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$(QED)_2$ in the Coulomb Gauge (*)

by

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ABSTRACT. — We give a construction of the Coulomb gauge for $(QED)_2$ in a finite volume, both on Fock space and as a functional integral. As an application we obtain the non-relativistic limit for the theory.

RÉSUMÉ. — On donne une construction de la jauge de Coulomb pour $(QED)_2$ dans un volume fini, dans l'espace de Fock et comme intégrale fonctionnelle. Comme application, on obtient la limite non relativiste de la théorie.

I. INTRODUCTION

In any gauge quantum field theory the Coulomb gauge plays a distinguished role. It is in this gauge that the classical field theory can be cast in Hamiltonian form and one can more or less apply standard quantization procedures. The resulting structure is however rather ill-defined and one usually makes a formal transformation to one of the covariant guages before studying the theory. For completeness it would be interesting to make this first step precise, that is to construct the theory in the Coulomb gauge and show that it is equivalent to the covariant gauges. It might also be useful as a practical matter it would be a way to establish Osterwalder-Schrader positivity for the covariant gauges. Finally the Coulomb gauge seems to be the natural setting for studying the non-relativistic limit of a theory.

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In this paper we construct the Coulomb gauge for quantum electrodynamics in the charge zero sector on a compact two-dimensional space-time. There are actually two constructions. The first is a construction of the Hamiltonian for the theory on Fock space. As an application we study the non-relativistic limit for the model. The second construction is in terms of functional integrals and is closely connected with the functional integral formulation of the Landau gauge. At the end we comment on the equivalence of the two constructions.

II. FOCK SPACE FORMULATION

The classical field equations for a fermion field ψ interacting with an electromagnetic field A_μ are

$$\Box A_\mu - \partial_\mu(\partial_\nu A^\nu) = ej_\mu$$
$$(i\gamma^\mu(\partial_\mu - iec^{-1}A_\mu) - mc)\psi = 0$$

where $j_\mu = \tilde{\psi}\gamma_\mu\psi$ is the current, e is the charge, m is the fermion mass, and c is the speed of light [1]. By the Coulomb gauge (or axial gauge) we mean that $A_1 = 0$. Then the equation for A_0 becomes

$$-\partial_1^2 A_0 = ej_0 .$$

One solves this for A_0, inserts it into the equation for ψ and gets an equation for ψ alone.

Suppose now that our space time is $\mathbb{R} \times S^1$ or $\mathbb{R} \times [-\pi, \pi]$ with periodic boundary conditions. Then $-\partial_1^2$ is invertible on the orthogonal complement of the constants in $L_2(S^1)$ and has the kernel

$$V(x-y) = (2\pi)^{-1} \sum_{k \in \mathbb{Z}, k \neq 0} e^{ik(x-y)} k^{-2}$$

which is the Coulomb potential for S^1. (On $\mathbb{R}^1$ we would have $V(x-y) = -\frac{1}{2}|x-y|$). We restrict attention to total charge zero, i. e. $\int j_0 = 0$, so j_0 is orthogonal to constants, and have as the solution of the equation for A_0

$$A_0(x, t) = e \int_{-\pi}^{\pi} V(x-y) j_0(t, y) dy .$$

[1] The notation is $x^\mu = (x^0, x^1) = (ct, x)$ and $\partial_\mu = \partial/\partial x^\mu$. We have $\Box = g^{\mu\nu}\partial_\mu\partial_\nu$ where the metric is $g^{\mu\nu} = \mathrm{diag}(1, -1)$. The γ^μ are the Dirac matrices for $g^{\mu\nu}$ and $\tilde{\psi} = \bar{\psi}\gamma^0$. We have $\hbar = 1$ throughout.

Inserting this expression for A_0 and introducing $\beta = \gamma^0$, $\alpha = \gamma^0\gamma^1$ we have

$$i\frac{\partial\psi}{\partial t}(t,x) = \left(-i\alpha c\frac{\partial}{\partial x} + \beta mc^2\right)\psi(t,x) + e^2\int_{-\pi}^{\pi}\psi(t,x)V(x-y)j_0(t,y)dy.$$

For the quantum problem one wants to solve this equation with data which satisfy the canonical anti-commutation relations. On a fermion Fock space based on $L_2(S^1) \oplus L_2(S^1)$ we define

$$\psi(x) = (2\pi)^{-\frac{1}{2}}\sum_{p\in\mathbb{Z}}\left(\frac{mc^2}{\omega_p}\right)^{\frac{1}{2}}(b_p u_p e^{ipx} + d_p^* v_p e^{-ipx})$$

where b_p, d_p are annihilation operators for particles and anti-particles of momentum p, and u, v satisfy

$$(\gamma^0\omega_p - \gamma^1 pc - mc^2)u_p = 0$$
$$(\gamma^0\omega_p - \gamma^1 pc + mc^2)v_p = 0$$

and are normalized so $\tilde{u}_p u_p = 1$, $\tilde{v}_p v_p = -1$. Here ω_p is defined by

$$\omega_p = (p^2c^2 + m^2c^4)^{\frac{1}{2}}.$$

Then formally we have as a solution of the equation for ψ :

$$\psi(t,x) = e^{iHt}\psi(x)e^{-iHt}.$$

Here

$$H = H_0 + H_I + E$$

$$H_0 = \int_{-\pi}^{\pi} :\psi^*\left(-i\alpha c\frac{\partial}{\partial x} + \beta mc^2\right)\psi: dx$$
$$= \sum_{p\in\mathbb{Z}}\omega_p(b_p^*b_p + d_p^*d_p)$$
$$H_I = \frac{e^2}{2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} j_0(x)V(x-y)j_0(y)dxdy.$$

The charge density is

$$j_0(x) = :\psi^*(x)\psi(x):$$

and E is a (possibly infinite) constant. We are only interested in the charge zero sector $\left(Q = \int j_0 = 0\right)$ where the introduction of Wick ordering on j_0 corresponds to a shift in H by an infinite constant.

Our goal is to give a precise definition of H, and we show that it is a bilinear form on a certain dense domain in Fock space. Previously, McBryan [*1*] [*2*] has shown that $j_\mu(x) = :\tilde{\psi}(x)\gamma_\mu\psi(x):$ is well-defined either as a bilinear form or as an operator-valued distribution.

The method we use is that of N_τ estimates [3]. One defines with $\mu_p = (p^2 + 1)^{\frac{1}{2}}$

$$N_\tau = \sum_p (b_p^* b_p + d_p^* d_p)\mu(p)^\tau .$$

Then for any wick ordered monomial

$$W = \sum_{p_1,\dots,p_{l+m}} w(p_1, \dots, p_{l+m}) b_{p_1}^* \dots b_{p_l}^* b_{p_{l+1}} \dots b_{l+m}$$

or with any b replaced by a d, we have:

$$\left\| \left(\prod_{i=1}^{l} (N_{\tau_i} + 1)^{-\frac{1}{2}} \right) W \left(\prod_{i=l+1}^{l+m} (N_{\tau_i} + 1)^{-\frac{1}{2}} \right) \right\| \leq \left\| \left(\prod_{i=1}^{l+m} \mu_{p_i}^{-\tau_i/2} \right) w \right\|_2 .$$

Let $\rho(k) = \hat{j}_0(k)$ be the Fourier coefficients of $j_0(x)$ so that $\rho(-k) = \rho(k)^*$ and

$$H_I = \frac{e^2}{2} \sum_{k \neq 0} \rho(k)^* \rho(k) k^{-2} .$$

Then we have

$$\rho(k) = \rho_+(k) + \rho_0(k) + \rho_-(k)$$

where

$$\rho_+(k) = (2\pi)^{-\frac{1}{2}} \sum_{p,q} w_+(p, q) \delta_{k,-p-q} b_p^* d_q^*$$

$$\rho_0(k) = (2\pi)^{-\frac{1}{2}} \sum_{p,q} w_0(p, q) \delta_{k,q-p} (b_p^* b_q - d_p^* d_q)$$

$$\rho_-(k) = (\rho_+(-k))^*$$

and where the kernels are

$$\begin{aligned} w_+(p, q) &= mc^2(\omega_p \omega_q)^{-\frac{1}{2}} \bar{u}_p v_q \\ &= (2\omega_p \omega_q)^{-\frac{1}{2}} (\omega_p \omega_q + pqc^2 - m^2 c^4)^{\frac{1}{2}} \operatorname{sgn}(\bar{u}_p v_q) \\ w_0(p, q) &= mc^2(\omega_p \omega_q)^{-\frac{1}{2}} \bar{u}_p u_q \\ &= (2\omega_p \omega_q)^{-\frac{1}{2}} (\omega_p \omega_q + pqc^2 + m^2 c^4)^{\frac{1}{2}} . \end{aligned}$$

LEMMA. — For $\tau > 1$ the following operators are bounded uniformly in $c \geq 1$, $k \in \mathbb{Z}$:

(a) $(N_\tau + 1)^{-1} \rho_+(k)$ and $(N_\tau + 1)^{-2} (\rho_+(k))^2$;
(b) $\rho_-(k)(N_\tau + 1)^{-1}$ and $(\rho_-(k))^2 (N_\tau + 1)^{-2}$;
(c) $(N_\tau + 1)^{-1} \rho_0(k)$ and $\rho_0(k)(N_\tau + 1)^{-1}$;

(*d*) $(N_\tau + 1)^{-1}[\rho_0(-k), \rho_+(k)]$ and $[\rho_-(-k), \rho_0(k)](N_\tau + 1)^{-1}$;
(*e*) $(N_\tau + 1)^{-\frac{1}{2}} : [\rho_-(-k), \rho_+(k)] : (N_\tau + 1)^{-\frac{1}{2}}$.

Proof: (*a.*) — We have by an N_τ estimate and $| w_+(p, q) | \leq \mathcal{O}(1)$

$$\begin{aligned} \| (N_\tau + 1)^{-1}\rho_+(k) \|^2 &\leq (2\pi)^{-1} \sum_{p,q} | w_+(p, q) |^2 \delta_{k,-p-q}\mu_p^{-\tau}\mu_q^{-\tau} \\ &\leq \mathcal{O}(1) \sum_p \mu_p^{-\tau}\mu_{k+p}^{-\tau} \\ &\leq \mathcal{O}(1). \end{aligned}$$

($\tau > 1$ is more than we need.) The estimate on $\rho_+(k)^2$ is similar.

(*b.*) — This follows from (*a.*) by taking adjoints.

(*c.*) — Since $\| (N_\tau + 1)^{-\frac{1}{2}}(N_0 + 1)^{\frac{1}{2}} \| < 1$ and $[N_0, \rho_0] = 0$ we have using $| w_0(p, q) | \leq \mathcal{O}(1)$

$$\begin{aligned} \| (N_\tau + 1)^{-1}\rho_0(k) \|^2 &\leq \| (N_\tau + 1)^{-\frac{1}{2}}\rho_0(k)(N_0 + 1)^{-\frac{1}{2}} \|^2 \\ &\leq \pi^{-1} \sum_{p,q} | w_0(p, q) |^2 \delta_{k,q-p}\mu_p^{-\tau} \\ &\leq \mathcal{O}(1). \end{aligned}$$

(*d.*), (*e.*) — A typical term in $[\rho_0(-k), \rho_+(k)]$ is

$$(2\pi)^{-1} \sum_{p,q} w_0(p, -k + p)w_+(-k - q, q)\delta_{p,-q}b_p^* d_q^*$$

and a typical term in $: [\rho_-(-k), \rho_+(k)] :$ is

$$(2\pi)^{-1} \sum_{p,q} w_+(p, -k - p)w_+(q, k - q)\delta_{2k,q-p}b_p^* b_q .$$

By N_τ estimates, $| w_+ | \leq \mathcal{O}(1)$, and $| w_0 | \leq \mathcal{O}(1)$ these are estimated by $\sum_{p,q} \mu_q^{-\tau}\mu_q^{-\tau} \leq \mathcal{O}(1)$. Q. E. D.

Now let Ω_0 be the Fock vacuum and formally take $E = -(\Omega_0, H_I\Omega_0)$ so that

$$H = H_0 + \frac{e^2}{2} \sum_{k \neq 0} k^{-2}[\rho(-k)\rho(k) - \| \rho(k)\Omega_0 \|^2].$$

THEOREM 1. — H is well defined as a bilinear form on $D(N_\tau^2) \times D(N_\tau^2)$ for $\tau > 1$.

Proof. — This is easily verified for H_0. For the second term it suffices to show

$$\| (N_\tau + 1)^{-2} [\rho(-k)\rho(k) - \| \rho(k)\Omega_0 \|^2](N_\tau + 1)^{-2} \| \leq \mathcal{O}(1)$$

to show that the sum over k converges.

We expand $\rho = \rho_+ + \rho_0 + \rho_-$ and obtain nine terms. The terms $\rho_+\rho_-$, $\rho_0\rho_-$, ρ_-^2, ρ_+^2, $\rho_+\rho_0$, and ρ_0^2 can all be estimated using the lemma. We also have $\rho_0\rho_+ = \rho_+\rho_0 + [\rho_0, \rho_+]$ and both terms can be estimated by the lemma. Similarly we treat $\rho_-\rho_0$. Since $\rho\Omega_0 = \rho_+\Omega_0$ the only remaining terms are

$$\rho_-(-k)\rho_+(k) - \| \rho_+(k)\Omega_0 \|^2 = \rho_+(k)\rho_-(-k) + : [\rho_-(-k), \rho_+(k)] :$$

and both of these terms are bounded by the lemma. (Note that $\rho_-(-k)\rho_+(k)$ alone would not have a good bound in k.) Q. E. D.

Remarks 1. — Our results on the domain for H are not optimal. Quite likely similar estimates would show that H is actually an operator (rather than a bilinear form) on a suitable domain. The real issue for further progress is however to show that H is bounded from below. As a corollary H would define a self-adjoint operator.

2. — The theorem should also work for $(QCD)_2$.

III. THE NON-RELATIVISTIC LIMIT

We show that H has the expected non-relativistic limit as $c \to \infty$. The model $(QED)_2$ seems to be unique in that this limit exists at the level of Hamiltonians rather than Green's functions, for example. In an earlier paper [*4*] we obtained the non-relativistic limit for the $\mathcal{P}(\varphi)_2$ model. The $\mathcal{P}(\varphi)_2$ results are weaker than the present results in that they only refer to two-particle phenomena, but stronger in that they could be carried out in infinite volume.

The non-relativistic Hamiltonian has the form

$$H_{NR} = H_{0,NR} + H_{I,NR}$$

where

$$H_{0,NR} = \sum_p p^2/2m(b_p^* b_p + d_p^* d_p)$$

and the interaction is the Coulomb interaction between charge densities ρ_{NR}:

$$H_{I,NR} = \frac{e^2}{2} \int \rho_{NR}(x) V(x - y) \rho_{NR}(y) dx dy$$
$$= \frac{e^2}{2} \sum_{k \neq 0} \rho_{NR}(-k) \rho_{NR}(k) k^{-2}$$

where $\rho_{\mathrm{NR}}(x) = \int (b^*(x)b(x) - d^*(x)d(x))dx$ and hence

$$\rho_{\mathrm{NR}}(k) = (2\pi)^{-\frac{1}{2}} \sum_{p,q} \delta_{k,q-p}(b_p^* b_q - d_p^* d_q) .$$

Let $\mathscr{D}_0$ be the domain in Fock space of vectors with a finite number of particles and wave functions in Schwartz space so $\mathscr{D}_0 \subset \mathrm{D}(\mathrm{N}_\tau^\alpha)$ for any $\tau, \alpha \geq 0$. Then making an adjustment of energy scale on each n particle subspace by nmc^2 we have:

THEOREM 2. — *On* $\mathscr{D}_0 \times \mathscr{D}_0$:

$$\lim_{c\to\infty} (\mathrm{H} - mc^2\mathrm{N}_0) = \mathrm{H}_{\mathrm{NR}}$$

Proof. — We have

$$(p^2c^2 + m^2c^4)^{\frac{1}{2}} - mc^2 = p^2/2m + \mathcal{O}(p^4c^{-2})$$

and from this it is straightforward that

$$\lim_{c\to\infty} (\mathrm{H}_0 - mc^2\mathrm{N}_0) = \mathrm{H}_{0,\mathrm{NR}} .$$

For the interaction terms we proceed as before. Instead of the estimate $|w_+| \leq \mathcal{O}(1)$ we now use for $0 \leq \alpha \leq 1$

$$\begin{aligned} |w_+(p, q)| &\leq \mathcal{O}(\omega_p^{-\alpha/2}\omega_q^{-\alpha/2} |p - q|^\alpha) \\ &\leq \mathcal{O}(c^{-2\alpha} |p - q|^\alpha) \end{aligned}$$

(For $\alpha = 1$, see [5], equation 3.2.17, and for general α take a convex combination with the old bound.) For α sufficiently small the $|p - q|^\alpha$ does not disturb any of our previous estimates and the $c^{-2\alpha}$ gives convergence to zero. Thus any term in H_I which has a ρ_+ or ρ_- converges to zero.

To complete the proof it suffices to show that

$$\lim_{c\to\infty} \frac{e^2}{2} \sum_{k\neq 0} (\rho_0(-k)\rho_0(k) - \rho_{\mathrm{NR}}(-k)\rho_{\mathrm{NR}}(k))k^{-2} = 0$$

But this follows from our previous estimates and the following estimate for α sufficiently small

$$\begin{aligned} \| (\rho_0(k) - \rho_{\mathrm{NR}}(k))(\mathrm{N}_\tau + 1)^{-1} \|^2 &\leq \mathcal{O}(1) \sum_{p,q} |1 - w_0(p, q)|^2 \delta_{k,q-p}\mu_p^{-\tau} \\ &\leq \mathcal{O}(1) \sum_{p,q} c^{-8\alpha} |p + q|^{4\alpha} \delta_{k,q-p}\mu_p^{-\tau} \\ &\leq \mathcal{O}(c^{-8\alpha}(1 + |k|)^{4\alpha}) . \end{aligned}$$

Here we have used

$$\begin{aligned} |1 - w_0(p, q)| &\le |1 - (w_0(p, q))^2| \\ &= |w_+(p, -q)|^2 \\ &\le c^{-4\alpha} |p + q|^{2\alpha}. \end{aligned}$$

IV. THE LANDAU GAUGE

In this section we give the functional integral formulation of the Landau gauge for $(\mathrm{QED})_2$. For simplicity we only consider the partition function, but the results can be adapted to Schwinger functions for currents and/or their generating functions.

The partition function in the Mathews-Salam formulation is formally

$$Z = \int \det(1 + S(ie\not{A}))d\mu(A)$$

Here $\not{A} = A_\mu \gamma_\mu$ where γ_μ are now Dirac matrices for Euclidean two-dimensional space-time satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ and $(\gamma_\mu)^* = \gamma_\mu$. Also $S = (\not{\partial} + m)^{-1}$ is the fermion propagator and $d\mu(A)$ is the Gaussian measure with mean zero and covariance

$$D_{\mu\nu} = (\delta_{\mu\nu} - \partial_\nu \partial_\nu / \Delta)(-\Delta)^{-1}.$$

Our aim is to explain the precise definition of Z following the original treatment of Seiler for $(\mathrm{Yukawa})_2$ [*6*] and the formulation for $(\mathrm{QED})_2$ due to Weingarten and Challifour [*7*], Weingarten [*8*], and Seiler [*9*] [*10*].

The Euclidean space-time is $S^1 \times S^1$ or $[-\pi, \pi] \times [-\pi, \pi]$ with periodic boundary conditions. In this case the fermion propagator is

$$S(x - y) = (2\pi)^{-2} \sum_{p \in \mathbb{Z}^2} (i\not{p} + m)^{-1} e^{ip(x-y)}.$$

We also consider the case of anti-periodic boundary conditions for fermions in which case the above sum is replaced by a sum over $\left(\mathbb{Z} + \frac{1}{2}\right)^2$. (Then spinors are sections of a vector bundle over S^1.)

Since $S\not{A}$ is not trace class one must make further modifications to define the determinant. We introduce

$$\mathscr{K}(A) = (-\Delta + m^2)^{\frac{1}{4}} S(ie\not{A})(-\Delta + m^2)^{-\frac{1}{4}}$$

which formally gives the same determinant. Then if A is a bounded function on S^1 we have that $\mathscr{K}(A)$ is a bounded operator on $L_2(S^1 \times S^1)$ in the class ∂_4 (i. e. $(\mathscr{K}^*\mathscr{K})^2$ is trace class), and hence the modified determinant

$\det_4(1 + \mathscr{K}(A))$ is well-defined. We formally restore the terms we have omitted by defining

$$\det_{\text{ren}}(1 + \mathscr{K}(A)) = \det_4(1 + \mathscr{K}(A)) \exp\left(\sum_{k=1}^{3} (-1)^{k+1} \operatorname{Tr}_{\text{ren}}(\mathscr{K}(A)^k)/k\right)$$

where $\operatorname{Tr}_{\text{ren}}$ is a renormalized trace. The precise definition of the renormalized trace depends on introducing and removing an ultraviolet cutoff. Using the lattice approximation one obtains [7]:

$$\det_{\text{ren}}(1 + \mathscr{K}(A)) = \det_4(1 + \mathscr{K}(A)) \exp\left(-\frac{1}{2}\int A_\mu(x)\Pi_{\mu\nu}(x-y)A_\nu(y)dxdy\right)$$

where with $S(p) = (i\not{p} + m)^{-1}$:

$$\Pi_{\mu\nu}(x-y) = -e^2(2\pi)^{-4}\sum_{p,k} e^{ik(x-y)} \operatorname{Tr}\left[S(p)\gamma_\mu(S(p+k) - S(p))\gamma_\nu\right].$$

The expression $\int A_\mu \Pi_{\mu\nu} A_\nu$ is well-defined for A_μ sufficiently smooth on S^1.

An important feature of the renormalized determinant is gauge invariance. For χ sufficiently smooth we have

$$\det_{\text{ren}}(1 + \mathscr{K}(A + \partial\chi)) = \det_{\text{ren}}(1 + \mathscr{K}(A)).$$

(In fact $\det_4$ is invariant and $\partial_\mu \Pi_{\mu\nu} = 0$). This can be proved in the lattice approximation.

Next we consider the integral over A_μ. To define the covariance $D_{\mu\nu}$ we must define an inverse for $-\Delta$. In $L_2(S^1 \times S^1)$ Laplacian is invertible on the orthogonal complement of the constants and we could define the inverse to be zero on the constants. Instead let $\mathscr{M}$ be the subspace of functions which are constant in x_1, invert $(-\Delta)$ on $\mathscr{M}^\perp$, and define it to be zero on $\mathscr{M}$. Thus we set

$$(-\Delta)^{-1}(x, y) = (2\pi)^{-2}\sum_k{}' e^{ik(x-y)}k^{-2}$$

where

$$\sum_k{}' = \sum_{k \in \mathbb{Z}^2, k_1 \neq 0}$$

Note that in the infinite volume limit the distinction will disappear.

We define A_μ as a Gaussian random variable with covariance $D_{\mu\nu}$ as follows. Let $F = F(h)$, the field strength, be a Gaussian random variable indexed by $h \in L_2(S^1 \times S^1)$ (real-valued) with mean zero and covariance I, i. e. F is white noise. The measure is denoted $d\nu(F)$. We define

$$A_\mu = -\varepsilon_{\mu\nu}\partial_\nu(-\Delta)^{-1}F.$$

Note that $\varepsilon_{\mu\nu}\partial_\nu(-\Delta)^{-1}$ is a bounded operator on $L_2(S^1 \times S^1)$ so A_μ is well-defined via the adjoint. Using $\varepsilon_{\mu\nu}\varepsilon_{\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}$ we see A_μ has covariance $D_{\mu\nu}$.

It is convenient to work with a cutoff version of these fields. Let $e_k(x) = (2\pi)^{-1}e^{ikx}$ and

$$F_k = F(\bar{e}_k)$$

which satisfy $\int \bar{F}_j F_k d\nu(F) = \delta_{jk}$. Then define C^∞ Gaussian random functions

$$F^N(x) = \sum_{|k|\leq N}{}' F_k e_k(x)$$

We can also use a real basis for the same subspace such as $C_k = 2^{-\frac{1}{2}}(e_k + e_{-k})$ and $S_k = 2^{-\frac{1}{2}}i^{-1}(e_k - e_{-k})$ with $k_1 > 0$. Then we have

$$F^N(x) = \sum_{|k|\leq N}{}'' a_k C_k(x) + b_k S_k(x)$$

where $a_k = F(C_k)$ and $b_k = F(S_k)$ and Σ'' means sum over $k_1 > 0$ only.

We also have that $A^N_\mu(x) = -\varepsilon_{\mu\nu}\partial_\nu(-\Delta)^{-1}F^N(x)$ are C^∞, Gaussian random functions with covariance:

$$D^N_{\mu\nu}(x-y) = (2\pi)^{-2}\sum_{|k|\leq N}{}' e^{ik(x-y)}(\delta_{\mu\nu} - k_\mu k_\nu k^{-2})k^{-2}$$

Now define

$$\begin{aligned} Z_N &= e^{E_N}\int \det_{\text{ren}}(1 + \mathscr{K}(A^N))d\nu(F) \\ &= e^{E_N}\int \det_{\text{ren}}(1 + \mathscr{K}(-\varepsilon_{\mu\nu}\partial_\nu(-\Delta)^{-1}F^N))d\nu(F), \end{aligned}$$

where

$$E_N = \frac{1}{2}\int \Pi_{\mu\nu}(x-y)D^N_{\mu\nu}(x-y)dxdy\,.$$

Since A^N is C^∞ the determinant is well defined. It is also bounded and so the integral converges. In fact we have [*8*] [*11*] for any A:

$$|\det_{\text{ren}}(1 + \mathscr{K}(A))| \leq 1\,.$$

The proof uses the lattice approximation and Osterwalder-Schrader positivity which requires anti-periodic boundary conditions for fermions. A bound for periodic boundary conditions was given by Ito [*12*]. The constant E_N is a vacuum energy renormalization and has the effect of Wick ordering the fields in $\int A_\mu \Pi_{\mu\nu} A_\nu$. Using the transversality of $\Pi_{\mu\nu}$ one obtains the bound $|E_N| = \mathcal{O}(\log N)$ as $N \to \infty$.

Using the above bounds as input one obtains the basic stability result. This is the existence of the limit:

$$Z = \lim_{N\to\infty} Z_N .$$

For the proof see Seiler [*9*] [*10*]. (The second reference has a simple proof for e/m small.)

V. THE COULOMB GAUGE-FUNCTIONAL INTEGRAL FORMULATION

The partition function Z_C for the Coulomb gauge is formally defined by

$$Z_C = \int \det(1 + S(ie\gamma_0\Phi))d\omega(\Phi)$$

where $d\omega$ is Gaussian with covariance

$$C = (-\partial_1^2)^{-1} .$$

Formally this is the same as the Landau gauge Z by the change of variables

$$A_\mu = \varepsilon_{\mu\nu}\partial_\mu\partial_1(-\Delta)^{-1}\Phi .$$

We want to make this statement precise.

The covariance C is positive definite on $\mathscr{M}^\perp \subset L_2(S^1 \times S^1)$ and we let Φ be a Gaussian random variable indexed by $\mathscr{M}^\perp$ with covariance C. The associated measure is denoted $d\omega(\Phi)$.

We also let $\Phi_k = \Phi(\bar{e}_k)$, $k_1 \neq 0$, which satisfies $\int \overline{\Phi}_j\Phi_k d\omega(\Phi) = k_1^{-2}\delta_{jk}$. Then

$$\Phi^N(x) = \sum_{|k|\leq N}{}' \Phi_k e_k(x)$$

is a smooth Gaussian random function with covariance

$$C^N(x-y) = (2\pi)^{-2} \sum_{|k|\leq N}{}' e^{ik(x-y)}k_1^{-2}$$

(Φ^N can also be written in a real basis as before.)

The cutoff partition function is defined by

$$Z_{C,N} = e^{E_N} \int \det_{\text{ren}}(1 + \mathscr{K}(\Phi^N, 0))d\omega(\Phi) .$$

The next result gives stability for the Coulomb gauge and equivalence to the Landau gauge.

THEOREM 3. — $\lim_{N\to\infty} Z_{C,N}$ exists and $Z_C = Z$.

Proof. — It suffices to show $Z_{C,N} = Z_N$. The first step is a change of variables. Using the definition of the integral of a cylinder function we have for suitable f:

$$\begin{aligned} \int f(F^N)d\nu(F) &= \int f\Big(\sum_{|k|\leq N}{}'' (a_k C_k + b_k S_k)\Big) \prod_{|k|\leq N}{}'' d\rho_1(a_k)d\rho_1(b_k) \\ &= \int f\Big(\sum_{|k|\leq N}{}'' (k_1 a_k C_k + k_1 b_k S_k)\Big) \prod_{|k|\leq N}{}'' d\rho_{(k_1^{-2})}(a_k)d\rho_{(k_1^{-2})}(b_k) \\ &= \int f(\pm\, \partial_1 \Phi^N)d\omega(\Phi)\,. \end{aligned}$$

Here $d\rho_\lambda$ is Gaussian on $\mathbb{R}$ with variance λ:

$$d\rho_\lambda(x) = (2\pi\lambda)^{-\frac{1}{2}} \exp(-\, x^2/2\lambda)dx\,.$$

Similarly we have for Z_N

$$\int \det_{ren}(1 + \mathscr{K}(-\,\varepsilon_{\mu\nu}\partial_\nu(-\,\Delta)^{-1}F^N))d\nu(F) = \int \det_{ren}(1 + \mathscr{K}(-\,\varepsilon_{\mu\nu}\partial_\nu\partial_1(-\,\Delta)^{-1}\Phi^N))d\omega(\Phi)\,.$$

(The extra derivatives cause no problem since everything is smooth.)

To complete the proof we need the identity

$$\det_{ren}(1 + \mathscr{K}(-\,\varepsilon_{\mu\nu}\partial_\nu\partial_1(-\,\Delta)^{-1}\Phi^N)) = \det_{ren}(1 + \mathscr{K}(\Phi^N, 0))\,.$$

But this is just the gauge invariance of $\det_{ren}$ for we have with $\Phi = \Phi^N$ and $A_\mu = -\,\varepsilon_{\mu\nu}\partial_\nu\partial_1(-\,\Delta)^{-1}\Phi$ that

$$\begin{aligned} (A_0, A_1) &= (-\,\partial_1^2(-\,\Delta)^{-1}\Phi, \partial_0\partial_1(-\,\Delta)^{-1}\Phi) \\ &= (\Phi, 0) + (\partial_0\chi, \partial_1\chi) \end{aligned}$$

where $\chi = \partial_0(-\,\Delta)^{-1}\Phi$.

VI. REMARKS

We explain why our two constructions of the Coulomb gauge are formally equivalent, ignoring finer points such as matching the vacuum energy counter terms. The determinant in Z_C can be formally written as an integral over anti-commuting fermion fields $\tilde{\psi}$, ψ to give

$$Z_C = \mathcal{N}^{-1} \int \exp\Big(-\int \tilde{\psi}(\partial\!\!\!/ + m)\psi - ie\int j_0\Phi\Big)d\tilde{\psi}d\psi d\omega(\Phi)$$

where $j_\mu = \tilde{\psi}\gamma_\mu\psi$ and $\mathcal{N}^{-1}$ is a normalization so that $Z_C = 1$ at $e = 0$.

If we now do the integral over Φ and use $C(x - y) = \delta(x_0 - y_0)V(x_1 - y_1)$ we obtain

$$Z_C = \mathcal{N}^{-1} \int \exp\left(- \int \tilde{\psi}(\not\partial + m)\psi \right) \exp\left(- \frac{e^2}{2} \int j_0(t, x)V(x-y)j_0(t, y)dxdy \right) d\tilde{\psi}d\psi.$$

Assuming periodic boundary conditions for the fermions this can be identified as a partition function for the Fock space theory at inverse temperature $\beta = 2\pi$

$$Z_C = \mathrm{Tr}\,[e^{-2\pi H}]/\mathrm{Tr}\,[e^{-2\pi H_0}].$$

This identification can be made in perturbation theory for example. More generally if the time coordinate had period β in the functional integral we would have inverse temperature β. Similar formulas exist connecting the Schwinger functions and thermal correlation functions. (For $\mathscr{P}(\varphi)_2$ this is even rigorous [*13*]).

To make the above connection precise the best bet seems to be to introduce a lattice approximation. However this does have the disadvantage of spoiling the nice integration over Φ above.

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