27ez + t_1, \quad (A.53)$$

$$g_{11}(z) := (2b + 3)z + t_5, \quad (A.54)$$

$$g_{12}(z) := -d_1z^2 - d_3z + 3, \quad (A.55)$$

$$g_{13}(z) := (2eb_3 - b_5 + 2ec_1 + c_3)z - 27e, \quad (A.56)$$

$$g_{14}(z) := 9z^2 - u_3, \quad (A.57)$$

$$g_{15}(z) := 3z^3 + (2et_2 + u_1)z, \quad (A.58)$$

$$g_{16}(z) := 3r_6z^3 + (6e + s_4)z^2 + t_2, \quad (A.59)$$

$$g_{17}(z) := d_3z^2 - f_1z + 12, \quad (A.60)$$

$$g_{18}(z) := (b_7 - c_5)z + 8ae, \quad (A.61)$$

$$g_{19}(z) := b_7z^3 - 3ez + 3a_9, \quad (A.62)$$

$$g_{20}(z) := 3z_8z^3 + s_6z^2 + 3z + t_4, \quad (A.63)$$

$$g_{21}(z) := 21ez + v_1, \quad (A.64)$$

$$g_{22}(z) := 6ez^2 + (2et_4 + u_2)z, \quad (A.65)$$

$$\Gamma_i(z) := l_iz^2 + m_iz + n_i. \quad (A.66)$$

The functions l_i, m_i, and n_i appearing in (A.66) are functions only of v for all $i=1, \ldots, 10$. We also have

$$r_i := \bar{a}_i, \quad \forall i = 1, \ldots, 9, \quad s_i := b_i, \quad \forall i = 1, \ldots, 7.$$
$$t_i := \bar{c}_i, \quad \forall i = 1, \ldots, 5, \quad u_i := \bar{d}_i, \quad \forall i = 1, \ldots, 3, \quad v_1 = \bar{f}_1. \tag{A.67}$$

Since g_2 is never the zero polynomial for any choice of the function a_7, we may solve (A.35) for Q in terms of N and P yielding

$$Q = g_2^{-1}(h_4 P + h_3 N + \Gamma_2), \tag{A.68}$$

We then substitute for Q in the remaining equations (A.34) to (A.43) obtaining a system of equations which may be written as

$$\sigma_i P = \lambda_i N + \Omega_i, \qquad (i = 1, \ldots, 9), \tag{A.69}$$

where

$$\sigma_1 := g_2 h_2 + h_1 h_4, \quad \lambda_1 := g_1 g_2 - h_1 h_3, \quad \Omega_1 := -g_2 \Gamma_1 - h_1 \Gamma_2, \tag{A.70}$$
$$\sigma_2 := g_2 g_3 - g_4 h_4, \quad \lambda_2 := g_2 h_5 + g_4 h_3, \quad \Omega_2 := g_4 \Gamma_2 + g_2 \Gamma_3, \tag{A.71}$$
$$\sigma_3 := g_2 g_7 + g_6 h_4, \quad \lambda_3 := g_2 g_5 - g_6 h_3, \quad \Omega_3 := -g_6 \Gamma_2 - g_2 \Gamma_4, \tag{A.72}$$
$$\sigma_4 := g_2 g_8 + g_9 h_4, \quad \lambda_4 := g_2 h_6 - g_9 h_3, \quad \Omega_4 := -g_9 \Gamma_2 + g_2 \Gamma_5, \tag{A.73}$$
$$\sigma_5 := g_2 g_{11} + h_4 h_7, \quad \lambda_5 := g_2 g_{10} - h_3 h_7, \quad \Omega_5 := -h_7 \Gamma_2 - g_2 \Gamma_6, \tag{A.74}$$
$$\sigma_6 := g_2 g_{13} + h_4 h_8, \quad \lambda_6 := -g_2 g_{12} - h_3 h_8, \quad \Omega_6 := -h_8 \Gamma_2 + g_2 \Gamma_7, \tag{A.75}$$
$$\sigma_7 := g_2 g_{14} + g_{15} h_4, \quad \lambda_7 := -g_2 g_{16} - g_{15} h_3, \quad \Omega_7 := -g_{15} \Gamma_2 + g_2 \Gamma_8, \tag{A.76}$$
$$\sigma_8 := g_2 g_{18} - g_{19} h_4, \quad \lambda_8 := g_2 g_{17} + g_{19} h_3, \quad \Omega_8 := g_{19} \Gamma_2 - g_2 \Gamma_9, \tag{A.77}$$
$$\sigma_9 := g_2 g_{21} + g_{22} h_4, \quad \lambda_9 := -g_2 g_{20} - g_{22} h_3, \quad \Omega_9 := -g_{22} \Gamma_2 + g_2 \Gamma_{10}, \tag{A.78}$$

We compute

$$\sigma_7 = 3(r_9 - 3r_7)z^3 + 9z^2 + [r_7 u_3 + r_9(2et_2 + u_1)]z - u_3, \tag{A.79}$$

and observe that σ_7 is never the zero polynomial. This result implies that we may always solve Eq. (A.69), $i = 7$, for P in terms of N namely

$$P = \sigma_7^{-1}(\lambda_7 N + \Omega_7). \tag{A.80}$$

Using the expression for P thus obtained we eliminate P from the remaining equations (A.69) obtaining a system of equations which may be written as

$$\gamma_i N = \beta_i, \qquad (i = 1, \ldots, 9, i \neq 7), \tag{A.81}$$

where

$$\gamma_i := \sigma_i \lambda_7 - \sigma_7 \lambda_i, \tag{A.82}$$
$$\beta_i := \sigma_7 \Omega_i - \sigma_i \Omega_7, \tag{A.83}$$

for $i = 1, \ldots, 9$, $i \neq 7$. In view of the preceding definitions it is clear that the γ_i and β_i are polynomials for all admissible values of i. It thus follows from (A.81) that N is always a rational function of z unless for some choice of the functions $a_1, \ldots, a_9$, $b_1, \ldots, b_7$, $c_1, \ldots, c_5$, $d_1, \ldots, d_3$, f_1. the polynomials $\gamma_i = 0$, for all $i = 1, \ldots, 9$, $i \neq 7$. However, an examination of the equations that must be satisfied by these functions as a consequence of the requirement that $\gamma_i = 0$, $\forall i = 1, \ldots, 9$, $i \neq 7$, shows that this system of equations has no solutions. We are then able to conclude that in the general case considered the function N is rational in z. It follows from Eqs (A.68) and (A.80) that the functions P and Q are also rational. The same result is obtained by a similar method in the special cases that result when one or more of the inequalities (A.13) (A.9) (A.6) and (A.4) fail to hold.

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