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# Some remarks on the index theorem approach to anomalies

by

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ABSTRACT. — A direct link beetween non abelian anomalies in gauge theories and gravitation and the family index theorem is given, by somewhat explicit computation of the fermionic path integral. Independence of the regularization procedure is discussed. The geometrical meaning of the Wess-Zumino term is enligthened.

RÉSUMÉ. — On met en évidence un lien direct entre les anomalies non abéliennes dans les théories de jauge et de la gravitation et le théorème de l'indice pour des familles par un calcul assez explicite de l'intégrale de chemins fermionique. On discute l'indépendance par rapport à la méthode de régularisation. On éclaire la signification géométrique du terme de Wess-Zumino.

## 1. INTRODUCTION

One loop computation of the effective action in field theory is known to exhibit pathologies, usually called anomalies. Among these, non abelian anomalies in gauge theory and gravitation have been recently given a topological interpretation by means of the family index theorem of Atiyah and Singer [*1*] [*2*] [*3*]. Such an interpretation is actually grounded on a

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computational coincidence; namely it happens that the anomaly obtained by physicists via some regularization scheme is cohomologous to a certain differential form whose geometrical and topological meaning is well known. Indeed, it turns out that this form is the transgression of the first Chern class of the virtual index bundle for the family of Dirac operators $\not{d}_A$, parametrized by the background gauge field A. The same is true for gravitational anomalies. In particular this explains why secondary invariants and the « Russian formula » of Stora [*4*] [*5*] [*6*] are so effective in dealing with this kind of anomalies, both in gauge theories and gravitation [*7*] [*16*] [*19*] [*20*].

Still, there are some puzzling questions, mainly related to physical applications, which we want to address in this paper. First of all we would like to shed some light as to why certain structures related to the zero modes of an operator (namely the index bundle) keep so effectively control of the invariance properties of the generating functional, which is constructed by physicists explicitely avoiding as far as possible the zero modes themselves. When « chiral » fermions are concerned, one has to modify the naive measure by inserting the product of Fermi fields which generate the space of zero modes. This is the same as inserting in the vacuum functional a section of the determinant of the index bundle $L = \det(\text{ind}\,\not{d})$ of the family of Dirac operators. The transformation properties of the regularized and renormalized vacuum functional $W_R$ are thereby completely determined, and $W_R$ is turned into a section of the line bundle L itself. This result is largely independent of the regularization scheme, provided it fulfills certain reasonable requirements. As a byproduct, one can easily show that the resulting anomalies are always transgressive, making contact with secondary invariants.

Besides, we will coment on a number of related topics. In particular the anomalies considered here prevent the existence of a single valued effective action and hence of any functional generating them (either locally depending on fields or not). So we see that the trouble in this contest is not related to locality. Of course there are milder anomalies (e. g. trace anomalies) which do not affect the effective action and can be treated via the methods of local cohomology as explained in [*6*] [*8*]. Still one can construct a path dependent local functional $\Gamma_{WZ}$ generating the anomaly, which coincides with the Wess-Zumino term, when certain assumptions are made.

## 2. GEOMETRICAL SET UP FOR EXTERNAL GAUGE THEORIES

To be definite, we consider a classical theory with action

$$S(\varphi, \psi) = \langle \psi \mid D_\varphi \psi \rangle, \tag{1}$$

where $\psi$ is some collection of « matter fields », $\varphi$ plays the role of an external interaction field interacting with $\psi$ via a linear differential operator $D_\varphi$, which is assumed to depend explicitely on $\varphi$ and $\langle \mid \rangle$ denotes an $L^2$ inner product on the relevant functional spaces. For instance, Yang-Mills theory is recovered by taking $\psi$ to be some G-multiplet ($G = SU(n)$) and $\varphi$ a G-gauge potential. If $\varphi$ is the metric field and $\psi$ some matter field, we get gravitation.

In all the examples of physical interest we have one further property; there exists an invariance group $\mathcal{G}$ acting on fields by

$$(\varphi, \psi) \to (g\varphi, g\psi) \tag{2}$$

and leaving the action (1) invariant, i. e. such that

$$S(g\varphi, g\psi) = S(\varphi, \psi) . \tag{2'}$$

For instance in Yang-Mills theory $\mathcal{G}$ is the group of gauge transformations, while in Einstein theory it is the group of vertical automorphisms of the principal spin bundle over space time M. In particular, in these cases, (2') holds because $\mathcal{G}$ acts unitarily on matter fields, i. e.

$$\langle g\psi \mid g\psi \rangle = \langle \psi \mid \psi \rangle , \tag{3}$$

and the operator $D_\varphi$ is equivariant with respect to this action, i. e.

$$D_\varphi g\psi = g D_{g\varphi} \psi . \tag{4}$$

Hereinafter we assume that (3) and (4) hold for our groups and operators.

The action (1) is a function $S : \Phi \times \Psi \to \mathbb{R}$ mapping the space $\Phi$ of interaction fields and the space $\Psi$ of matter fields into the reals. However $\mathcal{G}$-invariance implies that S is actually defined on the equivalence classes of fields in $\Phi \times \Psi$ defined by (2). More precisely, we can give $\Phi$ the structure of a Hilbert manifold and suitably restrict the group $\mathcal{G}$(*)in such a way that $\Phi \to \Phi/\mathcal{G}$ is a principal fibre bundle over an Hilbert manifold. We refer to [*9*] for a review on these bundles. As for matter fields, we take the quotient of $\Phi \times \Psi$ with respect to the action (2). The theory is then defined on the Hilbert vector bundle $\mathcal{E} = \Phi \times_{\mathcal{G}} \Psi$, with standard fibre $\Psi$ over the orbit space $\Phi/\mathcal{G}$. Clearly, $\mathcal{E}$ is associated to the principal orbit bundle $\Phi \to \Phi/\mathcal{G}$ and it is topologically non trivial if the orbit bundle itself is non trivial, that is when there is no global « gauge » fixing and Gribov ambiguity is present in the theory.

(*) I. e. we consider the pointed gauge group, i. e. the group of gauge transformations which equal the identity at a given point of M.

## 3. VACUUM FUNCTIONALS IN EXTERNAL « GAUGE » THEORIES

In the Euclidean path-integral approach to the quantization of such a theory, one assumes that M is a properly Riemannian manifold which is compact and without boundary. One then considers a vacuum functional of the form

$$W(\varphi) = N \int_{\Psi} \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp - \langle \psi \mid D_{\varphi}\psi \rangle , \tag{5}$$

where now $\varphi$ plays the role of an external current and $\psi$ are quantum fluctuations of the matter fields. If these are Fermi fields, we assume Berezin integration rules over anticommuting variables.

As it is well known, the functional integral (5) is to be considered as a formal expression, because integration over infinite dimensional spaces hardly has a mathematical meaning. Actual computations of (5) are grounded on the analogy with finite dimensional integrals [*10*]. As it is well known, if $\Psi$ was finite dimensional, one would get

$$W(\varphi) = N' [\det D_{\varphi}]^{\alpha}_{\cdot}$$

where $\alpha = \pm 1 (\pm 1/2)$ for complex (real) fermions and bosons respectively. Implementing such an analogy entails however three problems:

*a*) first of all one has to carefully avoid « zero modes » of the operator $D_{\varphi}$ and its adjoint $D_{\varphi}^{*}$, which will make the bosonic integral diverge and the fermionic one identically vanish.

*b*) Next one needs eigenvalues for the operator $D_{\varphi}$, as its determinant formally equals the product of eigenvalues. In physical applications, we can assume that $D_{\varphi}$ has countable eigenvalues, if any.

*c*) Finally, such an infinite product will seldom converge, so some regularization and renormalization scheme is needed. This can be done by means of a variety of techniques, whose common effect is to define a map $\mathcal{R} : \mathbb{C}^{\infty} \to \mathbb{C}$, which associates to the sequence of eigenvalues $\lambda_i(\varphi)$ a complex number

$$\det_R D_{\varphi} = \mathcal{R}(\lambda_1(\varphi), \lambda_2(\varphi), \ldots)$$

which is called the determinant of $D_{\varphi}$. The only property we shall require in the following is that $\det_R D_{\varphi}$ vanishes whenever one of the eigenvalues vanishes, as it should be for a « resonable » determinant.

When $D_{\varphi} : \Psi \to \Psi$ is an endomorphism of a Hilbert space, one can set up an eigenvalue problem $D_{\varphi}\psi = \lambda(\varphi)\psi$. Equivariance of $D_{\varphi}$ then tells us that the eigenvalues are $\mathcal{G}$-invariant; accordingly $\det_R D_{\varphi}$ is $\mathcal{G}$-invariant as well, for any regularization and renormalization procedure $\mathscr{R}$. But when $D_\varphi : E \to F$ maps some space of fields into another, as for the « chiral » Dirac operator, looking for eigenvalues is meaningless and some other tool is needed, as we shall see below.

Still one has to treat zero modes, which make our regularized determinant vanish identically. In physical problems, $D_\varphi$ is an elliptic operator between Hilbert spaces of sections of vector bundles on a compact manifolds, which has finite dimensional kernel and cokernel. So we see that problem *a)* is due to the integration at any $\varphi$ over finite dimensional subspaces of $\Psi$, namely the spaces $\ker D_\varphi$ and $\ker D_\varphi^*$ of zero modes of $D_\varphi$ and $D_\varphi^*$. Roughly speaking this is why one can tackle the problem via fibre bundles techniques, as we shall shortly see.

A naive approach to problem *a)* is to avoid integration over the kernels, which is tantamount to restrict the regularizing map $\mathscr{R}$ to $\mathbb{C}^\infty - \{0\}$ by simply forgetting the zero eigenvalue. But one might have troubles in identifying $\ker D_\varphi$ and $\ker D_\varphi^*$ in $\Psi$. Indeed, both their dimensions and their embedding in $\Psi$ depend on $\varphi$.

As for the embedding, some more information comes from the equivariance (4) implying that the kernel and the cokernel are themselves $\mathscr{G}$-equivariant on $\Phi$. In other words, $\psi$ is in $\ker D_\varphi$ if and only if $g\psi$ belongs to $\ker D_{g\varphi}$. Then the action of $\mathscr{G}$ induces an isomorphism between $\ker D_\varphi$ and $\ker D_{g\varphi}$; if $\psi_1(\varphi), \ldots, \psi_n(\varphi)$ span $\ker D_\varphi$ then the transformed set $g\psi_1(\varphi), \ldots, g\psi_n(\varphi)$ span $\ker D_{g\varphi}$. Although isomorphic, these spaces will be in general different subspaces of $\Psi$.

As for dimensions, one can prove semicontinuity [*11*], i. e.

$$\dim \ker D_{\varphi_0} \geq \dim \ker D_\varphi \tag{6}$$

for any $\varphi$ in a convenient neighborhood of $\varphi_0$. So dimensions can jump, but for equivariant operators jumping can occour only when $\varphi$ is varied transversally to the orbits. The same clearly holds for the adjoint operator $D_\varphi^*$. The dimensional jumpings of $\ker D_\varphi$ and $\ker D_\varphi^*$ are however forced by the index theorem [*12*] to satisfy

$$\dim \ker D_\varphi - \dim \ker D_\varphi^* = I, \tag{7}$$

where I is an integer independent of $\varphi$. If $D_\varphi$ is selfadjoint, the index I vanishes. Vanishing of the index also occours when the euclidean spacetime is odd dimensional.

To get one step further we will distinguish two cases, and to keep the following arguments simple we will sometimes assume that no dimensional jumping arise. As a word of caution, notice that this is not the case in physical situations [*13*].

## 4. COMPUTING PATH INTEGRALS FOR SELFADJOINT OPERATORS ON REAL BOSONS

Let us start with a simple example, showing how the naive approach to zero modes works in certain cases. Assume $D_\varphi : B \to B$ is a selfadjoint operator acting on a real Hilbert space B of boson quantum fluctuations. If $D_\varphi$ has zero modes, one splits the path integral as follows

$$\begin{aligned} W^B(\varphi) &= N \int_B \mathscr{D}b \exp - \langle b \mid D_\varphi b \rangle \\ &= N \int_{\ker D_\varphi} \mathscr{D}b_0 \int_{(\ker D_\varphi)^\perp} \mathscr{D}b_1 \exp - \langle b_1 \mid D_\varphi b_1 \rangle , \end{aligned}$$

where $b_0$ and $b_1$ are the projections of $b$ onto $\ker D_\varphi$ and $(\ker D_\varphi)^\perp$. Now the integral on $(\ker D_\varphi)^\perp$ does not contain zero modes any more and can be regularized and renormalized yielding

$$W^B(\varphi) = N (\det_R D'_\varphi)^{-1/2} \int_{\ker D_\varphi} \mathscr{D}b_0 = N W^B_R(\varphi) \int_{\ker D_\varphi} \mathscr{D}b_0, \tag{9}$$

where $D'_\varphi$ is $D_\varphi$ restricted to $(\ker D_\varphi)^\perp$. Expression (9) clearly diverges, because of the integration of $\ker D_\varphi$. So we change the naive measure $\mathscr{D}b_0$ into a compactedly supported volume form $\mathscr{D}'b_0(\varphi)$ such that

$$\int_{\ker D_\varphi} \mathscr{D}'b_0(\varphi) = 1/N . \tag{10}$$

acheiving our task of suppressing zero modes.

We now study $\mathscr{G}$-invariance of this procedure. Clearly $\det_R D'$ is $\mathscr{G}$-invariant, because D is equivariant and its eigenvalues are constant on $\mathscr{G}$-orbits. The only question is about the existence of a family of forms $\mathscr{D}'b_0(\varphi)$ such that

$$\int_{\ker D_\varphi} g^* \mathscr{D}'b_0(g\varphi) = \int_{\ker D} \mathscr{D}'b_0(\varphi) . \tag{11}$$

As the kernels are real spaces, the only trouble could come from orientation. If the action of $\mathscr{G}$ preserves orientation of the kernels, then (11) holds; otherwise it holds up to a sign.

This can be made more precise by taking the quotient by the action of $\mathscr{G}$. Equivariance of kernels implies that the collection of linear spaces $\ker D_\varphi$ descends to a vector bundle

$$\operatorname{Ker} D_\varphi \to \Phi/\mathscr{G},$$

over the orbit space $\Phi/\mathcal{G}$. Here (9) becomes

$$w^{\mathrm{B}}([\varphi]) = \mathrm{N} \det_{\mathrm{R}} \mathrm{D}'_{[\varphi]} \int_{\mathrm{Ker}\, \mathrm{D}_{[\varphi]}} \omega , \tag{12}$$

where $[\varphi]$ denotes a point in $\Phi/\mathcal{G}$, $\ker \mathrm{D}_{[\varphi]}$ is the fibre of the bundle Ker D at $[\varphi]$ and $\omega$ is some compactedly supproted volume form on $\ker \mathrm{D}_{[\varphi]}$. So we need integration along the fibres of the vector bundle Ker D. As it is well known (see e. g. Bott and Wu [*14*]), this operation yields a section of the orientation bundle of Ker D, i. e.

$$w^{\mathrm{B}} : \Phi/\mathcal{G} \to \det(\mathrm{Ker}\, \mathrm{D}) .$$

Now det (Ker D) is a real line bundle and the only obstruction against its triviality is in $\mathrm{Z}_2$, namely the orientability of Ker D. Pulling back $w^{\mathrm{B}}$ to $\Phi$, we get at most a sign ambiguity along any $\mathcal{G}$-orbit. If $\mathcal{G}$ is connected (e. g. in gauge theories over $\mathrm{S}^4$, for $\mathrm{G} = \mathrm{SU}(n)$, $n \geq 3$), such an ambiguity does not exist, because Ker D is orientable. Summing up we proved the following;

PROPOSITION. — Let $\mathrm{D}_\varphi : \mathrm{B} \to \mathrm{B}$ be a $\mathcal{G}$-equivariant selfadjoint operator on a real Hilbert space B. Then if $\mathcal{G}$ is connected, the regularized renormalized vacuum functional $\mathrm{W}^{\mathrm{B}}_{\mathrm{R}}(\varphi) = \mathrm{N}(\det_{\mathrm{R}} \mathrm{D}')^{-1/2}$ is $\mathcal{G}$-invariant.

Notice that when $\mathcal{G}$ is not connected (e. g. for $\mathrm{G} = \mathrm{SU}(2)$), the sign ambiguity may be treated by taking the absolute value, but such operation may destroy locality of functionals.

## 5. COMPUTING PATH INTEGRALS FOR CHIRAL FERMIONS

Let us now come to a more subtle case. Consider the Dirac operator $\not{d}_\varphi : \mathrm{E} \to \mathrm{F}$ mapping fermions of one « chirality » into the other. The formal vacuum functional reads

$$\mathrm{W}(\varphi) = \mathrm{N} \int_{\mathrm{E}\times\mathrm{F}} \mathcal{D}e\mathcal{D}f \exp - \langle f \mid \not{d}_\varphi e \rangle , \tag{13}$$

where $e$ and $f$ are now independent fields and Berezin integration rules are used. Besides zero modes, $\not{d}_\varphi$ has no eigenvalues, so we need some further trick to define (13).

Assume for simplicity that $\ker \not{d}^*_{\varphi_0} = 0$ at $\varphi_0$. In gauge theories over $\mathrm{S}^4$, it is sufficient to take $\varphi_0$ to be an instanton to get vanishing of $\ker \not{d}^*_{\varphi_0}$, as shown in [*15*]. Then equivariance implies that $\ker \not{d}^*_{g\varphi_0} = 0$ along the $\mathcal{G}$-orbit through $\varphi_0$ and semicontinuity (6) yields that $\ker \not{d}^*_\varphi = 0$ for any $\varphi$ in a whole tubular neighborhood $\mathrm{U}([\varphi_0])$ of the orbit through $\varphi_0$. Then zero modes in (13) come only from $\ker d_\varphi$, which for any $\varphi \in U([\varphi_0])$ has constant dimension $\dim \ker d_\varphi = I$. So we split our integral as follows

$$(14) \qquad W(\varphi) = N \int_{\ker d_\varphi} \mathcal{D}e_0 \int_{(\ker d_\varphi)^\perp \times F} \mathcal{D}e_1 \mathcal{D}f \exp - \langle f \mid d_\varphi e_1 \rangle ,$$

where $e_0, e_1$ denote the projections of $e$ onto $\ker d_\varphi$ and $(\ker d_\varphi)^\perp$. Now any $f$ can be uniquely written as $f = d_{\varphi_0} e'_1$, for $e'_1 \in (\ker d_{\varphi_0})^\perp$. Substituting into (14) we get

$$(15) \qquad W(\ ) = N \,«\det d_{\varphi_0}»\int_{\ker d_\varphi} \mathcal{D}e_0 \int_{(\ker d_\varphi)^\perp \times (\ker d_{\varphi_0})^\perp} \mathcal{D}e_1 \mathcal{D}e'_1 \exp - \langle e'_1 | d_{\varphi_0} d e_1 \rangle$$

where « $\det d_{\varphi_0}$ » denotes the Jacobian determinant of the (invertible) map $e'_1 \to d_{\varphi_0} e'_1$. By means of the Faddeev-Popov trick we set

$$«\det d_{\varphi_0}» = \int_{(\ker d_{\varphi_0})^\perp \times F} \mathcal{D}\xi \mathcal{D}\eta \exp - \langle \eta \mid d_{\varphi_0} \xi \rangle = A(\varphi_0),$$

getting

$$(16) \qquad W(\varphi) = NA(\varphi_0) \det{}_R (d^*_{\varphi_0} d_\varphi) \int_{\ker d_\varphi} \mathcal{D}e_0 ,$$

where now the regularized determinant is computed in terms of the operator

$$d^*_{\varphi_0} d_\varphi : (\ker d_\varphi)^\perp \to (\ker d_{\varphi_0})^\perp .$$

When restricted to such a domain, this operator has no zero modes in the whole tube $U([\varphi_0])$, since $\ker d^*_{\varphi 0} d_{g\varphi} = g \ker d^*_{g\,\varphi 0} d_\varphi = g \ker d_\varphi$. Obviously this operator fails to be equivariant when one keeps $\varphi_0$ fixed; so its determinant will be not $\mathcal{G}$-invariant.

Notice that $A(\varphi_0)$, and therefore $W(\varphi)$, is still a formal object, as one should expect, because for operators between different complex spaces the determinant can be defined up to a multiplicative constant. We can normalize this constant $A(\varphi_0)$ as suggested in [*16*], by taking $\xi_i$ be a complete orthonormal set in $(\ker d_{\varphi_0})^\perp$ and $\eta_i = d_{\varphi_0} \xi_i$ be orthogonal. Then we set $A_R(\varphi_0) = (\det d^*_{\varphi_0} d_{\varphi_0})^{1/2}$. This is $\mathcal{G}$-invariant and by suitably choosing N in (16) can be normalized to 1. So finally we have a well defined, but trivial functional

$$(17) \qquad W_R(l) = \det{}_R\, d^*_{\varphi_0} d_\varphi \int_{\ker d_\varphi} \mathcal{D}e_0 ,$$

which vanishes identically because the Berezin integral over $\ker d_\varphi$ vanishes. Now, these finite dimensional integral over anticommuting variables are really algebraic objects, which can be identified with elements of the linear space dual to the higher exterior power $\Lambda^I \ker d_\varphi$.

Indeed, the product of Fermi fields $e_0^1(\varphi) \dots e_0^I(\varphi)$ generating $\ker \mathscr{d}_\varphi$ is antisymmetric and therefore belongs to $\Lambda^I \ker \mathscr{d}_\varphi$. Since

$$\int_{\ker \mathscr{d}_\varphi} \mathscr{D}e_0 e_0^1(\varphi) \dots e_0^I(\varphi) = 1\,, \tag{18}$$

we can use the dual basis $e_0^{i*}(\varphi)$ ($i = 1, \dots, I$) and write

$$\int_{\ker \mathscr{d}_\varphi} \mathscr{D}e_0 = e_0^{1*}(\varphi) \dots e_0^{I*}(\varphi)\,.$$

That is, we identify the integral with an element of the space $(\Lambda^I \ker \mathscr{d})^*$. When we move to $g\varphi$, we have

$$\begin{aligned}\int_{\ker \mathscr{d}_{g\varphi}} \mathscr{D}e_0 &= g^{-1}e_0^{1*}(\varphi) \dots g^{-1}e_0^{I*}(\varphi) = \\ &= r(g^{-1})e_0^{1*}(\varphi) \dots e_0^{I*}(\varphi) = r(g^{-1}) \int_{\ker \mathscr{d}_\varphi} \mathscr{D}e_0\,,\end{aligned}$$

for some representation $r : \mathscr{G} \to \mathrm{Gl}(1, \mathrm{C})$.

Now the invariance of the whole naive path integral (including zero modes) implies that

$$\begin{aligned}W(g\varphi) &= (\det_R \mathscr{d}^*_{\varphi_0}\mathscr{d}_{g\varphi}) r(g^{-1}) e_0^{1*}(\varphi) \dots e_0^{I*}(\varphi) = \\ &= (\det_R \mathscr{d}^*_{\varphi_0}\mathscr{d}_\varphi) e_0^{1*}(\varphi) \dots e_0^{I*}(\varphi) = W(\varphi)\,.\end{aligned}$$

If one omits (umpaired) zero modes and sets

$$W_R(\varphi) = \det_R \mathscr{d}^*_{\varphi_0}\mathscr{d}_{\varphi'} \tag{19}$$

as it is usual in the physical literature, one gets

$$W_R(g\varphi) = r(g) W_R(\varphi), \qquad \text{for any} \qquad \varphi \in U([\varphi_0])\,. \tag{20}$$

Accordingly, the transformation properties of $W_R(\varphi)$ are entirely controlled by those of $\Lambda^I \ker \mathscr{d}_\varphi$. In other words, we see that the restricted measure, when zero modes are excluded, can very well depend on $\mathscr{G}$.

So far we have been working in the tube $U([\varphi_0])$, where $\ker \mathscr{d}^*_\varphi$ vanishes. We can extend such a procedure to the whole of $\Phi$, getting $W(\varphi) = 0$ whenever $\mathscr{d}^*_\varphi$ has zero modes. A more clever approach requires also the treatment of the zero modes of $\mathscr{d}^*_{\varphi'}$ but since these arise in pairs with further zero modes of $\mathscr{d}_\varphi$, this does not change eq. *n* (20). In any case, since $W_R(\varphi)$ is not $\mathscr{G}$-invariant, one cannot expect that it descends to a function on the orbit space $\Phi/\mathscr{G}$. Actually it goes down to a section on the complex line bundle $\Phi \times_{\mathscr{G}} \mathbb{C}$, obtained by taking the quotient of $\Phi \times \mathrm{C}$ with respect to the action of $\mathscr{G}$ given by $(\varphi, W) \to (g\varphi, r(g)W)$. One can easily check that this line bundle is the determinant line bundle $L = \det(\operatorname{ind} \mathscr{d})$ of the virtual index bundle of the family of Dirac operators on $\Phi/\mathscr{G}$. We refer to [*1*] [*12*] for more information on this bundle. Accordingly we have the following

PROPOSITION. — The vacuum functional $W_R$ given by (19) descends to a section $w : \Phi/\mathscr{G} \to \det(\operatorname{ind}\not{d})$, and it cannot be $\mathscr{G}$-invariant on unless the line bundle $\det(\operatorname{ind}\not{d})$ is trivial.

## 6. ON THE EXISTENCE OF THE EFFECTIVE ACTION AND THE WESS-ZUMINO TERM

Having computed $W_R(\varphi)$, one's next task is to define its logarithm $\Gamma(\varphi) = \log W_R(\varphi)$, i. e. the one-particle-irreducible generating functional. Clearly $\Gamma(\varphi)$ will not exist where $W_R(\varphi)$ vanishes, so we restrict our attention to the open subbundle $\Phi^c$ of $\Phi$ where $W_R(\varphi) \neq 0$. If $W_R(\varphi)$ is real, as for selfadjoint operators, then $\log W_R(\varphi)$ is a nice continuous functional on $\Phi^c$. But, as noticed in [*1*], when $W_R(\varphi)$ is complex valued as for the « chiral » Dirac operator, its logarithm may be not definable on a domain which is not simply connected.

There is a standard sheaf cohomological explanation for this phenomenon. Consider the exact sequence of sheaves

$$0 \to Z \to \mathscr{C} \xrightarrow{\exp 2\pi i} \mathscr{C}^* \to 1,$$

where $Z$, $\mathscr{C}$, $\mathscr{C}^*$ are the sheaves of germs of integer valued, complex valued, and never vanishing complex valued functions on $\Phi^c$. A segment of the long cohomology sequence reads

$$\ldots \to H^0(\Phi^c, \mathscr{C}) \to H^0(\Phi^c, \mathscr{C}^*) \xrightarrow{\delta^*} H^1(\Phi^c, Z) \to \ldots$$

So our vacuum functional $W_R \in H^0(\Phi^c, \mathscr{C}^*)$ cannot come form exponentiating a functional in $H^0(\Phi^c, \mathscr{C})$ unless $\delta^* W_R = 0$. In other words, the logarithm of $W_R$ does not exist.

One simple way of explaining this phenomenon is to notice that the one form $dW_R/W_R$ is smooth and closed on $\Phi^c$. but it might very well be not exact. In this case one can define a local potential for $dW_R/W_R$ by setting

$$\Gamma(\varphi) = \Gamma(\varphi_0) + \int_\gamma dW_R/W_R, \tag{21}$$

where $\gamma$ is a curve from $\varphi_0$ to $\varphi$ on $\Phi^c$. If we compute the integral along another curve $\gamma'$, then $\Gamma'(\varphi) = \Gamma(\varphi)$ as far as the loop $\gamma - \gamma$ is contractible. However if $\pi_1(\Phi^c) \neq 0$ and $\gamma' - \gamma$ is a non contractible loop, then $\Gamma'(\varphi) \equiv \Gamma(\varphi)$ and the integral above does not define a single valued function. What matters here is the phase of $W_R$. Indeed writing $W_R(\varphi) = \rho(\varphi)e^{i\theta(\varphi)}$, one has $\Gamma(\varphi) = \log \rho(\varphi) + i(\theta(\varphi) + 2n(\varphi)\pi)$. While $\log \rho(\varphi)$ is well defined on $\Phi^c$, one cannot be sure that $n(\varphi)$ is single valued.

As in [*2*], one can give sufficient conditions for such a pathology. Consider a loop $l$ through $\varphi_0$, $\varphi$ and lel $ml$ be the loop gotten by going along $l$ $m$ times. Then $n(\varphi) = m \deg (W_R / | W_R |)$, where $\deg (W_R / | W_R |)$ is the winding number of the phase of $W_R$ on $l$. Clearly, $n(\varphi)$ cannot be single valued if there exists at least one loop $l$ on which the winding number above does not vanish. This is the same as saying that the function $W_R / | W_R |$ cannot be extended to a continuous function on any disk D bounded by $l$, that is $W_R$ has to vanish at some point of D, for any such disk. In other terms $l$ has to be non trivial in $\pi_1(\Phi^c)$ and $W_R / | W_R |$ should be at least of degree one on $l$.

To make contact with the homotopy of the invariance group $\mathcal{G}$, recall that any loop $l$ in $\Phi$ can be continously deformed into a loop $l'$ entirely belonging to an orbit of $\mathcal{G}$ in $\Phi$ where $W_R \neq 0$. If $l'$ was contractible on this orbit, then the degree of $W_R / | W_R |$ would vanish on $l'$, because $W_R$ never vanish on the orbit. In the following we shall omit pull-backs and we will think of $\mathcal{G}$-orbits in $\Phi$ as $\mathcal{G}$ itself. As the orbits are homeomorphic to the group $\mathcal{G}$, we need a non contractible loop in $\mathcal{G}$ itself, i. e. $\pi_1(\mathcal{G}) \neq 0$, and $\deg (W_R / | W_R |) \neq 0$ along it. One might check this condition by studying the eigenvalues of $d^*_{\varphi_0} d_\varphi$ as done in [*2*] for gauge theories. One might intuitively picture such a situation as follows; $\mathcal{G}$-orbit in $\Phi$ have a « hole » encircling an orbit where $W_R$ vanishes. Moreover the phase of $W_R$ is at least of degree one on any loop on $l'$ around such a « hole ».

This topological situation has a representation in terms of differential forms. If $\pi_1(\mathcal{G}) \neq 0$, then also the first cohomology group $H^1(\mathcal{G})$ does not vanish, i. e. there are one forms on $\mathcal{G}$ which are closed but not exact. For instance, in gauge theories on $S^4$ one knows that $\pi_1(\mathcal{G}) = \pi_5 (SU(n)) = Z$ for $n \geq 3$. From the previous discussion, it is clear that if $W_R / | W_R |$, when restricted to an orbit in $\Phi^c$, has at least degree one on a non contractible loop in $\mathcal{G}$, then the one form

$$a = \delta W_R / W_R \tag{22}$$

on $\mathcal{G}$ (here $\delta$ is the differential on $\mathcal{G}$) is a non trivial element of $H^1(\mathcal{G})$. This one form is called the integrated anomaly. Clearly $\delta a = 0$, i. e. $a$ satisfies the Wess-Zumino consistency condition, but there is no single valued functional $\Gamma$ on $\Phi^c$ such that $a = \delta\Gamma$.

Summing up these anomalies prevent the existence of a single valued 1PI generating functional $\Gamma$, the ambiguity arising from the fact that it is path dependent as it is clear from (21). All what one can do is to enlarge the space of definition of $\Gamma$, including path dependence. We can restrict (21) to paths along $\mathcal{G}$-orbits in $\Phi^c$, i. e. we set

$$\Gamma(g\varphi, \gamma) = \Gamma(\varphi) + \int_{\mathbb{1}}^{g} a \tag{23}$$

where the integral is along a path $\gamma$ on $\mathcal{G}$ such that $\gamma(0)=\mathbb{1}, \gamma(1)=g$. Obviously $\Gamma(g\varphi, \gamma)$ will depend on the chosen path, but its differential will not, i. e. $\delta\Gamma(g\varphi, \gamma) = a$, yielding a potential for the anomaly. To get rid of multivaluedness, one can try to fix a standard path, by choosing the shortest one $\gamma_0$ between 1 and $g$ in the group $\mathcal{G}$. Formula (23) with $\gamma = \gamma_0$ now gives the Wess-Zumino term

$$\Gamma_{WZ}(\varphi, g) = \Gamma(g\varphi, \gamma_0). \tag{24}$$

One might ask to what extent the presence of anomalies depends on the regularization and renormalization procedure $\mathcal{R}$ adopted to define $W_R(\varphi)$. We can partially answer this question, as any continuous deformation $W_R(\varphi, t)$ of $W_R(\varphi)$ preserving its zeroes will give rise to the same obstructions. To be definite, assume that $\mathcal{R}_t$ is a continuous one parameter family of regularizing maps such that

$$W_R(\varphi, t) = N(\varphi, t)W_R(\varphi, 0),$$

with $N(\varphi, 0) = 1$ and $N(\varphi, t) \neq 0$ for any $t$ and $\varphi \in \Phi$. We will then have a one parameter family of anomalies

$$a(t) = \delta W_R(\varphi, t)/W_R(\varphi, t) = a(0) + \delta \log N(\varphi, t).$$

Now $\log N(\varphi, t)$ exists because $N(\varphi, t)$ does not vanish on the whole of $\Phi$; accordingly $a(t)$ will be cohomologous to $a(0)$ in $H^1(\varphi)$. So we will have exactly the same obstruction to the existence of the effective action, but different representations. Notice that such a redefinition of the anomaly occours when one changes the renormalization point within a given regularization scheme.

## 7. ANOMALIES AND TRANSGRESSIONS

As the anomaly vanishes whenever $W_R$ is $\mathcal{G}$-invariant or, which is the same, when the line bundle L is trivial, it is clear that there are relations between $a$ and the topology of L. This relation has been already noticed by several authors, as recalled in the introduction. Here we add a corollary, by explicitely proving that the anomaly is always transgressive.

From the homotopy sequence

$$\ldots \to \pi_{k+1}(\Phi) \to \pi_{k+1}(\Phi/\mathcal{G}) \to \pi_k(\mathcal{G}) \to \pi_k(\Phi) \to \ldots$$

of the orbit bundle $\Phi \to \Phi/\mathcal{G}$, we have an isomorphism

$$\pi_1(\mathcal{G}) = \pi_2(\Phi/\mathcal{G}), \tag{25}$$

because, being $\Phi$ contractible, all the homotopy groups $\pi_k(\Phi)$ vanish for $k \geq 1$. Hence a non contractible loop $l$ in $\mathcal{G}$ corresponds to a non contractible 2-sphere $\Sigma$ in the orbit space $\Phi/\mathcal{G}$. See [*2*] for more details.

As we already know, $\mathrm{W_R}(\bar{\varphi})$ descends to a section $w : \Phi/\mathscr{G} \to \mathrm{L}$. Now the winding number of $\mathrm{W_R}/|\mathrm{W_R}|$ on a loop $l$ on a $\mathscr{G}$-orbit in $\Phi$ is the same as that of $w([\varphi])/|w([\varphi])|$ on a loop $l'$ on the sphere $\Sigma$ corresponding to $l$. If we restrict L to this sphere, we see that it cannot be trivial unless the winding number of $w([\varphi])/|w([\varphi])$ on $l'$ vanishes. Actually, this winding number equals the first Chern number $\mathrm{C_1(L)}$ of the line bundle L, and can be given as the integral on $\Sigma$ of a differential form $c_1$, which is a representative of the first Chern class of L. Summing up we have the equality;

$$\deg(\mathrm{W_R}/|\mathrm{W_R}|) = 1/2\pi i \int \delta \mathrm{W_R}/\mathrm{W_R} = \int_\Sigma c_1 \tag{26}$$

Now, $\Sigma - \pi(l)$, where $\pi$ denotes the projection $\pi : \Phi \to \Phi/\mathscr{G}$, is diffeomorphic to a disk D in $\Phi$ bounded by $l$. Then we have

$$\int_\Sigma c_1 = \int_{\Sigma - \pi(l)} c_1 = \int_\mathrm{D} \pi^* c_1 ,$$

where $\pi^* c_1$ is the pull-back to $\Phi$ of the first Chern class $c_1$ of L. As it is well known, the two form $\pi^* c_1$ is exact on $\Phi$, that is there exists a one form $tc_1$, called the transgression of $c_1$, such that

$$\pi^* c_1 = dtc_1 .$$

Hence, by Stokes theorem, it follows that

$$\int_\mathrm{D} \pi^* c_1 = \int_\mathrm{D} dtc_1 = \int_l tc_1 .$$

So we see that, on any loop $l$ on a generic orbit in $\mathscr{G}$,

$$\int_l \delta \mathrm{W_R}/\mathrm{W_R} = \int_l tc_1 |_{v'}$$

where $|_v$ means restriction to that orbit. This implies that the anomaly $a = \delta \mathrm{W_R}/\mathrm{W_R}$ and the vertical restriction $tc_1 |_v$ are cohomologous in $\mathrm{H}^1(\mathscr{G})$. So we recover the result of [*1*], quite independently of the regularization procedure.

Recall that a cohomology class $[a] \in \mathrm{H}^1(\mathscr{G})$ is called transgressive on a principal $\mathscr{G}$-bundle P if one can find a representative a such that

1) $a$ is the restriction to an orbit of $\mathscr{G}$ in P of a form $\tilde{a}$ on P

2) the differential of $\tilde{a}$ is the pull-back to P of a form $b$ on the basis space $\pi(\mathrm{P})$ of P, that is $d\tilde{a} = \pi^* b$.

So in mathematical therms, one can summarize the construction above as follows.

PROPOSITION. — The anomaly $a = \delta \mathrm{W_R}/\mathrm{W_R}$ is always transgressive on $\Phi$ and can be represented (up to differentials) by vertically restricting the transgression $tc_1$ of the first Chern class of the determinant line bundle $L = \det(\text{ind}\, \not{d})$.

We refer to [1] for the representation of $tc_1$ in terms of differential forms, yielding a concrete expression for the anomaly. This is far from being trivial, as it involves computation of the topological « part » of the family index theorem.

## 8. CONCLUDING REMARKS

The main aim of this paper was to show that there are *a priori* reasons for identifying the fermionic vacuum functional $W_R(\varphi)$, as computed by physicists and given by (19), with the pull-back to $\Phi$ of a section of the line bundle $L = \det(\text{ind}\, d)$ on the orbit space $\Phi/\mathcal{G}$. Then the coincidence of the anomaly with the vertical restriction of the transgression $tc_1$ of the first Chern class of L easily follows by well known arguments.

This identification, first done in [1], in terms of $\zeta$-function regularization of determinants, seems to be quite independent of the regularization scheme, provided certain requirements are met. Instead it comes from the algebraic nature of the Berezin integration rules.

Being L a complex line bundle associated to the principal orbit bundle $\Phi \to \Phi/\mathcal{G}$, there is no surprise that its Chern class can be expressed in terms of a principal connection, i. e. a connection on $\Phi$. This gives a mathematical explanation as to why one can construct anomalies forgetting about fermions and their currents, even if some physical argument is still missing.

As for locality, notice that the transgression on $\Phi$ of $c_1(L)$ is a non local functional but, after vertical restriction, it yields a local expression for the anomaly, that is an expression depending polynomially on $\varphi$ and its derivatives. Although it is local, the anomaly prevents the existence of a single valued effective action, either local or not. Clearly, by the Poincaré lemma, a non local effective action can be defined in any contractible neighborhood of a given field $\varphi_0$, but the anomaly forbids it can be extended to a single valued functional even in a tube of orbits around $\varphi_0$.

However, by simply looking for anomalies $a$ which do not admit any local potential $\Gamma_{loc}$, i. e. looking for non trivial elements in the BRS cohomology, one gets exactly the same anomalies [4] [5] [6]. Uniqueness of this procedure is missing, as one knows that actual anomalies, as given by the index theorem and computed by physicists, are non trivial elements of the BRS cohomology, but BRS cohomology may contain more of them. In certain cases, e. g. for SU $(n)$ gauge theories with $n \geq 3$, one can actually prove unicity of the anomalies coming from topology by explicitely computing the first Chech cohomology group $H^1(\mathcal{G})$ [17]. It turns out that this is one dimensional over the reals; accordingly the anomaly class is unique up to a multiplicative constant, and restricting to « local » functionals has no real effect.

So, although the index theorem approach yields directly and uniquely the non abelian anomalies found by physicists, more work on the BRS cohomology seems worth. This may help in understanding in physical terms why one can construct anomalies forgetting about fermions. On the other hand cohomology of « local » functionals has some deep physical roots in the renormalization procedures and, besides, it seems to describe fairly well with certain anomalies, as the trace ones [*8*] [*18*], which don't seem to be entirely of topological nature, not to talk about U(1) and gravitational anomalies.

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