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The group structure of supergravity

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ABSTRACT. — An intrinsic description of the « group manifold approach » to supergravity is given. Emphasis is placed on some geometric structures which allow us to obtain a direct full covariant formulation. In particular, the geometric theory of partial differential equations allows us to give a dynamic description of space-time.

Some applications to physically interesting situations are discussed in detail.

RÉSUMÉ. — On donne une description intrinsèque du « modèle de variété de groupe » pour la supergravité. L'accent est placé sur quelques structures géométriques qui permettent de donner une formulation directe complètement covariante. De plus la théorie géométrique des équations différentielles nous permet d'obtenir une description dynamique de l'espace temps.

Quelques applications physiquement intéressantes sont également discutées en détail.

1. INTRODUCTION

In recent years geometric theories of supergravity were introduced in order to give unified models for the fundamental forces of Nature [*1*] [*2*]

[3] [4] [5] [6]. Although none of these theories has reached the status of a final unified theory many of them show interesting features and use novel ideas. For this reason it is desirable to deal with them by a standard method and a unified formalism capable of handling a broad range of applications and having formal power and consistency.

The so-called « group manifold approach », first introduced in ref [1] and then further developed in some other papers (see e. g. refs [2] [3]) seems to us to play a very distinguished role. In fact, in this model the fundamental structure is a « group » on which the physics is built by adopting a spirit close to the gauge theory one.

Cartan was the first to worry about gauge theories from a geometric point of view and introduced the notion of connection, the geometric counter part of the Yang-Mills fields of the physicists. He built an extension of conventional gravity by the use of torsion. In particular, although Cartan proposed torsion already in 1922 [7], its relevance was recognized only about 30 years later.

We underline, however, that the past formulations of the group manifold approach in supergravity seem to overlook a proper gauge theory, since the so-called « pseudoconnections » are not given as principal connections on principal fiber bundles. However, in this paper we shall present a detailed intrinsic exposition of « group model » emphasizing some fundamental structures which allow us to give a direct interpretation of such theory as a fully covariant gauge theory. Further, by using some geometric tools of formal differential equations we shall give a description of space-time as a dynamic variable.

So, the first step in the present formulation of the group manifold approach is to take the geometry of differential equations seriously and use it, beside the theory of geometric objects, instead of the conventional tensor calculus. For these reasons a large part of this paper is devoted to consider in some detail fundamental results about some geometric structures which support our theory.

It is natural to ask why formulate a gauge theory of supergravity on a Lie group G. The answer is that such a structure offers a natural environment where curved space-times can be recognized as dynamical variables. Parenthetically this aspect should be very profitable also in order to develop the problem of quantum fluctuations. In fact, another new result of the present formulation of the group manifold approach to supergravity is that in this unified framework space-time has the role of a dynamical variable via the embeddings of the vacuum space-time G/H in the (super) group G. These embeddings are sections of the gauge structure supporting the model, and are conditioned to be solutions of a suitable dynamic equation.

Let us comment also on the relation between the use of supermanifolds within this framework. In general, super-Lie groups, used in this contest, cannot be assimilated to graded Lie groups in the sense of Kostant ([8] [9] [10]). But, more precisely they are Lie groups where the usual Lie algebra has a natural structure of graded algebra. Actually, in order to give a satisfactory description of supergravity we can use these geometric objects without entering in the category of supermanifolds in the sense of Kostant. In fact, the use of such structures is not really very consistent with the group manifold approach and nevertheless leads to a description of physical fields that can be worked out in details.

Let us now present some considerations about the advantage of using an intrinsic completely covariant framework in a theoretical description of physics.

In the original paper of Einstein, general relativity was concerned with a very deep and beautiful relation between Riemannian geometry and gravitational fields. In this traditional setting, Riemannian manifolds are understood as differentiable manifolds with an additional structure provided by the metric tensor $g_{\mu\nu}$. Then, one requires a canonical connection with Christoffel symbol

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta} = \frac{1}{2} g^{\alpha j}\left(\frac{\partial g_{\beta j}}{\partial x_{\gamma}} + \frac{\partial g_{j\gamma}}{\partial x_{\beta}} - \frac{\partial g_{\gamma\beta}}{\partial x_{j}}\right)$$

which satisfies the relation $\dfrac{\partial g_{\mu\nu}}{\partial x_{\lambda}} - \Gamma^{\beta}_{\mu\lambda} g_{\beta\nu} - \Gamma^{\beta}_{\gamma\nu} g_{\mu\beta} = 0$.

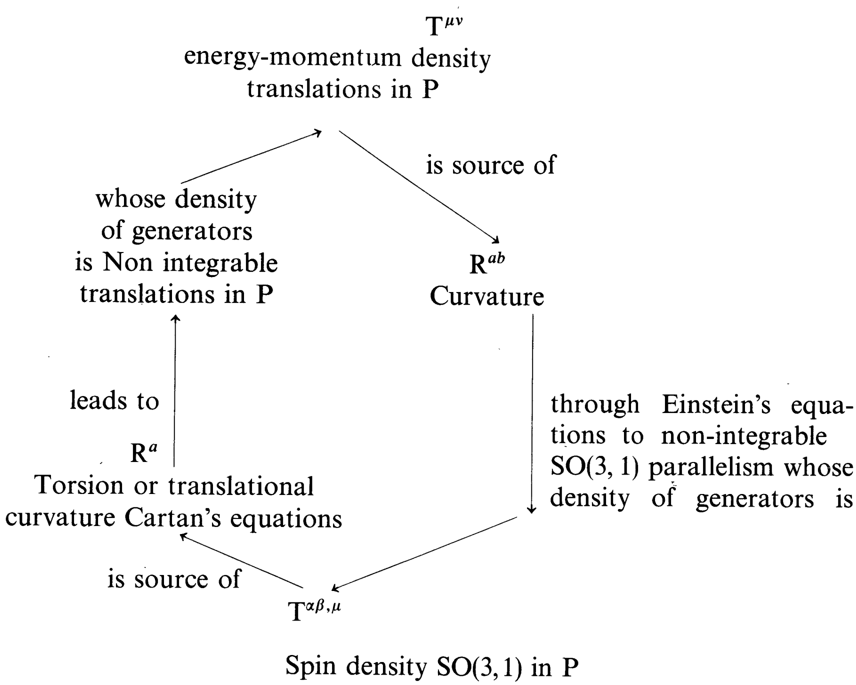
The corresponding Riemannian curvature is given by

$$R_{\alpha}{}^{\beta}{}_{\mu\sigma} = -\frac{\partial \Gamma^{\beta}_{\alpha\mu}}{\partial x_{\sigma}} + \frac{\partial \Gamma^{\beta}_{\alpha\sigma}}{\partial x_{\mu}} + \Gamma^{\tau}_{\alpha\mu}\Gamma^{\beta}_{\tau\sigma} - \Gamma^{\tau}_{\alpha\sigma}\Gamma^{\beta}_{\tau\mu}$$

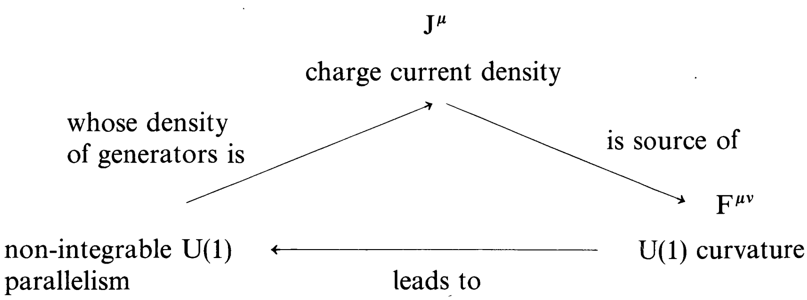
which satisfies the identity $R_{\alpha}{}^{\beta}{}_{\mu\sigma} = - R_{\alpha}{}^{\beta}{}_{\sigma\mu}$ and $g_{\mu\beta} R^{\beta}_{\nu\lambda\sigma} + g_{\beta\nu} R^{\beta}_{\lambda\mu\sigma} = 0$. The origin of these symmetries is different, although they look the same, since the antisymmetry in the last pair of indices simply reflects the anticommutativity in the Grassmannian algebra while the one in the first pair is related to the Lie algebra of SO(3, 1). But this is only an example, where the true meaning of the things is well understood only by using an intrinsic description: (g, Γ, R play the role of geometric objects that are sections of suitable fiber bundles.). Furthermore, to the local covariance, automatically assured in the language of fiber bundles, must be added the full covariance that requires additional functorial structures on the fiber bundles. In this framework one recognizes the usefulness of the so-called super-bundles of geometric objects (see refs. [11] [12] [13]).

Another observation concerns the use of connections which are not of Levi-Civita type, but that have some torsion. The logical desirability of torsion follows from a generalized « action-reaction » principle. We draw six boxes each containing some field which acts of the following box and obtain the following diagram:

Cartan-Einstein's gravity hexagon



Maxwell's triangle



Without torsion, the gravity diagram does not close and achieve full symmetry. A natural way to introduce it is through an action principle.

It is easy to show that: $R\sqrt{-g}d^4x = \frac{1}{2} R^{ab} \wedge \theta^c \wedge \theta^d \varepsilon_{abcd}$ so that the action of the gravitational field could be taken to be:

$$I_{grav.} = \frac{c^3}{16\pi G} \int_{M^4} R^{ab} \wedge \theta^c \wedge \theta^d \varepsilon_{abcd}.$$

We shall vary it by using a first order formalism, following Palatini's method. Therefore, ω^{ab} and θ^a in the Cartan equations:

$$d\omega^{ab} - \omega^{at} \wedge \omega_t^b = R^{ab}$$
$$d\theta^a - \omega^{at} \wedge \theta_t = R^a = 0$$

are considered to be independent variables. Variation of θ^a leads to:

(I) $$R^{ab} \wedge \theta^c \varepsilon_{abcd} = 0$$

which coincides with Einstein equations in vacuum.

Variation of ω^{ab} gives $R^c \wedge \theta^d - R^d \wedge \theta^c = 0$. If the vierbein is not singular it follows simply:

(I′) $$R^a = 0$$

which is the original equation defining ω^{ab} in terms of the derivative of the θ^d. If matter is present we have the action: $I = I_{grav.} + I_{matter}$. If the Lagrangian density of the matter contains ω^{ab} explicitly, as it happens with high spin fields, we find an additional term and the torsion will not vanish.

Before concluding this introduction we come to a final formal point.

Given a field configuration ω^{ab} and θ^a we can extend it to a configuration on a principal bundle by introducing an extra $\Lambda \in SO(3, 1)$ variable along the fibre and defining extended fields as:

(II) $$\omega_{ext}^{ab} = - (\Lambda^{-1} d\Lambda)^{ab} + (\Lambda^{-1})_s^a \omega_t^s \Lambda^{tb}$$
$$\Lambda_{ext}^a = (\Lambda^{-1}\theta)^a$$

In fact, (II) represents a generic, unspecified SO(3, 1) gauge transformation. For this reason we have also:

$$R_{ext}^{ab} = (d\omega^{ab} - \omega^{at} \wedge \omega_t^b)_{ext} = (\Lambda^{-1})^a{}_s R^s{}_t \Lambda^{tb}$$
$$R_{ext}^a = (\Lambda^{-1})^a = (\Lambda^{-1})^a{}_s R^s.$$

Then, we may write:

$$I_{grav.} = - \frac{c^3}{16\pi G} \int_{M^4} R_{ext}^{ab} \wedge \theta_{ext}^c \wedge \theta_{ext}^d \varepsilon_{abcd}.$$

The variational equations are now obtained by varying all fields independently and asking that I be stationary if integrated over a submanifold M^4

of the bundle. They look just the same as (I), (I′) with ω^{ab}_{ext}, θ^a_{ext} replacing ω^{ab}, θ^a. But curiously enough, the final content of the equation is just the same and one can give heuristic arguments in favour or (II) being the only solution, modulo inessential co-ordinate changes, of the variational equations. So, we may conceive the action principle and associated fields as living on the group manifold P, any 10-dimensional solution is actually « factorizable » into a trivial SO(3, 1) gauge transformation and the usual four-dimensional configuration. When this kind of mechanism is also obtained in more complicated theories, we can always factor out the gauge transformation of some subgroup $H \subset G$, where G is the full group (here $G = P$, $H = SO(3, 1)$). But this brings us to a final important comment. The set of forms ω^{ab}_{ext}, θ^a_{ext} would indicate that we are about to construct a theory which is Poincaré invariant.

Actually $I_{\text{grav.}}$ is only invariant under $H = SO(3, 1)$ although we started with the full set of G gauge fields.

So, we are faced with the problem of finding some principle, other than invariance, to restrict the choice of action. A number of interesting theories and possibly all the interacting ones, satisfy the following condition:

A) The Lagrangian density is a polynomial in the ω^A built using the operators d, $\wedge$ only without using the star Hodge operator $*$. In fact, in defining it we need to have a (pseudo) Riemannian structure on the manifold, but this cannot be given in any physical situation. For this reason it is wise to avoid it. This sometimes difficult; after all the Maxwell action must be written with the help of $*$; if A is the vector potential and $F = dA$ the field we have the action: $I = \frac{1}{8\pi}\int F \wedge F^*$. It is, however, possible to circumvent this obstacle in many different ways which will be examined later. Another, simple condition besides (A) is:

B) The variational equations must admit gauge null fields (flat space as solutions).

Therefore, they should be at least linear in the curvature R^A.

General relativity in all dimensions and supergravity in dimensions 4, 5, 6 satisfy A) and B). But also a number of physically unacceptable theories do so and must be ruled out by more conditions.

2. GRADED LIE ALGEBRAS AND SUPER LIE GROUPS

In this section we shall consider some fundamental notions of graded Lie algebras and introduce a notion of super Lie group which is not exactly that considered by Kostant in ref. [8]. Really, for a correct formulation of the « group manifold approach » it is not necessary to use the concept of supermanifold as given by Kostant which, even if dealt from the super-spaces point of view (see e. g. ref. [14]) presents some intriguing and not very well solved questions.

2.1. Fundamental functors on the category of graded vector spaces.

In this section we shall give a short account of fundamental definitions and results of graded Lie algebras by using the functorial language. This made in order to understand better the following considerations about super Lie groups from a completely covariant point of view.

Let $\mathbb{K}$ be the algebra of real $\mathbb{K} = \mathbb{R}$ or complex $\mathbb{K} = \mathbb{C}$ numbers. Let us introduce some fundamental categories.

1) $V_g(\mathbb{K})$ = *category of $\mathbb{Z}_2$-graded vector spaces over* $\mathbb{K}$ (1). A graded vector space V is one where one has fixed subspaces V_0 and V_1 called respectively the *even* and *odd* (*homogeneous*) *parts* of V such that $V = V_0 \oplus V_1$. Morphisms are graded linear maps with degree $\delta \in \mathbb{Z}_2$. Recall that $h \in \mathrm{Hom}_{V_g(\mathbb{K})}(V, W)$, with $V, W \in \mathrm{Ob}(V_g(\mathbb{K}))$, if $h(V_i) \subset W_{i+\delta}$, if the degree of h is $\delta = 0$ or $\delta = 1$ (2).

One has the functors: *i-degree* $P_i : V_g(\mathbb{K}) \to V_g(\mathbb{K})$, $P_i(V) = V_i$, $i = 0, 1$.

$V(\mathbb{K})$ is the sub-category of $\mathbb{K}$-vector spaces. (Any vector space has a natural $\mathbb{Z}_2$-grading $V = V \oplus \{0\}$). In particular, $\mathbb{K} \in \mathrm{Ob}(V(\mathbb{K}))$. If $v \in V_i$, we set $|v| = i$ and we call this integer the *degree* of v. Note, that Hom is a natural functor $V_g(\mathbb{K}) \times V_g(\mathbb{K}) \to V_g(\mathbb{K})$.

Similar considerations apply to the functor $\otimes$.

2) $A_g(\mathbb{K})$ = *category of graded algebras over* $\mathbb{K}$.

A graded algebra B is a graded vector space such that $B_iB_j \subseteq B_{i+j}$ and such that $1 \in B_0$. If x, y are homogeneous elements of $B \in \mathrm{Ob}(A_g(\mathbb{K}))$ one has $xy = (-1)^{|x||y|}yx$. In $A_g(\mathbb{K})$ can be distinguished a sub-category $A(\mathbb{K})$ of K-algebras such that $\mathrm{Ob}(A(\mathbb{K})) \subset \mathrm{Ob}(V(\mathbb{K}))$.

3) $\mathscr{L}A_g(\mathbb{K})$ = *category of graded Lie algebras.*

A graded Lie algebra is a graded vector space $\mathscr{G} = \mathscr{G}_0 \oplus \mathscr{G}_1$, together with a bilinear operation $[x, y]$ on $\mathscr{G}$ such that:

i) $[x, y] \in \mathscr{G}_{|x|+|y|}$; *ii)* $[x, y] = -(-1)^{|x||y|}[y, x]$; *iii)* (*graded Jacobi identity*)

$$(-1)^{|x||z|}[[x, y], z] + (-1)^{|y||x|}[[y, z], x] + (-1)^{|z||y|}[[z, x], y] = 0.$$

---

(1) Here $\mathbb{Z}_2 \equiv \{0, 1\}$ is an additive group with « addition » given by: $0 + 0 = 0, 0 + 1 = 1, 1 + 1 = 0, -1 = 1$.

(2) Ob (A) denotes the set of objects of a category A and Hom (A) denotes the set of morphisms of A.

Note, that $[x, y] = [y, x] \in \mathscr{G}_0$ if $x, y \in \mathscr{G}_1$. Further, if on $\mathscr{G}$ is chosen an adapted basis, the relations *ii*) and *iii*) can be written by using the *structure constants* $C^{\gamma}_{\alpha\beta}$:

ii') $C^{\gamma}_{\alpha\beta} = -(-1)^{|\alpha||\beta|}C^{\gamma}_{\beta\alpha}$;

iii') $(-1)^{|\alpha||\gamma|}C^{\omega}_{\alpha\delta}C^{\delta}_{\beta\gamma}+(-1)^{|\beta||\alpha|}C^{\omega}_{\beta\delta}C^{\delta}_{\gamma\alpha}+(-1)^{|\gamma||\beta|}C^{\omega}_{\gamma\delta}C^{\delta}_{\alpha\beta}=0.$

One has the following natural functors:

a) $j_0 : \mathscr{L}A_g(\mathbb{K}) \to \mathscr{L}A(\mathbb{K}) \equiv$ sub-category of Lie algebras, such that $\mathscr{G} \mapsto j_0(\mathscr{G}) = \mathscr{G}_0$;

b) $j_1 : \mathscr{L}A_g(\mathbb{K}) \to \mathscr{G}_A =$ sub-category of A-modules, (A = Lie algebra), such that $\mathscr{G} \mapsto j_1(\mathscr{G}) = \mathscr{G}_1$.

In the following table we list some important natural functors defined between the categories above defined.

TABLE 1. — *Some important functors regarding graded algebras.*

Functor	Definition	Name	Natural equivalences
$L: V_g(\mathbb{K}) \to A_g(\mathbb{K})$	$L(V) \equiv$ set of endomorphisms of V	*endo-morphism*	
$T: V_g(\mathbb{K}) \to A_g(\mathbb{K})$	$T(V) \equiv \bigoplus_{k\geq 0}(T^k(V) \equiv \underbrace{V\otimes\ldots\otimes V}_{k}), T^0(V) \equiv \mathbb{K}$	*tensor alg.*	
$S: V_g(\mathbb{K}) \to A_g(\mathbb{K})$	$S(V) \equiv T(V)/(x\otimes y+(-1)^{\lvert x\rvert\lvert y\rvert}y\otimes x)$	*symm. alg.*	$S \approx \otimes\circ(S\times\Lambda)\circ(\mathscr{F},\mathscr{F})\circ(j_0,j_1)$
$\Lambda: V_g(\mathbb{K}) \to A_g(\mathbb{K})$	$\Lambda(V) \equiv T(V)/(x\otimes y-(-1)^{\lvert x\rvert\lvert y\rvert}y\otimes x)$	*ext. alg.*	$\Lambda \approx \otimes\circ(\Lambda\times S)\circ(\mathscr{F},\mathscr{F})\circ(j_0,j_1)$
$\mathscr{L}: A_g(\mathbb{K}) \to \mathscr{L}A_g(\mathbb{K})$	$\mathscr{L}(A) \equiv$ Lie graded alg. with $[x, y] \equiv xy-(-1)^{\lvert x\rvert\lvert y\rvert}yx$	*gr. Lie alg.*	
$U: \mathscr{L}A_g(\mathbb{K}) \to A_g(\mathbb{K})$	$U(A) \equiv T(A)/(x\otimes y-(-1)^{\lvert x\rvert\lvert y\rvert}y\otimes x-[x, y])$	*univ. env. alg.*	$U \mapsto \mathscr{L}$
$\mathrm{Der}: A_g(\mathbb{K}) \to A_g(\mathbb{K})$	$\mathrm{Der}(A) \equiv$ graded derivations of the graded algebra A	*deriv. alg.*	
$C: \mathscr{I}_g(\mathbb{K}) \to A_g(\mathbb{K})$	$C(V, g) \equiv T(V)/(x\otimes x-g(x, x)1)$	*Clifford alg.*	$C \approx \otimes\circ(\Lambda\times C)\circ(j_0,j_1)$

In this table 1 $\mathscr{F} : A_g(K) \to V_g(K)$ is the forgetting functor and $U \vdash \mathscr{L}$ denotes that U is the left adjoint of $\mathscr{L}$ (For details on the functorial language see e. g. refs. [*15*] [*16*]). Further, $\mathscr{I}_g(\mathbb{K})$ denotes the category of isometric finite dimensional $\mathbb{Z}_2$-graded inner product spaces over $\mathbb{K}$, such that if $(V \equiv V_0 \oplus V_1, g) \in \mathrm{Ob}(\mathscr{I}_g(\mathbb{K}))$, one has $V_0 =$ the null space of g and on V_1, g is not degenerate.

Example (*The* $\mathbb{Z}_2$*-graded algebra of the derivations of a superspace*). —

Superspaces are to-day extensively used in unified field theories. They interpret in a fully covariant way the Kostant structures of the supermanifolds.

Let Sup(M) be the category of superspaces over a manifold M (See also ref. [*11*]). An object of Sup(M) is given by the following sequence $\Lambda B \to B \to M$ where $B \to M$ is a vector bundle over M. We have two natural functors defined on Sup(M):

a) $C_0^\infty : \mathrm{Sup}(M) \to A_g(F(M))$; $\Lambda B \to B \to M \mapsto C_0^\infty(\Lambda B) \equiv$ graded algebra over F(M) ([3]) of C^∞-sections of $\Lambda B \to M$ with compact support. The structure of $\mathbb{Z}_2$-graded algebra is with respect to the exterior product: $\alpha \wedge \beta = (-1)^{|\alpha||\beta|}\beta \wedge \alpha$.

b) $\mathscr{D}_{er} : \mathrm{Sup}(M) \to \mathscr{L}A_g(F(M))$; $\Lambda B \to B \to M \mapsto \mathscr{D}_{er}(\Lambda B) \equiv$ graded derivations of $C_0^\infty(\Lambda B)$. One has $\mathscr{D}_{er} = \mathscr{L} \circ D_{er} \circ C_0^\infty$.

2.2. **Super Lie groups.**

To an arbitrary graded Lie algebra $\mathscr{G} = \mathscr{G}_0 \oplus \mathscr{G}_1$ we cannot in a natural way associate a supermanifold which should be also a Lie group as one usually does for ungraded Lie algebras. In fact, the so-called « graded Lie group », introduced by Kostant [*8*], associated with a graded Lie algebra, is a supermanifold identifiable with a superspace $\Lambda B \to B \to G$ (where G is the Lie group corresponding to $\mathscr{G}_0$ and B is a suitable vector bundle over G) and has the total space ΛB which has not a natural structure of Lie group.

However, the motivation of why we are conduced to consider graded Lie algebras is that these arise from the problematic of field unification. In fact, the group structure of a unified field theory must be an extension of some sub-group corresponding to non unified fields. In particular, we see that central extensions of a Lie group are related to extensions of the corresponding algebra and to a $\mathbb{Z}_2$-grading.

We recall some fundamental definitions.

DEFINITION 2.1. — We say that an extension

$$1 \to H \to G \xrightarrow{i} K \xrightarrow{p} 1 \tag{1}$$

of a Lie group K by means of another Lie group H is *central* if $i(H)$ is contained in the center of G (This implies that H is commutative and that G contains a subgroup H′ isomorphic to H such that $G/H' \cong K$).

Taking the Lie algebras of the group sequence (1) we obtain the following exact sequence:

$$0 \to A(H) \xrightarrow{i_*} A(G) \xrightarrow{p_*} A(K) \to 0 \tag{2}$$

([3]) F(M) is the algebra of numerical functions over M.

of Lie algebras homomorphisms. The above sequence is an extension of Lie algebra with abelian kernel A(H). Then, it is well known (see e. g. ref. [*16*]) that if $s : A(K) \to A(G)$ is a section, that is a $\mathbb{K}$-linear map such that $p \circ s = id_{A(K)}$ we can define in $i_*(A(H))$ and hence in A(H) an A(K)-module structure by

$$\text{(M)} \qquad x \,.\, i_*(a) = [s(x), i_*(a)], \qquad a \in A(H),\ x \in A(K)$$

where [,] denotes the bracket in A(G). Since, A(H) is abelian the A(K)-action thus defined on A(H) does not depend upon the choice of sections. This A(K)-module structure on A(H) is called *induced by the extension.*

Now, we have the following

DEFINITION 2.2. — An *extension* of the Lie algebra h by an h-module A is an extension of Lie algebras $0 \to A \to \mathcal{G} \to h \to 0$, with abelian kernel such that the given h-module structure in A agrees with the one induced by the extension.

So, we have the following

PROPOSITION 2.1. — To any central extension of a group K : $1 \to H \to G \to K \to 1$ there corresponds an extension of the Lie algebra A(K) with the A(K)-module H where the action of A(K) on H is given by (M) :

$$(3) \quad 0 \to H \to A(G) \to A(K) \to 0, \quad A(G) \cong A(K) \oplus H\,.$$

Note, now, that on A(G) we can recognize a structure of $\mathbb{Z}_2$-graded Lie algebra if an additional structure is defined on H. In fact, we have the following

PROPOSITION 2.2. — Let $0 \to A \to \mathcal{G} \to h \to 0$ be an extension of the Lie algebra h by an h-module A. Let $\tau : h \to \mathcal{L} \circ L(A)$ denote the Lie algebra representation which gives to A the structure of h-module. Then, $\mathcal{G} \cong h \oplus A$ becomes a $\mathbb{Z}_2$-graded Lie algebra if on A is defined a bilinear map $\rho : A \times A \to h$ which satisfies the following requirements:

$$
\begin{array}{ll}
& i) \ \text{(symmetry)}\ \rho(u, v) = (v, u),\ u, v \in A\,; \\
& ii) \ \mathrm{Ad}\,(x)\,(\rho(u, v)) = [x, \rho(u, v)] = \rho(\tau(x)(u), v) + \rho(u, \tau(x)(v)) \\
(4) & \qquad\qquad \forall x \in h\,; \quad u, v \in A\,; \\
& iii) \ \tau(\rho(u, v))(w) + \tau(\rho(v, w))(u) + \tau(\rho(w, u))(v) = 0 \quad \forall u, v, w \in A.
\end{array}
$$

Proof. — In fact, we can define the bracket $[,]_\rho$ as $[x, y]_\rho \equiv [x, y] \in h$, $[x, u]_\rho \equiv -[u, x]_\rho = \tau(x)(u) \in A \;\; [u, v]_\rho \equiv \rho(u, v) \in h$.

Then, $\mathcal{G} = h \oplus A$ with this parenthesis becomes a $\mathbb{Z}_2$-graded Lie algebra. □

Example. — Let K be any Lie group and H be a vector space. Let $\tau : K \to \mathrm{Aut}\,(H)$ be a representation of K in H, so that H becomes a K-module.

Then, the exterior semidirect product (see e. g. refs. [*11*] [*17*]) $K \times_\tau H$ is a Lie group which is a central extension of K by means of H. Now, let us consider the semidirect product $A(K) \times_\tau H$ that is the Lie algebra with underlying vector space $A(K) \oplus H$ endowed with the following parenthesis: $[(a, x), (b, y)] = ([a, b], D\tau(a)(y) - D\tau(b)(y))$.

One can see that the split extension

$$0 \to H \to A(K) \times_\tau H \to A(K) \to 0 \tag{5}$$

is an extension of the Lie algebra A(K) by the A(K)-module H. Further, (5) is just the sequence of Lie algebras canonically associated to

$$1 \to H \to K \times_\tau H \to K \to 1 ;$$

Finally, if on H is defined a bilinear map $\rho : H \times H \to A(K)$ satisfying properties (4) the parenthesis $[,]_\rho$ gives to $A(K) \times_\tau H$ a structure of $\mathbb{Z}_2$-graded Lie algebra.

Now, we are able to give the following fundamental

DEFINITION 2.3. — A *super Lie group* is a Lie group such that on the tangent space T_eG of the unit $e \in G$ there exists beside the usual Lie algebra structure a structure of graded Lie algebra.

PROPOSITION 2.3. — Any central extension of a group K:

$$1 \to H \to G \to K \to 1$$

such that on H is defined a bilinear map $\rho : H \times H \to A(K)$ which satisfies the requirements (4) with $\tau: A(K) \to \mathscr{L} \circ L(H)$ the representation induced by the extension, is a super Lie group.

Proof. — This is a direct consequence of Proposition 2.2 and Definition 2.3. □

Examples. — 1) Let $M \overset{(g,\eta)}{\longrightarrow} T^0_2M \cong S^0_2M \oplus \Lambda^0_2M$ be a 2-dimensional oriented affine space with metric g and volume form η ([4]). Let $\underline{M}$ be the space of free vectors of M. The symmetry group of such structure is denoted by A_2. By choosing an adapted coordinate system on M we have a Lie group isomorphism $A_2 \cong M_2$, where M_2 is the group of Euclidean motions in $\mathbb{R}^2$, that is the group of matrices like $\begin{pmatrix} h & a \\ 0 & 1 \end{pmatrix}$ where $h \in SO(2)$ and $a \in \mathbb{R}^2$. Let us denote by T and SO(M) the subgroups of A_2 corresponding respectively to translations and proper rotations in M. T can be identified with $\underline{M}$. So, we put $T \equiv \underline{M}$.

([4]) Let X be an n-dimensional manifold and TX, T*X be respectively the tangent and cotangent bundle of X. Set $T^p_qX \equiv TX \otimes \overset{p}{\cdots} \otimes TX \otimes T^*X \otimes \overset{q}{\cdots} \otimes T^*X$. We denote by Λ^p_qX and S^p_qX respectively the skewsymmetric subbundle and symmetric subbundle of T^p_qX.

Then, we can see that A_2 is a central extension of SO(M) by means of $\underline{M}$:

$$1 \to \underline{M} \to A_2 \to SO(M) \to 1. \tag{6}$$

More precisely, $A_2 = SO(M) \times_\tau \underline{M}$ where $\tau : SO(M) \to \mathrm{Aut}(\underline{M})$ is given by $\tau(k)(h) = k^{-1}hk$. The Lie algebra sequence corresponding to (6) is $0 \to \underline{M} \to A(A_2) \to A(SO(M)) \to 0$, $A(A_2) \cong A(SO(M)) \oplus \underline{M}$, and it is naturally $\mathbb{Z}_2$-graded. In fact the structure constants C^k_{ij} are given by $C^k_{12}=0$, $C^k_{31}=-C^k_{13}=\delta^k_2$, $C^k_{23}=-C^k_{32}=\delta^k_1$. So, we can see that A_2 has a natural structure of super Lie group.

2) Let us consider the following central extension: $1 \to E \to GP \to P \to 1$ of the (*special*) *Poincaré group* P (= group of symmetry of the oriented Minkowskian space-time M) with the vector space E ≡ charge conjugation invariant part of the 4-dimensional complex space $S \equiv \Lambda \underline{M}^c$ of Dirac spinors; $P = SO(M) \times_\tau \underline{M}$, where $\underline{M}$ is the space of free vectors of M and SO(M) is the Lorentz group of M. τ is the natural representation of SO(M) on $\underline{M}$; $\tau(k)(h) = k^{-1}hk$.

Let SO(M) act on E by means of the natural spin representation. GP is called the *super-Poincaré group* (or *graded Poincaré group*). The Lie algebra A(GP) has the splitting $A(GP) \cong A(P) \oplus E$ and it is naturally endowed with a $\mathbb{Z}_2$-graded bracket as can be easily seen by considering the structure constants of GP:

$$C^{\mu\nu}{}_{\omega\alpha}{}^{\rho\delta}=\eta^{\mu\delta}\delta^\nu_\omega\delta^\rho_\alpha-\eta^{\mu\rho}\delta^\nu_\omega\delta^\delta_\alpha+\eta^{\nu\rho}\delta^\mu_\omega\delta^\delta_\alpha-\eta^{\nu\delta}\delta^\mu_\omega\delta^\rho_\alpha$$

$$C^{\mu\nu A}_{\ \ \ B}=\eta^{\nu A}\delta^\mu_B-\eta^{\mu A}\delta^\nu_B,\quad C^{\mu\nu}{}_a{}^b=-\frac{1}{2}(\sigma^{\mu\nu})_a{}^b,$$

$$C^{\mu\nu}{}_{\dot a}{}^{\dot b}=-\frac{1}{2}(\sigma^{\mu\nu})^{\dot b}{}_{\dot a},\quad C_{a\dot b\mu}=(\sigma_\mu)_{a\dot b}$$

where $\eta^{\mu\nu}=0$ if $\mu\neq\nu$, $\eta^{00}=-\eta^{11}=-\eta^{22}=-\eta^{33}=1$. $\sigma^{\mu\nu}=\frac{1}{2}[\gamma^\mu,\gamma^\nu]$, γ^μ are the Dirac matrices:

$$(\gamma^\mu)^a{}_b=(\gamma^\mu)^{\dot a}{}_{\dot b}=0,\qquad (\gamma^\mu)^a{}_{\dot b}=-(\sigma^\mu)^a{}_{\dot b},\qquad (\gamma^\mu)^{\dot a}{}_b=(\sigma^\mu)^{\dot a}{}_b,$$

where $(\sigma^\mu)_{a\dot b}$ are the elements of Pauli matrices where the indices are lowered by means of the antisymmetric matrices: $(\varepsilon_{ab})=(\varepsilon^{ab})$, $(\varepsilon_{\dot a\dot b})=(\varepsilon^{\dot a\dot b})$, $\varepsilon_{12}=\varepsilon_{\dot 1\dot 2}=1$. The greek and capital italic indices run from 0 to 3; italic indices (dotted and undotted) run from 1 to 2.

One can see that the symmetric bilinear map ρ: $E \times E \to A(P)$ is now just $\rho(u, v) = [u, v]$.

3. DIFFERENTIAL EQUATIONS, CONNECTIONS AND CURVATURE

In this section we shall consider connections and curvature directly related to differential equations. This is of particular interest, as it allows us to consider connections as particular constraints on the space of states in a gauge theory (See also refs. [*11*] [*18*] [*19*] [*20*] [*21*] [*22*]).

3.1. Fundamental results of formal theory in differential equations.

Let $\pi : W \to X$ be a fiber bundle over a manifold X and TW be the tangent bundle of W. We denote by vTW the vertical tangent subbundle of TW. Let $J\mathcal{D}^k(W)$ be the k-jet-derivative bundle of W [*11*] [*24*]. We have a natural injection

$$0 \to \pi^*_{k,0}(S^0_k X \otimes vTW) \overset{\mu_{(k)}}{\to} vTJ\mathcal{D}^k(W)$$

such that

$$(7) \quad 0 \to \pi^*_{k,0} S^0_k X \otimes vTW \overset{\mu_{(k)}}{\to} vTJD^k(W) \to \pi^*_{k,k-1} vTJ\mathcal{D}^{k-1}(W) \to 0$$

is an exact sequence of vector bundles over $J\mathcal{D}^k(W)$.

Let $\pi' : E \to X$ be another fiber bundle over X. We define the *symbol* of a differential operator $K. : C^\infty(W) \to C^\infty(E)$ characterized by a fiber bundle morphism $K : J\mathcal{D}^k(W) \to E$ the bundle map $\sigma_k(K) \equiv vT(K) \circ \mu_{(k)} : S^0_k X \otimes vTW \to vTE$. The *rth-prolongation* of K is the differential operator of order $k + r$ $K^{(r)} : C^\infty(W) \to C^\infty(J\mathcal{D}^r(E))$ characterized by the following fiber bundle morphism $K^{(r)} = J\mathcal{D}^r(K) \circ \beta_{r,k} : J\mathcal{D}^{k+r}(W) \to J\mathcal{D}^r(E)$, where $\beta_{r,k}$ is the canonical injection $J\mathcal{D}^{k+r}(W) \to J\mathcal{D}^r(J\mathcal{D}^k(W))$. The symbol $\sigma_{k+r}(K^{(r)})$ of $K^{(r)}$ is the rth-prolongation of the symbol $\sigma_k(K)$ and is given by $\sigma_{k+r}(K^{(r)}) : S^0_{k+r} X \otimes vTW \to vTJ\mathcal{D}^r(E)$.

In physics differential equations are generally obtained as kernel of differential operators. So, if f is a fixed section of E, a differential equation of order k is obtained by setting $E_k \equiv \ker_f K \subset J\mathcal{D}^k(W)$, and prolongated equations are $E_{k+r} = \ker_{D^r f} K^{(r)} \subset J\mathcal{D}^{k+r}(W)$. The corresponding *symbols* are $g_k = \ker \sigma_k(K)$ and $g_{k+r} = \ker \sigma_{k+r}(K^{(r)})$. Note, that differential equations can be defined independently on any differential operator, as subbundles of $J\mathcal{D}^k(W)$ over X for some $k \geq 0$. Then, the symbols g_k and their prolongations can be defined as $g_k \equiv \mu_{(k)}(S^0_k X \otimes vTW) \cap vTE_k$ and $g_{k+r} = \mu_{(k+r)}(S^0_{k+r} X \otimes vTW) \cap vTE_{k+r}$, being $E_{k+r} \equiv J\mathcal{D}^r(E_k) \cap J\mathcal{D}^{k+r}(W)$. g_{k+r} coincides with the kernel of the following compositions of homomorphisms:

$$(7') \quad S^0_{k+r} X \otimes vTW \to S^0_r X \otimes S^0_k X \otimes vTW \to S^0_r X \otimes (S^0_k X \otimes vTW/g_k).$$

Further, we have the following exact sequence of vector spaces over E_{k+r} :

$$(8) \qquad 0 \to \pi^*_{k+r,k} g_{k+r} \to vTE_{k+r} \to \pi^*_{k+r,k+r-1} vTE_{k+r-1}$$

A differential equation $E_k \subset J\mathcal{D}^k(W)$ is *formally integrable* if for $r \geq 0$,

g_{k+r+1} is a vector bundle over E_k and the map $\pi_{k+r+1,k+r} : E_{k+r+1} \to E_{k+r}$ is surjective.

Taking into account that the following propositions are equivalent:

(9) 1) Any prolongation E_{k+r}, $0 < r \leq h$ is an affine subbundle of $J\mathcal{D}^{k+r}(W) \mid E_{k+r-1}$ modeled on the vector bundle $\pi^*_{k+r,k+r-1} g_{k+r}$ over E_{k+r-1};

(9) 2) g_{k+r-1} is a vector bundle over E_k and $E_{k+r+1} \to E_{k+r}$ is surjective for $0 < r \leq h$;

We try that the formal integrability is equivalent to say that any prolongation E_{k+r}, $r \geq 1$ has a structure of affine subbundle of $J\mathcal{D}^{k+r}(W)$ over E_{k+r+1} with associated vector bundle $\pi^*_{k+r,k+r-1} g_{k+r} \to E_{k+r-1}$.

Associated to any differential equation $E_k \subset J\mathcal{D}^k(W)$ one recognizes a family of vector spaces $\{ H^{m-j,j}_q \}_{q \in E_k}$, $m \geq k, j, k \in \mathbb{N}$, where $H^{m-j,j}$ is the cohomology of $\Lambda^0_j X \otimes g_{m-j}$ (*Spencer cohomology*) of the first Spencer complex:

(10) $$g_m \xrightarrow{\delta^0} T^*X \otimes g_{m-1} \xrightarrow{\delta^1} \Lambda^0_2 X \otimes g_{m-2} \xrightarrow{\delta^2} \dots \to \Lambda^0_r X \otimes g_{m-r} \to \dots \to \Lambda^0_{m-k} X \otimes g_k \xrightarrow{\delta^{m-k}} \Lambda^0_{m-k+1} X \otimes S^0_{k+1} X \otimes vTW$$

with the coboundaries δ^m the vector morphisms canonically induced by the coboundaries

$$d^m : \begin{cases} d^0 : S^0_m V \to T^*V \otimes S^0_{m-1} V & \text{(canonical monomorphism)} \\ d^j : \Lambda^0_j V \otimes S^0_{k-j} V \to \Lambda^0_{j+1} V \otimes S^0_{k-j-1} V, & d^j(\omega \otimes u) = (-1)^j \omega \wedge d^0(u), \end{cases}$$

being V a vector space.

If $H^{m,j} = 0$ E_k is called *involutive*. If $H^{m,j} = 0$ for $m \geq k$, $0 \leq j \leq r$, E_k is called *r-acyclic*. Further, one can see that there exists an integer $k_0 \geq k$ depending on the dimension of X, the dimension of W and k such that E_{k_0} is involutive. If $E_k \subset J\mathcal{D}^k(W)$ is a differential equation such that the map $E_{k+1} \to E_k$ is surjective and g_{k+1} is a vector bundle over E_k then there is a morphism over E_k:

$$\mathcal{k}(E_k) : E_{k+1} \to C^2 \equiv \{ \Lambda^0_2 X \otimes vTE_k \} / \delta(TX \otimes g_{k+1})$$

such that one has the following exact sequence:

(11) $$E_{k+2} \to E_{k+1} \underset{0 \circ \pi_{k+1,k}}{\overset{\mathcal{k}(E_k)}{\rightrightarrows}} C^2 ,$$

$\mathcal{k}(E_k)$ is called *curvature* of E_k. More precisely, $\mathcal{k}(E_k)$ has values into Spencer cohomology space

$$H^{k,2} \equiv \frac{\ker \{ \delta : \Lambda^0_2 X \otimes g_k \to \Lambda^0_3 X \otimes S^0_{k-1} X \otimes vTW \}}{\delta(T^*X \otimes g_{k+1})} \subset C^2 .$$

Further, iff g_{k+2} is a vector bundle over E_k one has the following exact sequence of vector bundles over E_k :

$$0 \rightarrow g_{k+2} \rightarrow T^*X \otimes g_{k+1} \rightarrow \Lambda_2^0 X \otimes vTE_k \rightarrow C^2 .$$

We have the following important

THEOREM 3.1. — If $E_k \subset J\mathscr{D}^k(W)$ is a differential equation such that the map $E_{k+1} \rightarrow E_k$ is surjective, g_{k+1} is a vector bundle over E_k and g_k is 2-acyclic, then E_k is formally integrable.

Proof. — Since g_k is 2-acyclic we have that $H^{k,2} = 0$, so the exactness of sequence (11) sayies that $E_{k+2} \rightarrow E_{k+1}$ is surjective. Further, one can prove that if g_k is 2-acyclic and g_{k+1} is a vector bundle then g_{k+r}, $r \geq 1$ is a vector bundle over E_k. Then, proceeding by induction and by using the fact (10) we conclude that E_k is formally integrable. □

Finally, we have the following criterion of integrability.

THEOREM 3.2. — Let $E_k \subset J\mathscr{D}^k(W)$ be a differential equation. Then, there is an integer $k_0 \equiv k + h \geq k$ depending only on k, the dimension of X and the dimension of W, such that E_{k_0} is involutive and such that if g_{k+r+1} is a vector bundle over E_k and $\pi_{k+r+1,k+r}: E_{k+r+1} \rightarrow E_{k+r}$ is surjective for $0 \leq r \leq h$, then E_k is formally integrable.

A remarkable application of above theorem is an alternative form of the Frobenius theorem for r-distributions $\mathbb{E} \subset TX$ over a manifold X.

THEOREM 3.3. — Let $\mathbb{E} \subset TX$ be a r-distribution on X. Let $\{ \xi_\alpha \}_{\alpha=1,\dots,r}$ be r independant vector fields defined on $U \subset X$ which span the distribution at each point of U. Then, $\mathbb{E}$ is completely integrable (involutive) iff the following first order linear system

$$\xi_\alpha . \phi \equiv \xi_\alpha^i \frac{\partial \phi}{\partial x_i} = 0$$

is involutive formally integrable.

Let us, now, consider another important concept.

DEFINITION 3.1. — A differential equation $E_k \subset J\mathscr{D}^k(W)$ is *completely integrable* if for any $q \in E_k$ there exists a local section $s : U \subset X \rightarrow W$, such that $D^k s(x) = q$, where $\pi_k(q) = x$, and $D^k s(U) \subset E_k$ (This implies that the map $E_{k+r} \rightarrow E_k$, $r > 0$ is surjective).

The formal integrability does not assure the completely integrability (see e. g. ref. [*20*]) and a completely integrable differential equation is not necessarily formally integrable [*18*]. However, in the analytic case the formal integrability implies the completely integrability.

However, the differential equations to which we are interested in this section are that associated to k-connections and are characterized by submanifolds E_k of $J\mathscr{D}^k(W)$ diffeomorphic to $\pi_{k,k-1}(E_k) \equiv E_{k-1}$.

So, in the following of this section we shall consider a theorem of integrability for such systems. Let us, first give some useful definition.

DEFINITION 3.2. — 1) The *k-order sesquiholonomic prolongation* of W is the kernel $\overset{\cup}{J}\mathscr{D}^k(W)$ of the following double flèche:

$$\begin{array}{ccc} J\mathscr{D}J\mathscr{D}^{k-1}(W) & \longrightarrow & J\mathscr{D}J\mathscr{D}^{k-2}(W) \\ \downarrow & & \\ J\mathscr{D}^{k-1}(W). & & \end{array}$$

2) The *k-order semiholonomic prolongation* of W is the kernel $\bar{J}\mathscr{D}^k(W)$ of the following double flèche:

$$\begin{array}{ccc} J\mathscr{D}\bar{J}\mathscr{D}^{k-1}(W) & \longrightarrow & J\mathscr{D}\bar{J}\mathscr{D}^{k-2}(W) \\ \downarrow & & \\ \bar{J}\mathscr{D}^{k-1}(W). & & \end{array}$$

$$\bar{J}\mathscr{D}(W) = J\mathscr{D}(W).$$

One has: (*a*) $J\mathscr{D}^k(W) \subset \overset{\cup}{J}\mathscr{D}^k(W) \subset \bar{J}\mathscr{D}^k(W)$; (*b*) $\overset{\cup}{J}\mathscr{D}(W) = J\mathscr{D}(W) = \bar{J}\mathscr{D}(W)$: (*c*) $\bar{J}\mathscr{D}^2(W) = \overset{\cup}{J}\mathscr{D}^2(W)$.

Similarly, for an equation $E_k \subset J\mathscr{D}^k(W)$ we set $\overset{\cup}{E}_{k+r} \equiv J\mathscr{D}^r(E_k) \cap \overset{\cup}{J}\mathscr{D}^{k+r}(W)$ and $\bar{E}_{k+r} \equiv J\mathscr{D}^r(E_k) \cap \bar{J}\mathscr{D}^{k+r}(W)$. One has $E_{k+r} \subset \overset{\cup}{E}_{k+r} \subset \bar{E}_{k+r}$. Then, we have the following fundamental theorem of integrability.

THEOREM 3.4. — Let $E_k \subset J\mathscr{D}^k(W)$ be a differential equation diffeomorphic to its projection $\pi_{k,k-1}(E_k) \equiv E_{k-1} \subset J\mathscr{D}^{k-1}(W)$. Then, E_k is completely integrable iff $E_{k+1} = \overset{\cup}{E}_{k+1}$.

Proof. — Note, that as $E_k \cong E_{k-1}$ one has also that $\overset{\cup}{E}_{k+1} \cong E_k$. Then, since $E_{k+1} \subset \overset{\cup}{E}_{k+1}$, $E_{k+1} \to E_k$ is surjective iff $\overset{\cup}{E}_{k+1} = E_{k+1}$. On the other hand the diffeomorphism $E_{k-1} \cong E_k$ implies that E_k can be written in local coordinates as

$$y^j_{i_1\ldots i_k} = F^j_{i_1\ldots i_k}(x^\alpha, y^j, \ldots, y^j_{i_1\ldots i_{k-1}}).$$

So, E_k identifies a distribution $\mathbb{E}$ which since $E_{k+1} \to E_k$ is assumed surjective and $g_k = 0$, the Theorem 3.3 assures to be completely integrable. □

3.2. **Curvature of a connection.**

The concept of curvature of a connection on a fiber bundle can be related to that of curvature map of a differential equation.

DEFINITION 3.3. — 1) Let $\pi : W \to X$ be a fiber bundle.

A *k-connection* on W is a *k*-order differential equation $C_k \subset J\mathscr{D}^k(W)$ diffeomorphic to $J\mathscr{D}^{k-1}(W)$ by means of the projection

$$\pi_{k,k-1} : J\mathscr{D}^k(W) \to J\mathscr{D}^{k-1}(W).$$

2) C_k is called *flat* if C_k is completely integrable.

3) We say that a fiber bundle $\pi : W \to X$ is *flat* if one can define on it a flat 1-connection.

A *k*-connection is equivalent to assign an affine fiber bundle morphism $\ulcorner : J\mathscr{D}^k(W) \to \pi^*_{k-1,0}(S^0_k X \otimes vTW)$ over $J\mathscr{D}^{k-1}(W)$ such that $\overset{v}{D}\ulcorner = 1$, or to assign a section $\urcorner : J\mathscr{D}^{k-1}(W) \to J\mathscr{D}^k(W)$ to the affine bundle $\pi_{k,k-1}$ modeled on the vector bundle $\pi^*_{k-1,0}(S^0_k X \otimes vTW)$.

$\overset{v}{D}\ulcorner$ denotes the vertical derivative of $\ulcorner$ with respect to the fiber bundle structure $\pi_{k,k-1}$ of $J\mathscr{D}^k(W)$. So that 1 denotes the identity map

$$\pi^*_{k-1,0}(S^0_k X \otimes vTW) \to \pi^*_{k-1,0}(S^0_k X \otimes vTW).$$

One has

$$\ulcorner = \mathrm{id}_{J\mathscr{D}^k(W)} - \urcorner \circ \pi_{k,k-1} \qquad \text{and} \qquad \ulcorner \circ \urcorner = 0.$$

If $W \equiv E$ is a vector bundle over X, then $\pi_s : J\mathscr{D}^s(E) \to X$ are vector bundles for $k \geq s \geq 0$ and we can define a *linear k-connection* as a *k*-connection $C_k \subset J\mathscr{D}^k(E)$ which is also a linear differential equation.

This is equivalent to assign a section $\urcorner : J\mathscr{D}^{k-1}(E) \to J\mathscr{D}^k(E)$ which is a fiber homomorphism of vector bundles over X (See also ref. [*24*]).

Assigned a *k*-connection on W we can define *k-th absolute derivative* of a section $s : X \to W$ as $\overset{(k)}{\nabla} s \equiv \ulcorner \circ D^k s : X \to S^0_k X \otimes vTW$. If $W = E$ is a vector bundle one has $\overset{(k)}{\nabla} s : X \to S^0_k X \otimes E$. A first order linear connection determines a splitting of the following exact sequence of vector bundles over:

$$0 \to T^*X \otimes E \underset{\ulcorner}{\rightleftarrows} J\mathscr{D}(E) \underset{\urcorner}{\rightleftarrows} E \to 0.$$

So a linear connection identifies a splitting $J\mathscr{D}(E) \cong T^*X \otimes E \oplus_X \urcorner(E)$. More generally any first order connection can be identified with a splitting of the following exact sequence of vector bundle over W:

$$0 \to vTW \underset{\ulcorner}{\rightleftarrows} TW \underset{\urcorner}{\rightleftarrows} \pi^*TX \to 0. \tag{12}$$

(See also ref. [*11*]). So, a connection identifies a splitting

$$TW \cong vTW \oplus_W \urcorner(\pi^*TX).$$

More precisely, the relation between the connection $\ulcorner : J\mathscr{D}(W) \to T^*X \otimes vTW$ and the splitting (12) is obtained by means of the canonical embedding of $J\mathcal{D}(W)$ in the derivative space [*24*] $\mathcal{D}(W) \cong T^*X \otimes TW$. Therefore, the following diagram

$$\begin{array}{ccc} \mathcal{D}(W) & \xrightarrow{\check{\ulcorner} \equiv \mathrm{id}_{T^*X} \otimes \check{\ulcorner}} & T^*X \otimes vTW \\ \uparrow & \nearrow_{\ulcorner} & \\ J\mathcal{D}(W) & & \end{array}$$

is commutative. In particular, for a vector bundle $W = E$ one has:

$$\begin{array}{ccc} \mathcal{D}(E) & \xrightarrow{\check{\ulcorner}} & T^*X \otimes vTE \cong T^*X \otimes E \oplus E \\ \uparrow & & \downarrow \\ J\mathcal{D}(E) & \xrightarrow{\qquad \ulcorner \qquad} & T^*X \otimes E\,. \end{array}$$

Moreover, by considering the sequence (7) we check that a k-connection $\urcorner$ determines a splitting $vTJD^k(W) \cong S^0_kX \otimes vTW \oplus vT(\urcorner)(vTJ\mathcal{D}^{k-1}(W))$.

Note. — If E and F are vector bundles on X and C_k, C'_k are linear k-connections on E and F respectively, we can canonically induce linear k-connections $C^{\oplus}_k$, $C^{\otimes}_k$ on the Withney sum $E \oplus F$ and on the tensor product $E \otimes F$ respectively. More precisely, $C^{\oplus}_k = C_k \oplus C'_k \subset J\mathcal{D}^k(E \oplus F) \cong J\mathcal{D}^k(E) \oplus J\mathcal{D}^k(F)$ and $C^{\otimes}_k = J\mathcal{D}^k(p)(C^{\oplus}_k) \subset J\mathcal{D}^k(E \otimes F)$, where p is the canonical projection $E \oplus F \cong E \times F \to E \otimes F$.

Finally, given a first order linear connection $C_1 \subset J\mathcal{D}(E)$, there is a unique linear connection $C^*_1 \subset J\mathcal{D}(E^*)$ on the dual bundle $E^* \to X$ such that the following sequence

$$0 \to C_1 \times_X C^*_1 \hookrightarrow J\mathcal{D}(E \times E^*) \underset{0 \circ \pi_1}{\overset{\langle , \rangle_*}{\rightrightarrows}} T^*X$$

of vector bundles over X is exact, where $\langle , \rangle_*$ is the homomorphism $J\mathcal{D}(E \times E^*) \to J\mathcal{D}(X \times \mathbb{R}) \cong T^*X$ induced by the pairing $\langle , \rangle$.

DEFINITION 3.4. — Let $\mathcal{P} = (P, X, \pi; G)$ be a principal fiber bundle over X. A connection $C_1 \equiv \urcorner(P) \subset J\mathcal{D}(P)$ on P determines a A(G)-valued differential form $\urcorner\omega \equiv \mathrm{id}_{T^*P} \otimes \psi \circ \check{\ulcorner} : P \to T^*P \otimes A(G)$, where $\psi_p : vT_pP \to A(G)$ is the isomorphism induced by the action map $\phi : G \times P \to P$. Then, C_1 is a *principal connection* if $\phi^*_a \urcorner\omega = ada^{-1} \circ \urcorner\omega$, $\forall a \in G$. $\urcorner\omega$ is called the *Ehresmann connection* (See also ref. [*11*]). So, a principal connection is defined by the following commutative diagrams:

$$\begin{array}{ccc} P & \xrightarrow{\urcorner\omega} & T^*P \otimes A(G) \\ & \searrow_{\ulcorner} & \uparrow{\scriptstyle \mathrm{id}_{T^*P} \otimes \psi} \\ & & T^*P \otimes vTP \end{array} \qquad \begin{array}{ccc} T^*P \otimes A(G) & \xleftarrow{\mathbb{B}(\phi_a^{-1}) \equiv T^*(\phi_a) \otimes \mathrm{ad}(a)} & T^*P \otimes A(G) \\ {\scriptstyle \mathrm{ad}(a)^* \urcorner\omega}\uparrow & & \uparrow{\scriptstyle \urcorner\omega} \\ P & \xrightarrow{\qquad \phi_a \qquad} & P \end{array}$$

DEFINITION 3.5. — Given a k-connection $C_k = \urcorner(J\mathscr{D}^{k-1}(W)) \subset J\mathscr{D}^k(W)$ we define *curvature* of C_k the composition map

$$_\neg R \equiv pr \circ \urcorner^{(1)} | C_k : C_k \to \overset{\cup}{J}\mathscr{D}^{k+1}(W)/J\mathscr{D}^{k+1}(W)$$

where $pr : \overset{\cup}{J}\mathscr{D}^{k+1}(W) \to \overset{\cup}{J}\mathscr{D}^{k+1}(W)/J\mathscr{D}^{k+1}(W)$ is the canonical projection on the quotient manifold.

2) We say that a connection C_k is *flat* if $C_k \subset \ker(_\neg R)$. One can prove that this definition is equivalent to that given in Definition 3.3/2 (See next Theorem 3.5/4).

If $\pi : W \equiv E \to X$ is a vector bundle, the curvature $_\neg R$ of a linear connection $C_1 \equiv \urcorner(E) \subset J\mathscr{D}(E)$ is a morphism of vector fiber bundles over X: $_\neg R : C_1 \to \Lambda_2^0 X \otimes E$. In fact by using the following commutative diagram where the rows and the columns are exact

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S_2^0 X \otimes E & \to & J\mathscr{D}^2(E) & \longrightarrow & J\mathscr{D}(E) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T_2^0 X \otimes E & \to & \overset{\cup}{J}\mathscr{D}^2(E) & \longrightarrow & J\mathscr{D}(E) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Lambda_2^0 X \otimes E & \to & \overset{\cup}{J}\mathscr{D}^2(E)/J\mathscr{D}^2(E) & \to & 0 & &
\end{array}$$

we recognize the canonical isomorphism $\overset{\cup}{J}\mathscr{D}^2(E)/J\mathscr{D}^2(E) \cong \Lambda_2^0 X \otimes E$. So, taking into account that $C_1 \equiv \urcorner(E)$, $\urcorner \equiv$ linear, we can also identify $_\neg R$ with a section $\bar{R} : X \to \Lambda_2^0 X \otimes E \otimes E^*$. Then, taking a local coordinate system $\{ x^\alpha, y^j \}$ on E, the local expression of $_\neg\bar{R} \equiv {_\neg R}$ is as follows:

$$_\neg R = \sum_{\substack{1 \le \alpha, \beta \le n = \dim X \\ 1 \le j, i \le m = \dim \text{fiber} E}} {_\neg R}^j_{\alpha\beta i} dx^\alpha \wedge dx^\beta \otimes e_j \otimes \theta^i$$

with $R^j_{\alpha\beta i} = \dfrac{\partial \Gamma^j_{\beta i}}{\partial x_\alpha} - \dfrac{\partial \Gamma^j_{\alpha i}}{\partial x_\beta} + \Gamma^j_{\alpha h}\Gamma^h_{\beta i} - \Gamma^j_{\beta h}\Gamma^h_{\alpha i} : X \to \mathbb{R}$ where $\Gamma^j_{\beta i} \equiv \Gamma^j_\beta \circ e_i :$ $X \to \mathbb{R}$, with $\{ e_j \}_{1 \leqslant j \leqslant m}$ a basis of $C^\infty(E)$ such that the following diagram is commutative

$$\begin{array}{ccc}
E \cong 0^* vTE & \longrightarrow & vTE \\
{\scriptstyle e_j} \uparrow & & \uparrow {\scriptstyle \partial y_j} \\
X & \xrightarrow[0]{\hspace{2em}} & E
\end{array}$$

with 0 the zero section. $\{ \theta^j \}_{1 \le j \le m}$ is the dual basis of $\{ e_j \}$. Further,

$\Gamma^j_\beta = -\urcorner^j_\beta \equiv -y^j_\beta \circ \urcorner : E \to \mathbb{R}$. The proof is obtained directly taking into account that

$$y^j_{\alpha\beta} \circ J\mathscr{D}(\urcorner) \circ \urcorner = \frac{\partial \Gamma^j_\beta}{\partial x_\alpha} + \frac{\partial \Gamma^j_\beta}{\partial y_i} \Gamma^i_\alpha .$$

Let us, now, consider the fundamental theorem of integrability for connections.

THEOREM 3.5. — Let $\pi : W \to X$ be a fiber bundle. Any k-connection $C_k = \urcorner(J\mathscr{D}^{k-1}(W)) \subset J\mathscr{D}^k(W)$ on W has the following properties:

1) $g_{k+r} = 0, \forall r \geq 0$.

2) C_k is not, in general, a formally integrable differential equation.

3) The set of solutions of C_k is characterized by the set of sections $C^\infty(W)$ with zero k-absolute derivation.

4) C_k is flat (completely integrable) (and also formally integrable) iff ${}_\urcorner R = 0$.

Proof. — 1) If we prove that $g_k = 0$, from exact sequence (7′) will follow that $g_{k+r} = 0$, $r > 0$. Now, a connection $\urcorner$ gives a splitting $v\mathrm{T}(\urcorner)$ on the left of the exact sequence (7). So, if there is a vector $v \neq 0$ belonging to $v\mathrm{TC}_k = v\mathrm{T}(\urcorner)(v\mathrm{TJ}\mathscr{D}^{k-1}(W))$ and $S^0_k X \otimes v\mathrm{TW}$, its image in $v\mathrm{TJ}\mathscr{D}^{k-1}(W)$ under $v\mathrm{T}(\pi_{k,k-1})$ should not be zero, as $\urcorner$ is a diffeomorphism onto C_k and taking into account that $v\mathrm{T}(\pi_{k,k-1}) \circ v\mathrm{T}(\urcorner) = \mathrm{id}_{v\mathrm{TJ}\mathscr{D}^{k-1}(W)}$, in contrast with the fact that sequence (7) is exact. We conclude that

$$g_k \equiv (S^0_k X \otimes v\mathrm{TW}) \cap v\mathrm{TC}_k = 0 .$$

2) In general C_k is not formally integrable, in fact from sequence (8) we check that the map $v\mathrm{TC}_{k+r} \to v\mathrm{TC}_{k+r-1}$ is injective for $r \geq 1$.

3) If $s : X \to W$ is a section of W, one has $\overset{(k)}{\nabla} s = D^k s - \Gamma \circ D^{k-1} s$. So, $\overset{(k)}{\nabla} s = 0$ iff $D^k s = \Gamma \circ D^{k-1} s$, that is iff s is a solution of C_k.

4) Finally, the coincidence of the flatness with the zero curvature ${}_\urcorner R = 0$ is obtained by using Theorem 3.4 and the fact that $\urcorner^{(1)}\urcorner(J\mathscr{D}^{k-1}(W)) = \overset{\cup}{C}_{k+1}$, where $\overset{\cup}{C}_{k+1}$ is the first sesquiholonomic prolongation of C_k:

$$\overset{\cup}{C}_{k+1} \equiv J\mathscr{D}(C_k) \cap \overset{\cup}{J}\mathscr{D}^{k+1}(W). \qquad \square$$

The following theorem gives a more enlightening geometric meaning to flat connections.

THEOREM 3.6. — 1) If $C_k \subset J\mathscr{D}^k(W)$ is a flat k-connection on W then $(D^k s)^* v\mathrm{TC}_k \to X$ is a flat vector bundle, $s \in C^\infty(W)$, $D^k s : X \to C_k$.

2) In particular if $W \equiv E$ is a vector bundle and C_k is a flat k-linear connection on E then $C_k \to X$ is a flat vector bundle.

Proof. — 1) Let us consider first the Spencer operator ([5])

$$\mathbf{D} : \underline{\Lambda^0_r X \otimes J\mathscr{D}^k(vTW)} \rightarrow \underline{\Lambda^0_{r+1} X \otimes J\mathscr{D}^{k-1}(vTW)}$$

which is characterized by the following conditions:

a) the sequence $0 \rightarrow \underline{vTW} \rightarrow \underline{J\mathscr{D}^k(vTW)} \overset{\mathbf{D}}{\rightarrow} \underline{T^*X \otimes J\mathscr{D}^{k-1}(vTW)}$ is exact;

b) $\mathbf{D}(\alpha \wedge s) = d\alpha \wedge \pi_{k,k-1} \circ s + (-1)^q \alpha \wedge \mathbf{D}s$, $\forall s \in \underline{\Lambda^0_p X \otimes J\mathscr{D}^k(vTW)}$, $\alpha \in \underline{\Lambda^0_q X}$.

We have $\mathbf{D}^2 = 0$, $\mathbf{D}\big(\underline{\Lambda^0_r X \otimes vTC_{k+p+1}}\big) \subset \underline{\Lambda^0_{r+p} X \otimes vTC_{k+p}}$ an the map δ : $\Lambda^0_r X \otimes S^0_k X \otimes vTW \rightarrow \Lambda^0_{r+1} X \otimes S^0_{k-1} X \otimes vTW$ is just the restriction of $-\mathbf{D}$. Set $C_{m,r} \equiv (\Lambda^0_r X \otimes vTC_{m+1})/\delta(\Lambda^0_{r-1} X \otimes g_{m+2})$.

Then, the diagram

$$\begin{array}{ccccccc}
0 \rightarrow & \underline{\Lambda^0_r X \otimes g_{m+2}} & \longrightarrow & \underline{\Lambda^0_r X \otimes vTC_{m+2}} & \longrightarrow & \underline{\Lambda^0_r X \otimes vTC_{m+1}} & \rightarrow 0 \\
 & \downarrow{\scriptstyle -\delta} & & \downarrow{\scriptstyle \mathbf{D}} & & \downarrow{\scriptstyle \mathbf{D}} & \\
0 \rightarrow & \underline{\delta(\Lambda^0_r X \otimes g_{m+2})} & \rightarrow & \underline{\Lambda^0_{r+1} X \otimes vTC_{m+1}} & \rightarrow & \underline{C_{m,r+1}} & \longrightarrow 0
\end{array}$$

induces an operator $\tilde{\mathbf{D}}$ which factors through $\underline{C_{m,r}}$ and therefore defines an operator $\tilde{\mathbf{D}} : \underline{C_{m,r}} \rightarrow \underline{C_{m,r+1}}$. Thus, we are able to write the second Spencer complex for C_k:

$$0 \rightarrow \Theta \rightarrow \underline{C_{k,0}} \rightarrow \underline{C_{k,1}} \rightarrow \underline{C_{k,2}} \rightarrow \ldots$$

Taking, now, into account that C_k is formally integrable and that $g_{m+1} = 0$, $1 \geq 0$, we try that $\Lambda^0_r X \otimes vTC_{k+1} \rightarrow \Lambda^0_r X \otimes vTC_{k+2}$ is an isomorphism, and the second Spencer complex is given by (by using the fact that $C_{k+1} \cong C_k$ (see Theorem 3.5)):

$$0 \rightarrow \Theta \rightarrow \underline{vTC_k} \underset{\hat{\mathbf{D}}}{\rightarrow} \underline{T^*X \otimes vTC_k} \underset{\hat{\mathbf{D}}}{\rightarrow} \underline{\Lambda^0_2 X \otimes vTC_k} \rightarrow \ldots.$$

Thus, $\hat{\mathbf{D}}^2 = 0$ and if $fs \in \underline{vTC_k}$, where f is a function, then we have $\hat{\mathbf{D}}(fs) = df \wedge s + f\hat{\mathbf{D}}s$. It follows that $\sigma(\hat{\mathbf{D}}) = id$, in other words, $\hat{\mathbf{D}}$ is a connection on vTC_k. Further, $\hat{\mathbf{D}}^2 = 0$ is the curvature of $\hat{\mathbf{D}}$. Therefore, as vTC_k admits a flat linear connection it is a flat vector bundle.

2) In this case we can report the above considerations by substituting vTW by E and vTC_k by C_k. □

Let us, now, consider the concept of torsion associated to any linear connection on a manifold.

([5]) If $F \rightarrow X$ is a fiber bundle, we denote the sheaf of sections of F by $\underline{F}$. Further, by abuse of notations, we shall write in this Proof vTC_h, vTW and $\underline{vTJ\mathscr{D}^k(W)} \cong J\mathscr{D}^k(vTW)$ instead of $(D^h s)^* vTC_h$, $s^* vTW$ and $(D^k s)^* vTJ\mathscr{D}^k(W) \cong J\mathscr{D}^k(s^* vTW)$ respectively, where s is any global solution of C_k.

DEFINITION 3.6. — 1) Let $C_1 \subset J\mathcal{D}(TX)$ be a linear connection on the manifold X. Then, the *torsion* of C_1 is the following morphism of vector bundles over X given by ${}_{\neg}T \equiv d|C_1^* : C_1^* \to \Lambda_2^0 X$, where $d : J\mathcal{D}(T^*X) \to \Lambda_2^0 X$ is a morphism of vector bundles over X which identifies the exterior differentiation of differential forms on X and C_1^* is the dual linear connection canonically associated to C_1 :

2) A connection is *symmetric* if $C_1^* \subset \ker(d)$.

Of course as $C_1^* = \urcorner^*(T^*X)$, $\urcorner^* =$ linear, we can identify ${}_{\neg}T$ with a section ${}_{\neg}\bar{T} : X \to \Lambda_2^0 X \otimes TX$. Let $\{ x^\alpha, \dot{x}_\alpha \}$ be a local coordinate system on T^*X and let $\{ x^\alpha, \dot{x}_\alpha, \dot{x}_{\alpha\beta} \}$ be the corresponding coordinate system on $J\mathcal{D}(T^*X)$. Then, the local expression of ${}_{\neg}\bar{T} \equiv {}_{\neg}T$ is as follows:

$${}_{\neg}T = {}_{\neg}T^k_{ij} dx^i \wedge dx^j \otimes \partial x_k, \qquad {}_{\neg}T^k_{ij} \equiv \left(\Gamma^k_{ij} - \Gamma^k_{ji}\right) : X \to \mathbb{R},$$

where

$$\Gamma^k_{ij} \equiv \ulcorner_{ij} \circ dx^k, \qquad \ulcorner_{ij} \equiv \dot{x}_{ij} \circ \urcorner^* : T^*X \to \mathbb{R}.$$

The connection C_1 is symmetric iff $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Note. — Let $\pi_i : E_i \to X, i = 1, 2, 3$ be vector fiber bundles with a bilinear vector bundle morphism over X $(,) : E_1 \times E_2 \to E_3$. Then, we can define the following operator (*exterior product*)

$$\wedge : \Lambda_p^0 X \otimes E_1 \times \Lambda_q^0 X \otimes E_2 \to \Lambda_{p+q}^0 X \otimes E_3$$

given by

$$(\alpha \otimes u) \wedge (\beta \otimes v) = (\alpha \wedge \beta) \otimes (u, v).$$

In the following TAB. 2 we report some useful operators related to vector fiber-valued differential forms endowed with linear connections on the vector fiber bundles. Note, that all the operators reported in TAB. 2 are directly expressed by means of the so-called covariant exterior differential.

DEFINITION 3.7. — Let $\pi : E \to X$ be a vector bundle over X endowed with a linear connection $C_1 \equiv \urcorner(E) \subset J\mathcal{D}(E)$. The *covariant exterior differential* is the vector fiber bundle morphism over X: ${}_{\neg}d \equiv pr_3 \circ {}_{\neg}j$: $J\mathcal{D}(\Lambda_p^0 X \otimes E) \to \Lambda_{p+1}^0 X \otimes E$, where ${}_{\neg}j$ is the isomorphism induced by $\urcorner$: $J\mathcal{D}(\Lambda_p^0 X \otimes E) \cong \Theta^p \oplus \Lambda_{p+1}^0 E$, being Θ^p a vector bundle over X; pr_2 is the projection on the second space, e. g. the following diagram

$$\begin{array}{ccc} J\mathcal{D}(\Lambda_p^0 X \otimes E) & \xrightarrow{\ {}_{\neg}d\ } & \Lambda_{p+1}^0 X \otimes E \\ \Vert\wr & \nearrow_{pr_2} & \\ \urcorner^j \Theta^p \oplus \Lambda_{p+1}^0 X \oplus E & & \end{array}$$

is commutative.

In local coordinates $\{ x^{\alpha}, \dot{x}_{i_1 \ldots i_p} \otimes y^j, x_{\alpha i_1 \ldots i_p} \otimes y^j \}$ on $J\mathscr{D}(\Lambda^0_p X \otimes E)$ we get for any E-valued p-differential form

$$\alpha = \alpha^j_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \otimes \partial y_j$$
$${}_{\urcorner}d\alpha = [\partial x_{[\alpha} . \alpha^j_{i_1 \ldots i_p]} + \Gamma^j_{[\alpha s} \alpha^s_{i_1] \ldots i_p}] dx^{\alpha} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} \otimes \partial y_j .$$

Let us, now, characterize principal connections on a principal fiber bundle by means of some additional geometric objects. Let us, first, give the following

DEFINITION 3.8. — The *absolute differential* induced by a connection $\urcorner$ on $\pi : W \to X$ is the first order differential operator ${}_{\urcorner}\nabla$:

$$J\mathscr{D}(\Lambda^0_p W \otimes V) \to \Lambda^0_{p+1} W \otimes V, \qquad (V = \text{vector space}),$$

given by means of the following commutative diagram:

$${}_{\urcorner}\nabla \equiv (\Lambda^0_{p+1}(\pi) \otimes \mathrm{id}_V) \circ (\Lambda^0_{p+1}(\overline{\urcorner}) \otimes \mathrm{id}_V) \circ d$$

$$\begin{array}{ccc} J\mathscr{D}(\Lambda^0_p W \otimes V) & \xrightarrow{\ {}_{\urcorner}\nabla\ } & \Lambda^0_{p+1} W \otimes V \\ {\scriptstyle d} \downarrow & & \uparrow {\scriptstyle \Lambda^0_{p+1}(\pi) \otimes \mathrm{id}_V} \\ \Lambda^0_{p+1} W \otimes V & \xrightarrow[\Lambda^0_{p+1}(\overline{\urcorner}) \otimes \mathrm{id}_V]{} & \Lambda^0_{p+1} X \otimes V \end{array}$$

TABLE 2. — *Some useful operator related to the covariant exterior differential assoc. to a conn.* $\urcorner$ (1).

Name	Definition	Properties	Appl. to E = TX
Cov. ext. diff.	${}_{\urcorner}d : J\mathscr{D}(\Lambda^0_p X \otimes E) \to \Lambda^0_{p+1} X \otimes E$		${}_{\urcorner}T = {}_{\urcorner}d\eta$ (2)
Cov. Lie Deriv.	${}_{\urcorner}\mathscr{L}_Y : J\mathscr{D}(\Lambda^0_p X \otimes E) \to \Lambda^0_p X \otimes E$ ${}_{\urcorner}\mathscr{L}_Y . \alpha \equiv {}_{\urcorner}\mathscr{L}_Y \circ D\alpha$ $= Y \lrcorner ({}_{\urcorner}d\alpha) + {}_{\urcorner}d(Y \lrcorner \alpha)$	$[{}_{\urcorner}\mathscr{L}_X, {}_{\urcorner}\mathscr{L}_Y]$ $= {}_{\urcorner}\mathscr{L}_{[X,Y]} + {}_{\urcorner}R(X, Y)$ $[{}_{\urcorner}\mathscr{L}_X, {}_{\urcorner}d] = X \lrcorner {}_{\urcorner}R \wedge$ ${}_{\urcorner}d^2 = {}_{\urcorner}R \wedge$ ${}_{\urcorner}d {}_{\urcorner}R = 0$ (Bianchi id.)	${}_{\urcorner}d {}_{\urcorner}T = {}_{\urcorner}R \wedge \eta$
Cov. abs. diff. (E=TX)	${}_{\urcorner}\nabla$: $J\mathscr{D}(\Lambda^0_p X \otimes TX) \to \Lambda^0_{p+1} X \otimes TX$ ${}_{\urcorner}\nabla\alpha \equiv {}_{\urcorner}\nabla \circ D\alpha$ $= {}_{\urcorner}d\alpha - \alpha \circ ({}_{\urcorner}T \wedge \underbrace{\eta \wedge \ldots \wedge \eta}_{p-1})$ (p-degree of α)		${}_{\urcorner}\nabla {}_{\urcorner}R = - {}_{\urcorner}R \circ ({}_{\urcorner}T \wedge \eta)$ ${}_{\urcorner}\nabla {}_{\urcorner}T = {}_{\urcorner}d {}_{\urcorner}T$ $- {}_{\urcorner}T \circ ({}_{\urcorner}T \wedge \eta)$

(1) In this table it is assumed that the vector fiber bundle $E \to X$ has a vector fiber bundle morphism over $X (,) : E \times_X E \to E$ which allows to define the exterior product for E-valued differential forms on X.

(2) $\eta : X \to T^*X \otimes TX$ is the TX-valued 1-form such that $\eta(Y) = Y$.

where d is the exterior differential naturally extended for V-valued differential forms (See e. g. ref. [*25*]).

THEOREM 3.7. — The curvature $\urcorner R$ of a principal connection

$$C_1 \equiv \urcorner(P) \subset J\mathscr{D}(P),$$

on a principal fiber bundle $\mathscr{P} \equiv (P, X, \pi : G)$ can be identified with a map

$$\urcorner R \equiv \urcorner\Omega \circ \Gamma^{-1} = (\urcorner\nabla\urcorner\omega)\circ\Gamma^{-1} : C_1 \to \Lambda^0_2 P \otimes A(G).$$

In the following table 3 we report some useful standard formulas for Ehresmann connections.

TABLE 3. — *Ehresmann connection and related objects for any principal connection.*

Name	Definition	Local expression	Properties	Case $P=\mathscr{E}(X)$ (1)
Ehresmann conn.	$\urcorner\omega$	$\urcorner\omega = \sum_{1\le i\le m=\dim A(G)} \urcorner\omega^i \otimes Z_i$		
Curvature	$\urcorner\Omega \equiv \urcorner\nabla\urcorner\omega$	$\urcorner\Omega = \urcorner\Omega^s \otimes Z_s$ $\urcorner\Omega^s = d\omega^s + \sum_{i<j} \omega^i \wedge \omega^j C^s_{ij}$	$\urcorner\Omega = d\urcorner\omega + \frac{1}{2}[\urcorner\omega, \urcorner\omega]$ (2) $\urcorner\nabla\urcorner\Omega = 0$	
Torsion				$\urcorner\Theta \equiv \urcorner\nabla\theta$: $\mathscr{E}(X) \to \Lambda^0_2\mathscr{E}(X)\otimes\mathbb{R}^n$ $\urcorner\Omega = d\urcorner\omega + \urcorner\omega\wedge\urcorner\omega$ $\urcorner\Theta = d\theta + \urcorner\omega \wedge \theta$ $\urcorner\nabla\urcorner\Omega = 0$ $\urcorner\nabla\urcorner\Theta = \urcorner\Omega \wedge \theta$

(1) $\mathscr{E}(X)$ = principal bundle of linear reference frames on X. $\theta : \mathscr{E}(X) \to T^*\mathscr{E}(X) \otimes \mathbb{R}^n$ is the canonical $\mathbb{R}^n$-valued horizontal 1-form on $\mathscr{E}(X)$.

(2) $[\urcorner\omega, \urcorner\omega] = [\urcorner\omega^i \otimes Z_i, \urcorner\omega^j \otimes Z_j] = 2\,\urcorner\omega^i \wedge \urcorner\omega^j \otimes C^k_{ij} Z_k$, C^k_{ij} = structure constants of G. This is an example of « exterior product » with respect to the bracket of A(G).

As an application we shall consider principal connections on homogeneous spaces. This will allow us to obtain the classic Maurer-Cartan equation for Lie groups as a consequence of existence of a flat connection on a G-principal bundle canonically associated to any Lie group.

DEFINITION 3.2. — Let G be a n-dimensional Lie group. Let H be a closed Lie subgroup of G. A *H-principal connection* on G is a principal connection $\urcorner$ on the principal fiber bundle $H \to G \xrightarrow{\pi} G/H$.

For such connections we have the following

PROPOSITION 3.1. — To assign a G-invariant H-principal connection on G is equivalent to assign a splitting of the vector space $\mathscr{F}(A(G)) \equiv A(G)$ that is

$$0 \to A(H) \underset{s_\Gamma}{\rightleftarrows} A(G) \underset{s_\urcorner}{\rightleftarrows} M \to 0, \qquad M \equiv A(G/H),$$

such that

$$\text{(13)} \qquad \mathrm{Ad}\,(A(H))(s_\urcorner(M)) \subset s_\urcorner(M).$$

Further, ${}_\urcorner\Omega(\sigma(Z), \sigma(Z')) = [Z, Z']_{A(H)}$, $\sigma(Z)$, $\sigma(Z')$ = invariant vector fields on G corresponding to $Z, Z' \in M$ and $[Z, Z']_{A(H)}$ is the component of $[Z, Z']$ into A(H).

2) Further, the H-principal connection $\urcorner$ on G associated to such a splitting s_Γ is the G-invariant connection ${}_\urcorner\omega : G \to T^*G \otimes A(H)$ given by: ${}_\urcorner\omega(e)(Z) = s_\Gamma(Z)$, $\forall Z \in T_eG$.

3) Any splitted extension of a Lie group K by means of a Lie group H, $1 \to H \to G \overset{s}{\rightleftarrows} K \to 1$, which determines a splitting of the corresponding vector space: $A(G) \cong A(H) \oplus M$, $M \equiv s_*(A(K))$, such that condition (13) is verified, determines a G-invariant H-principal connection on G. Then, iff M is a Lie sub-algebra of A(G), then ${}_\urcorner\Omega(\sigma(Z), \sigma(Z')) = 0$, $\forall Z, Z' \in M$. Note that, in general, we do not assume that s is a group-homomorphism but it is a pointed differentiable map $s : K \to G$ such that $\pi_G \circ s = \mathrm{id}_K$, being π_G the canonical epimorphism $\pi_G : G \to K$.

THEOREM 3.8. — For any n-dimensional Lie group G the following propositions are equivalent:

1) There exists a canonical A(G)-valued differential 1-form ω on G which satisfies the following equation $d\omega + \frac{1}{2}[\omega, \omega] = 0$ (*Maurer-Cartan equation*).

2) There exists a canonical completely integrable G-principal connection on G.

3) If $\{\omega^i\}$ is a basis of G-invariant differential forms then they satisfy the following equation: $d\omega^i + C^i_{kj}\omega^k \wedge \omega^j = 0$.

Proof. — Let us consider the following principal bundle

$$G \overset{\mathrm{id}}{\to} G \to G/G = 1\,.$$

Since one has $J\mathscr{D}(G) = 0 = T^*1 \otimes TG$, we have a canonical G-principal connection $C_1 \subset J\mathscr{D}(G)$. Or, in other words there is a canonical G-invariant horizontal space on any point $a \in G$ given by $H_a = \{0\}$. To this connection we associate the Ehresmann connection ${}_\urcorner\omega : G \to T^*G \otimes A(G)$

which coincides with the canonical 1-form ω of G, that is the differential 1-form, A(G)-valued, given by $T^*(L_{s-1})(\omega(s)(X_s)) \in T_e G$, $\forall X_s \in T_s G : {}_\neg\omega \equiv \omega$. Then, we try also

$$0 = {}_\neg R = d\omega + \frac{1}{2}[\omega, \omega]$$

Finally, in the basis $\{ \omega^i \}$ above equation reduces in the form of point (3). □

PROPOSITION 3.2. — The following propositions are equivalent:

1) A G-invariant H-principal connection on the Lie group G and a G-invariant linear connection on G/H are given.

2) A splitting on the left of the exact sequence of vector spaces

$$0 \to A(H) \to A(G) \to A(G/H) \to 0 \tag{14}$$

is given such that condition (13) is verified and a Ad(A(H))-invariant bilinear form $\alpha : A(G/H) \times A(G/H) \to A(G/H)$ is also assigned.

Proof. — In fact assign a G-invariant H-principal connection is equivalent to assign a splitting of exact sequence (14) plus condition (13). On the other hand to assign a G-invariant linear connection on G/H is equivalent to assign an Ad(A(H))-invariant bilinear form on any sub-space M of A(G) such that $A(G) \cong A(H) \oplus M$ and $Ad(A(H))(M) = M$.

The relation between H-principal connection on a Lie group and graded Lie algebras is given by the following

PROPOSITION 3.3. — The assignment of a G-invariant H-principal connection on the Lie group G gives to G the structure of a super Lie group if on A(G/H) is defined a symmetric bilinear map ρ:

A(G/H) × A(G/H) → A(H) such that

$$\left\{ \begin{array}{l} Ad(X)(\rho(u, v)) = \rho(Ad(X)(u), v) + \rho(u, Ad(X)(v)) \\ Ad(\rho(u, v))(w) + Ad(\rho(v, w))(u) + Ad(\rho(w, u))(v) = 0 . \end{array} \right. \tag{15}$$

Proof. — In fact a G-invariant H-principal connection on G,

$$C_1 \equiv \neg(G/H) \subset J\mathscr{D}(G),$$

gives a splitting of the vector space $A(G) \cong A(H) \oplus M \equiv A(G)_0 \oplus A(G_1$, where $M \equiv s_\neg(A(G/H))$, is a Ad(A(H))-invariant vector subspace of A(G). Further, let us define on A(G) a symmetric bilinear map ρ:

$$A(G/H) \times A(G/H) \to A(H)$$

such that condition (15) is satisfied. Then, on A(G) can be defined the following bracket: $\urcorner[,]: A(G) \times A(G) \rightarrow A(G)$ given by

$$
\urcorner[X,Y] \equiv \begin{cases} [X,Y] \in A(H), & \text{if } X,Y \in A(H) \\ \rho(\bar{\pi}(X),\bar{\pi}(Y)) \in A(H), & \text{if } X,Y \in M \\ (\bar{\pi}: A(G) \rightarrow A(G/H) \text{ is the canonical projection}) & \\ [X,Y] \in M, & \text{if } X \in A(H), \ Y \in M, \end{cases}
$$

where [,] is the usual parenthesis of A(G). In fact, we have

i) $\urcorner[X,Y] \in A(G)_{|X|+|Y|}$;

ii) $\urcorner[X,Y] = -[X,Y] = -(-1)^{|X||Y|}\,\urcorner[X,Y]$, if $X,Y \in A(H)$

$\urcorner[X,Y] = \rho(\bar{\pi}(X),\bar{\pi}(Y)) = \urcorner[Y,X] = -(-1)^{|X||Y|}\,\urcorner[Y,X]$, if $X,Y \in M$,

$\urcorner[X,Y] = -[Y,X] = -(-1)^{|X||Y|}\,\urcorner[Y,X]$, if $X \in A(H)$, $Y \in M$.

iii) Further one can easy see that also the graded Jacobi identity is satisfied. □

4. THE GROUP MANIFOLD APPROACH TO UNIFIED GRAVITY

In this section we shall enter in the details of the geometric formulation of the « group model » for supergravity. We shall, in particular, emphasize that this can be made by using the general framework for gauge theories developed in ref. [*11*]. This allows us to better understand the full covariance content and to distinguish which are the characterizing structural elements of the present model.

4.1. Some fundamental geometric objects on a Lie group.

Let us consider a super Lie group G with graded structure constants. Let us give the following fundamental definition.

DEFINITION 4.1. — A *pseudoconnection* on G is a first order connection $C_1 \equiv \urcorner(E) \subset J\mathscr{D}(E)$ on the trivial vector fiber bundle $\pi: E \equiv G \times A(G) \rightarrow G$ such that C_1 is freely generated by $\tilde{C}_1$ and A(G), that is $C_1 \cong \tilde{C}_1 \times A(G)$, where $\tilde{C}_1$ is a *m*-dimensional sub-fiber bundle of $T^*G \otimes A(G)$ over G, $m = \dim G$.

PROPOSITION 4.1. — Any pseudoconnection on G is characterized by means of a A(G)-valued differential 1-form on G:

$$\urcorner\mu : G \rightarrow T^*G \otimes A(G).$$

Proof. — In fact, as $J\mathscr{D}(E) \cong A(G) \times T^*G \otimes A(G)$ we get that any first order connection $\urcorner : E \to J\mathscr{D}(E)$ on E can be written as $\urcorner = (\tilde{\urcorner}, \mathrm{id}_{A(G)})$, where $\tilde{\urcorner} : G \times A(G) \to T^*G \otimes A(G)$.

On the other hand if $\urcorner$ is a pseudoconnection the partial derivation of $\tilde{\urcorner}$ with respect to A(G), $D_2\tilde{\urcorner}$, must be zero: $D_2\tilde{\urcorner} = 0$. Therefore $\tilde{\urcorner}$ is identified with a section $\mu : G \to T^*G \otimes A(G)$ of the fiber bundle $T^*G \otimes A(G) \to G$ and $\urcorner$ looks like $\urcorner = {}_{\neg}\mu \times \mathrm{id}_{A(G)}$. □

Note. — As a pseudoconnection on G is characterized by a differential form valued in a (trivial) vector fiber bundle endowed with a connection, we are able to apply the results resumed in table 2. In particular, we have the following

PROPOSITION 4.2. — Let α be any p-differential form A(G)-valued on G. Then, the covariant exterior differential of α with respect to a pseudoconnection $\urcorner$ of G is given by

$$ {}_{\neg}d\alpha = d\alpha + \frac{1}{2}\,[{}_{\neg}\mu, \alpha] $$

where ${}_{\neg}\mu$ is the A(G)-valued form characterizing $\urcorner$.

THEOREM 4.1. — 1) The curvature ${}_{\neg}R$ of a pseudoconnection on G is given by ${}_{\neg}R = {}_{\neg}d\,{}_{\neg}\mu = d\,{}_{\neg}\mu + \frac{1}{2}\,[{}_{\neg}\mu, {}_{\neg}\mu]$

2) $d^2\alpha = {}_{\neg}R \wedge \alpha$.

3) $d\,{}_{\neg}R + \frac{1}{2}\,[{}_{\neg}\mu, {}_{\neg}R] = 0$ (*Bianchi identity for pseudoconnections*).

Proof. — These results are direct consequence of Proposition 4.1 and results reported in Table 2. □

PROPOSITION 4.3. — The fiber bundle $\pi : E \equiv G \times A(G) \to G$ is a principal fiber bundle with respect to the natural action of A(G) on E considering A(G) as a commutative additive group. Then, a pseudoconnection $\urcorner$ on G becomes a principal connection on $(E, G, \pi ; A(G))$.

Proof. — The action of A(G) on E is $\phi : (u ; a, v) \mapsto (a, u + v)$. The Lie algebra of A(G) is just $T_0A(G) = A(G)$. Further, since for any $u \in A(G)$ the map $\phi_u : E \to E$ is an affine map with derivative the identity of the space, we get that for the horizontal spaces H_0 is satisfied the requirement of A(G)-covariance. In fact, $H_{(a,u+v)} = H_{(a,u)}$. □

PROPOSITION 4.4. — Any H-principal connection on G determines a unique pseudoconnection on G.

Proof. — In fact we have the following commutative diagram

$$\begin{array}{ccc} G & \overset{\urcorner\mu}{\dashrightarrow} & T^*G \otimes A(G) \\ & \searrow^{\urcorner\omega} & \uparrow j \\ & & T^*G \otimes A(H) \end{array}$$

where j is the canonical injection. So, we can give the following

DEFINITION 4.2. — A *reductif pseudoconnection* ${}_G\urcorner$ on G is a pseudo-connection on G such that there is a H-principal connection ${}_H\urcorner$ on G for some closed sub-group $H \subset G$ with Ehresmann connection ${}_{H\urcorner}\omega$ such that the following diagram

$$\begin{array}{ccc} G & \overset{{}_G\urcorner\mu}{\longrightarrow} & T^*G \otimes A(G) \\ & \searrow^{{}_H\urcorner\omega} & \downarrow p_H \\ & & T^*G \otimes A(H) \end{array}$$

is commutative, where p_H is the projection induced by means of the splitting $A(G) \cong A(H) \oplus M$ induced by means of the connection ${}_H\urcorner$ (6).

PROPOSITION 4.5. — For a reductif pseudoconnection $C_1 \equiv \urcorner(E) \subset J\mathscr{D}(E)$ we have that the associated form $\urcorner\mu$ and the Ehresmann connection $\urcorner\omega$ are related by the equation $\urcorner\omega(e) \circ \urcorner\mu = \urcorner\omega$ ($\circ \equiv$ composition on the valuation spaces). In particular, $\urcorner\mu$ is reductif iff

$$\langle \phi_a^* \urcorner\omega(e) \circ \urcorner\mu, X \rangle = \mathrm{ad}(a^{-1}) \langle \urcorner\omega(e) \circ \urcorner\mu, X \rangle, \qquad \forall a \in H .$$

Example. — Let us consider the super Poincaré group GP. By taking an adapted coordinate system we have the following 9×9 matrix representation of GP:

$$\begin{pmatrix} \Lambda(S)^{ab} & \left[\frac{1}{2}\lambda C\gamma^a S\right]^b & P^a \\ 0 & S^{ab} & \lambda^a \\ 0 & 0 & 1 \end{pmatrix}$$

where $(S^{ab}) \in SL(2, \mathbb{C})$, $\Lambda : SL(2, \mathbb{C}) \to SO(3,1)$ is the usual homomorphism, (P^a) is a four-vector, (λ^a) is an anticommuting spinor and C is the charge conjugation matrix ($C \equiv \gamma^2 \circ {}^-$, where $^-$ is complex conjugation (7)).

(6) Note, that the pseudoconnections of Proposition 4.4 are of a particular reductif type that we shall call *fully reductif pseudoconnections*.

(7) Recall that if ψ is a controvariant 1-spinor one has $C : (\overset{1}{\psi}, \overset{2}{\psi}, \overset{3}{\psi}, \overset{4}{\psi}) \mapsto (-\overline{\psi}^4, \overline{\psi}^3, \overline{\psi}^2, -\overline{\psi}^1)$ in a standard Dirac representation of Clifford algebra (see ref. [*11*]).

Then, a pseudoconnection $\urcorner$ on $G \equiv GP$ is locally characterized by the following A(G)-valued 1-form on G:

$$\urcorner\mu = \frac{1}{2}\omega^{\alpha\beta} \otimes J_{\alpha\beta} + \theta^{\alpha} \otimes P_{\alpha} + \psi^{a} \otimes Q_{a} : G \to T^{*}G \otimes A(G),$$

where $\omega^{\alpha\beta}$, θ^{α}, ψ^{a} are 1-forms on G. $J_{\alpha\beta}$ generate the SL(2, $\mathbb{C}$) sub-group, P_{α} are the bosonic translations and Q_{a} the fermionic ones. Further, the corresponding graded Lie algebra is given by the following parenthesis:

$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} \quad (^8)$$
$$[J_{ab}, P_{c}] = \eta_{bc}P_{a} - \eta_{ac}P_{b}$$
$$[Q_{a}, Q_{b}] = (C\gamma^{c})_{ab}P_{c}$$
$$[J_{ab}, Q_{c}] = (\sigma_{ab})_{c}^{d}Q_{d} \quad (^8)$$

The curvature of $\urcorner\mu$ is locally given by

$$\urcorner R = \frac{1}{2}\Omega^{ab} \otimes J_{ab} + T^{a} \otimes P_{a} + I^{a} \otimes Q_{a} \tag{16}$$

with

$$\Omega^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega_{c}^{b}$$
$$T^{a} = d\theta^{a} + \omega^{ac} \wedge \theta_{c} + \frac{1}{2}\psi^{b} \wedge \psi^{c}(C\gamma^{a})_{bc}$$
$$I^{a} = d\psi^{a} + \frac{1}{2}\omega^{bc} \wedge \psi^{d}(\sigma_{bc})_{d}^{a}$$

The Bianchi identity looks like

$$d\Omega^{ab} + \omega_{c}^{a} \wedge \Omega^{cb} - \Omega^{ac} \wedge \omega_{c}^{b} = 0$$
$$dT^{a} + \omega^{ac} \wedge T_{c} - \Omega^{ac} \wedge \theta_{c} + \psi^{b} \wedge I^{c}(C\gamma^{a})_{bc} = 0$$
$$dI^{a} + \frac{1}{2}\omega^{bc} \wedge I^{d}(\sigma_{bc})_{d}^{a} - \frac{1}{2}\Omega^{bc} \wedge \psi^{d}(\sigma_{bc})_{d}^{a} = 0$$

that are formally the same of ones obtained in the formalism of principal connection on principal bundles with super Poincaré group as structure group (see e. g. ref. [*26*] [*27*]).

If the pseudoconnection $\urcorner$ is G-invariant then we get from (16) the Maurer-Cartan equations for $\urcorner\mu$:

$$d\omega^{ab} + \omega^{ac} \wedge \omega_{c}^{b} = 0$$
$$d\theta^{a} + \omega^{ac} \wedge \theta_{c} + \frac{1}{2}\psi^{b} \wedge \psi^{c}(C\gamma^{a})_{bc} = 0$$
$$d\psi^{a} + \frac{1}{2}\omega^{bc} \wedge \psi^{d}(\sigma_{bc})_{d}^{a} = 0$$

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(8) $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$; $\sigma_{ab} = \frac{1}{4}[\gamma_{a}, \gamma_{b}]$.

where ω^{ab}, θ^{a}, ψ^{b} are G-invariant differential forms on G. Further, if $\urcorner$ is a fully reductif pseudoconnection on G for the Lie subgroup

$$H \equiv SL(2, \mathbb{C}) \subset G$$

we get $\theta^{a} = \psi^{a} = 0$ and $_{\urcorner}\mu$ and $_{\urcorner}R$ reduce to the following expressions:

$$_{\urcorner}\mu = \frac{1}{2}\omega^{\alpha\beta} \otimes J_{\alpha\beta}, \quad R = \frac{1}{2}\Omega^{ab} \otimes J_{ab}.$$

4.2. The group model as a gauge theory.

Recall [*11*] that a k-order gauge continuum system over a basis manifold M is a couple $G(M) \equiv (\mathscr{B}, E_k)$ where: (1) $\mathscr{B} \equiv (\mathscr{P}, \mathscr{C} : \mathbb{B})$ is a super-bundle. of geometric objects [*11*] [*12*] over M having as basic bundle the principal bundle over M $\mathscr{P} \equiv (P, M, \pi_P; H)$, as total bundle a suitable fiber bundle $\{ \pi : C \to M \} \equiv \mathscr{C}$ over M containing the fiber bundle of connections over P; and $\mathbb{B}$ is a covariant functor $\mathbb{B} : \mathscr{C}(P) \to \mathscr{C}(C)$, where $\mathscr{C}(P)$ (resp. $\mathscr{C}(C)$) is the category whose objects are open subbundles of P (resp. C), and whose morphisms are the local fiber bundle automorphisms between those objects such that:

i) if $B \mid U \in \mathrm{Ob}(\mathscr{C}(P)) \Rightarrow \mathbb{B}(B \mid U) = \pi^{-1}(U) \in \mathrm{Ob}(\mathscr{C}(C))$;

ii) if $f \in \mathrm{Hom}(\mathscr{C}(P))$ with $f = (f_P, f_M) P \mid U \to P \mid U'$ then $\mathbb{B}(f) \in \mathrm{Hom}(\mathscr{C}(C))$, and satisfies: $(a) \pi \circ \mathbb{B}(f) = f_M \circ \pi$;

(b) if $B \mid U \in \mathrm{Ob}(\mathscr{C}(P))$, $\quad U' \subset U \Rightarrow \mathbb{B}(f) \mid \pi^{-1}(U') = \mathbb{B}(f) \mid P \mid U'$;

(2) E_k is a k-order differential equation on $\pi : E_k \subset J\mathscr{D}^k(C)$. An *internal constraint* of G(M) is a subbundle C_0 of C over M.

DEFINITION 4.3. — A gauge continuum system G(M) has a *blow up* structure if the fiber bundle of principal connections C(P) over P has a canonical non trivial embedding in the configuration bundle, which coincides with a fiber bundle of connections over another fiber bundle over P or M. So, in a gauge continuum system with blow up structure C(P) is present as a non trivial internal constraint.

Then, we are able to give the following fundamental definition.

DEFINITION 4.4. — *A group model gauge theory* is a k-order gauge continuum system $G(M) \equiv (\mathscr{B}, E_k)$ with blow up structure where:

1) The gauge structure is given by means of the following principal fiber bundle: $\mathscr{P} \equiv 1 \to H \to G \overset{\pi_G}{\to} G/H \to 1$, where H is a closed sub-group of a super Lie group G. G is called also *group symmetry of vacuum* and H is the *gauge structure group*. $G/H \equiv M$ is called the *vacuum space-time* (that is also the basis manifold).

A splitting on the left of $\mathscr{P} \equiv 1 \to H \to G \overset{\pi_G}{\underset{\leftarrow}{\to}} G/H \to 1$, that is a map $s: G/H \to G$ such that $\pi_G \circ s = \mathrm{id}_{G/H}$ is just a section of π_G; the image $N \equiv s(G/H) \subset G$ is called a *space-time.*

2) The total bundle C is given by means of the fiber bundle of pseudo-connections on G. So, C can be identified with the fiber bundle π : $C \equiv T^*G \otimes A(G) \to G/H$.

The blow, up structure is given by means of the embedding $\pi_G^* C(P) \to C$ over G. □

3) The functor $\mathbb{B}: \mathscr{C}(P) \to \mathscr{C}(C)$ is given by $\mathbb{B}(\phi) \equiv T^*(\phi^{-1}) \otimes f(\phi^{-1})$, where T* is the cotangent functor and $f(\phi)$ is calculated as given in Appendix A 1, for any $\phi \in \mathrm{Hom}(\mathscr{C}(P))$.

4) The *dynamic equation* is a sub-fiber bundle E_k of $J\mathscr{D}^k(C)$, $(k = 2h)$, identified with the set of points $u \equiv D^k c(\pi_k(u)) \in J\mathscr{D}^k(C)$ such that:

a) the sections $c: G/H \to C$ of π are factorizable into $c = {}_\neg\mu \circ s$, where $s: G/H \to G$ is a section of π_G and ${}_\neg\mu; G \to C$ is a pseudoconnection on G, reductif with respect to the H-principal connection identified by s;

b) $D^k{}_\neg\mu(s(x)) \in \overline{E}_k \subset J\mathscr{D}^k(\overline{E})$, $x \equiv \pi_k(u)$, where $\overline{E}$ denotes the fiber bundle $\pi_{\overline{E}}: \overline{E} \equiv T^*G \otimes A(G) \to G$ and $\overline{E}_k$ is a differential equation on the fiber bundle $\overline{E}$ identified by means of a variational principle by using a H-invariant Lagrangian density

$$J\mathscr{D}^h(\overline{E}) \overset{\Omega}{\to} \Lambda_p^0 J\mathscr{D}^h(\overline{E}), \qquad p = \dim G/H.$$

Further, $\overline{E}_k$ is obtained by requesting that the action $I_A[{}_\neg\mu] \equiv \int_{A \subset N} (D^h{}_\neg\mu)^*\Omega$, (A is any submanifold with smooth boundary of a space-time N of G), should be stationary under all independent variations in the space-time N and fields. It is then requested that there must exist a flat solution $({}_\neg R = 0)$. There, exist other solutions beside the « flat ». (Let (17) be this set of conditions).

A solution c of E_k is a section $c: G/H \to C$ that can be factorized as $c = {}_\neg\mu \circ s$ for some section s of π_G and pseudoconnection ${}_\neg\mu$ such that $D^k c = J\mathscr{D}^k(s, \mathrm{id}) \circ (D^k{}_\neg\mu) \circ s : G/H \to E_k$, $D^k{}_\neg\mu(N) \subset \overline{E}_k$ and ${}_\neg\omega \equiv pr \circ {}_\neg\mu$: $G \to T^*G \otimes A(H)$, where pr is the canonical projection pr :

$$A(G) \cong A(H) \oplus M \to A(H)$$

identified by s, is a H-principal connection. Then $N \equiv s(G/H)$ is called a *dynamic space-time.*

Remark. — A very important category of unified theories can be identified by imposing some internal constraints. Set, for any closed subgroup $K \subset G$,

$$J\mathscr{D}^h(\overline{E})^K \equiv \{ u \in J\mathscr{D}^h(\overline{E}) \mid \exists K\text{-invariant section}$$

$$\mu \in C^\infty(\overline{E}) \mid D^h\mu(\pi_h(u)) = u \} \subset J\mathscr{D}^h(\overline{E}),$$

where $\pi_h : J\mathscr{D}^h(\overline{E}) \to G$ is the canonical projection.

Then, there exists a natural fiber bundle morphism

$$\overset{h}{{}_K\pi} : J\mathscr{D}^h(\overline{E})^K \to J\mathscr{D}^h(\overline{E}/K)$$

over ${}_K\pi : \overline{E} \to \overline{E}/K$ given by $\overset{h}{{}_K\pi} : u \equiv D^h\mu(x) \mapsto D^h\mu/K([x]_K)$, where μ/K is the unique section of $\overline{E}/K \to G/K$ corresponding to the K-invariant section μ, anx $[x]_K$ is the image of $x \in G$ under the canonical projection $\pi_K : G \to G/K$. Further, one can see that given a section μ of $\pi_E : \overline{E} \to G$ there exists a μ/K, section of ${}_K\pi_{\overline{E}} : \overline{E}/K \to G/K$, such that the following diagram

$$\begin{array}{ccc} \overline{E} & \xrightarrow{K\pi} & \overline{E}/K \\ \mu \uparrow & & \downarrow \mu/K \\ G & \xrightarrow[\pi_K]{} & G/K \end{array} .$$

is commutative iff $D^h\mu : G \to J\mathscr{D}^h(\overline{E})^K$.

Finally, a section $\mu : G \to \overline{E}$ is a K-invariant solution to $\overline{E}_k$ iff μ/K is a solution of the *reduced* differential equation $(\overline{E}_k)^K \equiv \overset{k}{{}_K\pi}({}^K\overline{E}_k)$, where ${}^K\overline{E}_k \equiv \overline{E}_k \cap J\mathscr{D}^k(\overline{E})^K$.

A *supersymmetry* of a group model gauge theory $G(M) \equiv (\mathscr{B}, E_k)$ is defined by the diffeomorphism $\phi_a : G \to G$, induced by the multiplication law of G, such that: (*i*) $a \notin H^{\cdot} \equiv H - \{e\}$; (*ii*) $\phi_a \in \mathrm{Hom}(\mathscr{C}(P))$; (*iii*) the natural diffeomorphism $\mathbb{B}(\phi_a)$ of $T^*G \otimes A(G)$ identifies a diffeomorphism of $J\mathscr{D}^k(C)$ that is a diffeomorphism of E_k too (symmetry of E_k).

So, a supersymmetry transforms any solution c of E_k into a new solution $c' \equiv \phi_a^* c \equiv \mathbb{B}(\phi_a^{-1}) \circ c \circ \overline{\phi}_a$, where $\overline{\phi}_a$ is the diffeomorphism on G/H induced by ϕ_a.

Furthermore, as

$$c' \equiv \mathbb{B}(\phi_a^{-1}) \circ {}_\neg\mu \circ s \circ \overline{\phi}_a = \mathbb{B}(\phi_a^{-1}) \circ {}_\neg\mu \circ \phi_a \circ \phi_a^{-1} \circ s \circ \overline{\phi}_a = (\phi_a^* {}_\neg\mu) \circ (\phi_a^* s),$$

we try that $\phi_a^* {}_\neg\mu$ must be a solution of $\overline{E}_k$ on the space-time $\phi_a^* s(G/H)$. Therefore, ϕ_a is a symmetry (supersymmetry) of $\overline{E}_k$ also.

So, we can give the following

DEFINITION 4.5. — 1) Given a group model gauge theory $G(M) = (\mathscr{B}, E_k)$ we call K-*reduction* of G(M), K = closed subgroup of G, the gauge continuum system given by $G(M)_K \equiv (\mathscr{B}, (E_k)^K)$ where $(E_k)^K \subset J\mathscr{D}^k(C)$ is the dynamic equation such that the corresponding reductif pseudoconnections

$\urcorner\mu : G \to T^*G \otimes A(G)$ are K-invariant solutions of $\overline{E}_k$, e. g. $\urcorner\mu$ is a solution of ${}^K\overline{E}_k$. We call $(E_k)^K$ the K-*reduction* of E_k.

2) We call *canonical reduction* of $G(M) \equiv (\mathscr{B}, E_k)$ the K-reduction of G(M) with K = H.

3) A group model gauge theory $G(M) \equiv (\mathscr{B}, E_k)$ is K-*supersymmetric* if E_k is K-supersymmetric, that is the set of supersymmetries coincides with $K \subset G - H^{\cdot}$.

Note. — A good category of unified theory can be obtained by considering $\overline{E}_{2h} \subset J\mathscr{D}^k(\overline{E})$, $(h=1,\ k=2h)$, described by a Lagrangian density like

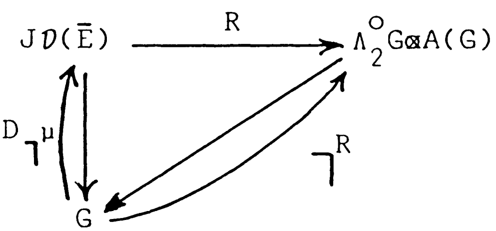
$$\Omega \equiv \sum_s \frac{1}{s!} \overset{[s]}{R} \underline{\Delta} \overset{[s]}{\nu},$$

where:

a) $\overset{[s]}{R} : J\mathscr{D}(\overline{E}) \to \Lambda^0_{2s}G \otimes A(G)$ is the differential operator on $\overline{E}$ given by

$$\overset{[s]}{R} \equiv [\ldots \underbrace{[[R, R], R], \ldots, R]}_{s}$$

being $R : J\mathscr{D}(\overline{E}) \to \Lambda^0_2 G \otimes A(G)$ the fiber bundle morphism over G which defines the curvature for pseudoconnections, e. g. one has the following commutative diagram:



for any pseudoconnection $\urcorner\mu$.

If $\{ Z_i \}$ is a graded basis for A(G) one has

$$\begin{aligned}\overset{[s]}{R} &= R^{A_1} \wedge \ldots \wedge R^{A_s} \otimes [\ldots [[Z_{A_1}, Z_{A_2}], Z_{A_3}], \ldots, Z_{A_s}] \\ &= R^{A_1} \wedge \ldots \wedge R^{A_s} \otimes C^{B_1}_{A_1A_2} C^{B_2}_{B_1A_3} \ldots C^{B_{s-1}}_{B_{s-2}A_s} Z_{B_{s-1}}\end{aligned}$$

where $R^{A_i} : J\mathscr{D}(\overline{E}) \to \Lambda^0_2 G$.

b) $\overset{[s]}{\nu} : J\mathscr{D}(\overline{E}) \to \Lambda^0_{p-2s}G \otimes A(G)^*$ is a 0-order differential operator that can be factorized as follows: ($p = \dim G/H$)

$$\begin{array}{ccc} C^\infty(\overline{E}) & \xrightarrow{\ \overset{[s]}{\nu}\ } & C^\infty(\Lambda^0_{p-2s}G \otimes A(G)^*) \\ \downarrow{\scriptstyle \urcorner\!\!\!\mu \otimes \psi} & & \nearrow{\scriptstyle \langle , \rangle} \\ C^\infty(\Lambda^0_{p-2s}G \otimes A(G)) \otimes A(G)^* \otimes A(G)^* & & \end{array}$$

More precisely $\overset{[s]}{v}$ is given by

$$\neg\mu \overset{\not{h}\otimes\psi}{\longrightarrow} \underbrace{[\ldots[\neg\mu, \neg\mu], \neg\mu], \ldots, \neg\mu]}_{(p-2s)} \otimes \Psi(\neg\mu)$$
$$\equiv \omega \otimes v \otimes \alpha \otimes \beta \mapsto \omega \langle v, \alpha \rangle \otimes \beta.$$

So, by taking a basis $\{ \xi^A \}$ in $A(G)^*$ one can write $\overset{[s]}{v}$ as follows:

$$\overset{[s]}{v} . \neg\mu = C_{AD_1 \ldots D_{p-2s}} \neg\mu^{D_1} \wedge \ldots \wedge \neg\mu^{D_{p-2s}} \otimes \xi^A = \overset{[s]}{v}_A \otimes \xi^A ,$$

where $C_{AD_1 \ldots D_{p-2s}} \in \mathbb{K}(G) \equiv$ set of $\mathbb{K}$-functions on G.

c) $\underline{\Delta}$ is the bilinear vector fiber bundle morphism over G

$$\underline{\Delta} : \Lambda^0_{2s}G \otimes A(G) \times_G \Lambda^0_{p-2s}G \otimes A(G)^* \to \Lambda^0_p G$$

given by

$$(\alpha \otimes v, \beta \otimes \gamma) \mapsto (\alpha \otimes v)\underline{\Delta}(\beta \otimes \gamma) = \alpha \wedge \beta \langle \gamma, v \rangle \ (^9).$$

If $\{ \xi^A \}$ is the dual basis of $\{ Z_A \}$ we can write Ω as follows:

$$\Omega . \neg\mu = \sum_s \frac{1}{s!} \neg R^{A_1} \wedge \ldots \wedge \neg R^{A_s} (\overset{[s]}{v}_{B_{s-1}} . \neg\mu) C^{B_1}_{A_1 A_2} \ldots C^{B_{s-1}}_{B_{s-2} A_s}$$

For sake of semplicity we set

$$\overset{[s]}{v}_{A_1 \ldots A_s} . \neg\mu \equiv C^{B_1}_{A_1 A_2} \ldots C^{B_{s-1}}_{B_{s-2} A_s} \overset{[s]}{v}_{B_{s-1}}$$
$$= C^{B_1}_{A_1 A_2} \ldots C^{B_{s-1}}_{B_{s-2} A_s} C_{B_{s-1} D_1 \ldots D_{p-2s}} \neg\mu^{D_1} \wedge \ldots \wedge \neg\mu^{D_{p-2s}}$$
$$\equiv C_{A_1 \ldots A_s D_1 \ldots D_{p-2s}} \neg\mu^{D_1} \wedge \ldots \wedge \neg\mu^{D_{p-2s}}$$

with $C_{A_1 \ldots A_s D_1 \ldots D_{p-2s}} \in \mathbb{K}(G)$.

So, we have that the expression of the Lagrangian density is as follows:

$$\Omega . \neg\mu = \sum_s \frac{1}{s!} R^{A_1} \wedge \ldots \wedge R^{A_s} (\overset{[s]}{v}_{A_1 \ldots A_s} . \neg\mu) \tag{18}$$

for any pseudoconnection $\neg\mu$.

We can, now, calculate the differential equation for extremals of the action integral

$$I_A[\neg\mu] \equiv \int_A (D\neg\mu)^*\Omega = \int_A (D\neg\mu)^*\sigma[\Omega]$$

where A is any submanifold with smooth boundary of a space-time of G and $\sigma[\Omega]$ is the Cartan form associated to Ω (see ref. [*11*]).

(9) In other words the symbols [,] and $\underline{\Delta}$ used to define Ω are « exterior products » with respect to the graded bracket of A(G) and the pairing $\langle , \rangle$ respectively (see Note before Definition 3.7).

Let $_\neg\tilde{\mu}$ be a one-parameter family of sections of $\pi_{\overline{\mathrm{E}}} : \overline{\mathrm{E}} \to \mathrm{G}$ with $_\neg\tilde{\mu}_0 = {}_\neg\mu$, and let $_\neg\xi \equiv \partial \mathrm{D}_\neg\tilde{\mu}$ the tangent vector field along $\mathrm{D}_\neg\mu$ to $\mathrm{D}_\neg\tilde{\mu}$ at $t = 0$, so that $_\neg\xi(a) \in v\mathrm{T}_{\mathrm{D}_\neg\mu(a)}\mathrm{J}\mathscr{D}(\overline{\mathrm{E}})$ is the tangent vector to the curve $\mathrm{D}_\neg\tilde{\mu}_t(a)$. Then, by using the formula

$$\partial[(\mathrm{D}_\neg\tilde{\mu})^*\sigma[\Omega]] = (\mathrm{D}_\neg\mu)^*({}_\neg\xi \lrcorner d\sigma[\Omega] + d(\mathrm{D}_\neg\mu)^*({}_\neg\xi \lrcorner \sigma[\Omega]),$$

and supposing that $d\mathrm{I}_\mathrm{A}[{}_\neg\tilde{\mu}] = d\int_\mathrm{A} (\mathrm{D}_\neg\tilde{\mu})^*\sigma[\Omega]$ vanishes for all variations $_\neg\tilde{\mu}$ of $_\neg\mu$ which agree with $_\neg\mu$ outside a compact subset of $\mathring{\mathrm{A}}$ we have that $_\neg\xi$ has compact support and by Stokes' theorem we get

$$\int_\mathrm{A} (\mathrm{D}_\neg\mu)^*({}_\neg\xi \lrcorner d\sigma[\Omega]) = 0\,.$$

Since this must hold for all vertical fields $_\neg\xi$ along of compact support, we conclude that on $\mathring{\mathrm{A}}$

$$(\mathrm{D}_\neg\mu)^*(\xi \lrcorner d\sigma[\Omega]) = 0\,. \tag{19}$$

In the following sections we shall give some examples. In order the language to be accessible also to readers non-well expert in differential geometry we shall use the formulation in coordinates and a less technical exposition.

4.3. **General relativity.**

General relativity can be cast in the form (18) by setting

$$\overset{[s]}{v} = 0, \quad s \neq 1, \quad \overset{[1]}{v}_\mathrm{A} = \varepsilon_{abcd}\theta^c \wedge \theta^d, \qquad \text{if} \qquad \mathrm{A} \equiv (a, b)\,,$$

and is seen to obey (19) as expected since we already know that Minkowski space is a solution; also pure gravity in higher dimensions obeys (19). It stands to reason to say that also the third principle in (17) is satisfied, with the exception of gravity in three-dimension, where in fact the field equations imply that space is flat. We refer to this condition as « rigidity » as opposed to « softness », satisfying (17). In fact, soft manifolds are object of mathematical investigation. We recall that field equations for n-dimensional gravity appear in the form:

$$\mathrm{R}^{a_1} \wedge \theta^{a_2} \wedge \ldots \wedge \theta^{a_{n-2}}\varepsilon_{bca_1\ldots a_{n-2}} = 0$$
$$\mathrm{R}^{a_1a_2} \wedge \theta^{a_3} \wedge \ldots \wedge \theta^{a_{n-1}}\varepsilon_{a_1\ldots a_n} = 0.$$

From these it is easy to show that $\mathrm{R}^a = 0$, even under the assumption, implicit in our description, that they are written in a $n(n+1)/2$ dimensional differentiable manifold. Next we parametrize the curvature as:

$$R^{ab} = \frac{1}{2}\theta^s \wedge \theta^t R^{ab}{}_{st} + \omega^{pq} \wedge \theta^t R^{ab}{}_{tpq} + \frac{1}{2}\omega^{pq} \wedge \omega^{em} R^{ab}{}_{(pq)(em)}$$

and it is equally easy to prove, if we treat the ω^{ab} as independent forms, $R^{ab}{}_{AB} = 0$ if either/or A, B = (p, q).

What is meant is that gauge transformations along the $SO(1, n-1) = H$ sub-group are also interpretable as flows induced by appropriate tangent vectors and that the theory factorizes in the sense described in the Introduction.

4.4. Supergravity N = 1, d = 4.

This theory uses the GP gauge fields, already discussed. The curvatures are also written

$$(20)\qquad \begin{aligned} R^{ab} &= d\omega^{ab} - \omega^{at} \wedge \omega_t{}^b \\ R^a &= d\theta^a - \omega^a{}_t \wedge \theta^t \\ \rho &= d\psi - \frac{i}{2}\omega^{ab} \wedge \Sigma_{ab}\psi \end{aligned}$$

where ψ is now a Majorana spinor one-form [10]. Supergravity must contain ordinary gravity whenever we set $\psi = 0$, it must invariant under SO(1, 3) gauge transformations. All this suggest trying an action of the kind:

$$I = \int (R^{ab} \wedge \theta^c \wedge \theta^d \varepsilon_{abcd} + A\bar{\psi} \wedge \gamma_5\gamma_a\rho \wedge \theta^a + BR^{ab} \wedge \bar{\psi} \wedge \Sigma^{cd}\psi\varepsilon_{abcd}$$

that is again of type corresponding to a Lagrangian density like (18) with $\overset{[s]}{v} = 0$ unless $s = 1$. The Lagrangian density can be also written putting in evidence that v_A is a « co-adjoint » multiplet:

$$\Omega . \neg\mu \equiv R^{ab} \wedge v_{ab} + R^a \wedge v_a + \bar{\rho} \wedge n + \bar{n} \wedge \rho,$$

(see e. g. ref. [3]). The variational equations are now:

$$(21)\qquad \begin{aligned} \varepsilon_{abcd}R^c \wedge \theta^d - B\bar{\psi} \wedge \Sigma^{cd}\rho\varepsilon_{abcd} &= 0 \\ \varepsilon_{abcd}R^{bc} \wedge \theta^d + 4\bar{\psi} \wedge \gamma_5\gamma_a\rho &= 0 \\ 2\gamma_5\gamma_a\rho \wedge \theta^a - \gamma_5\gamma_a\psi \wedge R^a + \frac{B}{4}\varepsilon_{abcd}\Sigma^{cd}\psi \wedge R^{ab} &= 0 . \end{aligned}$$

---

(10) $\Sigma_{ab} = \frac{i}{4}[\gamma_a, \gamma_b]$. The signature is $(1, -1, -1, -1)$. Recall that a Majorana spinor ψ is defined by: $C\bar{\psi} = \psi$.

The remarkable fact now happens that if expand in the θ^a, ω^{ab}, ψ treated as independent forms, we obtain all $R^A = 0$, unless we set the theory factorizes the Lorentz group and reduces to ordinary supergravity. The rigidity of the $B \neq 0$ theory can be seen as follows. Equations (20) scale under the replacement:

$$(22)\qquad \begin{aligned} &d \to \tau d, \quad \omega^{ab} \to \tau\omega^{ab}, \quad \theta^a \to \kappa\theta^a \\ &\psi \to \sqrt{\kappa\tau}\psi, \quad R^{ab} \to \tau^2 R^{ab}, \quad \rho \to \tau^{3/2}\kappa^{1/2}\rho . \end{aligned}$$

On the action (20) this means that B scales as $B \to \frac{\tau}{\kappa} B$. Therefore a variation of τ and κ such that $\tau\kappa = 1$ yields the condition:

$$(23)\qquad R^{ab} \wedge \bar{\psi} \wedge \Sigma^{cd}\psi\varepsilon_{abcd} = 0$$

which is a particular consequence of equations (21). If we insert in (23) the generic expression:

$$R^{ab} = \frac{1}{2}\theta^s \wedge \theta^t \wedge R^{ab}{}_{st} + \theta^t \wedge (\bar{\psi}T^{ab}{}_t) + \theta^t \wedge \bar{T}^{ab}_t\psi + \text{terms in } \bar{\psi},\ \psi$$

we find that the space-time part must vanish. The theory contains therefore a trivial gravity. But this means that through Bianchi's identities and the other equations, we have $R^A = 0$. We must set $B = 0$. In general, in building action similar to (20) according to (17) we must include forms which have the same dimension as the standard Einstein term:

$$R^{a_1 a_2} \wedge \theta^{a_3} \wedge \ldots \wedge \theta^{a_n}\varepsilon_{a_1 \ldots a_n}$$

under the scaling law (22) or of the de Sitter cosmological term:

$$\Lambda\theta^{a_1} \wedge \ldots \wedge \theta^{a_n}.$$

In this case, we must give to Λ a dimension $(\kappa/\tau)^2$. Although this rule has not been proved rigorously, it seens to be well-respected in all the cases examined so far. In place of (20) we have now

$$I = \int (R^{ab} \wedge \theta^c \wedge \theta^d\varepsilon_{abcd} + 4\bar{\psi} \wedge \gamma_5\gamma_a\rho \wedge \theta^a) .$$

The resulting variational equations are such that we obtain a set of algebric relations between curvature compononents (inner) along space-time and outer curvature components along normal directions. The effect of this relation is that it is possible to extend a solution given initially on space-time on the whole group manifold. This possibility has some analogies to the standard Cauchy problem, the space-time configuration is a set of initial data for a system of differential equations on the group manifold. This possibility and the fact that the lifted solution is unique has been called « rheonomy » in refs. [*2*] [*3*].

The rheonomic lifting can be seen as a space-time transformation of the fields. Indeed, once we reach another space-time imbedded in the group manifold we can operate on it with a suitable diffeomorphism and bring it on the initial space-time. The old solution is thus mapped into a new one. This procedure is equivalent to on-shell supersymmetry transformations. Related to this discussion is the role of Bianchi identities. These pose constraints on the curvature as calculated from a set of potential one-forms ω^{A}. The rheonomic conditions appear therefore as an additional set of constraints on the curvatures and as a set of differential equations on the potentials, when these are considered on the whole group manifold.

If we insert the rheonomic conditions into the Bianchi identities, we see that they are compatible only if the inner equations hold. By inner equations, here we mean the complete set of field equations on space-time.

This kind of relation implies that it is in general impossible to lift rheonomically an arbitrary field configuration, only field solutions on space-time can be lifted to the whole group manifold.

This corresponds to the standard result, already well known in the conventional approach to supersymmetry, that supersymmetry transformations admit an infinitesimal algebra which closes only on shell. Should we want to lift an arbitrary field configuration one has to relax the rheonomy conditions so that they do not imply any more the field equations. This is done by inserting extra (auxiliary) fields into the parametrization of the curvatures. The Bianchi identities then merely state the supersymmetry transformation rules and determine the outer derivatives of the auxiliary fields.

From this discussion and from the concrete examples which will be discussed, we then can add another principle:

(.) The non-trivial solutions of the variational equations must satisfy the rheonomy condition, i. e. the outer curvature components must be completely determined by the inner components (Such theories are called *rheonomic symmetrical.* Really they are supersymmetric group model gauge theories).

This set of conditions guarantees that the set of classical solutions on a space-time manifold will have a closed algebra of supersymmetric transformations. Infinitesimal transformations can be reached as follows.

The effect of an infinitesimal diffeomorphism is given by the Lie derivative:

$$(24) \quad {}_\neg\mathscr{L}_{X}{}_\neg\mu = X\lrcorner({}_\neg d\,{}_\neg\mu) + {}_\neg d(X\lrcorner\,{}_\neg\mu) = X\lrcorner\,{}_\neg R + {}_\neg d(X\lrcorner\,{}_\neg\mu)$$
$$= 2X^{B}R^{D}_{BC}dx^{C} \otimes Z_{D} + [{}_\neg d(X\lrcorner\,{}_\neg\mu]^{D} \otimes Z_{D}.$$

In this formula all components of the curvature appear explicitly. The procedure them implies that we must replace all outer components of the curvature as functions of the inner ones, using the rheonomic conditions. In this way the infinitesimal change is given entirely in terms of the space-time components of the fields and we obtain a transformation which is properly defined. The transformation defined by (24) and these substitutions form an algebra which closes on-shell. In fact, if we define the Lie derivatives through (24) we see that closure implies the Bianchi identities. The rheonomy principle gives a direct construction of on-shell symmetries on space-time.

But is it possible to extend these symmetries to off-shell configurations or better to symmetries of the action? There is no clear-cut extension of Eq. (24) to off-shell configurations or rather this extension is given only modulo the field equations. However, in the cases where this extension has been found it is seen to imply that the derivative of the Lagrangian form $d\Omega$ vanishes on G. Obviously, this is equivalent to say that the action, as computed on any space-time manifold, does not depend on it.

This discussion can be exemplified in $N = 1$ supergravity. Here the field equations (21) are:

$$
\begin{aligned}
&\varepsilon_{ijkl} R^k \wedge \theta^1 = 0 \\
&R^{ab} \wedge \theta^k \varepsilon_{abkl} + 2\bar{\psi} \wedge \gamma_5 \gamma_1 \rho = 0 \qquad (25)\\
&2\gamma_5\gamma_m\rho \wedge \theta^m - \gamma_5\gamma_m\psi \wedge R^m = 0\,.
\end{aligned}
$$

The solution of the outer equations (that is taking into account rheonomy) are:

$$
\begin{aligned}
&R^a = R^a_{mn}\theta^m \wedge \theta^n \\
&\rho = \frac{1}{2}\rho_{mn}\theta^m \wedge \theta^n - \frac{1}{2}\left(\gamma_m\gamma_k R^k_{mn} - \frac{i}{3}\gamma_m\Sigma^{\tau s}\gamma_k R^k_{\tau s}\right)\psi \wedge \theta^m \\
&R^{ab} = \frac{1}{2}R^{ab}{}_{mn}\theta^m \wedge \theta^n - \frac{1}{2}\varepsilon^{abrs}\bar{\psi}\gamma_5\gamma_m\rho_{rs} \wedge \theta^m - \frac{1}{2}\varepsilon^{trs[a}\bar{\psi}\gamma_5\gamma_t\rho_{rs}\theta^{b]} \qquad (26)
\end{aligned}
$$

In considering these equations, we should stress that they express the $\psi\theta$ and $\psi\psi$ components of the curvature entirely in terms of $\theta\theta$ components. They can easily be checked by inserting them into (26).

Finally, we have the inner field equations relating $\theta\theta$ components only:

$$
\begin{aligned}
&R^k_{mn} = 0 \\
&R^{km}{}_{lm} - \frac{1}{2}\delta^k_l R^{mn}{}_{mn} = 0 \qquad (27)\\
&\gamma^m\rho_{mn} = 0, \qquad \varepsilon^{abcd}\gamma_b\rho_{cd} = 0\,.
\end{aligned}
$$

Besides these equations, we have to take into account the Bianchi identities:

$$
\begin{aligned}
&dR^{ab} - \omega^{at} \wedge R^b_t + \omega_t{}^b \wedge R^{at} = 0 \\
&dR^a - \omega^{at} \wedge R_t - i\bar{\psi}\gamma^a \wedge \rho + R^{ab} \wedge \theta_b = 0 \\
&d\rho + \frac{i}{2}\Sigma_{ab}\omega^{ab} \wedge \rho - \frac{i}{2}\Sigma_{ab}R^{ab} \wedge \psi = 0\,.
\end{aligned}
$$

These are seen to imply that

$$\overset{*}{\rho}^{ab} = \frac{1}{2}\varepsilon^{abrs}\rho_{rs} = i\gamma_5\rho^{ab},$$

so that the Rarita-Schwinger spinor ρ^{ab} is self-dual on-shell. The expression for the curvature is then much simpler if all on-shell conditions and Bianchi identities are taken into account:

$$R^a = 0$$
$$\rho = \frac{1}{2}\rho_{mn}\theta^m \wedge \theta^n$$
$$R^{ab} = \frac{1}{2}R^{ab}{}_{mn}\theta^m \wedge \theta^n + i\bar{\psi}\gamma_m\rho^{ab}\theta^m.$$

From these we can recover the infinitesimal transformations:

$$\delta_\varepsilon\theta^a = d\varepsilon^a - \omega^{ab} \wedge \varepsilon_b - i\bar{\psi} \wedge \gamma^a\varepsilon + \mathscr{L}_\varepsilon R^a$$
$$\delta_\varepsilon\omega^{ab} = d\varepsilon^{ab} - \omega^{at} \wedge \varepsilon_t{}^b + \omega_t{}^b \wedge \varepsilon^{at} + \mathscr{L}_\varepsilon R^{ab}$$
$$\delta_\varepsilon\psi = d\varepsilon - \frac{i}{2}\omega^{ab} \wedge \Sigma_{ab}\varepsilon - \frac{i}{2}\Sigma_{ab}\varepsilon^{ab} \wedge \psi + \mathscr{L}_\varepsilon\rho.$$

There is no need to consider the ε^a and ε^{ab} terms since they yield trivial co-ordinate and Lorentz transformations. The spinor term leads to the familiar transformation rules:

$$\delta_\varepsilon\theta^a = -i\bar{\psi} \wedge \gamma^a\varepsilon$$
$$\delta_\varepsilon\omega^{ab} = i\bar{\varepsilon} \wedge \gamma_m\rho^{ab}\theta^m$$
$$\delta_\varepsilon\psi = d\varepsilon - \frac{i}{2}\omega^{ab} \wedge \Sigma_{ab}\varepsilon.$$

Their closure on-shell is guaranteed by the Bianchi identities.

4.5. Supergravity in five dimensions.

The same techniques can be applied also to other theories. A non trivial example is provided by supergravity in five dimensions [*28*]. Here we have a four-component spinor ξ beside the vierbein θ^a, the spin connection ω^{ab} and a scalar 1-form B. The curvature components are given by:

$$R^{ab} = d\omega^{ab} - \omega^{at} \wedge \omega_t{}^b, \qquad R^a = d\theta^a - \omega^{at} \wedge \theta_t - \frac{i}{2}\bar{\xi} \wedge \gamma^a\xi$$
$$R^{\otimes} = dB - i\bar{\xi} \wedge \xi, \qquad \rho = d\xi + \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \xi \ (^{11})$$

(11) The signature of the metric is $(1, -1, -1, -1, -1)$. $\gamma_{ab} \equiv \frac{1}{2}[\gamma_a, \gamma_b] \equiv -2i\Sigma_{ab}$.

Notice that ξ is not a Majorana spinor, according to the general rules. The manifold M is now embedded in the group $\overline{\mathrm{SU}}(2,2/1)$, a contraction of SU(2, 2/1). A detailed analysis (see ref. [*28*]) leads to the first order action with Lagrangian density:

$$\Omega = \frac{1}{3}\mathrm{R}^{ab} \wedge \theta^i \wedge \theta^j \wedge \theta^k \varepsilon_{abijk} + \mathrm{R}^{ab} \wedge \theta_a \wedge \theta_b \wedge \mathrm{B} + i\mathrm{R}^a \wedge \bar{\xi}\gamma_a\xi \wedge \mathrm{B} +$$

$$+ \frac{i}{2}\mathrm{R}^{\otimes} \wedge \bar{\xi} \wedge \xi \wedge \mathrm{B} - 2i\mathrm{R}^{\otimes} \wedge \bar{\xi} \wedge \gamma_a\xi \wedge \theta^a - \frac{1}{2}(\bar{\xi} \wedge \xi)\wedge(\bar{\xi} \wedge \xi)\wedge\mathrm{B} +$$

$$+ \bar{\xi} \wedge \xi \wedge (\bar{\xi} \wedge \gamma^a\xi) \wedge \theta_a + 4\bar{\xi} \wedge \Sigma_{ab}\xi \wedge \theta^a \wedge \theta^b + \frac{1}{4}\mathrm{R}^{\otimes} \wedge \mathrm{R}^{\otimes} \wedge \mathrm{B} +$$

$$+ \mathrm{R}^a \wedge \mathrm{R}_a \wedge \mathrm{B} - (1 - \eta)\mathrm{R}^a \wedge \mathrm{R}^{\otimes} \wedge \theta_a$$

and to the variational equations:

(28.1) (Einstein eqs.)

$$\varepsilon_{abijk}\mathrm{R}^{bi} \wedge \theta^j \wedge \theta^k + 2i(\bar{\xi} \wedge \xi) \wedge \mathrm{R}_a - i\left(\frac{3}{2} - \frac{n}{2}\right)\mathrm{R}^{\otimes} \wedge \bar{\xi} \wedge \gamma_a\xi$$

$$-4(\bar{\rho} \wedge \Sigma_{ab}\xi + \bar{\xi} \wedge \Sigma_{ab}\rho) \wedge \theta^b + i(1-\eta)(\bar{\rho} \wedge \xi - \bar{\xi} \wedge \rho) \wedge \theta^a + 2\eta\mathrm{R}_a \wedge \mathrm{R}^{\otimes} = 0$$

(28.2) (Maxwell eqts.)

$$n\mathrm{R}^{ab} \wedge \theta_a \wedge \theta_b - \frac{i}{2}(3-\eta)\mathrm{R}^a \wedge \bar{\xi} \wedge \gamma_a\xi + \frac{3i}{2}\mathrm{R}^{\otimes} \wedge \bar{\xi} \wedge \xi +$$

$$- i\frac{3+\eta}{2}(\bar{\rho} \wedge \gamma_a\xi - \bar{\xi} \wedge \gamma_a\rho) \wedge \theta^a + \eta\mathrm{R}^a \wedge \mathrm{R}_a - \frac{3}{4}\mathrm{R}^{\otimes} \wedge \mathrm{R}^{\otimes} = 0$$

(28.3) (Rarita-Schwinger eq.)

$$4\Sigma_{ab}\rho \wedge \theta^a \wedge \theta^b - 4\Sigma_{ab}\xi \wedge \mathrm{R}^a \wedge \theta^b + i(1-\eta)\xi \wedge \mathrm{R}^a \wedge \theta_a$$

$$- \frac{i}{2}(\eta + 3)\gamma_a\xi \wedge \mathrm{R}^{\otimes} \wedge \theta^a = 0, \qquad (\eta = \pm 1).$$

If we project now Eqts. (28) along the normal components we find the constraints:

$$\mathrm{R}^{ab} = \mathrm{R}^{ab}{}_{ij}\theta^i \wedge \theta^j + \varepsilon^{abijk}\left\{ \bar{\rho}_{jk}\left(\Sigma_{ml} - \frac{i}{4}(1-\eta)\delta_{ml}\mathbf{1}\right)\xi + \text{c. c.}\right\} \wedge \theta^m +$$

$$+ \frac{i}{2}\mathrm{F}^{ab}\bar{\xi} \wedge \xi - \frac{i}{8}(1 + \eta)\varepsilon^{abijk}\mathrm{F}_{ij}\bar{\xi} \wedge \gamma_k\xi$$

$$\mathrm{R}^a = -\frac{\eta}{4}\varepsilon^{abcdf}\mathrm{F}_{bc}\theta_d \wedge \theta_f$$

$$R^{\otimes} = F_{ab}\theta^a \wedge \theta^b$$
$$\rho = \rho_{ab}\theta^a \wedge \theta^b + \left(\frac{1}{2}\gamma_a\xi F^{ab} - \frac{i}{8}(1-\eta)\varepsilon^{bacdf}F_{ac}\Sigma_{df}\xi\right) \wedge \theta_b$$
$$\bar{\rho} = \bar{\rho}_{ab}\theta^a \wedge \theta^b + \left(\frac{1}{2}\bar{\xi}\gamma_a F^{ab} + \frac{i}{8}(1-\eta)\varepsilon^{bacdf}F_{ac}\bar{\xi}\Sigma_{df}\right) \wedge \theta^b .$$

4.6. Supergravity d = 6.

This is the simplest one containing a non-trivial (that is not all potentials are one-forms) free differential algebra given by:

$$R^{ab} = d\omega^{ab} - \omega^{at} \wedge \omega_t{}^b = \dashv d\omega^{ab}$$
$$R^a = d\theta^a - \omega^{at} \wedge \theta_t - \frac{i}{2}\bar{\psi} \wedge \gamma^a\psi = \dashv d\theta^a - \frac{i}{2}\bar{\psi} \wedge \gamma^a\psi$$
$$\rho = d\psi + \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \psi = \dashv d\psi$$
$$R^{\otimes} = dB - \frac{i}{2}\bar{\psi} \wedge \gamma^a\psi \wedge \theta_a \qquad (a, b, t = 1, \dots, 6)\ (^{12})$$

where the connection ω^{ab}, the vierbein θ^a and the gravitino ψ are gauge potentials for the six-dimensional super-Poincare group. B is a 2-differential form. We have therefore strengths with three indices F^{abc}.

The Clifford algebra is given by:

$$[\gamma_a, \gamma_b] = 2\eta_{ab}, \qquad \gamma_1^+ = \gamma_1, \qquad \gamma_r^+ = -\gamma_r \qquad i = 2, \dots, 6,$$
$$\gamma_7 \equiv \gamma_1\gamma_2 \dots \gamma_6, \qquad \gamma_7^+ = \gamma_7, \qquad \gamma_7^2 = 1, \qquad \varepsilon_1 \dots r_6 = 1,$$
$$\gamma_a^T = C\gamma_a C^{-1}, \qquad C = C^T = C^{-1}, \qquad \gamma^{b_1 \dots b_r}\varepsilon_{a_1 \dots a_k b_1 \dots b_r} = (-1)^{r(r-1)/2}\, r!\, \gamma_7\gamma_{a_1 \dots a_k}\ (^{13})$$

We require that the gravitino is Weyl: $\gamma_7\psi = \psi$, so that the Fierz identity holds: $\gamma_a\psi \wedge \bar{\psi}\gamma^a\psi = 0$ and the only non vanishing currents are: $\bar{\psi} \wedge \gamma^a\psi$, $\bar{\psi} \wedge \gamma^{abc}\psi$. The three-index current is self-dual: $\bar{\psi} \wedge \gamma_{abc}\psi = \frac{1}{6}\varepsilon_{abcpqr}\bar{\psi} \wedge \gamma^{pqr}\psi$.

The action must remain invariant under the following scaling properties: $\omega^{ab} \to \omega^{ab}$, $\theta^a \to e\theta^a$, $\psi \to \sqrt{e}\psi$, $R^{ab} \to R^{ab}$, $R^a \to eR^a$, $B \to e^2B$.

---

(12) $\dashv d$ denotes the covariant exterior differential with respect to the fully reductif pseudo-connection $\dashv$ for the Lie subgroup SL(2, $\mathbb{C}$). The signature of the metric is (1, −1, −1, − 1, − 1, − 1).

(13) The matrix $\gamma_{a_1 \dots a_k}$ is the antisymmetrized in all indices of the product $\gamma_{a_1} \dots \gamma_{a_k}$.

Furthermore one requires exact gauge invariance under the following gauge transformation: $B \to B + d\chi$.

If we impose rheonomy and all the outher conditions on the theory we find two choices:

$$I = \iint \Big\{ R^{ab} \wedge \frac{1}{4}\varepsilon_{abc_1\ldots c_4}\theta^{c_1} \wedge \ldots \wedge \theta^{c_4} + 3iR^a\overline{\psi}\gamma^b\psi \wedge \theta_b \wedge \theta_a - 3iR^{\otimes}\overline{\psi} \wedge \gamma^a\psi \wedge \theta_a - i\overline{\rho}\gamma_{abc} \wedge \psi \wedge \theta^a \wedge \theta^b \wedge \theta^c + (\text{terms c.}) \pm 2\sqrt{3}R^a \wedge R^{\otimes} \wedge \theta_a \Big\}.$$

The variational equations are then:

(Torsion eq.) $\varepsilon_{abijkl}R^i \wedge \theta^j \wedge \theta^k \wedge \theta^1 - \xi R^{\otimes} \wedge \theta_a \wedge \theta_b = 0$

(Einstein eq.) $\varepsilon_{abcdef}\ R^{ab} \wedge \theta^c \wedge \theta^d \wedge \theta^e - 6iR^a \wedge \theta_a \wedge \overline{\psi} \wedge \gamma_f\psi - 6i\overline{\psi}\gamma^b \wedge \psi R_{[f} \wedge \theta_{b]} + 3i(\overline{\rho} \wedge \gamma^b\psi - \overline{\psi} \wedge \gamma^b\rho) \wedge \theta_b \wedge \theta_f - 3i(\overline{\rho} \wedge \gamma_{abf}\psi + \overline{\psi}\gamma_{abf} \wedge \rho) \wedge \theta^a \wedge \theta^b + 2\xi R_f \wedge R^{\otimes} = 0$

(Maxwell eq.)

$$\xi R^{ab} \wedge \theta_a \wedge \theta_b - i\left(3 + \frac{\xi}{2}\right)(\overline{\rho} \wedge \gamma^a\psi - \overline{\psi} \wedge \gamma^a\rho) \wedge \theta_a - 3iR^a \wedge \overline{\psi} \wedge \gamma_a\psi + \xi R^a \wedge R_a = 0$$

(Gravitino eq.)

$$\left(-3 + \frac{\xi}{2}\right)R^a \wedge \gamma^b\psi \wedge \theta_a \wedge \theta_b + \left(3 + \frac{\xi}{2}\right)R^{\otimes} \wedge \gamma_a\psi \wedge \theta^a - 2\gamma_{abc}\rho \wedge \theta^a \wedge \theta^b \wedge \theta^c + 3\gamma_{abc}\psi \wedge R^a \wedge \theta^b \wedge \theta^c = 0.$$

From ref. [*29*] one gets

$$R^a = -\frac{1}{12}\xi\varepsilon^{abcpqr}F_{pqr}\theta_b \wedge \theta_c$$
$$R^{\otimes} = F_{abc}\theta^a \wedge \theta^b \wedge \theta^c$$
$$\xi = \pm 2\sqrt{3}.$$

In order to satisfy the general requirement that any realistic theory must have the same number of Fermi and Bose degrees of fredom, we assume the following self-duality conditions for F^{abc}:

$$F^{abc} = F^{*abc} = \frac{1}{6}\varepsilon^{abcpqr}F_{pqr}.$$

The following type of identity is then very useful in the computations. If A_{abc}, B_{abc} have both the same duality type then: $\varepsilon_{abcpqr}A^{abc}B^{pqr} = 0$. In particular $F_{apq}\overline{\psi} \wedge \gamma^{ars}\psi\varepsilon^{pq}{}_{rslm} = 0$. Having this in mind it is possible to construct a Lagrangian satisfying the axiom of the theory. The Bianchi identities are given by:

$$\neg d\mathrm{R}^{ab} = 0$$

$$\neg d\mathrm{R}^{a} + \mathrm{R}^{ab} \wedge \theta_b + \frac{i}{2}(\bar{\rho} \wedge \gamma^a \psi - \bar{\psi} \wedge \gamma^a \rho) = 0$$

$$\neg d\rho + \frac{1}{4}\gamma_{ab}\mathrm{R}^{ab} \wedge \psi = 0$$

$$d\mathrm{R}^{\otimes} + \frac{i}{2}(\bar{\rho} \wedge \gamma^a \psi - \bar{\psi} \wedge \gamma^a \rho) \wedge \theta_a + \frac{i}{2}\bar{\psi} \wedge \gamma^a \psi \wedge \mathrm{R}_a = 0$$

It can be checked that duality follows only if we take into account the normal components of the equations. On space-time the theory is consistent with a generic strength but admits no supersymmetry unless we assume self-duality. This raises a number of interesting questions.

On space-time the supersymmetry holds only if we restrict the manifold of the solutions with a condition which does not follow from variation of the fields, we need the gravitino components. This implies that the set of theories which can be described by the present scheme is somewhat larger than the usual set of supergravities.

The group manifold approach regard all variational equations on the same footing and therefore also normal components have the same states as the space-time ones.

4.7. The CJS (Cremmer-Julia-Scherk) theory in eleven dimension.

From the physical point of view this theory appears the most important of the supergravities. The theory is based on the free differential algebra

$$\mathrm{R}^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b = \neg d\omega^{ab}$$

$$\mathrm{R}^{a} = d\theta^{a} - \omega^{at} \wedge \theta_t - \frac{i}{2}\bar{\psi} \wedge \gamma^a \psi = \neg d\theta^a - \frac{i}{2}\bar{\psi} \wedge \gamma^a \psi$$

$$\rho = d\psi + \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \psi = \neg d\psi$$

$$\mathrm{R}^{\square} = d\mathrm{A} - \frac{1}{2}\bar{\psi} \wedge \gamma^{a_1 a_2}\psi \wedge \theta_{a_1} \wedge \theta_{a_2}$$

where θ^a and ω^{ab} are the conventional vierbein and spin connection for the Poincaré group, the ψ are Majorana and generate the supertranslations. The potential A is a three-form. We have therefore strengths with four indices F^{abcd}. The action is then given by means of the following Lagrangian density:

$$(29)\quad \Omega = -\frac{1}{9} R^{a_1a_2} \wedge \theta^{a_3} \wedge \ldots \wedge \theta^{a_{11}} \varepsilon_{a_1 \ldots a_{11}} + \frac{7i}{30} R^a \wedge \theta_a \wedge \bar{\psi}\gamma^{b_1 \ldots b_5} \psi \wedge \theta^{b_6} \wedge \ldots \wedge \theta^{b_{11}} \varepsilon_{b_1 \ldots b_{11}} + 2\bar{\rho} \wedge \gamma_{a_1 \ldots a_8} \psi \wedge \theta^{a_1} \wedge \ldots \wedge \theta^{a_8}$$
$$- 84 R^{\square} \wedge (i\bar{\psi} \wedge \gamma_{a_1 \ldots a_5} \wedge \psi \wedge \theta^{a_1} \wedge \ldots \wedge \theta^{a_5} - A \wedge \psi \wedge \gamma^{a_1a_2}\psi \wedge \theta_{a_1} \wedge \theta_{a_2})$$
$$+ \frac{1}{4} \bar{\psi} \wedge \gamma^{a_1a_2}\psi \wedge \bar{\psi} \wedge \gamma^{a_3a_4}\psi \wedge \theta^{a_5} \wedge \ldots \wedge \theta^{a_{11}} \varepsilon_{a_1 \ldots a_{11}}$$
$$- 210 \bar{\psi} \wedge \gamma^{a_1a_2}\psi \wedge \bar{\psi} \wedge \gamma^{a_3a_4}\psi \wedge \theta_{a_1} \wedge \ldots \wedge \theta_{a_4} \wedge A + (c.)$$
$$- 840 R^{\square} \wedge R^{\square} \wedge A - \frac{1}{330} F_{a_1 \ldots a_4} F^{a_1 \ldots a_4} \theta^{c_1} \wedge \ldots \wedge \theta^{c_{11}} \varepsilon_{c_1 \ldots c_{11}}$$
$$+ 2 F_{a_1 \ldots a_4} R^{\square} \wedge \theta_{a_5} \wedge \ldots \wedge \theta_{a_{11}} \varepsilon^{a_1 \ldots a_{11}}.$$

We do not report here all the variational equations associated with the above Lagrangian density (see ref. [*1*]).

The normal components are seen to satisfy rheonomy as a consequence of the choice of the coefficients in the action. The resulting equations are:

$$R^a = 0, \qquad R^{\square} = F_{m_1 \ldots m_4} \quad \theta^{m_1} \wedge \ldots \wedge \theta^{m_4},$$
$$\rho = \rho_{ab} \quad \theta^a \wedge \theta^b + \rho^{(\text{on-shell})}_{(\psi\theta)}.$$

In (29) we see that the last term violates the scheme which we have followed so far, it contains a set of space-time scalars which ultimately represent the strengths and allowed us to form the dual of the form $R^{\square}$. We can in fact regard the scalars as zero forms in the free differential algebra, the duality is actually achieved in the tangent space and is not dangerous. We are allowed to use the general method explained before.

In fact the same procedure allows us to discuss the electromagnetic field coupled to gravity in any dimension. The Bianchi identities appear as:

$$\daleth d R^{ab} = d R^{ab} - \omega^{at} \wedge R_t^b - \omega_c^b \wedge R^{at} = 0$$
$$\daleth d R^a + R^{ab} \wedge \theta_b - i\bar{\psi} \wedge \gamma^a \rho = d R^a - \omega^{at} \wedge R_t + R^{ab} \wedge \theta_b - i\bar{\psi} \wedge \gamma^a \rho = 0$$
$$\daleth d\rho + \frac{1}{4} \gamma_{ab} \psi \wedge R^{ab} = d\rho + \frac{i}{2} \Sigma_{ab} \omega^{ab} \wedge \rho + \frac{1}{4} \gamma_{ab} \psi \wedge R^{ab} = 0$$
$$d R^{\square} - \bar{\psi} \wedge \gamma^{a_1a_2} \rho \wedge \theta_{a_1} \wedge \theta_{a_2} + \bar{\psi} \wedge \gamma^{a_1a_2} \psi \wedge R_{a_1} \wedge \theta_{a_2} = 0.$$

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APPENDIX A

PULL-BACK OF A(G)-VALUED DIFFERENTIAL FORMS ON A LIE GROUP OR PRINCIPAL FIBER BUNDLE

When a differential form α on a manifold M is valued on a vector space V the usual way to define the pull-back $\phi^*\alpha$ of α via a diffeomorphism ϕ of M is by means of the following commutative diagram:

$$\begin{array}{ccc} \Lambda_p^0 M \otimes V & \xleftarrow{\Lambda_p^0(\phi)\otimes id_V} & \Lambda_p^0 M \otimes V \\ \phi^*\alpha \uparrow & & \uparrow \alpha \\ M & \xrightarrow[\phi]{} & M \end{array}$$

In fact, $\Lambda_p^0(\phi) \otimes id_V$ is the fiber diffeomorphism of $\Lambda_p^0 M \otimes V$ over ϕ^{-1}, canonically associated to ϕ.

However, if M is a Lie group G (or a G-principal fiber bundle P), and $V \equiv A(G)$, since $A(G) = T_eG$ (or $A(G) \cong vT_pP$) we have no *a priori* reasons to consider that to any diffeomorphism ϕ of G (resp. P) the canonical map associated to A(G) is just $id_{A(G)}$. In fact, we should see that in some circumstances the situation is completely different.

(A.1) Let $\alpha : G \to \Lambda_p^0 G \otimes A(G)$ be a A(G)-valued differential form on a Lie group G. Let $\phi : G \to G$ be any diffeomorphism of G. The pull-back of α by means of ϕ is defined by means of the following commutative diagram:

$$\begin{array}{ccc} \Lambda_p^0 G \otimes A(G) & \xleftarrow{\Lambda_p^0(\phi)\otimes f(\phi)} & \Lambda_p^0 G \otimes A(G) \\ \phi^*\alpha \uparrow & & \uparrow \alpha \\ G & \xrightarrow[\phi]{} & G \end{array}$$

where $f(\phi)$ is a map canonically associated to ϕ. In order to see which concrete map $f(\phi)$ is, we shall distinguish two cases: (*a*) $\phi(e) \neq e$. As $A(G) = T_eG$ we can see that the map canonically associated to ϕ is just

$$T_eG \xrightarrow{T(\phi)} T_{\phi(e)}G \xrightarrow{T(\phi^{-1})} T_eG .$$

Therefore, $f(\phi) = id_{A(G)}$ (Note, also, that to go back from $T_{\phi(e)}G$ to T_eG we shall use the group law $\psi : G \times G \to G$, but since in general G is not commutative we have none criterion to choice the left or right multiplication !)

These considerations can be applied, for example, if $\phi = R_a$ or $\phi = L_a$ that are the so-called right and left translations respectively.

(*b*) $\phi(e) = e$. In these cases $T(\phi)$ is an isomorphism of A(G). So, $f(\phi) = T(\phi^{-1}) \neq id_{A(G)}$. For example, if $\phi = j_a$, $a \in G$, $j_a : b \mapsto a^{-1}ba$, we check that $f(j_a) = \mathrm{ad}\,(a^{-1})$.

(A.2) Note, that if $\omega : G \to T^*G \otimes A(H)$ is a Ehresmann connection on the H-principal bundle $\pi_G : G \to G/H$ for some closed subgroup H of G, we have that the pull-back of ω by means of any principal fiber bundle diffeomorphism ϕ of π_G is given by means of the following commutative diagram:

$$\begin{array}{ccc} T^*G \otimes A(H) & \xleftarrow{T^*(\phi)\otimes \bar{f}(\phi)} & T^*G \otimes A(H) \\ \phi^*\omega \Big\uparrow & & \Big\uparrow \omega \\ G & \xrightarrow[\phi]{} & G \end{array}$$

where the map $\bar{f}(\phi)$ is canonically associated to ϕ.

On the other hand as $A(H) = vT_eG \subset T_eG = A(G)$ and since $T(\phi)$ maps vertical spaces into vertical ones, we get $\bar{f}(\phi) = f(\phi) \mid A(H)$, where $f(\phi)$ is the map given in (A.1).

(A.3) Finally, let us consider the pull-back of a A(H)-valued differential form α on a H-principal fiber bundle $(P, M, \pi; H)$. Let $\psi : H \times P \to P$ be the action map of H over P. For any principal fiber bundle diffeomorphism of P the pull-back of α is given by the following commutative diagram:

$$\begin{array}{ccc} \Lambda^0_q P \otimes A(H) & \xleftarrow{F(\phi)\equiv\Lambda^0_q(\phi)\otimes f(\phi)} & \Lambda^0_q P \otimes A(H) \\ \phi^*\alpha \Big\uparrow & & \Big\uparrow \alpha \\ P & \xrightarrow[\phi]{} & P \end{array}$$

To investigate the structure of $F(\phi)$ we note that A(H) can be identified with vertical tangent spaces of P. So, we get the following commutative diagram:

$$\begin{array}{ccc} (\Lambda^0_q P)_p \otimes T_eH & \xrightarrow{F(\phi^{-1})\equiv\Lambda^0_q(\phi^{-1})\otimes T(\psi^{-1}_{\phi(p)}\circ\phi\circ\psi_p)} & (\Lambda^0_q P)_{\phi(p)} \otimes T_eH \\ \mathrm{id}\otimes T(\psi_p) \Big\| & & \Big\| \mathrm{id}\otimes T(\psi^{-1}_{\phi(p)}) \\ (\Lambda^0_q P)_p \otimes vT_pP & \xrightarrow[\Lambda^0_q(\phi^{-1})\otimes vT(\phi)]{} & (\Lambda^0_q)_{\phi(p)} \otimes vT_{\phi(p)}P \end{array}$$

So, in general $f(\phi) \neq \mathrm{id}_{A(H)}$. But, if $\phi = \psi_a$, for some $a \in H$, then since $(\psi^{-1}_{\psi_a(p)} \circ \psi_a \circ \psi_p) = \mathrm{id}_H$, we have that $f(\phi) = \mathrm{id}_{A(H)}$.

As a consequence of above considerations we try that if $\alpha : G \to \Lambda^0_p G \otimes A(G)$ is a A(G)-valued differential form on G, α is G-invariant (e. g. invariant for left or right translations) iff in any basis $\{ Z_A \}$ of A(G) the p-forms $\alpha^A : G \to \Lambda^0_p G$ in the linear representation $\alpha = \alpha^A \otimes Z_A$ are G-invariant.

APPENDIX B

SOME OBSERVATIONS ON THE PRINCIPAL CONNECTIONS AND PSEUDOCONNECTIONS

Let $\pi : P \to M$ be a G-principal bundle over M. Let $\phi : G \times P \to P$ be the action map of G on P. A *principal connection* on P can be characterized as a G-invariant connection $C_1 \equiv \urcorner(P) \subset J\mathcal{D}(P)$. More precisely, the differential equation C_1 is invariant under the natural action of G on $J\mathcal{D}(P)$, e. g. the following diagram

$$\begin{array}{ccccc} C_1 & \hookrightarrow & J\mathcal{D}(P) & \xrightarrow{\pi_{1,0}} & P \\ {\scriptstyle J\mathcal{D}(\phi_a)|C_1}\downarrow & & {\scriptstyle J\mathcal{D}(\phi_a)}\downarrow & & \downarrow{\scriptstyle \phi_a} \\ C_1 & \hookrightarrow & J\mathcal{D}(P) & \underset{\pi_{1,0}}{\to} & P \end{array}$$

is commutative for any $a \in G$.

This is equivalent to say that the section $\urcorner : P \to J\mathcal{D}(P)$ is G-invariant, that is

$$\phi_a^* \urcorner \equiv J\mathcal{D}(\phi_a) \circ \urcorner \circ \phi_a^{-1} = \urcorner$$

or that the splitting map $\bar{\Gamma} : TP \to vTP$ gives the following commutative diagram

$$\begin{array}{ccc} TP & \xrightarrow{\bar{\Gamma}} & vTP \\ {\scriptstyle T(\phi_a)}\downarrow & & \downarrow{\scriptstyle vT(\phi_a)} \\ TP & \underset{\bar{\Gamma}}{\longrightarrow} & vTP \end{array}$$

for any $a \in G$.

Furthermore, there exists a unique section $\urcorner/G : P/G \to J\mathscr{P}(P)/G$ such that the following diagram

$$\begin{array}{ccc} J\mathcal{D}(P) & \xrightarrow{\check{\pi}_G} & J\mathcal{D}(P)/G \\ {\scriptstyle \urcorner}\uparrow & & \uparrow{\scriptstyle \urcorner/G} \\ P & \underset{\pi_G}{\longrightarrow} & P/G \end{array}$$

is commutative, where π_G and $\check{\pi}_G$ are the canonical projections.

Now, if $\urcorner \equiv \urcorner\mu \times id_{A(G)}$ is a pseudoconnection on a Lie group G we can see that really it is a principal connection on the A(G)-principal bundle $\pi_E : E \equiv G \times A(G) \to G$. In fact, we can easily see that $\phi_u^* \urcorner = \urcorner$, $\forall u \in A(G)$ by means of the following commutative diagram:

$$\begin{array}{ccc} J\mathcal{D}(E) \doteq T^*G \otimes A(G) \times A(G) & \xrightarrow{J\mathcal{D}(\phi_u) = id_{T^*G \otimes A(G)} \times (+u)} & J\mathcal{D}(E) \cong T^*G \otimes A(G) \times A(G) \\ \uparrow{\scriptstyle \urcorner \equiv \urcorner\mu \times id_{A(G)}} & & \uparrow{\scriptstyle \urcorner \equiv \urcorner\mu \times id_{A(G)}} \\ E \equiv G \times A(G) & \underset{\phi_u = id_G \times (+u)}{\longrightarrow} & E \equiv G \times A(G) \end{array}$$

Furthermore, we can see that the unique section $\urcorner/A(G)$ associated to $\urcorner$ is just the A(G)-valued differential form $\neg\mu$. In fact one has the following commutative diagram:

$$\begin{array}{ccc} J\mathscr{D}(E) \cong T^*G \otimes A(G) \times A(G) & \xrightarrow{\pi_{A(G)} = pr_1} & J\mathscr{D}(E)/A(G) \cong T^*G \otimes A(G) \\ \uparrow \scriptstyle{\urcorner = \neg\mu \times id_{A(G)}} & & \uparrow \scriptstyle{\urcorner/A(G) = \neg\mu} \\ E \equiv G \times A(G) & \xrightarrow[\pi_{A(G)} = \pi_A]{} & G \cong E/A(G) \end{array}$$

Finally, the morphism $\ulcorner : J\mathscr{D}(E) \to T^*G \otimes vTE$ corresponding to a pseudoconnection $C_1 \equiv \urcorner(E) \subset J\mathscr{D}(E)$, $E \equiv G \times A(G)$, is given by the following fiber morphism over G, characterized by $\neg\mu$:

$$\ulcorner = \check{\ulcorner} \times 0 : J\mathscr{D}(E) \cong T^*G \otimes A(G) \times A(G) \to T^*G \otimes vTE \cong T^*G \otimes A(G) \times A(G)$$

where $\check{\ulcorner} \equiv id_{T^*G\otimes A(G)} - \neg\mu \circ \pi_{\bar{E}} : T^*G \otimes A(G) \to T^*G \otimes A(G)$ is a morphism over G, 0 is the zero map $A(G) \to A(G)$ and $\pi_{\bar{E}}$ the canonical projection $\pi_{\bar{E}} : \bar{E} \equiv T^*G \otimes A(G) \to G$.

Let us, now, conclude this appendix with a generalization of the concept of pseudoconnections in order to consider A(G)-valued p-differential forms not limited to $p = 1$. These forms admit the usual supersymmetric grading. Here, we shall prove that such objects can be considered as particular connections of order p on the trivial fiber bundle π_E:

$$\pi_E : E \equiv G \times A(G) \to G.$$

DEFINITION B1. — A *p-pseudoconnection* on a Lie supergroup G is a semiholonomic p-connection $\bar{C}_p \equiv \urcorner(\bar{J}\mathscr{D}^{p-1}(E)) \subset \bar{J}\mathscr{D}^p(E)$ on the trivial fiber bundle $\pi_E : E \equiv G \times A(G) \to G$ such that $\bar{C}_p$ is freely generated by $\tilde{\tilde{C}}_p$ and $\bar{J}\mathscr{D}^{p-1}(E)$, that is $\bar{C}_p \cong \tilde{\tilde{C}}_p \times \bar{J}\mathscr{D}^{p-1}(E)$, where $\tilde{\tilde{C}}_p$ is a m-dimensional sub-fiber bundle of $\Lambda^0_p G \otimes A(G) \to G$, ($m = \dim G$).

Note that $\Lambda^0_p G \otimes A(G)$ can be considered a sub-fiber bundle of $\bar{J}\mathscr{D}^p(E)$ as we have the following diagram of exact sequences of vector fiber bundles over G:

$$\begin{array}{ccccccccc} 0 & \to & T^0_p G \otimes A(G) & \to & \bar{J}\mathscr{D}^p(E) & \underset{\urcorner}{\rightleftarrows} & \bar{J}\mathscr{D}^{p-1}(E) & \to & 0 \\ & & \uparrow & & & & & & \\ & & \Lambda^0_p G \otimes A(G) & & & & & & \\ & & \uparrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

A semiholonomic p-connection gives a splitting $\bar{J}\mathscr{D}^p(E) \cong \bar{J}\mathscr{D}^{p-1}(E) \times T^0_p G \otimes A(G)$. Then, we can see that a p-pseudoconnection $\bar{\urcorner}$ on G has the following structure:

$$\bar{\urcorner} = (id_{\bar{J}\mathscr{D}^p} - 1_{(E)}, \neg\mu \circ \bar{\pi}_{p-1}),$$

where $\neg\mu$ is a section of $\Lambda^0_p G \otimes A(G) \to G$ and $\bar{\pi}_{p-1}$ is the canonical projection

$$\bar{J}\mathscr{D}^{p-1}(E) \to G.$$

So, a A(G)-valued p-differential form $\alpha : G \to \Lambda^0_p G \otimes A(G)$ can be identified with a p-pseudoconnection on G $\bar{\urcorner} = (id_{k\mathscr{D}^p} - 1_{(E)}, \alpha \circ \bar{\pi}_{p-1})$.

Of course a pseudoconnection on G (see Definition 4.1), is a p-pseudoconnection with $p = 1$.

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