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Maxwell's equations in the Debye potential formalism

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ABSTRACT. — We propose a method for computing the electromagnetic test-field created by a given distribution of charges and currents using the Debye potential formalism in curved space-times. All this framework has been applied explicitly to the Schwarzschild case.

RÉSUMÉ. — On propose une méthode de calcul du champ électromagnétique créé par une distribution donnée de charges et de courants en utilisant le formalisme du potentiel de Debye dans un espace temps courbe. Ce cadre est appliqué explicitement au cas de Schwarzschild.

I. INTRODUCTION

The advantages of the Debye or two-component Hertz potential formalism for computing the electromagnetic field in a source-free region in curved space-times was presented in Cohen and Kegeles work [1]. They showed that the Hertz formalism could be extended to all curved space-times, and that the Debye formalism could be extended to every algebraically special geometry, in the sense of Petrov, which admits a shear-

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free congruence of nul geodesics along the repeated principal direction of the Weyl tensor (in the electromagnetic test-field approximation).

In this work, we propose a way to incorporate the part related with the electromagnetic field sources into this formalism in such a way that it is possible to compute the Debye potential created by a given distribution of charges and currents and, through it, the corresponding electromagnetic field. We shall see that in the space-times where the source free Debye formalism can be applied, this extension will also be feasible.

Our approach is based on a generalization to curved space-times of the potential current formalism introduced by Laporte and Uhlenbeck [2], and used by Nisbet [3], for the Minkowskian case. Stephani [4] also presented a method to deal with the Maxwell equations with sources in the Debye potentials formalism in General Relativity. However, our approach presents the advantage of stressing the geometrical meaning of the objects we are dealing with; this makes our procedure clearer than Stephani's and also easier to handle from the computational viewpoint.

In part II we prepare the Debye potential formalism, by using the exterior calculus in Riemannian manifolds [5], in order to obtain the complete differential equation for the potential in the Newman-Penrose (NP) framework. The stream potentials and the 1-forms associated with the gauge transformations of the third kind, which were already present in the special relativistic treatment of the problems, are generalized to curved space-times and their relation with the electromagnetic 4-current will be shown.

In part III, we explain how to compute the stream potential from a given 4-current. Finally, in section IV we will apply this method to the Schwarzschild space-time.

II. MAXWELL'S EQUATIONS, HERTZ'S AND DEBYE'S POTENTIALS

The Maxwell equations in the framework of General Relativity and using the differential forms language can be written as

$$df = 0 \tag{2.1 a}$$

$$\delta f = -4\pi j \tag{2.1 b}$$

where f is the electromagnetic tensor (a 2-form), j the electromagnetic current (a 1-form), d the exterior derivative operator and δ the coderivative [5] [6].

Using the Poincaré lemma in (2.1 a) [7] we can define a 1-form A (four potential), related with f by $f = dA$. The Lorentz gauge can be written as

$$\delta A = 0 \tag{2.2}$$

and the Maxwell equations boil down to the following system of second order differential equations

$$\Delta A = -4\pi j \quad (2.3)$$

where $\Delta = d\delta + \delta d$ is the Laplace-Beltrami operator. Using again the Poincaré lemma in (2.2) we can define a 2-form π , called the Hertz potential [3] such that

$$A = \delta\pi \quad (2.4)$$

The four-current satisfies the continuity equation

$$\delta j = 0$$

and so, it is possible to find a 2-form q , called the stream-potential [2] such that

$$4\pi j = \delta q \quad (2.5)$$

Inserting (2.4) and (2.5) into (2.3), we obtain

$$\delta[\Delta\pi + q] = 0 \quad (2.6)$$

whose most general solution is

$$\Delta\pi + q = \delta\tilde{G}$$

where G is an arbitrary 1-form, and « \sim » is the Hodge operator [5]. However, the new stream potential q' defined by

$$q' = q - \delta\tilde{G}$$

generates the same 4-current as q .

Therefore, we can write without any loss of generality that the solution of (2.6) is

$$\Delta\pi + q = 0 \quad (2.7)$$

which is the equation that the Hertz potential has to satisfy.

The steps that have to be carried out in order to find the electromagnetic field f created by a 4-current j are the following:

- i) From a given 4-current j one has to find the stream potential q , as a particular solution of (2.5).
- ii) From (2.7) we can compute the Hertz potential π corresponding to the stream potential q once the suitable boundary conditions have been given. The latter are derived from those of the actual physical problem (generally defined in terms of the electromagnetic field components).
- iii) Eventually, we can obtain the electromagnetic field by combining the A and π definitions. That is to say:

$$f = d\delta\pi \quad (2.8)$$

To solve (2.7) is still very involved (as much as to solve (2.3)). Nonetheless,

the fact that physics is contained in the electric and magnetic fields and not in the Hertz potential shall give us a lot of freedom, that will allow us to simplify the problem as we will see in what follows.

Let us consider the following transformations:

$$\pi' = \pi + \delta\tilde{\Gamma} + d\Lambda \quad (2.9)$$

$$q' = q + \delta\tilde{G} + dL \quad (2.10)$$

where Γ , Λ , G and L are 1-form. These transformations leave eq. (2.7) invariant.

$$\Delta\pi' + q' = 0 \quad (2.11)$$

provided that the following relations between G , L , Γ and Λ

$$\Delta\Gamma + G = d\xi_1 \quad (2.12 a)$$

$$\Delta\Lambda + L = d\xi_2 \quad (2.12 b)$$

are fulfilled (where ξ_1 and ξ_2 are two arbitrary functions).

In this case, the electromagnetic tensor will be obtained by substituting (2.9) in (2.8)

$$f = d\delta[\pi' - (\delta\tilde{\Gamma} + d\Lambda)]$$

but, taking into account the properties of the operators, d and δ and (2.12) we can easily derive the following relation

$$f = d[\delta\pi' + L] \quad (2.13)$$

Appealing to the Hodge operator properties, the space of 2-forms with complex values on \mathcal{M}_4 can be separated in direct sum of the self-dual and antiself-dual 2-form subspaces, corresponding to the eigenvalues $-i$ and $+i$ of the Hodge operator, respectively.

So, $\forall \pi \in \Lambda^2(\mathcal{M}_4, \mathbb{C})$ we can write

$$\pi = \frac{1}{2}(\overset{\star}{\pi} + \overset{\circ}{\pi}) \quad (2.14)$$

where $\overset{\star}{\pi} = \pi + i\tilde{\pi}$ is the self-dual part and $\overset{\circ}{\pi} = \pi - i\tilde{\pi}$ the anti-self dual one.

In particular, a real 2-form is completely determined when one of the two parts is given (for instance the self-dual one). Indeed $\pi \in \Lambda^2(\mathcal{M}_4, \mathbb{R}) \Rightarrow \overset{\circ}{\pi} = \overset{\star}{\pi}$. Using the decomposition (2.14), it can easily be seen that eq. (2.11) is equivalent to

$$\frac{1}{2}[\Delta\overset{\star}{\pi}' + \overset{\star}{q} + (dQ)\overset{\star}{\pi}] + \frac{1}{2}[\Delta\overset{\circ}{\pi}' + \overset{\circ}{q} + (dK)\overset{\circ}{\pi}] = 0$$

where $Q = (L - iG)$ and $K = \overline{Q}$.

The first three and the last three terms are respectively self-dual and antiself-dual bivectors. We can split (2.11) in two equations: one for the

self-dual part and the other for the antiself-dual one. The latter will be the complex conjugate of the former. So,

$$\Delta \overset{\star}{\pi}' + \overset{\star}{q} + (dQ)^\star = 0 \tag{2.15}$$

$$\Delta \overset{\circ}{\pi}' + \overset{\circ}{q} + (dK)^\circ = 0 \tag{2.16}$$

and one has to solve only one of them (the first one for instance) since $\overset{\circ}{\pi}'$ is the complex conjugate of $\overset{\star}{\pi}'$.

The electromagnetic tensor of (2.13) will be now

$$f = \frac{1}{2} d[\delta \overset{\star}{\pi}' + Q] + \text{complex conjugate} \tag{2.17}$$

Working in the framework of the NP formalism [8] and after choosing the null tetrad (m, \bar{m}, l, k) [9, § 3.2] we have automatically a basis of the bivectors space we will denote by: $U, W, V, \bar{U}, \bar{W}, \bar{V}$ [9, § 3.4] respectively, $Z^I, \bar{Z}^I; I, J = 0, 1, 2$). The first three elements become a basis of the self-dual bivectors subspace and the last three a basis of the antiself-dual one.

The expression of the Hertz potential in this basis is

$$\pi' = \frac{1}{2} \sum_{r=0}^2 \psi_r Z^r + \frac{1}{2} \sum_{r=0}^2 \bar{\psi}_r \bar{Z}^r.$$

Now we can wonder if it is always possible to choose Q so that the corresponding Hertz potential (which is the solution of the differential equation with the suitable boundary conditions) will have a unique non-null component. That is to say

$$\overset{\star}{\pi}' = \psi Z^I \tag{2.18}$$

The 0-form ψ is the complex Debye potential which coincides with the two real Debye potentials of the classical electromagnetism [3]. For certain types of space-time, Cohen and Kegeles showed that in the study of the free fields, this choice was possible, We shall have to study the restrictions that arise when we introduce the stream potentials. Inserting (2.18) into (2.15) we obtain

$$\psi(\Delta Z^I)_{ab} + (\Delta \psi)Z^I_{ab} - 2\psi'^c Z^I_{ab;c} + \overset{\star}{q}_{ab} + (dQ)^\star_{ab} = 0 \tag{2.19}$$

This equation (which holds for self-dual bivectors) is equivalent to say that the projections of the left-hand side on the three elements of the bivector basis ($Z^J, J = 0, 1, 2$) must be zero. But, since there is only one value of $J = I_0$ such that $\langle Z^I, Z^{I_0} \rangle \neq 0$ (*), there will be only one projection that will contain the second derivatives of ψ (due to the term $(\Delta \psi)Z^I$).

(*) If $A, B \in \Lambda^2, \langle A, B \rangle = \frac{1}{2} A_{\alpha\beta} B^{\alpha\beta}$.

The other two will only contain first derivatives and we shall have to select a suitable Q in order to make these two projections vanish. The first equation, which will include second order derivatives of ψ , is called the « Debye potential equation ».

The 1-form Q can be splitted in two parts: the first part Q_ψ will have to cancel the parts that depend on ψ and the second one will do the same with q , so that

$$Q = Q_\psi + Q_q$$

The existence condition of Q_ψ and Q_q relies on the choice of π^* we make, for, according to (2.18). Z^I can be any of the three bivectors of the basis.

For a given Z^I , the two equations the 1-form Q_ψ must satisfy are:

$$\psi \langle \Delta Z^I, Z^{I_0} \rangle - Z \langle \nabla_\psi Z^I, Z^{I_0} \rangle + \langle Z dQ_\psi, Z^{I_0} \rangle = 0 \quad (2.20)$$

and the two of Q_q are

$$\langle \overset{*}{q} + (dQ_q)^*, Z^{I_0} \rangle = 0 \quad (2.21)$$

where I_0 can take the two values different from I_0 .

The Debye potential equation will be then obtained by inserting the values of Q_ψ and Q_q in the expression

$$\langle \Delta(\psi Z^I) + \overset{*}{q} + (dQ)^*, Z^{I_0} \rangle = 0 \quad (2.22)$$

From now on we will use NP formalism [9, § 7] to write all expressions.

a) If we choose $\pi^* = \psi U$ it is easy to show that if the space-time is such that the spin coefficients λ and ν and the Weyl tensor component Ψ_3 are zero [8] then Q_ψ exists as a solution of the two equations (2.20) and is:

$$Q_{\psi_a} = 2(-\mu \bar{m}_a + \pi l_a) \psi \quad (2.23)$$

As far as Q_q is concerned, one can see that if we make $Q_q^1 = Q_q^3 = 0$ and taking into account that $\lambda = \nu = \Psi_3 = 0$, the two equations (2.22) become reduced to

$$q_2 + 2(-\bar{\delta} + \bar{\tau} - \alpha - \bar{\beta})Q_q^4 = 0 \quad (2.24 a)$$

$$-q_1 + (-\bar{\delta} - \bar{\beta} + \alpha + \pi + \bar{\tau})Q_q^2 + (D - \bar{\rho} - \rho + \bar{\varepsilon} + \varepsilon)Q_q^4 = 0. \quad (2.24 b)$$

The first one is a first order differential equation and thus its compatibility is assured. Since we can get Q_q^4 from the first equation, the same reasoning is valid for the second one. Therefore, if we take $Q_q^1 = Q_q^3 = 0$ and Q_q^2 and Q_q^4 are particular solutions of (2.24), equations (2.21) become identities.

With these values of Q_ψ and Q_q we may write the Debye potential equation

$$\begin{aligned} & \{ (D - \varepsilon + \bar{\varepsilon} - \bar{\rho})(\Delta - 2\gamma - \mu) - (\delta - \bar{\alpha} - \beta + \bar{\pi})(\bar{\delta} - 2\alpha - \pi) \} \psi = \\ & = -\frac{1}{2} q_0 + (-D - \bar{\varepsilon} + \varepsilon + \bar{\rho})Q_q^2 + \chi Q_q^4. \quad (2.25) \end{aligned}$$

b) If we make the choice $\pi^* = \psi W$ we will have to carry out the same steps as in the previous case and the Debye potential equation will be

$$\begin{aligned} & \{ -(\Delta - \gamma - \bar{\gamma} + \bar{\mu} - \mu)D + (\bar{\delta} + \bar{\beta} - \alpha - \bar{\tau} - \pi)\delta \} \psi = \\ & = \frac{1}{2} \{ q_1 - (-\Delta + \mu - \bar{\mu} + \bar{\gamma} + \gamma)Q_q^3 - (D - \bar{\rho} + \rho + \bar{\varepsilon} + \varepsilon)Q_q^4 \} \end{aligned} \quad (2.26 a)$$

where, now Q_ψ is the following

$$Q_\psi = 2(-\pi m - \bar{\tau} \bar{m} + \rho l + \mu k)\psi \quad (2.26 b)$$

with the condition $\lambda = \nu = \chi = \sigma = 0$. As far as Q_q is concerned, taking into account the previous condition and making $Q_q^1 = Q_q^2 = 0$, the other two components will be a particular solution of the equations

$$\frac{1}{2} q_2 + (-\bar{\delta} + \bar{\tau} - \alpha - \bar{\beta})Q_q^4 = 0 \quad (2.27 a)$$

$$\frac{1}{2} q_0 - (-\delta + \bar{\pi} + \beta + \bar{\alpha})Q_q^3 = 0 \quad (2.27 b)$$

c) When $\pi' = \psi V$ we can use « the modified calculus » [9, § 7.3] applied to option a). The Debye potential equation is:

$$\begin{aligned} & \{ (\Delta - \bar{\gamma} + \gamma + \bar{\mu})(D + 2\varepsilon + \rho) - (\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})(\delta + 2\beta + \tau) \} \psi = \\ & = -\frac{1}{2} q_2 - (-\Delta + \bar{\gamma} - \gamma - \bar{\mu})Q_q^1 - \nu Q_q^3. \end{aligned} \quad (2.28 a)$$

where Q_ψ takes the value

$$Q_\psi = 2(\rho m - \tau k)\psi \quad (2.28 b)$$

with the condition that $\chi = \sigma = \Psi_1 = 0$. A particular solution of (2.21) for Q_q can be obtained making $Q_q^2 = Q_q^4 = 0$ and taking Q_q^1 and Q_q^3 as particular solutions of the following equations

$$\frac{1}{2} q_0 - (-\delta - \bar{\pi} + \beta + \bar{\alpha})Q_q^3 = 0 \quad (2.29 a)$$

$$-q_1 + (\delta + \beta - \bar{\alpha} + \bar{\pi} + \tau)Q_q^1 + (-\Delta + \mu - \bar{\mu} + \bar{\gamma} + \gamma)Q_q^3 = 0 \quad (2.29 b)$$

III. STREAM POTENTIALS

The calculation method of the stream potential q as a particular solution of (2.5) has to be deeply studied since it is a crucial step in the whole process.

If we express the equation in coordinates we obtain

$$\partial_\beta \hat{q}^{\beta\alpha} = \hat{j}^\alpha 4\pi \quad \alpha, \beta = 0, 1, 2, 3. \quad (3.1)$$

where $\hat{q} = \sqrt{-g} q^{\alpha\beta}$, $\hat{j}^\alpha = \sqrt{-g} j^\alpha$ and $g = \det(g_{\alpha\beta})$.

As we are only searching for a particular solution of this equation we will choose $\hat{q}^{ij} = 0$ ($i, j = 1, 2, 3$). We may write (3.1) as

$$4\pi \hat{j}^0 = \partial_i \hat{q}^{i0} \quad (3.2 a)$$

$$4\pi \hat{j}^i = \partial_0 \hat{q}^{0i} \quad (3.2 b)$$

Since in what follows we will use the Fourier transform in order to solve the Debye potential differential equations, we are interested in obtaining the Fourier transform of the stream potential q components which are different from zero.

Let us expand them

$$\hat{j}^\alpha(x^\mu) = \int_{-\infty}^{+\infty} d\omega e^{-i\omega t} \hat{j}^\alpha(\omega, x^k) \quad (3.3 a)$$

$$\hat{q}^{0i}(x^\mu) = \int_{-\infty}^{+\infty} d\omega e^{-i\omega t} \hat{q}^{0i}(\omega, x^k) \quad (3.3 b)$$

Inserting these expressions in (3.2 b) and taking the coordinate associated with the subindex « 0 » as the t variable used in the Fourier transform, we obtain

$$\frac{1}{4\pi} \cdot \hat{q}^{0i}(\omega, x^k) = \frac{i}{\omega} j^i(\omega, x^k) + \hat{K}^i(x^k) \delta(\omega) \quad (3.4)$$

where \hat{K}^i are three functions that will be determined by inserting (3.4) in (3.2). Working along these lines and isolating the term that contains \hat{K}^i we obtain

$$[\partial_i \hat{K}^i(x^k)] \delta(\omega) = - \int_{-\infty}^{+\infty} dt e^{i\omega t} \left[\hat{j}^0(x^\mu) - \frac{i}{\omega} \partial_0 \hat{j}^0(x^\mu) \right] \quad (3.5)$$

where we have used the inverse Fourier transform and the continuity equation.

To interpret this last equation we will place ourselves in two different situations:

a) If we assume that $j^\alpha(x^\mu) = j^\alpha(\omega, x^k) e^{-i\omega_1 t}$ the right hand side of (3.5) is zero and we can thus take $\hat{K}^i \equiv 0$.

b) If, on the contrary, $j(x^\mu) = A(x^k)$ (3.5) becomes:

$$\partial_i \hat{K}^i(x^k) = - \sqrt{-g} A(x^k)$$

and, therefore, $\hat{K}^i(x^k)$ will be a particular solution of this equation.

The appearance of the term $\hat{K}^i(x^k)$ in the stream potential Fourier transform expression is, thus, related with the static part of the four-current

component j^0 ; that is to say, it is related with the static part of the charge density.

Finally, as far as Q_q is concerned it is necessary to say that the space-time has to be specified in order to try to find a particular solution of the differential equations that have to be satisfied (these equations have been written in the previous section).

IV. DEBYE POTENTIAL IN A SCHWARZSCHILD SPACE-TIME

Now, we will study the particular case when the space-time, in which the charge and current distribution moves and in which we will measure the electromagnetic test-field, is a Schwarzschild spacetime with the metric:

$$ds^2 = a^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) - adt^2 \tag{4.1}$$

where
$$a = 1 - \frac{2M}{r}$$

The null tetrad we will use is the usual one:

$$\begin{aligned} m^\alpha &= \frac{1}{\sqrt{2r}} \left\{ 0, 0, 1, \frac{-i}{\sin\theta} \right\} \\ l^\alpha &= \frac{1}{2} \{ 1, -a, 0, 0 \} \\ k^\alpha &= \{ a^{-1}, 1, 0, 0 \}. \end{aligned} \tag{4.2}$$

The spin coefficients different from zero are:

$$\rho = -\frac{1}{r}; \quad \gamma = -\frac{M}{2r^2}; \quad \mu = -\frac{a}{\sqrt{2r}}; \quad \alpha = -\beta = -\frac{1}{2\sqrt{2r}} \cot\theta \tag{4.3}$$

a) Homogeneous equation.

We will study the Debye potential differential equation when q is zero and so we can choose Q_q to be also zero.

In all three options the Debye potential satisfies second order partial differential equations. Using the null tetrad (4.2) one immediately obtains the differential equations, in terms of the coordinates (t, r, θ, φ) , which can be decoupled.

Appealing to the behaviour of the self-dual bivectors basis elements U, V, W under rotations we have that for every option, it will be advisable to express the Debye potential angular part in terms of the spin-s spherical

harmonics with the suitable spin weight [10] [11], namely $s = -1$ for a), $s = 0$ for b) and $s = +1$ for c).

This is the reason why, from now on, with the subindex s in the Debye potential we will denote what option we are working in. The expansion we propose is then

$${}_s\psi(t, r, \theta, \varphi) = \int_{-\infty}^{+\infty} d\omega \sum_{l,m} {}_s\mathcal{R}_{lm}(\omega, r) {}_sY_{lm}(\theta, \varphi) e^{-i\omega t} \quad (4.4)$$

Inserting (4.4) into (2.25), (2.26) and (2.28) (making $q \equiv Q_q \equiv 0$) and for a given value of l, m and w we obtain the ordinary differential equations for the radial parts ${}_s\mathcal{R}_{lm}(\omega, r)$ of the potential [16] [12]

$$\left\{ \frac{d^2}{dy^2} + \alpha^2 - V_s(\omega, y) \right\} {}_sU_{lm}(\omega, y) = 0 \quad (4.5)$$

where

$$y = \frac{r}{2M} + \ln\left(\frac{r}{2M} - 1\right) \quad (4.6)$$

$$V_s = l(l+1) \frac{x-1}{x^3} + s^2 \frac{4x-3}{4x^4} - is\alpha \frac{2x-3}{x^2} \quad (4.7)$$

$$x = \frac{r}{2M}, \quad \alpha = 2M\omega$$

and the three new unknown variables ${}_sU_{lm}(\omega, r)$ are related with the former by

$${}_s\mathcal{R}_{lm}(\omega, x) = \left(\frac{x}{x-1}\right)^{-s/2} {}_sU_{lm}(\omega, x) \quad (4.8)$$

At first sight, equations (4.5) have a simple appearance but this is so because the complicated part is in the functions V_s , which will be called effective potentials.

In the case $s = 0$, we have the special feature that V_0 is a real function of r and so gives us a simpler differential equation. On the other hand, this equation has been studied using numerical methods [13], the JWKB approximation [14] [12] [15] or by means of series expansions [16].

In the other two cases $s = \pm 1$, it is difficult to find solutions (for instance, by means of the JWKB method) of (4.5). Nevertheless, we can avoid the straight search for the solutions, since, as we will see in what follows, it is possible to find them from the $s=0$ case solutions.

In the source-free case, the electromagnetic tensor can be obtained

$$f = \frac{1}{2} \left\{ d[\delta\pi^* + Q_\psi] + \text{c. c.} \right\}$$

For each choice of $\pi^*(-{}_1\psi, U, {}_0\psi, W, {}_+1\psi, V)$ we can find the associated 1-form

${}_s Q_\psi$ (where the subindex s points out in what option we are working) by following the path described in chapter II.

The free options have to give the same electromagnetic tensor f . Therefore we can write the following expressions:

$$d[\delta(-{}_1\psi \cdot U) + {}_{-1}Q_\psi] + \text{c. c.} = d[\delta({}_0\psi \cdot W) + {}_0Q_\psi] + \text{c. c.} \quad (4.9 a)$$

$$d[\delta({}_+{}_1\psi \cdot V) + {}_+{}_1Q_\psi] + \text{c. c.} = d[\delta({}_0\psi \cdot W) + {}_0Q_\psi] + \text{c. c.} \quad (4.9 b)$$

$$d[\delta(-{}_1\psi \cdot U) + {}_{-1}Q_\psi] + \text{c. c.} = d[\delta({}_+{}_1\psi \cdot V) + {}_+{}_1Q_\psi] + \text{c. c.} \quad (4.9 c)$$

Using the expansion (4.4) of the Debye potential ${}_s\psi$, together with (2.23), (2.26 b) and (2.28 b) we get from (4.9 a, b) and for each value of l, m and ω , the following relations

$${}_{-1}\mathcal{R}_{lm}(\omega, r) = -2\sqrt{2}\lambda^{-1/2}r\mathcal{D}_0\mathcal{R}_{lm}(\omega, r) \quad (4.10)$$

$${}_+{}_1\mathcal{R}_{lm}(\omega, r) = -\sqrt{2}\lambda^{-1/2}ar\mathcal{D}_0^+\mathcal{R}_{lm}(\omega, r) \quad (4.11)$$

where

$$\mathcal{D} \equiv \partial_r - i\omega a^{-1}; \quad \mathcal{D}^+(\omega) = \mathcal{D}(-\omega) \quad (4.12)$$

and from (4.9 c) we obtain the well-known Teukolsky identities in the Debye potential formalism, which were developed by Pechenich, Kearney and Cohen [17].

To end up, it is easy to prove that the relations (4.10) and (4.11) are such that if we start from two linearly independent solutions in the $s=0$ case, ${}_0\mathcal{R}_{lm}^{1,2}$, the two solutions ${}_+{}_1\mathcal{R}_{lm}^{1,2}$ we will obtain are also linearly independent; the same happens for ${}_{-1}\mathcal{R}_{lm}^{1,2}$.

b) Stream potential q and 1-form Q_q versus j .

In order to obtain the differential equation of the Debye potential in the b) choice we have seen that the 1-form Q_q has to obey the relations (2.21) or the equations (2.27 a, b) if we take into account that we have chosen $Q_q^1 = Q_q^2 = 0$. In the Schwarzschild space-time these equations are:

$$\frac{1}{2}q_2 = \bar{\delta}Q_q^4 \quad (4.13 a)$$

$$\frac{1}{2}q_0 = -\delta Q_q^3 \quad (4.13 b)$$

To find a particular solution of these equations is very easy, since:

i) The differential operators δ and $\bar{\delta}$ act only on the angular part and are proportional to the operators $\hat{\sigma}$ and $\hat{\bar{\sigma}}$ (when $s=0$) of the spin- s spherical harmonics [11]. These relations are:

$$\delta = -\frac{1}{\sqrt{2r}}\bar{\sigma}_{s=0}; \quad \bar{\delta} = -\frac{1}{\sqrt{2r}}\hat{\sigma}_{s=0} \quad (4.14)$$

ii) The components on the null-tetrad of the 1-form Q_q and the components of the stream potential q on the bivector basis can be developed in spin- s spherical harmonics:

$$Q_q^a(x^\mu) = \int_{-\infty}^{+\infty} d\omega e^{-i\omega t} \sum_{lm} Q_{qlm}^a(\omega, r)_s Y_{lm}(\theta, \varphi) \quad (4.15)$$

$$q_I(x^\mu) = \int_{-\infty}^{+\infty} d\omega e^{-i\omega t} \sum_{lm} q_{Ilm}(\omega, r)_s Y_{lm}(\theta, \varphi) \quad (4.16)$$

Inserting the latter expressions into (4.13) and appealing to the properties of the spin- s spherical harmonics we obtain:

$$Q_{qlm}^3(\omega, r) = -\frac{r}{\sqrt{2}\lambda^{1/2}} q_{0lm}(\omega, r) \quad (4.17 a)$$

$$Q_{qlm}^4(\omega, r) = -\frac{r}{\sqrt{2}\lambda^{1/2}} q_{2lm}(\omega, r) \quad (4.17 b)$$

If we consider (3.4) we can write out:

$$\begin{aligned} q_{0lm}(\omega, r) &= -2 \left\{ \frac{i}{\omega} j_{lm}^2(\omega, r) + K^2(r)\delta(\omega) \right\} \\ q_{1lm}(\omega, r) &= \frac{1}{2} \left\{ \frac{i}{\omega} \left[-j_{lm}^4(\omega, r) + \frac{a}{2} j_{lm}^3(\omega, r) \right] \right. \\ &\quad \left. + \delta(\omega) \left[-K_{lm}^4(r) + \frac{a}{2} K_{lm}^3(r) \right] \right\} \\ q_{2lm}(\omega, r) &= a \left\{ \frac{i}{\omega} j_{lm}^1(\omega, r) + K_{lm}^1(r)\delta(\omega) \right\} \end{aligned} \quad (4.18)$$

where

$$\begin{aligned} j_{lm}^a(\omega, r) &= \int d\Omega_s \bar{Y}_{lm}(\theta, \varphi) j^a(\omega, x^k). \\ K_{lm}^a(\omega, r) &= \int d\Omega_s \bar{Y}_{lm}(\theta, \varphi) K^a(x^k). \end{aligned}$$

and by substitution of (4.18) into (4.17) we, eventually, obtain

$$\begin{aligned} Q_{qlm}^3(\omega, r) &= \frac{\sqrt{2}r}{\lambda^{1/2}} \left\{ \frac{i}{\omega} j_{lm}^2(\omega, r) + K_{lm}^2(r)\delta(\omega) \right\} \\ Q_{qlm}^4(\omega, r) &= -\frac{\sqrt{2}r a}{\lambda^{1/2} 2} \left\{ \frac{i}{\omega} j_{lm}^1(\omega, r) + K_{lm}^1(r)\delta(\omega) \right\} \end{aligned} \quad (4.19)$$

The problem is, thus, solved in a Schwarzschild space-time. From a

given four-current j we are able to compute the stream potential q by using (4.18), and with (4.19) we can get the 1-form Q_q . The procedure shown in this section can also be used to solve the problem in choices a) and c).

c) Electromagnetic field.

The electromagnetic tensor f (a 2-form) can be developed

$$f = \frac{1}{2} [\phi_1 Z^1 + \bar{\phi}_1 \bar{Z}^1] \tag{4.20}$$

From (2.17) we obtain

$$\begin{aligned} \phi_0 &= -\frac{1}{2} q_0 + 2(-\rho + \Delta)\bar{\psi} - \delta\bar{Q}_q^3. \\ \phi_1 &= -\frac{1}{2} \{ q_1 + \bar{q}_1 + (2\alpha - \delta)\bar{\delta}\psi \}. \\ \phi_2 &= -\frac{1}{2} q_2 + 2(\mu + \Delta)\bar{\delta}\psi - \delta\bar{Q}_q^4. \end{aligned} \tag{4.21}$$

One observer at rest in the gravitational field measures the electric and magnetic fields:

$$\begin{aligned} E_\alpha &= f_{\alpha\beta} u^\beta \\ B_\alpha &= -\frac{1}{2} \varepsilon_{\alpha\beta\gamma\delta} u^\beta f^{\gamma\delta} \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} u^\alpha u_\alpha &= -1 & u^\alpha &= (a^{-1/2}, 0, 0, 0) \\ B^\alpha u_\alpha &= E^\alpha u_\alpha = 0 \end{aligned}$$

which may be expressed in terms of ϕ_1 .

$$\begin{aligned} \vec{E} &= \frac{1}{2} \left[a^{-1/2} \phi_2 \vec{m} - \frac{1}{2} a^{1/2} \phi_0 \vec{\bar{m}} + \phi_1 \vec{c} \right] + \text{c. c.} \\ \vec{B} &= \frac{1}{2} \left[a^{-1/2} \phi_2 \vec{m} - \frac{1}{2} a^{1/2} \phi_0 \vec{\bar{m}} + \phi_1 \vec{c} \right] + \text{c. c.} \end{aligned}$$

where the 3-vector \vec{c} is the projection of the 4-vector $-a^{-1/2}l + \frac{1}{2}a^{1/2}k$ on the space which is tangent to the spatial hypersurface, in the same way that \vec{m} , $\vec{\bar{m}}$, \vec{E} and \vec{B} are the projections of the four-vectors m , \bar{m} , E and B .

V. CONCLUSIONS

In this work we have managed to connect the electromagnetic field and its sources by using the Debye potential formalism. The introduction

of the source related terms does not modify the possibility of choosing between the three orientations of the Hertz potential. The problem of finding the electromagnetic field created by a given distribution of charges and currents boils down to finding the solution of a non-homogeneous second order differential equation and afterwards, obtaining the field by straightforward differentiation of the potential. The homogeneous part of this equation coincides with that of reference [1] and gives the Debye potential associated to the free electromagnetic field. The remaining one is the source related part and to obtain it for each four-current, the only obstacle is finding a particular integral of a system of two first order partial differential equations with two unknown functions. This has to be done for each specific space-time.

In the Schwarzschild space-time case, we have developed explicitly this formalism and obtained the complete differential equation of the Debye potential. The treatment of the source related terms in option *b*) (in Chapter II) has been specially simple due to the use of the spin-*s* spherical harmonics, once the usual separation of variables in the NP formalism has been made. Thus, from the knowledge of two linearly independent solutions of the homogeneous equation, which satisfy the boundary conditions of the particular physical problem, we are able to compute the potential through the Green function method.

It is well known that the differential equation satisfied by the free Debye potential in a Kerr space-time in option *b*) cannot be separated [1]. So, if we want to develop our theory, at least at first order in the angular momentum per unit of mass, *a*, we have to use option *a*) or *c*) (this will be made in a forthcoming paper). When *a* is sufficiently small so that terms of order higher than the first are negligible, but *M* is allowed to be large, Schwarzschild's solution is the zeroth order of the Kerr's one [18] [19]. Thus, in this case, using a perturbative method we will be able to build the solutions of the Debye potential homogeneous equation from the corresponding solutions in the Schwarzschild case. But, these solutions, as we have seen in chapter IV, can be obtained from the solution of option *b*).

Teukolsky's treatment of Maxwell's equations [20] shows that the second order differential equation for the component ϕ_1 of the electromagnetic tensor [21] cannot be splitted through separation of variables in the Kerr case. Therefore, in those physical situations when it is necessary to know this component [12], our treatment should be most convenient.

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