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Anomalous terms in gauge theory: relevance of the structure group

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ABSTRACT. — After recalling the results of reference [*1*] on anomalous terms in gauge theory, we apply the algorithm producing these terms to two examples of structure group, respectively $G = SU(3) \times SU(2) \times U(1)$ and $G = U(1) \times U(1) \times \ldots \times U(1) = (U(1))^N$ (or more generally G = an arbitrary finite dimensional abelian Lie group). These two examples illustrate very clearly the influence of the structure group on the cohomology describing the anomalous terms.

RÉSUMÉ. — Après un bref rappel des résultats sur les termes anomaux dans les théories de jauge décrits dans [*1*], nous appliquons l'algorithme permettant leur calcul explicite à deux exemples de groupes de structure, respectivement $G = SU(3) \times SU(2) \times U(1)$ et $G = U(1) \times \ldots \times U(1) = (U(1))^N$, (ou, plus généralement, G = un groupe de Lie abélien de dimension finie arbitraire). Ces deux exemples illustrent particulièrement bien l'influence du groupe de structure sur la cohomologie décrivant ces termes anomaux.

1. INTRODUCTION

It has been realized by Becchi, Rouet and Stora [2] that the renormalization program for gauge theories involves cohomological problems which also appear in the framework of current algebra [3].

The classical fields entering gauge theories are connections in a principal fibre bundle $P = P(V, G)$, where V is space-time and G is the structure group; (we shall denote by $\mathfrak{g}$ the Lie algebra of G). We denote by $\mathscr{C}$ the space of connections on P; $\mathscr{C}$ is an affine subspace of the space of $\mathfrak{g}$-valued 1-forms on P.

The group of gauge transformations is the group of automorphisms of P which induce the identity mapping on V. We shall denote it by $\mathrm{Aut}_V(P)$ and its Lie algebra by $\mathrm{aut}_V(P)$. Equivalently, an element of $\mathrm{Aut}_V(P)$ is a map $\gamma : P \to G$ with the equivariance property $\gamma(pg) = g^{-1}\gamma(p)g$, for any $p \in P$ and $g \in G$. Similarly an element of $\mathrm{aut}_V(P)$ is a map $\xi : P \to \mathfrak{g}$ satisfying $\xi(pg) = ad(g^{-1})\xi(p)$, for any $p \in P$ and $g \in G$.

By pull-back, there is a right action of $\mathrm{Aut}_V(P)$ on $\mathscr{C}$. Therefore there is, correspondingly, a linear left action W of $\mathrm{Aut}_V(P)$ on functionals on $\mathscr{C}$; we denote by w the associated (infinitesimal) action of $\mathrm{aut}_V(P)$.

Physically, one is interested in invariant functionals on $\mathscr{C}$, (i. e. functionals on $\mathscr{C}/\mathrm{Aut}_V(P)$). However, in the quantization process, either of the gauge field [2], or of fermions in an external classical fields [3], non invariant steps are required, and thus invariance is not ensured for the quantum theory.

For antisymmetric multilinear forms on $\mathrm{aut}_V(P)$ with values in functionals on $\mathscr{C}$, one defines an antiderivation δ, mapping the p-forms into the $(p + 1)$-forms and satisfying $\delta^2 = 0$, by the following formula:

$$\delta\Phi(\xi_0, \ldots, \xi_p) = \sum_{0 \leq k \leq p} (-1)^k w(\xi_k)\Phi(\xi_0, \ldots, \hat{\xi}_k, \ldots, \xi_p) + \sum_{0 \leq r \leq s \leq p} (-1)^{r+s}\Phi([\xi_r, \xi_s], \xi_0, \ldots, \hat{\xi}_r, \ldots, \hat{\xi}_s, \ldots, \xi_p).$$

Alternatively, one introduces the « ghost field » χ by the following construction: Define χ to be the identity mapping of $\mathrm{aut}_V(P)$ on itself, considered as an element of the space $\mathrm{aut}_V(P) \otimes \Lambda(\mathrm{aut}_V(P))^*$ equipped with its natural bracket.

Then antisymmetric multilinear forms on $\mathrm{aut}_V(P)$ can be written as « polynomials » in the « components » of χ by using the identity

$$(\chi^{\alpha_1} \Lambda \ldots \Lambda \chi^{\alpha_p})(\xi_1, \ldots, \xi_p) = \det(\chi^{\alpha_r}(\xi_s)).$$

The action of δ reduces on χ to the familiar rule: $\delta\chi = -1/2\,[\chi, \chi]$.

On the other hand, the action of δ on the components of the generic connection form A on P reduces to the formula: $\delta A = -d\chi - [A, \chi] = -\nabla_A \chi$ with the convention that χ anticommutes with differential 1-forms on P and obvious notations.

It is worth noticing here that χ lifts to the Maurer-Cartan forms $\tilde{\chi}$ on $\mathrm{Aut}_V(P)$ when one represents forms on $\mathrm{aut}_V(P)$ as differential forms on $\mathrm{Aut}_V(P)$ by the usual transformation; $\tilde{\chi}$ is canonically a differential 1-form on $P \times \mathrm{Aut}_V(P)$ with values in $\mathfrak{g}$. On the other hand a connection A on P also defines a $\mathfrak{g}$-valued differential 1-form $\tilde{A}$ on $P \times \mathrm{aut}_V(P)$ by $\tilde{A}(p, \gamma) = A^\gamma(p) (= \gamma(p)^{-1} A(p) \gamma(p) + \gamma^{-1}(p) d\gamma(p))$ and then the exterior differential in the direction of $\mathrm{Aut}_V(P)$ just induces the action of δ on χ and A [*4*].

Invariance of a functional Γ on $\mathscr{C}$ reads $\delta\Gamma = 0$, while the Wess-Zumino consistency condition for the anomaly Δ reads $\delta\Delta = 0$, where Δ is a local polynomial in the fields [*5*] which is of degree one in χ (i. e. Δ is a linear form on $\mathrm{aut}_V(P)$ with values in functionals on $\mathscr{C}$ which is local [*5*]). However, as well known, anomalies of the form $\Delta = \delta\Gamma$, where Γ is a local functional on $\mathscr{C}$, are in fact spurious [*2*]. Thus the problem is of cohomological nature.

At the price of introducing a reference connection A_0 [*5*], [*6*] it is easy to write $\Delta = \int_V Q$ (V is space-time) and to rewrite $\delta\Delta = 0$ as $\delta Q + dQ' = 0$ where Q and Q′ are differential forms on V. When P is trivial one chooses as A_0 the connection which vanishes in the section corresponding to the given trivialisation. In order to avoid inessential complications, we shall suppose here that this is the case.

We will use the natural bidegree:

bidegree = (d-degree, δ-degree)
 = (degree of form on V, ghost number).

If Q satisfies $\delta Q + dQ' = 0$, we say that Q is a δ-cocycle modulo d. Similarly, when $\Delta = \delta \int_V L$ i. e. $Q = \delta L + dL'$, we say that Q is a δ-coboundary modulo d, and then, Δ is spurious.

Thus solving the consistency equation is equivalent to finding the δ-cohomology modulo d in bidegree $(n, 1)$, $n = \dim(V)$. It is also known that the δ-cohomology modulo d in bidegree $(n - 1, 2)$ corresponds to anomalous Schwinger terms in equal time commutation relations of currents [*7*]. In [*1*], we have completely determined the δ-cohomology modulo d for any bidegree in the class of the natural objects generated by the fields of the theory (B. R. S. algebra). As shown in [*1*], assumption on the dimension of space-time may be avoided by working at the level of the universal B. R. S. algebra.

Our aim is to apply our results to some specific examples of interest:

For all bidegrees, we shall exhibit the possible anomalous terms for gauge theories with structure group $G = (U(1))^N$, or more generally $G =$ an arbitrary finite dimensional abelian Lie group, and $G = SU(3) \times SU(2) \times U(1)$, representative of various aspects of the problem.

2. COMPUTATIONAL ALGORITHM FOR ANOMALOUS TERMS

It was shown in [1] how the δ-cohomology modulo d is obtained from the δ-cohomology. We describe the δ-cohomology in step 1.

A. Step 1 : the δ-cohomology.

We assume that $\mathfrak{g}$ is a reductive Lie algebra of rank r, (i. e. $\mathfrak{g}$ is the direct product of an abelian Lie algebra with a semi-simple Lie algebra).

Choose r linearly independent homogeneous primitive invariant forms ω^i, $(i = 1, 2, \ldots, r)$, together with associated transgressed invariant polynomials $\tau(\omega^i)$ on $\mathfrak{g}$.

The δ-cohomology is then the tensor product of the free graded commutative algebra generated by the ω^i, (which is the algebra of invariant exterior forms on $\mathfrak{g}$ and which identifies with the cohomology of $\mathfrak{g}$ [8], [9]), and the symmetric algebra generated by the $\tau(\omega^i)$, (which is the algebra of invariant polynomials on $\mathfrak{g}$ [10], [9]). This identification is done through the compositions $\omega^i \to \omega^i(\chi)$ and $\tau(\omega^i) \to \tau(\omega^i)(F)$, $(F = dA + 1/2\,[A, A])$; so ω^i comes with its degree $m_i = 2n_i - 1$ and bidegree $(0, m_i)$, while $\tau(\omega^i)$ is given the degree $2n_i$ and bidegree $(2n_i, 0)$, (remembering that $\tau(\omega^i)$ is a polynomial of degree n_i on $\mathfrak{g}$), and the δ-cohomology is the free graded (in fact bigraded) commutative algebra generated by the ω^i and the $\tau(\omega^i)$ equipped with these degrees.

Practically, write $\mathfrak{g} = (\mathfrak{u}(1))^M \times \mathfrak{g}_1 \times \ldots \times \mathfrak{g}_N$ where the $\mathfrak{g}_k$ are simple Lie algebras of rank r_k respectively $\left(\text{so we have } r = M + \sum_{k=1}^{N} r_k\right)$.

We have to choose for each factor $\mathfrak{g}_k$, a basis of primitive forms ω^i_k, $(i = 1, 2, \ldots, r_k)$. The form ω^i_k is of degree $m_i = 2n_i - 1$ and $\tau(\omega^i_k)$ is an invariant polynomial of degree n_i on $\mathfrak{g}_k$ which identifies with an element of degree $2n_i$ (bidegree $(2n_i, 0)$; $\tau(\omega^i_k)(F)$) in the δ-cohomology. The values of m_i for simple Lie algebras are given in table 1 [9]: [11].

For instance, if $\mathfrak{g}_k = \mathfrak{su}(p)$, then rank $(\mathfrak{g}_k) = p - 1$ and we may take

$$\omega^i_k(\chi) = \frac{(-1)^{n_i - 1}}{C^{n_i}_{2n_i - 1}} \operatorname{tr}(\chi_k^{2n_i - 1}), \quad \tau(\omega^i_k)(F) = \operatorname{tr}(F_k^{n_i})$$

TABLE 1.

G	Type	r	$m_i (i = 1 \dots r)$
SU(N), $N \geqq 2$	A_{N-1}	$N - 1$	$3, 5, 7, \dots, 2N - 1$
SO(2N + 1), $N \geqq 2$ Sp(2N), $N \geqq 2$	B_N C_N	N	$3, 7, \dots, 4N - 5, 4N - 1$
SO(2N), $N \geqq 4$	D_N	N	$3, 7, \dots, 4N - 5, 2N - 1$
	G_2 F_4 E_6 E_7 E_8	2 4 6 7 8	3, 11 3, 11, 15, 23 3, 9, 11, 15, 17, 23 3, 11, 15, 19, 23, 27, 35 3, 15, 23, 27, 35, 39, 47, 59

where n_i takes the values 2,3, ... p, and where F_k is the part of the field strength associated to $\mathfrak{g}_k$ (and similarly for χ_k).

For the case of orthogonal groups SO(p), primitive elements are given by similar formula, (notice that F is expressed as an antisymmetric matrix, so $\operatorname{tr}(F^{2s+1}) = 0$ which explains the jump of the m_i). Moreover, when $p = 2q$, the determinant of F, which is an invariant polynomial, can be written as $\det(F) = (\mathrm{Pf}(F))^2$, where the pfaffian Pf(F) is an independent (of $\operatorname{tr}(F^{2s})$) invariant polynomial of degree q given by

$$\mathrm{Pf}(F) = 1/q!\, \Sigma\, \varepsilon(\sigma)\, F_{\sigma(1)\sigma(2)} \dots F_{\sigma(2q-1)\sigma(2q)} .$$

This explains the occurence of a (new) primitive form of degree $2N - 1$ in the table 1.

Finally, for each abelian factor $\mathfrak{u}(1)_m$ (in $(\mathfrak{u}(1))^M$) take one couple $\omega_m(\chi) = \chi_m$ and $\tau(\omega_m)(F) = F_m$ with degree $(\omega_m) = 1$.

B. The generalized transgression formula.

In order to motivate step 2 and step 3, let us reproduce lèmma 7.2 of [1]. Our aim is to construct δ-cocycles modulo d starting from products of primitive δ-cocycles. Some steps in this direction appear in references [12]. Thus consider $X = \prod_i \tau(\xi_i)(F)\,\omega_0(\chi) \dots \omega_n(\chi)$, ξ_i and ω_j being primitive forms. By [13], there are invariant $L_p(A, F)$ such that

$$\tau(\omega_p)(F) = dL_p(A, F) = (d + \delta)\, L_p(A + \chi, F),$$

and therefore $L_p(\chi, 0) = \omega_p(\chi)$. Then we have:

$$(d + \delta) \prod_i \tau(\xi_i)(F) L_0(A + \chi, F) \dots L_n(A + \chi, F) =$$

$$\sum_{p=0}^{n} (-1)^p \prod_i \tau(\xi_i)(F)\tau(\omega_p)(F) . L_0(A + \chi, F) \dots \hat{L}_p \dots L_n(A + \chi, F).$$

Expanding both sides in decreasing δ-degree yields a number of equations starting with $\delta X = 0$ and exhibiting explicit δ-cocycles modulo d:

$$\begin{aligned} &\delta X = 0 \\ &\delta Q_1 + dX = 0 \\ &\delta Q_2 + dQ_1 = 0 \\ &\text{-------------} \\ &\delta Q_{2r+1} + dQ_{2r} = 0 \\ &\delta Q_{2r+2} + dQ_{2r+1} = d_r X \end{aligned} \tag{*}$$

where $d_r X$ is obtained by the first non vanishing contribution of the right hand side, $2r + 1$ being *the smallest degree* of the primitive forms ω_p entering X:

$$d_r X = \sum_p (-1)^p \prod_i \tau(\xi_i)(F)\, \tau(\omega_p)(F) \omega_0(\chi) \dots \hat{\omega}_p(\chi) \dots \omega_n(\chi) . \tag{**}$$

$$\{\text{with } p \text{ such that } \deg(\omega_p) = 2r + 1\}$$

Furthermore, it is easy to see that if degree $(\xi_i) < 2r + 1$, for some ξ_i, then χ is a δ-coboundary modulo d. We are led to the definitions of Step 2.

C. Step 2 : the δ-cohomology modulo d.

Let us define P^{2r+1} and P_r by P^{2r+1} = Space of primitive forms of degree $2r + 1$, P_r = Space of primitive forms of degree $\geqq 2r + 1$. Thus we have $P_r = \bigoplus_{k \geqq r} P^{2k+1}$.

Set $\mathscr{I}_r = S\tau(P_r) \otimes \Lambda P_r$; $\mathscr{I}_r$ is the algebra generated by the primitive forms of degree $\geqq 2r + 1$ and their transgressions. Let E_r^r be the subspace of $\mathscr{I}_r$ of the elements « containing » explicitely at least one form of degree $2r + 1$ or its transgression, i. e.

$$E_r^r = \Big(\bigoplus_{m+n \geqq 1} S^m \tau(P^{2r+1}) \otimes \Lambda^n P^{2r+1}\Big) \otimes \mathscr{I}_{r+1} .$$

Finally write $E_r = \bigoplus_{k \geqq 0} E_{r+k}^{r+k}$ so, $E_r = E_r^r \oplus E_{r+1}$.

Define d_r on E_r to be the unique antiderivation satisfying $d_r\omega = \tau(\omega)$ if ω is of degree $2r + 1$, $d_r\omega = 0$ if ω is of degree $> 2r + 1$ and $d_r\tau(\omega) = 0$

for any primitive form ω of degree $\geqq 2r + 1$. This definition reproduces the expression appearing in (**) and implies that we have $d_r^2 = 0$.

It was shown in [1] how to reduce the computation of the δ-cohomology modulo d to the one of the δ-cohomology modulo d for even d-degree: If $Q^{r,s}$ is a δ-cocycle modulo d of bidegree (r, s), we have $\delta Q^{r,s} + dQ^{r-1,s+1} = 0$ where $Q^{r-1,s+1}$ (defined up to d of something) is also a δ-cocycle modulo d of bidegree $(r - 1, s + 1)$; this induces a well defined linear mapping in cohomology, $\partial : H^{r,s}(\delta, \operatorname{mod}(d)) \to H^{r-1,s+1}(\delta, \operatorname{mod}(d))$, which is an isomorphism whenever r is odd [1]. Thus we have with obvious notations an isomorphism, $\partial : H^{\mathrm{odd},*}(\delta, \operatorname{mod}(d)) \xrightarrow{=} H^{\mathrm{even},*}(\delta, \operatorname{mod}(d))$, and we only need to compute for instance $H^{\mathrm{even},*}(\delta, \operatorname{mod}(d))$.

We have [1]: $H_+^{\mathrm{even},*}(\delta, \operatorname{mod}(d)) \cong \bigoplus_{r \geqq 0} E_r/N_r$, where the quotients E_r/N_r are given by $E_r/N_r \cong \bigoplus_{k \geqq 0} E_{r+k}^{r+k}/d_{r+k}(E_{r+k}^{r+k})$ and where the index $+$ in $H_+^{\mathrm{even},*}(\delta, \operatorname{mod}(d))$ means restriction to strictly positive degrees, i. e.

$$H_+^{\mathrm{even},*}(\delta, \operatorname{mod}(d)) = \bigoplus_{r+s \geqq 1} H^{2r,s}(\delta, \operatorname{mod}(d)).$$

These isomorphisms are realized by the procedure of Step 3.

D. Step 3 : construction of representative δ-cocycles modulo d.

The previous isomorphism between $\bigoplus_r E_r/N_r$ and $H_+^{\mathrm{even},*}(\delta, \operatorname{mod}(d))$ is realized in [1], by going, for instance, from an element $X \in E_r$, which is a δ-cocycle, to the δ-cocycle modulo d Q_{2r} in the chain of equations (*) of B.

More precisely we choose for each s, a supplementary Σ_s of $d_s(E_s^s)$ in E_s^s and a basis in Σ_s. Thus $E_r/N_r \simeq \bigoplus_{k \geqq 0} \Sigma_{r+k}$, and we associate to each basis element the corresponding δ-cocycle modulo d Q_{2r} in the chain (*). Taking their cohomology classes yields independent elements of $H(\delta, \operatorname{mod}(d))$; finally, by doing this for each r, we get a basis of $H_+^{\mathrm{even},*}(\delta, \operatorname{mod}(d))$.

Practically, in order to obtain Q_{2r} from

$$X = \left(\prod_i \tau(\xi_i)(F)\right)\omega_0(\chi) \ldots \omega_n(\chi) \in E_r,$$

write for each L_p as in B.

$L_p(A + \chi, F) = Q^0(A, F) + Q^1(A, F, \chi) + \ldots$, where Q^k is of δ-degree k; then replace each L_p by this expansion in

$$\prod_i \tau(\xi_i)(F)\, L_0(A + \chi, F) \ldots L_n(A + \chi, F)$$

and extract the term of δ-degree $\sum_{p=0}^{n}$ degree $(\omega_p) - 2r$ in the product.

Since Σ_s appears in E_r/N_r for all $0 \leqq r \leqq s$, we see that, if $X \in E_s^s$ we have to apply this procedure $s + 1$ time.

Moreover, from the definition of ∂ and the isomorphism

$$\partial : H^{odd,*}(\delta, \mathrm{mod}\,(d)) \overset{\cong}{\to} H^{even,*}(\delta, \mathrm{mod}\,(d)),$$

we see that all δ-cocycles modulo d appearing in the chain (*) give both $H^{even,*}(\delta, \mathrm{mod}\,(d))$ and $H^{odd,*}(\delta, \mathrm{mod}\,(d))$ when X runs over Σ_s, for all s.

3. FIRST EXAMPLE : $G = SU(3) \times SU(2) \times U(1)$, $(\mathcal{G} = su(3) \times su(2) \times u(1))$

For the first factor $\mathfrak{g}_1 = \mathfrak{su}(3)$, $r_1 = 2$ and we choose as basis of primitive forms the two primitive forms

$$\xi = -\,1/3\,\mathrm{tr}\,(\chi_1^3) \text{ of degree } 3$$
$$\zeta = 1/10\,\mathrm{tr}\,(\chi_1^5) \text{ of degree } 5,$$

together with the two δ-cocycles corresponding to their transgressions

$$x = \mathrm{tr}\,(F_1^2) \text{ of degree 4, (bidegree (4,0))},$$
$$z = \mathrm{tr}\,(F_1^3) \text{ of degree 6, (bidegree (6,0))}.$$

We have $x = dL_\xi(A_1, F_1)$ with

$$\begin{aligned} L_\xi(A_1 + \chi_1, F_1) &= Q_\xi^0 + Q_\xi^1 + Q_\xi^2 + Q_\xi^3 \\ &= \mathrm{tr}(A_1F_1 - 1/3\,A_1^3) + \mathrm{tr}\,(\chi_1(F_1 - A_1^2)) - \mathrm{tr}\,(\chi_1^2A_1) - 1/3\,\mathrm{tr}\,(\chi_1^3). \end{aligned}$$

Similarly $z = dL_\zeta(A_1, F_1)$ with

$$\begin{aligned} L_\zeta(A_1 + \chi_1, F_1) &= Q_\zeta^0 + Q_\zeta^1 + Q_\zeta^2 + Q_\zeta^3 + Q_\zeta^4 + Q_\zeta^5 \\ &= \mathrm{tr}\,(A_1F_1^2 - 1/2\,A_1^3F_1 + 1/10\,A_1^5) \\ &+ \mathrm{tr}\,(\chi_1(F_1^2 - 1/2\,A_1^2F_1 - 1/2\,F_1A_1^2 - 1/2\,A_1F_1A_1 - 1/2\,A_1^4)) \\ &+ 1/2\,\mathrm{tr}\,(\chi_1^2A_1^3 + \chi_1A_1\chi_1A_1^2 - (\chi_1^2A_1 + \chi_1A_1\chi_1 + A_1\chi_1^2)F_1) \\ &+ 1/2\,\mathrm{tr}\,(\chi_1^3A_1^2 + A_1\chi\,A_1\chi_1^2 - \chi_1^3F_1) + 1/2\,\mathrm{tr}\,(\chi_1^4A_1) + 1/10\,\mathrm{tr}\,(\chi_1^5). \end{aligned}$$

This displays the decomposition $L = Q^0 + Q^1 + \ldots$ of Step 3.

For the second factor $\mathfrak{g}_2 = \mathfrak{su}(2)$, $r_2 = 1$ and we choose as basic primitive form $\sigma = -\,1/3\,\mathrm{tr}\,(\chi_2^3)$ of degree 3 with the corresponding $s = \mathrm{tr}\,(F_2^2)$ of degree 4, (bidegree (4,0)). So, the same formula as above for ξ, x, applies and we obviously have: $s = dL_\sigma(A_2, F_2)$, with

$$\begin{aligned} L_\sigma(A_2 + \chi_2, F_2) &= Q_\sigma^0 + Q_\sigma^1 + Q_\sigma^2 + Q_\sigma^3 \\ &= \mathrm{tr}\,(A_2F_2 - 1/3\,A_2^3) + \mathrm{tr}\,(\chi_2(F_2 - A_2^2)) - \mathrm{tr}\,(\chi_2^2A_2) - 1/3\,\mathrm{tr}\,(\chi_2^3). \end{aligned}$$

Finally for the abelian factor $\mathfrak{u}(1)$ take $\theta = \chi_3$ as basic primitive form and corresponding $t = F_3$. We have: $F_3 = dL_\theta(A_3, F_3)$, with

$$L_\theta(A_3 + \chi_3, F_3) = Q_\theta^0 + Q_\theta^1 = A_3 + \chi_3 .$$

In accordance with Step 2, we know that in E_0^0 either θ or t must appear. In other words, the general term in E_0^0 is a linear combination of terms of the form $P(z, x, s, t)\,\Lambda(\xi, \zeta, \sigma)\,\theta$ and terms of the form $t\,P(z, x, s, t)\,\Lambda(\xi, \zeta, \sigma)$ where P runs over the polynomials in z, x, s, t and $\Lambda(\xi, \zeta, \sigma)$ runs over the exterior algebra over the 3-dimensional space spanned by ξ, ζ, σ. Since $d_0\theta = t$, the terms of the second type get cancelled in $E_0^0/\mathrm{Im}\,d_0$ and we may choose the linear span in E_0^0 of the terms of first type as Σ_0.

In E_1^1, the θ and t do not appear and, at least one of the ξ, σ, x, s must appear. The general term in E_1^1 is a linear combination of terms of the following types:

1) $R(z, x, s)$ multiplied by one of the ξ, σ, $\xi\sigma$, $\xi\zeta$, $\sigma\zeta$, $\xi\sigma\zeta$,
2) $R(z, x, s)$ multiplied by one of the x, s, $s\zeta$, $s\zeta$,

where R runs over the polynomials in z, x, s. Since $d_1(\xi) = x$, $d_1\sigma = s$ and $d_1\zeta = 0$, the terms of type 2 are killed in $E_1^1/\mathrm{Im}\,d_1$. Moreover, since $d_1(R\xi\sigma) = R(x\sigma - s\xi)$, in each term of type 1, $s\xi$ may be replaced by $x\sigma$ in $E_1^1/\mathrm{Im}\,d_1$. Therefore, we may choose as supplementary Σ_1 of $\mathrm{Im}\,d_1$ in E_1^1 the linear span of the elements of the types $R(z, x, s)\xi\sigma$, $R(z, x, s)\xi\sigma\zeta$, $R(z, x, s)\sigma$, $R(z, x, s)\sigma\zeta$, $S(z, x)\xi$, $S(z, x)\xi\zeta$, where R runs over the polynomials in z, x, s and S runs over the polynomials in z, x only.

In E_2^2, only ζ and z can appear so E_2^2 is generated by $T(z)\zeta$ and $T(z)z$ where T runs over the polynomials in z.

Since $d_2(\zeta) = z$ (and $d_2(z) = 0$), we choose as supplementary Σ_2 to $\mathrm{Im}\,d_2$ the subspace of E_2^2 generated by the $T(z)\zeta$.

Finally we know that we have:

$$\begin{aligned} H_+^{\mathrm{even},*}(\delta, \mathrm{mod}\,(d)) &\cong (E_0/N_0) \oplus (E_1/N_1) \oplus (E_2/N_2) \\ &\cong (\Sigma_0 \oplus \Sigma_1 \oplus \Sigma_2) \oplus (\Sigma_1 \oplus \Sigma_2) \oplus \Sigma_2 , \end{aligned}$$

and we shall exhibit corresponding representative δ-cocycles modulo d by using the procedure of Step 3.

The elements of the first sum $(\Sigma_0 \oplus \Sigma_1 \oplus \Sigma_2)$ are δ-cocycles and thus define trivially δ-cocycles modulo d (i. e. $r = 0$ in the generalized transgression formula).

The elements of the second sum $(\Sigma_1 \oplus \Sigma_2)$ yield δ-cocycles modulo d through the use of the generalized transgression formula ($r = 1$ in Step 3).

Finally the elements of the third term Σ_2 yield δ-cocycles modulo d for $r = 2$ in Step 3.

We summarize the results in the table 2 where the rows correspond to independent (in $H_+^{\mathrm{even},*}(\delta, \mathrm{mod}\,(d))$) δ-cocycles modulo d of given ghost degrees (δ-degrees); the columns corresponding to the parts coming from $r = 0$, $r = 1$ and $r = 2$ in Step 3. As noticed above, the table of elements of $H^{\text{odd},*}(\delta, \text{mod},(d))$ is readily obtained from a similar calculation.

TABLE 2.

Ghost Number	E_0/N_0	E_1/N_1	E_2/N_2
1	$P(z,x,s,t)\theta$	$+ S(z,x)\,C^{2,1} + R(z,x,s)\,C'^{2,1}$	$+ T(z)\,C^{4,1}$
2			
3	$S(z,x)\xi + R(z,x,s)\sigma$	$+ T(z)\,C^{2,3}$	
4	$P(z,x,s,t)\xi\theta + P'(z,x,s,t)\sigma\theta$	$+ R(z,x,s)\,C^{2,4}$	
5	$T(z)\zeta$		
6	$P(z,x,s,t)\zeta\theta + R(z,x,s)\xi\sigma$	$+ S(z,x)\,C^{2,6} + R(z,x,s)\,C'^{2,6}$	
7	$P(z,x,s,t)\xi\sigma\theta$		
8	$S(z,x)\zeta\xi + R(z,x,s)\zeta\sigma$		
9	$P(z,x,s,t)\zeta\xi\theta + P'(z,x,s,t)\zeta\sigma\theta$	$+ R(z,x,s)\,C^{2,9}$	
10			
11	$R(z,x,s)\zeta\xi\sigma$		
12	$P(z,x,s,t)\zeta\xi\sigma\theta$		

where

$$C^{2,1} = Q^1_\xi, \quad C'^{2,1} = Q^1_\sigma, \quad C^{2,3} = Q^3_\zeta, \quad C^{4,1} = Q^1_\zeta,$$

$$C^{2,4} = Q^1_\xi Q^3_\sigma + Q^2_\xi Q^2_\sigma + Q^3_\xi Q^1_\sigma,$$

$$C^{2,6} = Q^5_\zeta Q^1_\xi + Q^4_\zeta Q^2_\xi + Q^3_\zeta Q^3_\xi + Q^2_\zeta Q^4_\xi + Q^1_\zeta Q^5_\xi,$$

$$C'^{2,6} = Q^5_\zeta Q^1_\sigma + Q^4_\zeta Q^2_\sigma + Q^3_\zeta Q^3_\sigma + Q^2_\zeta Q^4_\sigma + Q^1_\zeta Q^5_\sigma,$$

$$C^{2,9} = Q^3_\zeta Q^3_\xi Q^3_\sigma + Q^4_\zeta (Q^3_\xi Q^2_\sigma + Q^2_\xi Q^3_\sigma) + Q^5_\zeta (Q^3_\xi Q^1_\sigma + Q^2_\xi Q^2_\sigma + Q^1_\xi Q^3_\sigma),$$

and P, R, S, T are polynomials in (z, x, s, t), (z, x, s), (z, x) and z respectively, with $z = \operatorname{tr} F_1^3$, $x = \operatorname{tr} F_1^2$, $s = \operatorname{tr} F_2^2$ and $t = F_3$.

4. SECOND EXAMPLE: THE ABELIAN CASE

The abelian case in easy because any δ-cocycle modulo d is equivalent (modulo *a* δ-coboundary modulo *d*) to some δ-cocycle or, more precisely, to some δ-cohomology class; indeed we have $E_0 = E_0^0$ and $E_r = 0$ for $r \geqq 1$. Therefore we have $H_+^{\text{even},*}(\delta, \text{mod}(d)) \cong E_0/N_0 = E_0^0/d_0(E_0^0)$.

Nevertheless, this case is interesting since it is known [*12*] that while anomalies in even dimension always come from invariants, Schwinger terms (resp. anomalies in odd dimension) of different type can appear when, for instance, at least two U(1) factors are present.

We describe here the δ-cohomology modulo d in the general abelian case

$$\mathfrak{g} = \mathbb{R}^N = (\mathfrak{u}(1))^N.$$

In this case, we have primitive forms $\theta_1, \theta_2, \ldots, \theta_N$ of degree 1 and their transgressions $F_1, F_2, \ldots, F_N$. Thus identifying $\tau(\mathbb{R}^N)$ with $\mathbb{R}^N$ we have:

$$E_0 = E_0^0 = \bigoplus_{m+n \geqq 1} (S^m\mathbb{R}^N \otimes \Lambda^n\mathbb{R}^N).$$

Moreover, the sequences

$$\ldots \xrightarrow{d_0} S^m\mathbb{R}^N \otimes \Lambda^n\mathbb{R}^N \xrightarrow{d_0} S^{m+1}\mathbb{R}^N \otimes \Lambda^{n-1}\mathbb{R}^N \xrightarrow{d_0} \ldots$$

are exact sequences for $m + n \geqq 1$, (see in [1], for instance).

In this case, Step 3 is not required, so we shall content ourselves with the calculation of the dimensions

$$h^{m,n} = \dim H^{2m,n}(\delta, \bmod (d)) = \dim H^{2m+1,n-1}(\delta, \bmod (d)),$$

(notice that we have $h^{m,0} = 0$ for any m).

We have: $H^{2m,n}(\delta, \bmod (d)) \cong S^m\mathbb{R}^N \otimes \Lambda^n\mathbb{R}^N / d_0(S^{m-1}\mathbb{R}^N \otimes \Lambda^{n+1}\mathbb{R}^N)$.

It follows that we have, (by exactness):

$$\begin{aligned}\dim (S^m\mathbb{R}^N \otimes \Lambda^n\mathbb{R}^N) &= \dim (d_0(S^m\mathbb{R}^N \otimes \Lambda^n\mathbb{R}^N)) \\ &\quad + \dim (d_0(S^{m-1}\mathbb{R}^N \otimes \Lambda^{n+1}\mathbb{R}^N)) = h^{m,n} + h^{m-1,n+1}.\end{aligned}$$

Introducing the Poincaré series

$$\sum_{m+n \geqq 1} x^m y^n \dim (S^m\mathbb{R}^N \otimes \Lambda^n\mathbb{R}^N) = \left(\frac{1+y}{1-x}\right)^N - 1$$

and

$$\sum_{m+n \geqq 1} x^m y^n h^{m,n} = h(x, y)$$

the above equation reads:

$$\left(\frac{1+y}{1-x}\right)^N - 1 = h(x, y) + x/y\, h(x, y),$$

(which is meaningful because $h(x, 0) = 0$!).

It follows that $h(x, y)$ is given by $h(x, y) = \dfrac{y}{x+y}\left(\left(\dfrac{1+y}{1-x}\right)^N - 1\right)$ or equivalently $h(x, y) = \dfrac{y}{1-x} \displaystyle\sum_{p=0}^{N-1} \left(\frac{1+y}{1-x}\right)^p$.

5. CONCLUSION

We have shown, especially in the two examples developed here, how the interplay of the various differentials (d and δ) which naturally appear in the cohomological presentation of the anomaly problem in gauge theory, brings a description of its solution as an enriched version of the (classical) cohomology of the Lie algebra of the structure group.

In particular, in addition to the already known anomalous terms derived from invariant polynomials, there appear for sufficiently high ghost number, new types of anomalous terms which we can write down explicitely.

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