ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 43, nº 3 (1985), p. 251-267
<http://www.numdam.org/item?id=AIHPA_1985__43_3_251_0>

Ann. Inst. Henri Poincaré,
Vol. 43, n° 3, 1985, p. 251-267. *Physique théorique*

Canonical realizations of Lie algebras associated with foliated coadjoint orbits

by

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ABSTRACT. — Using the natural affine connection on leaves of a Lagrangean foliation induced by the Bott connection, we give a formula for the symplectomorphism stated in the theorem of Kostant and Weinstein. Applying this symplectomorphism to a certain type of foliated coadjoint orbits we are able to construct canonical realizations of Lie algebras. We demonstrate our method for the case of $gl(n, \mathrm{R})$.

RÉSUMÉ. — En utilisant la connexion affine naturelle induite par la connexion de Bott sur les feuilles d'un feuilletage Lagrangien, nous donnons une formule exprimant le symplectomorphisme décrit dans le théorème de Kostant-Weinstein. L'application de cette formule à un certain type d'orbites coadjointes feuilletées permet la construction de réalisations canoniques d'algèbres de Lie. Nous illustrons notre méthode en l'appliquant au cas de $gl(n, \mathrm{R})$.

0. INTRODUCTION

Canonical realizations of Lie algebras are used for studying physical systems with symmetries in the framework of the canonical formalism [*1*] [*4*]. They are especially useful in connection with the method of collective variables, e. g. in nuclear physics [*3*]. Moreover, they play a role in purely mathematical investigations, (e. g. in connection with the Gelfand-Kirillov conjecture), [2] [4] [5] [6] [7] [8]. A classical (quantum-mechanical) canonical realization of a Lie algebra $\mathfrak{G}$ is a homomorphism

$$f : \mathfrak{G} \to D(H_{2n})(\mathfrak{D}(H_{2n})), \tag{1}$$

where $D(H_{2n})(\mathfrak{D}(H_{2n}))$ is the quotient field of the symmetric (enveloping) algebra of the $2n + 1$-dimensional Heisenberg algebra H_{2n}, (with the identification $1 = \mathbb{1}$).

In the language of differential geometry a classical canonical realization of $\mathfrak{G}$ is a homomorphism

$$f : (\mathfrak{G}, [\ ,\ ]) \to (C^{\infty}(P), \{\ ,\ \}), \tag{2}$$

where (P, ω) is a symplectic manifold and $\{\ ,\ \}$ the associated Poisson bracket in $C^{\infty}(P)$. When a Lie group G acts canonically on P, $g^*\omega = \omega$, then—under some cohomological assumptions [9] [10]—one can construct a homomorphism (2) by integrating the forms $\hat{v} \lrcorner \omega$, $f_v = \int \hat{v} \lrcorner \omega$, where $\hat{v}$ denotes the fundamental vector field generated by $v \in \mathfrak{G}$ (Lie algebra of G). We will deal with P being an orbit $\mathfrak{O}$ of the coadjoint representation of G. Then these assumptions are always fulfilled and (2) is given by

$$f_v(x^*) := x^*(v), \tag{3}$$

where $x^* \in \mathfrak{O} \subset \mathfrak{G}^*$, $v \in \mathfrak{G}$. To obtain a realization in the sense of (1), one has to find Darboux coordinates (p, q) on $\mathfrak{O}$ such that f_v are (rational) polynomials in p and q. An adequate geometrical concept for this purpose is that of Lagrangean foliations (LF) of orbits.

In chapter 1 we consider LF's of arbitrary symplectic manifolds. Using the natural affine structure on leaves of a LF—induced by the Bott connection—we define a local symplectomorphism Φ between a foliated symplectic manifold with transversal Lagrangean submanifold (LSM) and the cotangent bundle over this LSM, (Theorem 1). Φ coincides with the symplectomorphism whose existence and uniqueness was stated in the Kostant-Weinstein theorem [11] [12] [13]. In our approach Φ is given explicitely, provided the exponential mapping of the affine connection is known.

In chapter 2 we introduce a certain class of foliations of coadjoint orbits, which are—in a sense—generated by the group action and for which the affine structure on leaves becomes especially transparent, (Theorem 2). In the special case (Proposition 3), when a leaf is an open subset of an affine subspace of $\mathfrak{G}^*$, the natural affine structure is induced by $\mathfrak{G}^*$ and, consequently, the exponential mapping is explicitely given (see (21)). This situation is always realized, when $\mathfrak{G}$ admits a certain decomposition into subalgebras with respect to $x^* \in \mathfrak{O}$, (see (22)).

In chapter 3 we derive formulae for canonical realizations associated with this type of foliations, (Proposition 4). In chapter 4 we demonstrate our method for the case of two orbit types of $gl(n, \mathrm{R})$.

1. LAGRANGEAN FOLIATIONS OF SYMPLETIC MANIFOLDS

Let (P, ω) be a symplectic manifold. A submanifold is called Lagrangean, iff

i) dim L = 1/2 dim P and

ii) $i^*\omega = 0$, where $i : \mathrm{L} \hookrightarrow \mathrm{P}$ is the embedding of L into P. A foliation $\mathfrak{F} = \{ \mathrm{F}_\lambda \}_{\lambda \in \Lambda}$ is called Lagrangean if the leaves F_λ are Lagrangean, or, equivalently, if the corresponding (integrable) distribution D is Lagrangean:

i) dim D = 1/2 dim P and

ii) $\omega_p(\mathrm{X}, \mathrm{X}') = 0$, $\mathrm{X}, \mathrm{X}' \in \mathrm{D}_p \subset \mathrm{T}_p\mathrm{P}$.

The standard example of a LF is the fibration of any cotangent bundle.

It is well-known that the leaves of a LF $\mathfrak{F}$ are equipped with a natural curvature free and torsionless affine connection, given by the partial covariant derivative

$$\nabla_{\mathfrak{X}'}\mathfrak{X} := \omega^{-1} \circ \mathscr{L}_{\mathfrak{X}'}(\mathfrak{X} \lrcorner \omega), \tag{4}$$

where $\mathfrak{X}, \mathfrak{X}' \in \Gamma(\mathrm{D})$. One obtains this connection starting from the Bott connection [*14*] on TP/D, taking its dual connection on (TP/D)* and transporting it to $\mathrm{D} \cong (\mathrm{TP/D})^*$ via ω. Using $d\omega = 0$ and the fact that D is Lagrangean one gets that curvature and torsion vanish. The geometrical sense of this connection can be illustrated as follows:

Let $\mathrm{U} \subset \mathrm{P}$ be a neighbourhood, $\mathfrak{F}_{\restriction_\mathrm{U}}$ the corresponding foliation of U and $\mathrm{Q} = \mathrm{U}/\sim$ the space of leaves of $\mathfrak{F}_{\restriction_\mathrm{U}}$. Obviously, U can be chosen such that

i) Q is a manifold,

ii) $\chi : \mathrm{U} \to \mathrm{Q}$ is a differentiable surjection,

iii) the leaves of $\mathfrak{F}_{\restriction_\mathrm{U}}$ are simply connected.

Because of *iii)* and the fact that the connection is flat we have a path-independent operator of parallel transport on every leaf of $\mathfrak{F}_{\restriction_\mathrm{U}}$

$$\tau_p^{p'} : \mathrm{D}_p \to \mathrm{D}_{p'}, \qquad \chi(p) = \chi(p').$$

Let $\mathfrak{X}$, $\mathfrak{X}' \in \Gamma(D)$ with $\mathfrak{X}_{p'} = \tau_p^{p'} \mathfrak{X}_p$. Then $\nabla_{\mathfrak{X}'} \mathfrak{X} = 0$ and, consequently, for every χ-related field $\mathfrak{Y} \in \Gamma(TU)$ we get

$$\begin{aligned} 0 &= \omega(\nabla_{\mathfrak{X}'} \mathfrak{X}, \mathfrak{Y}) \\ &= \mathfrak{Y} \lrcorner \mathscr{L}_{\mathfrak{X}'}(\mathfrak{X} \lrcorner \omega) \\ &= \mathfrak{X}'(\omega(\mathfrak{X}, \mathfrak{Y})) - \mathfrak{Y}(\omega(\mathfrak{X}, \mathfrak{X}')) - \omega(\mathfrak{X}, [\mathfrak{X}', \mathfrak{Y}]) \\ &= \mathfrak{X}'(\omega(\mathfrak{X}, \mathfrak{Y})) . \end{aligned}$$

This means that $\omega(\mathfrak{X}, \mathfrak{Y})$ is constant on every leaf.

PROPOSITION 1.

$$\left(\tau_{p'}^{p}(X_p) = X_{p'}\right) \Leftrightarrow \begin{pmatrix} \omega(X_p, Y_p) = \omega(X_{p'}, Y_{p'}) \\ \text{for all Y satisfying} \\ d\chi(Y_p) = d\chi(Y_{p'}) \end{pmatrix} . \tag{5}$$

□

Obviously, (5) may be also used to define the natural affine connection [15], see Fig. 1.

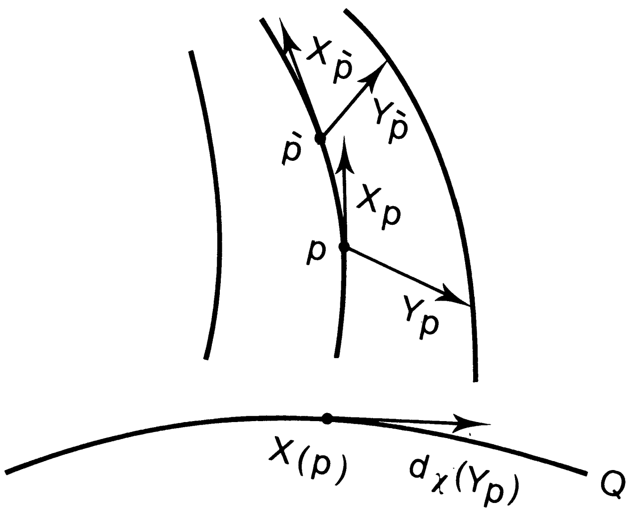

FIG. 1.

Now, let N be a submanifold of P, transversal to $\mathscr{F}$, dim N = 1/2 dim P, and $D_{\restriction N}$ be the restriction of D to N. Then the exponential mapping of the above partial affine connection defines a diffeomorphism between a neighbourhood W of the zero-section of $D_{\restriction N}$ and a neighbourhood, U := exp (W), of N in P:

$$\exp_{\restriction N} : D_{\restriction N} \supset W \rightarrow U \subset P . \tag{6}$$

Since $D_{\restriction N}$ is a vector bundle, it is obvious that we can choose W such that for every $n \in \mathrm{N}$ $\mathrm{W} \cap \mathrm{D}_n$ is contractible. This implies that the foliation $\mathfrak{F}_{\restriction u}$ has the following properties:

i) N intersects with every leaf of $\mathfrak{F}_{\restriction u}$ exactly at one point and, therefore, N can be identified with the space of leaves of $\mathfrak{F}_{\restriction u}$.

ii) The canonical projection $\chi : \mathrm{U} \to \mathrm{N}$ is a differentiable surjection.

iii) The leaves of $\mathfrak{F}_{\restriction u}$ are contractible and, particularly, simply connected.

The affine structure on leaves of $\mathfrak{F}_{\restriction u}$ and the choice of N determine a vertical vector field on U:

$$\mathrm{U} \ni p \to \mathfrak{Z}_p := \tau^p_{\mathfrak{X}(p)} \circ \exp^{-1}_{\chi(p)}(p) \in \mathrm{D}_p , \tag{7}$$

where $\exp_n : \mathrm{D}_n \to \mathrm{F}_n$ (leaf through $n \in \mathrm{N}$). Moreover, let us introduce a 1-form ϑ corresponding to $\mathfrak{Z}$:

$$\vartheta := \mathfrak{Z} \lrcorner\, \omega . \tag{8}$$

LEMMA 1. — Let $\mathfrak{X} \in \Gamma(\mathrm{D})$. Then

$$\mathscr{L}_{\mathfrak{X}} \vartheta = \mathfrak{X} \lrcorner\, \omega . \tag{9}$$

Proof. — It is sufficient to show that (9) holds, if we apply both its sides to a χ-related field $\mathfrak{Y}$. We have

$$\begin{aligned}(\mathscr{L}_{\mathfrak{X}}\vartheta)(\mathfrak{Y}) &= \mathfrak{X}(\omega(\mathfrak{Z}, \mathfrak{Y})) - \mathfrak{Y}(\omega(\mathfrak{Z}, \mathfrak{X})) - \omega(\mathfrak{Z}, [\mathfrak{X}, \mathfrak{Y}]) \\ &= \mathfrak{X}(\omega(\mathfrak{Z}, \mathfrak{Y})) \\ &= \mathfrak{X}(f),\end{aligned}$$

where $f = \omega(\mathfrak{Z}, \mathfrak{Y})$ may be regarded as a superposition

$$p \xrightarrow{f_1} \exp^{-1}_{\chi(p)}(p) \xrightarrow{f_2} -\exp^{-1}_{\chi(p)}(p) \lrcorner\, (\mathfrak{Y} \lrcorner\, \omega)_{\chi(p)} .$$

Of course, f_2 is a linear map. Finally, using (5) and the fact that for a curvature free and torsionless affine connection holds [*16*]

$$(d \exp_n)_{\mathrm{X}}(\mathrm{X}') = \tau_n^{\exp_n(\mathrm{X})}(\mathrm{X}'), \qquad \mathrm{X}, \mathrm{X}' \in \mathrm{D}_n ,$$

we have

$$\begin{aligned}\mathfrak{X}_p(f) = (df_2 \circ df_1)(\mathfrak{X}_p) &= df_2(\tau_p^{\chi(p)}(\mathfrak{X}_p)) \\ &= -\tau_p^{\chi(p)}(\mathfrak{X}_p) \lrcorner\, (\mathfrak{Y} \lrcorner\, \omega)_{\chi(p)} \\ &= \omega_p(\mathfrak{X}_p, \mathfrak{Y}_p) .\end{aligned}$$ □

Let $\omega_{\mathrm{N}} := i^*\omega$, where $i : \mathrm{N} \hookrightarrow \mathrm{P}$. Obviously, N is Lagrangean iff $\omega_{\mathrm{N}} = 0$.

PROPOSITION 2.

$$d\vartheta = \omega - \chi^*\omega_{\mathrm{N}} . \tag{10}$$

Proof. — We put $\eta = d\vartheta - \omega + \chi^*\omega_{\mathrm{N}}$. Then η is invariant under leaf-preserving local 1-parameter groups of diffeomorphisms. Really, if $\mathfrak{X} \in \Gamma(\mathrm{D})$, then—using Lemma 1 and the fact $\chi_*\mathfrak{X} = 0$—we have

$$\mathscr{L}_{\mathfrak{X}}\eta = d\mathscr{L}_{\mathfrak{X}}\vartheta - d(\mathfrak{X} \lrcorner\, \omega) + (i \circ \chi)^*\mathscr{L}_{(i\circ\chi)_*\mathfrak{X}}\omega = 0 .$$

Thus, it's sufficient to show that $\eta_n = 0$ for $n \in \mathrm{N}$. We have $\mathrm{T}_n\mathrm{P} = \mathrm{T}_n\mathrm{N} \oplus \mathrm{D}_n$.

1. If $\mathfrak{X}, \mathfrak{X}' \in \Gamma(\mathrm{D})$, then $\eta(\mathfrak{X}, \mathfrak{X}') = 0$, because $\mathfrak{F}$ is Lagrangean.
2. For $\mathrm{X} \in \mathrm{T}_n\mathrm{N}$ and $\mathrm{Y} \in \mathrm{T}_n\mathrm{N}$ equality $\eta_n(\mathrm{X}, \mathrm{Y}) = 0$ follows from $i^*\eta = di^*\vartheta - i^*\omega + i^* \circ \chi^* \circ i^*\omega = 0$, where we used $i^*\vartheta = 0$ and $i \circ \chi \circ i = i$.
3. If $\mathfrak{X} \in \Gamma(\mathrm{D})$ and $\mathfrak{Y}$ is χ-related, $\mathfrak{Y}_n \in \mathrm{T}_n\mathrm{N}$, then—using Lemma 1 we have

$$d\vartheta(\mathfrak{X}, \mathfrak{Y}) = \mathfrak{Y} \lrcorner \mathscr{L}_{\mathfrak{X}} \vartheta = \omega(\mathfrak{X}, \mathfrak{Y}) \quad \text{and} \quad \chi^*\omega_{\mathrm{N}}(\mathfrak{X}, \mathfrak{Y}) = 0 . \qquad \square$$

Now we are going to define a diffeomorphism Φ, which we will use in the following chapters. Since ω is nondegenerate, it defines the bundle isomorphism $j : \mathrm{T}^*\mathrm{N} \to \mathrm{D}_{\restriction \mathrm{N}}$, given by

$$\rho = i^*(j(\rho) \lrcorner \omega_{\pi(\rho)}), \tag{11}$$

where $\rho \in \mathrm{T}^*\mathrm{N}$, $\pi : \mathrm{T}^*\mathrm{N} \to \mathrm{N}$. Then $\mathrm{V} := j^{-1}(\mathrm{W})$ is a neighbourhood of the zero-section in $\mathrm{T}^*\mathrm{N}$.

THEOREM 1. — The mapping $\Phi : \mathrm{T}^*\mathrm{N} \supset \mathrm{V} \to \mathrm{U} \subset \mathrm{P}$, defined by

$$\Phi := \exp_{\restriction \mathrm{N}} \circ j, \tag{12}$$

is a diffeomorphism satisfying

i) $\Phi_{\restriction \mathrm{N}} = \mathrm{id}$, where N is identified with the zero-section in $\mathrm{T}^*\mathrm{N}$.
ii) Φ maps fibres into leaves:

$$\pi = \chi \circ \Phi . \tag{13}$$

iii)
$$\Phi^*\vartheta = \Theta ,$$

where Θ is the canonical symplectic 1-form on $\mathrm{T}^*\mathrm{N}$.
iv) The restriction Φ_n of Φ to any fibre $\mathrm{T}_n^*\mathrm{N}$ is an affine mapping.

Proof. — Obviously, Φ is a diffeomorphism satisfying *i*) and *ii*). For $\mathrm{X} \in \mathrm{T}_\rho\mathrm{T}^*\mathrm{N}$ we get

$$\begin{aligned}(\Phi^*\vartheta)(\mathrm{X}) = \vartheta_{\Phi(\rho)}(d\Phi(\mathrm{X})) &= \omega_{\Phi(\rho)}(\mathfrak{Z}_{\Phi(\rho)}, d\Phi(\mathrm{X})) \\ &= \omega_{\pi(\rho)}(\exp^{-1}_{\pi(\rho)}(\Phi(\rho)), d\pi(\mathrm{X})) \\ &= \rho(d\pi(\mathrm{X})) \\ &= \Theta_\rho(\mathrm{X}) ,\end{aligned}$$

where we used (5), (11) and (13).

Since for a curvature free and torsionless connection the exponential mapping is affine, Φ_n is affine as a superposition of affine mappings. $\square$

If N is Lagrangean, $\omega_{\mathrm{N}} = 0$, it follows from (10) and (14) that $\Phi^*(\omega) = d\Theta = \Omega$, where Ω is the canonical symplectic form on $\mathrm{T}^*\mathrm{N}$. Thus

COROLLARY 1. — If N is Lagrangean, then Φ is the symplectomorphism, whose existence and uniqueness was stated in the theorem of Kostant and Weinstein [*11*] [*12*] [*13*].

A similar explicit construction of the Kostant-Weinstein symplectomorphism was given in [22] [23] [24].

Formula (12) shows that in order to get Φ explicitely one has to know the exponential mapping. That means in the general case that one has to integrate the geodesic equations of first order:

$$\frac{d}{dt}\gamma(t) = \tau_n^{\gamma(t)}(X), \tag{15}$$

with $\gamma(0) = n$ and $X \in D_n$. Then $\exp_n(X) = \gamma(1)$.

In the next chapter we distinquish a certain type of foliations of coadjoint orbits for which the problem of solving (15) becomes trivial.

2. LAGRANGEAN FOLIATIONS OF COADJOINT ORBITS

For a brief introduction to coadjoint orbits we refer to [8] [9] [10]. Let $\mathfrak{G}$ be the Lie algebra of the Lie group G and $\mathfrak{O}_{x^*}(G_{x^*})$ the coadjoint orbit (resp. stabilizer) through (resp. of) $x^* \in \mathfrak{G}^*$. The G-invariant symplectic 2-form ω on $\mathfrak{O}_{x^*}$, [8], is given by

$$\omega_{y^*}(\hat{v}, \hat{w}) := y^*([v, w]), \tag{16}$$

where $y^* \in \mathfrak{O}_{x^*}$ and $\hat{v}$, $\hat{w}$ are the fundamental vector fields generated by $v, w \in \mathfrak{G}$. Since $T_{y^*}\mathfrak{O}_{x^*} \subset T_{y^*}\mathfrak{G}^* \cong \mathfrak{G}^*$, we may regard vectors tangent to the orbit as functionals, $\alpha = \hat{v}_{y^*} = \mathrm{ad}'v(y^*)$. Therefore,

$$\omega_{y^*}(\alpha, \hat{w}) = -\,\mathrm{ad}'v(y^*)(w) = -\,\alpha(w), \tag{17}$$

We will deal with local LF's of orbits, which are generated by the group action in the following sense:

Let F be a connected Lagrangean submanifold of $\mathfrak{O}_{x^*}$, $x^* \in F$. Let $\mathcal{N}$ be a submanifold of G transversal to $\mathcal{M} := \{ g \in G : \mathrm{Ad}'g(x^*) \in F \}$ at $e \in G$. Then

$$\begin{aligned} \dim \mathcal{M} &= \dim F + \dim G_{x^*} = 1/2(\dim G + \dim G_{x^*}) \\ &= 1/2 \dim \mathfrak{O}_{x^*} + \dim G_{x^*} \end{aligned} \tag{18a}$$

$$\dim \mathcal{N} = \dim G - \dim \mathcal{M} = 1/2(\dim G - \dim G_{x^*}) = 1/2 \dim \mathfrak{O}_{x^*} \tag{18b}$$

Suppose F and $\mathcal{N}$ are chosen sufficiently small, so that

$$\varphi : \mathcal{N} \times F \to \varphi(\mathcal{N} \times F) \subset \mathfrak{O}_{x^*},$$

defined by $\varphi(n, y^*) := \mathrm{Ad}'n(y^*)$, is a diffeomorphism. Then

$$\mathfrak{F} = \{ F_n \}_{n \in \mathcal{N}} = \{ \varphi(n, F) \}_{n \in \mathcal{N}} \tag{19}$$

is by the G-invariance of ω a Lagrangean foliation of a neighbourhood of x^*.

The subspace $\mathfrak{N}_n := \left\{ v \in \mathfrak{G} : \frac{d}{dt}\restriction_{t=0} e^{tv} \cdot n \in \mathrm{T}_n\mathcal{N} \subset \mathrm{T}_n\mathrm{G} \right\}$ of $\mathfrak{G}$ generates the space tangent to $\mathcal{N}$ at n. Thus, for every $y^* \in \mathrm{F}_n$ we have

$$\mathrm{T}_{y^*}\mathfrak{O}_{x^*} = \{ \hat{v}_{y^*} : v \in \mathfrak{N}_n \} \oplus \mathrm{T}_{y^*}\mathrm{F}_n . \tag{20}$$

Now, let $h_n : \mathfrak{G}^* \supset \mathrm{F}_n \to \mathfrak{N}_n^*$ be the restriction of functionals to $\mathfrak{N}_n$:

$$h_n(y^*) := y^*\restriction_{\mathfrak{N}_n}, \qquad y^* \in \mathrm{F}_n .$$

THEOREM 2. — Let $y^* \in \mathrm{F}_n$. Then the mapping h_n is an affine diffeomorphism of neighbourhoods of y^* in F_n and $h_n(y^*)$ in $\mathfrak{N}_n^*$.

Proof. — To see that Φ is a local diffeomorphism we show that $(dh_n)_{y^*}$ is an isomorphism and then apply the inverse function theorem. Using (18) we see that $\dim \mathrm{T}_{y^*}\mathrm{F}_n = \dim \mathfrak{N}_n^*$. It remains to show that $(dh_n)_{y^*}$ is injective. Since h_n is the restriction of a linear mapping, we have $(dh_n)_{y^*}(\alpha) = \alpha\restriction_{\mathfrak{N}_n}$, $\alpha \in \mathrm{T}_{y^*}\mathrm{F}_n \subset \mathfrak{G}$. Now, if $(dh_n)_{y^*}(\alpha) = 0$, then—using (17)—$0 = \alpha(v) = \omega_{y^*}(\hat{v}_{y^*}, \alpha)$, [illegible] $\mathfrak{N}_n$. From the nondegenerateness of ω we obtain $\alpha = 0$.

To see that h_n is affine, we prove that any geodesic line $\gamma(t)$ in F_n, with $\gamma(0) = y^*$ and $\dot{\gamma}(0) = \alpha$, is mapped by h_n onto a straight line in $\mathfrak{N}_n^*$.

By the construction of $\mathfrak{F}$ we have $(d\chi)_{y^*}(\hat{v}) = (d\chi)_{z^*}(\hat{v})$, for $y^*, z^* \in \mathrm{F}_n$. Using this we obtain from (5) that

$$\dot{\gamma}(t)(v) = -\,\omega_{\gamma(t)}(\dot{\gamma}, \hat{v}) = -\,\omega_{\gamma(0)}(\alpha, \hat{v}) = \alpha(v)$$

and, therefore,

$$h_n(\gamma(t)) = (y^* + t \cdot \alpha)\restriction_{\mathfrak{N}_n} . \qquad \square$$

Theorem 2 gives a simple description of the natural affine structure on F_n for the type of foliations considered in this chapter, see Fig. 2.

Especially, if F is an open subset of an affine subspace of $\mathfrak{G}^*$ (and the other leaves by linearity of $\mathrm{Ad}'n(\cdot)$, too), then the affine structure of $\mathfrak{G}^*$ restricted to F_n coincides with the natural affine structure of the leaf F_n and, therefore,

$$\exp_{y^*}(\alpha) = y^* + \alpha . \tag{21}$$

This situation is characterized by

PROPOSITION 3. — Let F be a connected Lagrangean submanifold of $\mathfrak{O}_{x^*}$. Then F is an open subset of an affine subspace of $\mathfrak{G}^*$ iff

$$\mathfrak{M}(y^*) = \{ v \in \mathfrak{G} : \hat{v}_{y^*} \in \mathrm{D}_{y^*} \}$$

is independent of $y^* \in \mathrm{F}$. In this case the space $\mathfrak{M} = \mathfrak{M}(y^*)$, $y^* \in \mathrm{F}$, is a Lie subalgebra.

Proof. — We show that $\mathrm{T}_{y^*}\mathrm{F} = \mathfrak{M}(y^*)^\perp = \{ \alpha \in \mathfrak{G}^* : \alpha(v) = 0,\ v \in \mathfrak{M}(y^*) \}$. By (18) we have $\dim \mathrm{T}_{y^*}\mathrm{F} = \dim \mathfrak{M}(y^*)^\perp$. Moreover, $\mathrm{T}_{y^*}\mathrm{F} \subset \mathfrak{M}(y^*)^\perp$.

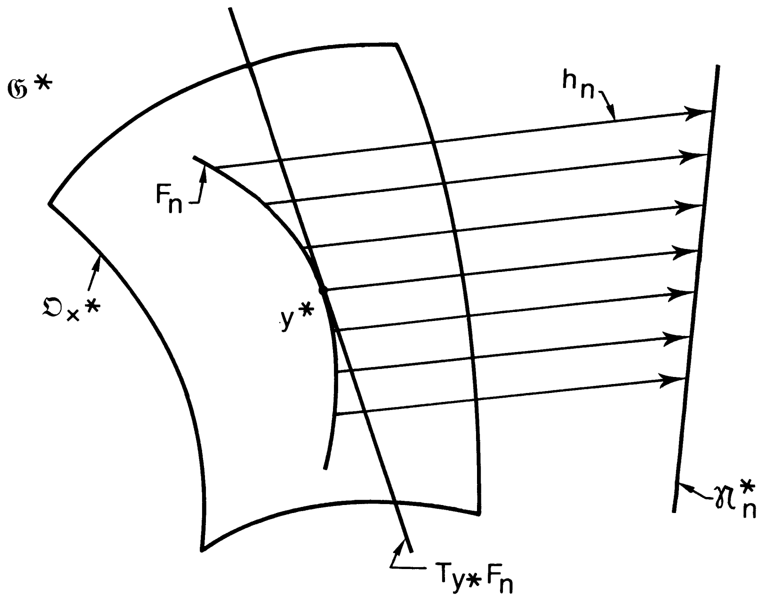

FIG. 2.

Really, if $\alpha \in T_{y^*}F$ and $v \in \mathfrak{M}(y^*)$, then $0 = \omega_{y^*}(\hat{v}_{y^*}, \alpha) = \alpha(v)$. Thus, if F is affine, then $\mathfrak{M}(y^*)^{\perp}$ and, therefore, also $\mathfrak{M}(y^*)$ are independent of y^*, and vice versa. In this case $\hat{v}$ and $\hat{w}$, with $v, w \in \mathfrak{M}$, are tangent to F and, consequently, $[\hat{v}, \hat{w}]_{y^*} = - \widehat{[v, w]}_{y^*}$, too. This implies that $[v, w] \in \mathfrak{M}$. □

Conversely, if $\mathfrak{M} \subset \mathfrak{G}$ is a subalgebra satisfying

i) $\dim \mathfrak{M} = 1/2 \dim \mathfrak{D}_{x^*} + \dim G_{x^*}$,

ii) $x^*([\mathfrak{M}, \mathfrak{M}]) = 0$,

then $F := \mathrm{Ad}' e^{\mathfrak{M}}(x^*)$ is a Lagrangean submanifold of $\mathfrak{D}_{x^*}$. Moreover, it is an open subset of an affine subspace of $\mathfrak{G}^*$. First we show that $\mathfrak{G}_{x^*} \subset \mathfrak{M}$: Obviously, $\{ v \in \mathfrak{M} : \hat{v}_{x^*} = 0 \} = \mathfrak{M} \cap \mathfrak{G}_{x^*} \subset \mathfrak{G}_{x^*}$. Moreover, by the isotropy of $\{ \hat{v}_{x^*} \in T_{x^*}\mathfrak{D}_{x^*}, v \in \mathfrak{M} \}$—see *ii)*—we have

$$\dim \{ \hat{v}_{x^*} \in T_{x^*}\mathfrak{D}_{x^*}, v \in \mathfrak{M} \} \leqslant 1/2 \dim \mathfrak{D}_{x^*},$$

and, therefore, we have

$$\begin{aligned} \dim \{ v \in \mathfrak{M} : \hat{v}_{x^*} = 0 \} &= \dim \mathfrak{M} - \dim \{ \hat{v}_{x^*} \in T_{x^*}\mathfrak{D}_{x^*}, v \in \mathfrak{M} \} \\ &\geqslant \dim \mathfrak{M} - 1/2 \dim \mathfrak{D}_{x^*} = \dim \mathfrak{G}_{x^*}. \end{aligned}$$

Thus $\mathfrak{G}_{x^*} = \{ v \in \mathfrak{M} : \hat{v}_{x^*} = 0 \}$. Therefore F is a submanifold of $\mathfrak{D}_{x^*}$ of dimension $\dim F = \dim \mathfrak{M} - \dim \mathfrak{G}_{x^*} = 1/2 \dim \mathfrak{D}_{x^*}$. Using *ii)* and the fact that $\mathfrak{M}$ is a subalgebra we obtain that F is Lagrangean. By Proposition 3, F is an open subset of an affine subspace of $\mathfrak{G}^*$.

3. CANONICAL REALIZATIONS RELATED TO COADJOINT ORBITS

In this chapter we define—using results of chapters 1 and 2—a local symplectomorphism Ψ between the orbit $\mathfrak{O}_{x^*}$ and a linear symplectic space $\mathfrak{N} \oplus \mathfrak{N}^*$. Then we use Ψ to express the canonical realization (3) in terms of canonical coordinates on $\mathfrak{N} \oplus \mathfrak{N}^*$.

We consider orbits $\mathfrak{O}_{x^*}$ for which $\mathfrak{G}$ admits the following decomposition with respect to x^* :

$$\mathfrak{G} = \mathfrak{N} \oplus \mathfrak{M}, \tag{22}$$

satisfying

i) $\mathfrak{N}$ and $\mathfrak{M}$ are subalgebras of $\mathfrak{G}$,

ii) $\dim \mathfrak{N} = 1/2 \dim \mathfrak{O}_{x^*}$ and, consequently,

$$\dim \mathfrak{M} = 1/2 \dim \mathfrak{O}_{x^*} + \dim \mathrm{G}_{x^*},$$

iii) $x^*([\mathfrak{N}, \mathfrak{N}]) = x^*([\mathfrak{M}, \mathfrak{M}]) = 0.$

We denote by $\pi_{\mathfrak{N}} : \mathfrak{G} \to \mathfrak{N}$ the canonical projection and by $\mathscr{N} := \operatorname{Exp}(\mathfrak{N})$ and $\mathscr{M} := \operatorname{Exp}(\mathfrak{M})$ connected subgroups of G. Putting $\mathrm{N} := \mathrm{Ad}'\, \mathfrak{N}(x^*)$ and $\mathrm{F} := \mathrm{Ad}'\, \mathscr{M}(x^*)$ we define in the neighbourhood of x^* a LF

$$\mathfrak{F} = \{ \mathrm{Ad}'\, n(\mathrm{F}) \}_{n \in \mathscr{N}},$$

which is of the type (19). Moreover, its leaves are by Proposition 3 open subsets of affine subspaces of $\mathfrak{G}^*$. Of course, N is a LSM transversal to $\mathfrak{F}$.

Remark. — If a subalgebra $\mathfrak{M}$ satisfies (22) and additionally $\mathrm{Ad}\, \mathrm{G}_{x^*}(\mathfrak{M}) \subset \mathfrak{M}$, then there exists a global G-invariant foliation of $\mathfrak{O}_{x^*}$, whose Lagrangean distribution is given by $\mathrm{D}_{\mathrm{Ad}' g(x^*)} = \{ \widehat{\mathrm{Ad}' g(v)}_{\mathrm{Ad}' g(x^*)} : v \in \mathfrak{M} \}$.

Now we consider the following diagram:

$$\begin{array}{ccccccccc}
\mathfrak{N} \oplus \mathfrak{N}^* & & & & & & \mathrm{N} \times \mathfrak{G}^* & & \\
& & & & & & \cup & & \\
\cong \mathrm{T}^*\mathfrak{N} & \xleftarrow{\mathrm{Exp}^*} & \mathrm{T}^*\mathscr{N} & \xleftarrow{\varphi^*} & \mathrm{T}^*\mathrm{N} & \xrightarrow{j} & \mathrm{VN} & \xrightarrow{\exp} & \mathfrak{O}_{x^*} \subset \mathfrak{G}^* \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\mathfrak{N} & \xrightarrow{\mathrm{Exp}} & \mathscr{N} & \xrightarrow{\varphi = \mathrm{Ad}'(.)(x^*)} & \mathrm{N} & \xrightarrow{\mathrm{id}} & \mathrm{N} & \xrightarrow{\mathrm{id}} & \mathrm{N}
\end{array} \tag{23}$$

Here $\Phi = \exp \circ j$ is the symplectomorphism defined by (12). Since diffeomorphisms of base manifolds induce symplectomorphisms of cotangent bundles, we get a local symplectomorphism $\Psi : \mathfrak{N} \oplus \mathfrak{N}^* \to \mathfrak{O}_{x^*}$, defined by

$$\Psi := \exp \circ j \circ (\mathrm{Exp}^* \circ \varphi^*)^{-1}. \tag{24}$$

PROPOSITION 4. — We have

$$\Psi(w, w^*) = \mathrm{Ad}' e^w \{ x^* - \pi^*_{\mathfrak{N}} \circ \xi^{-1}(\mathrm{ad}' w)(w^*) \}, \tag{25}$$

where $\xi(z) = (e^z - 1)/z$ and $e^w \equiv \mathrm{Exp}(w)$.

Proof. — We put $\alpha = - \pi_{\mathfrak{N}} \circ \xi^{-1}(\mathrm{ad}' w)(w^*)$. Since $\alpha \in \mathfrak{M}^\perp$ and F is affine, it follows that $x^* + \alpha \in \mathrm{F}$ and $\mathrm{Ad}' e^w(x^* + \alpha) \in \mathrm{F}_{\mathrm{Ad}' e^w(x^*)}$.

We notice that

$$d\,\mathrm{Exp}_w(u) = \widehat{\widehat{\xi(\mathrm{ad}\, w)(u)}}, \tag{26}$$

see [*16*], where $\hat{\hat{v}}$ denotes the fundamental vector field on $\mathcal{N}$ with respect to the left multiplication by $\mathcal{N}$. Moreover,

$$d\varphi(\hat{\hat{v}}) = \hat{v}, \tag{27}$$

$v \in \mathfrak{N}$, by the construction of $\mathfrak{F}$. Using these facts, we show that

$$\mathrm{Exp}^* \circ \varphi^* \circ j^{-1} \circ \exp^{-1} \circ \Psi^{-1} = \mathrm{id}_{\mathrm{T}^*\mathfrak{N}},$$

where—of course—for Ψ one has to insert the right hand side of (25). Let $u \in \mathrm{T}_w\mathfrak{N} \cong \mathfrak{N}$, then we get

$$
\begin{aligned}
&(\mathrm{Exp}^* \circ \varphi^* \circ j^{-1} \circ \exp^{-1} (\Psi(w, w^*)))(u) \\
&\quad = (\mathrm{Exp}^* \circ \varphi^* \circ j^{-1}(\mathrm{Ad}' e^w(\alpha)))(u) && \text{(see (21))} \\
&\quad = (\varphi^* \circ j^{-1}(\mathrm{Ad}' e^w(\alpha)))(d\,\mathrm{Exp}_w(u)) \\
&\quad = (\varphi^* \circ j^{-1}(\mathrm{Ad}' e^w(\alpha)))(\widehat{\widehat{\xi(\mathrm{ad}\, w)(u)}}) && \text{(see (26))} \\
&\quad = (j^{-1}(\mathrm{Ad}' e^w(\alpha)))(\widehat{\xi(\mathrm{ad}\, w)(u)}) && \text{(see (27))} \\
&\quad = \omega_{\mathrm{Ad}' e^w(x^*)}(\mathrm{Ad}' e^w(\alpha), \xi(\mathrm{ad}\, w)(u)) && \text{(see (11))} \\
&\quad = - \mathrm{Ad}' e^w(\alpha)(\xi(\mathrm{ad}\, w)(u)) && \text{(see (17))} \\
&\quad = - \alpha(\xi(- \mathrm{ad}\, w)(u)) \\
&\quad = \xi^{-1}(\mathrm{ad}' w)(w^*)(\xi(- \mathrm{ad}\, w)(u)) \\
&\quad = w^*(\xi^{-1}(- \mathrm{ad}\, w) \circ \xi(- \mathrm{ad}\, w)(u)) \\
&\quad = w^*(u). \qquad \square
\end{aligned}
$$

Now, let $(e_1, \ldots, e_n)$ be a basis in $\mathfrak{N}$ and $(\mathrm{Q}_i, \mathrm{P}_i)$ the induced canonical coordinates on $\mathfrak{N} \oplus \mathfrak{N}^*$. Then $(\mathrm{Q}_i \circ \Psi^{-1}, \mathrm{P}_i \circ \Psi^{-1})$ are canonical coordinates on $\mathfrak{O}_{x^*}$. We obtain from (25)

PROPOSITION 5. — The canonical realization (3) expressed in canonical coordinates induced by Ψ is given by

$$f_v(\mathrm{Q}_i, \mathrm{P}_i) = \mathrm{Ad}' e^w \{ x^* - \pi^*_{\mathfrak{N}} \circ \xi^{-1}(\mathrm{ad}' w)(w^*) \} (v), \tag{28}$$

where $w = \Sigma \mathrm{Q}_i e_i$, $w^* = \Sigma \mathrm{P}_i (e_i)^*$. $\square$

This realization is polynomial, iff all operators $\mathrm{ad}' w$ are nilpotent on $\mathfrak{G}^*$.

Let $i : \mathfrak{N} \to \mathfrak{G}$ be the natural embedding. A simple calculation shows that (28) takes the form

$$f_v(Q_i, P_i) = \mathrm{Ad}' e^w \{ x^* - (i \circ \pi_{\mathfrak{N}})^* \circ \xi^{-1}(\mathrm{ad}' w)(\pi_{\mathfrak{N}}^* w^*) \} (v) . \qquad (29)$$

4. CANONICAL REALIZATIONS OF $gl(n, \mathrm{R})$

Every functional x^* on $gl(n, \mathrm{R})$ can be represented by a real $n \times n$-matrix $X \in M_{n \times n} : x^* = \mathrm{Tr}\, X(\cdot)$. This identification implies an identification of Ad- and Ad'-orbits. Here we consider orbits consisting of matrices conjugated to diagonal matrices

$$X = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

Now, inserting

$$w = Q \quad \text{and} \quad \pi_{\mathfrak{N}}^* w^* = \mathrm{Tr}\, P(\cdot) \qquad (30)$$

into (29), we get

$$f_v(Q_i, P_i) = \mathrm{Tr} \{ e^Q (X - (i \circ \pi_{\mathfrak{N}})^+ \circ \xi^{-1}(\mathrm{ad}\, Q)(P)) e^{-Q} v \} , \qquad (31)$$

where $(i \circ \pi_{\mathfrak{N}})^+$ is the operator adjoint to $i \circ \pi_{\mathfrak{N}}$ with respect to the bilinear form $\mathrm{Tr}\,(XY)$.

4.1. The case of minimal orbit dimension.

Let $\lambda \equiv \lambda_1 = \lambda_2 = \ldots = \lambda_{n-1} \neq \lambda_n$. Then the stabilizer of X is $GL(n-1, \mathrm{R}) \times \mathrm{R}_*$. Thus

$$\mathfrak{O}_x = GL(n, \mathrm{R})/(GL(n-1, \mathrm{R}) \times \mathrm{R}_*) \qquad (32)$$

and $\dim \mathfrak{O}_x = 2(n-1)$.

A decomposition (22) of $gl(n, \mathrm{R})$ with respect to X is given by

$$\begin{aligned} \mathfrak{N} &:= \{ Y \in M_{n \times n} : Y_{ij} = 0, \quad i = 1, \ldots, n-1 \quad \text{or} \quad i = j = n \} \\ \mathfrak{M} &:= \{ Y \in M_{n \times n} : Y_{ni} = 0, \quad i = 1, \ldots, n-1 \} . \end{aligned} \qquad (33)$$

As a basis in $\mathfrak{N}$ we take $\{ E_{ni} \}_{i=1,\ldots,n-1}$, where $\{ E_{ij} \}$ is the standard basis in $M_{n \times n}$.

Then

$$Q = \sum_{i=1}^{n-1} Q_i E_{ni} ,$$

$$(i \circ \pi_{\mathfrak{N}})^+(E_{ij}) = \begin{cases} E_{in} & \text{if } j = n \text{ and } i < n , \\ 0 & \text{otherwise} , \end{cases}$$

and, therefore,

$$P = \sum_{i=1}^{n-1} P_i E_{in}, \qquad \text{see (30)}.$$

PROPOSITION 6. — The canonical realization (29) related to the minimal orbit $\mathfrak{O}_x$ is given by

$$\begin{aligned} f_{ij} &= Q_i P_j + \lambda \cdot \delta_{ij} \\ f_{ni} &= - P_i \\ f_{in} &= Q_i(Q_k P_k + \lambda - \lambda_n) \\ f_{nn} &= - Q_k P_k + \lambda_n, \qquad i, j = 1, \ldots, n-1, \end{aligned} \tag{34}$$

where $f_{ij} \equiv f_{E_{ij}}$, summation over k.

Proof. — Formulae (34) are obtained from (31), using that in this case $\xi^{-1}(\text{ad } Q)(P) = P$. □

4.2. The case of maximal orbit dimension.

Let $\lambda_i \neq \lambda_j$, $i, j = 1, \ldots, n$, $i \neq j$. Then the stabilizer of X is $(R_*)^n$ and

$$\mathfrak{O}_x = GL(n, R)/(R_*)^n, \qquad \dim \mathfrak{O}_x = 1/2n(n-1).$$

A decomposition (22) of $gl(n, R)$ is given by:

$$\begin{aligned} \mathfrak{N} &:= \{ Y \in M_{n \times n} : Y_{ij} = 0, \quad i \leqslant j \}, \\ \mathfrak{M} &:= \{ Y \in M_{n \times n} : Y_{ij} = 0, \quad i > j \}. \end{aligned} \tag{35}$$

As a basis in $\mathfrak{N}$ we take $\{ E_{ij} \}_{i>j}$. Then

$$Q = \sum_{i>j} Q_{ij} E_{ij},$$

$$(i \circ \pi_{\mathfrak{N}})^+(E_{ij}) \equiv \pi_{\triangledown}(E_{ij}) = \begin{cases} E_{ij} & \text{if } i < j \\ 0 & \text{otherwise}, \end{cases}$$

$$P = \sum_{i>j} P_{ij} E_{ji}, \qquad \text{see (30)}.$$

PROPOSITION 7. — The canonical realization (29) related to the maximal orbit $\mathfrak{O}_x$ is given by

$$f_{ij}(Q_{kl}, P_{kl}) = \{ e^Q(X - \pi_{\triangledown} \circ \xi^{-1}(\text{ad } Q)(P)) e^{-Q} \}_{ji}. \qquad □$$

This realization is polynomial, because Q and ad Q are nilpotent. It becomes simpler, if we perform a canonical transformation induced by a change of coordinates in $\mathfrak{N}$:

$$q = 1 - e^{-Q}. \tag{37}$$

The relation for the canonically conjugated momenta follows from

$$\operatorname{Tr} P dQ = \operatorname{Tr} p dq. \tag{38}$$

But $$dq = - de^{-Q} = e^{-Q}\xi(\operatorname{ad} Q)(dQ).$$

Thus $$P = \pi_{\nabla}(\xi(- \operatorname{ad} Q)(p)e^{-Q})$$

and $$p = \pi_{\nabla}(\xi^{-1}(- \operatorname{ad} Q)(P)e^{Q}). \tag{39}$$

Substituting (37) and (39) into (36) we get:

PROPOSITION 8. — Let $q = \sum_{i>j} q_{ij}E_{ij}$ and $p = \sum_{i>j} p_{ij}E_{ji}$. Then the canonical realization (29) is in coordinates (q, p) given by

$$f_{ij}(q_{kl}, p_{kl}) = \{ (1 - q)^{-1}(X - p + \pi_{\nabla}(q \cdot p))(1 - q) \}_{ji}. \tag{40}$$

□

Of course, this realization is also polynomial. For example, if $n = 3$, we put

$$q = \begin{bmatrix} 0 & 0 & 0 \\ q_1 & 0 & 0 \\ q_2 & q_3 & 0 \end{bmatrix}, \qquad p = \begin{bmatrix} 0 & p_1 & p_2 \\ 0 & 0 & p_3 \\ 0 & 0 & 0 \end{bmatrix}$$

and get

$$\begin{aligned}
f_{11} &= p_1q_1 + p_2q_2 + \lambda_1 \\
f_{12} &= p_3q_2 + q_1(p_1q_1 + \lambda_1 - \lambda_2) \\
f_{13} &= q_2(p_1q_1 + p_2q_2 + p_3q_3 + \lambda_1 - \lambda_3) + q_1q_3(p_1q_1 + \lambda_1 - \lambda_2) \\
f_{21} &= - p_1 + p_2q_3 \\
f_{22} &= - p_1q_1 + p_3q_3 + \lambda_2 \\
f_{23} &= - q_3(p_1q_1 - p_2q_2 - p_3q_3 - \lambda_2 + \lambda_3) - p_1q_2 \\
f_{31} &= - p_2 \\
f_{32} &= - p_3 \\
f_{33} &= - p_2q_2 - p_3q_3 + \lambda_3 .
\end{aligned}$$

For the canonical realization (40) we can give a recurrency formula due to passing from $gl(n, R)$ to $gl(n + 1, R)$. For this purpose we take $\{ E_{ij} \}_{i,j=1,\dots,n+1}$ as a basis in $gl(n + 1, R)$, $\{ E_{ij} \}_{i,j=1,\dots,n}$ as a basis in the subalgebra $gl(n, R)$.

Let

$$\tilde{q} = \sum_{j<i\leq n} q_{ij}\mathrm{E}_{ij} + \sum_{i=1}^{n} q_i\mathrm{E}_{n+1,i}$$

and

$$\tilde{p} = \sum_{j<i\leq n} p_{ij}\mathrm{E}_{ji} + \sum_{i=1}^{n} p_i\mathrm{E}_{i,n+1}\,.$$

Then we get

$$(1-\tilde{q})^{-1}(\tilde{\mathrm{X}}-\tilde{p}+\tilde{\pi}_{\nabla}(\tilde{q}\cdot\tilde{p}))(1-\tilde{q})$$

$$= \left[\begin{array}{c|c} (1-q)^{-1} & 0 \\ \hline \vec{q}(1-q)^{-1} & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} \mathrm{X}-p+\pi_{\nabla}(q\cdot p) & \begin{array}{c}-\vec{p}\\ +q\vec{p}\end{array} \\ \hline 0 & \lambda_{n+1} \end{array}\right] \cdot \left[\begin{array}{c|c} (1-q) & 0 \\ \hline -\vec{q} & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} (1-q)^{-1}(\mathrm{X}-p+\pi_{\nabla}(q\cdot p))(1-q)+\vec{p}\,\vec{q} & -\vec{p} \\ \hline \vec{q}(1-q)^{-1}(\mathrm{X}-p+\pi_{\nabla}(q\cdot p))(1-q)+\vec{q}(\vec{p}\,\vec{q}-\lambda_{n+1}) & \begin{array}{c}-\vec{q}\cdot\vec{p}\\ +\lambda_{n+1}\end{array} \end{array}\right]$$

where $\vec{q}\cdot\vec{p}$ is a number and $\vec{p}\,\vec{q}$ a matrix, $\vec{q} = (q_i)$ and $\vec{p} = (p_i)$. Thus, the realization F of $gl(n+1, \mathrm{R})$ is obtained from the realization f of $gl(n, \mathrm{R})$ and $(q_i, p_i)_{i=1,\dots,n}$ in the following form:

$$\begin{aligned} \mathrm{F}_{ij} &= q_i p_j + f_{ij} \\ \mathrm{F}_{n+1,i} &= -\,p_i \\ \mathrm{F}_{i,n+1} &= q_i(p_k q_k - \lambda_{n+1}) + q_k f_{ik} \\ \mathrm{F}_{n+1,n+1} &= -\,p_k q_k + \lambda_{n+1}, \qquad i,j,k = 1,\dots,n, \quad \text{summation over } k. \end{aligned}$$

This recurrency formula is similar to that obtained by Havliček and Lassner [*17*]. We found that the underlying geometry of such recurrencies is that of symplectic structures on bundles associated with $\mathrm{G} \to \mathrm{G}/\mathrm{G}_x$, see [*18*].

5. FINAL REMARKS

It would be interesting to apply our method to other classical Lie groups, too. One could try to classify orbittypes admitting a decomposition (22). Canonical realizations related to those orbittypes are just given by formula (29). Up to now we have—besides for $gl(n, \mathrm{R})$—only results for $sp(n, \mathrm{R})$, see [*19*].

One could also try to generalize our method—at least to the case of realizations related to foliations (19). In this case in general (21) does not hold and one has to solve (15) to find the symplectomorphism Φ explicitely.

The material contained in this paper is based on two KMU-preprints [*20*] [*21*], where one can find further technical details.

ACKNOWLEDGMENTS

We would like to thank Dr. W. Lassner (Leipzig) and Prof. D. Lehmann (Lille) for stimulating discussions.

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(Manuscrit reçu le 8 novembre 1984)

(Version révisée, reçue le 30 mai 1985)