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Central decomposition of Poincaré-invariant nets of local field algebras and absence of spontaneous breaking of the Lorentz group

by

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ABSTRACT. — We study reducible, Poincaré-invariant representations of nets of local field algebras and prove a number of structure results, some of which are generalizations of previous work on nets of observable algebras [*1*] and some of which are quite new. Using these we examine the central decomposition of such nets, study the spontaneous breaking of the Lorentz group symmetry under such decompositions into pure phases, and consider the significance of the modular automorphism groups of the wedge algebras.

RÉSUMÉ. — On étudie des représentations réductibles et Poincaré-invariantes de réseaux d'algèbres de champs locales et on démontre des résultats de structure, dont quelques-uns sont des généralisations d'un travail antérieur [*1*] sur les réseaux d'algèbres d'observables et quelques-uns sont tout à fait nouveaux. En les employant, on examine la décomposition centrale de tels réseaux, on étudie la brisure spontanée du groupe de Lorentz dans cette décomposition en phases pures, et on considère la signification des groupes d'automorphismes modulaires des algèbres associées aux régions de l'espace-temps en forme de coin.

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I. INTRODUCTION

In this paper we reexamine in the light of more recent results the old problem [1] of the decomposition of a net of local algebras (of observables or « fields » [2]), in a Hilbert space $\mathcal{H}$ with the subspace $P_0\mathcal{H}$ of translation-invariant vectors having dimension $\dim(P_0\mathcal{H}) > 1$, into a direct integral of irreducible nets of local algebras in pure phases, i. e. in Hilbert spaces $\mathcal{H}(\zeta)$ with $\dim(P_0\mathcal{H}(\zeta)) = 1$. (That such a situation can arise in concrete models is now well known—see e. g. [3] [4] [5] and references given therein.) Postponing any detailed definitions, we may sketch our intentions as follows. If $\{ \mathcal{F}(\mathcal{O}) \}$ is a Poincaré-invariant net of local field algebras (which may contain the net $\{ \mathcal{A}(\mathcal{O}) \}$ of observable algebras in the usual manner [2]) in a Hilbert space $\mathcal{H}$ with $\dim(P_0\mathcal{H}) > 1$, then the von Neumann algebra $\mathcal{F}$ generated by $\cup \mathcal{F}(\mathcal{O})$ is reducible. Considering the special case where $\{ \mathcal{F}(\mathcal{O}) \}$ coincides with $\{ \mathcal{A}(\mathcal{O}) \}$, Araki showed [1] that the central decomposition of $\mathcal{F}$:

$$\mathcal{F} = \int_S^{\oplus} \mathcal{F}(\zeta) d\nu(\zeta), \qquad \mathcal{H} = \int_S^{\oplus} \mathcal{H}(\zeta) d\nu(\zeta), \tag{1.1}$$

where S is the spectrum of the center of $\mathcal{F}$, yields a direct integral of irreducible « representations » of $\mathcal{F}$, in each of which the translation subgroup of the Poincaré group is implemented by a strongly continuous unitary group satisfying the spectrum condition. However, he presented an example of such a system for which in each Hilbert space $\mathcal{H}(\zeta)$ the Lorentz group is not unitarily implementable, i. e. the Lorentz group symmetry is spontaneously broken in the pure phases of the theory.

First of all, we verify that the above-mentioned results hold in the more general case of a net of field algebras satisfying Bose and Fermi statistics. And we clarify a number of technical points that were passed over in [1] but that do require attention. Furthermore, we present in the Appendix a new class of systems for which the Lorentz group symmetry is spontaneously broken in the pure phases. Other results in [1] are improved and/or extended as well.

Further results that go beyond the matters dealt with in [1] have been motivated by the following new observation: the center of $\mathcal{F}$ is equal to the center of each wedge algebra $\mathcal{F}(W)$, where W is a Poincaré transform of $W_R = \{ x \in \mathbb{R}^4 \mid |x^0| < x^1 \}$. With this fact in hand, we can show that a certain property that can be formulated entirely in the framework of the algebraic relativistic quantum theory entails that the Lorentz symmetry cannot be spontaneously broken under the decomposition (1.1). This property, which is explained in detail in Section 5, requires that the modular automorphism groups of the wedge algebras $\mathscr{F}(W)$ must be equal to the automorphism groups induced by the action of the appropriate Lorentz velocity transformations in the given representation of the Poincaré group. From this property one can also conclude that the wedge algebras satisfy (twisted) duality. Also of interest is another consequence of the equality of the center of $\mathscr{F}$ with the center of $\mathscr{F}(W)$, W any wedge region: all of the modular automorphism groups of the wedge algebras are inner automorphisms of the algebra $\mathscr{F}$. Thus, if the coincidence of these modular automorphism groups with the Lorentz velocity transformations does not hold, then there are many additional symmetries of the theory to be understood. If it does hold, however, we show that it is conserved in the decomposition (1.1), i. e. holds in each pure phase.

Before launching into the details of this work, we would like to embed it into a somewhat broader context that is admittedly somewhat speculative. The condition above is *prima facie* weaker than the so-called special condition of duality, discovered by Bisognano and Wichmann [*6*] [*7*] and found very useful in various applications [*8*] [*9*] [*10*], which holds whenever there is a quantum field $\varphi(x)$ in $\mathscr{H}$ with a cyclic, Poincaré-invariant vector, which transforms under the same representation of the Poincaré group as $\{ \mathscr{F}(\mathcal{O}) \}$, and which is associated to $\{ \mathscr{F}(\mathcal{O}) \}$ in the sense that the elements of $\mathscr{F}(W_R')$ commute weakly on a suitable dense domain with all field operators $\varphi(f)$ with test function f having support contained in W_R (see [*6*] [*7*] and Section 6 for further details). On the other hand, by results of [*11*] and what is shown below, one can conclude that if there is such a field in $\mathscr{H}$ (which need *not* necessarily commute weakly with the net in the above sense), the Lorentz group symmetry cannot be spontaneously broken under the decomposition (1.1). (This will be explained in Section 6.) It is not impossible that the above mentioned condition on the modular automorphism groups of the wedge algebras (with the concomitant combination of analyticity properteis and the geometry of the Lorentz group) could be the touchstone in the algebraic relativistic quantum theory for the implicit presence of a quantum field associated to the net—the condition is necessary and implies results (duality of wedge algebras, no spontaneous breaking of Lorentz group) that do not always hold in the general algebraic setting. This possibility should be investigated further.

II. DEFINITIONS AND NOTATION

To begin we must establish notation and formulate some definitions. Although we shall state all definitions and results for four space-time dimensions, they also hold *mutatis mutandis* in two and three space-time dimensions. Let W_R denote the right wedge defined by $W_R \equiv \{ x \in \mathbb{R}^4 \mid |x^0| < x^1 \}$, where x^0 is the time coordinate. Then the set of all wedges is

$$\mathscr{W} \equiv \{ W_{R,\lambda} \mid \lambda \in \mathscr{P}_+^\uparrow \},$$

where $W_{R,\lambda}$ is the image of W_R under the Poincaré transformation $\lambda \in \mathscr{P}_+^\uparrow$. Let $\mathscr{K}$, the set of all double cones, be the set of the interiors of all intersections of a forward light cone with a backward light cone.

$\mathscr{H}$ will denote an infinite-dimensional separable complex Hilbert space, $\mathscr{B}(\mathscr{H})$ the algebra of all bounded operators on $\mathscr{H}$, $U(\mathscr{P}_+^\uparrow)$ a strongly continuous, unitary representation of (the universal covering group of) the Poincaré group $\mathscr{P}_+^\uparrow$ that satisfies the spectrum condition requiring positivity of the energy [1] [2], and P_0 the projection in $\mathscr{H}$ onto the subspace of all vectors invariant under $\{ T(a) \}_{a \in \mathbb{R}^4}$, where $T(a) = U(1, a)$ is the Abelian subgroup of $U(\mathscr{P}_+^\uparrow)$ implementing the translations. $P_0\mathscr{H}$ can, of course, be multidimensional.

Given this structure, a Poincaré-invariant net of local field algebras $\{ \mathscr{F}(\mathcal{O}) \}$ is a map $\mathcal{O} \to \mathscr{F}(\mathcal{O})$ from the open subsets of $\mathbb{R}^4$ to the von Neumann algebras on $\mathscr{H}$ that satisfies the following properties.

(1) Isotony: $\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathscr{F}(\mathcal{O}_1) \subset \mathscr{F}(\mathcal{O}_2)$.

(2) Locally generated: for any open $\mathcal{O} \subset \mathbb{R}^4$, $\mathscr{F}(\mathcal{O})$ is the smallest von Neumann algebra containing $\{ \mathscr{F}(\mathcal{Q}_0) \mid \mathcal{Q}_0 \in \mathscr{K}, \overline{\mathcal{Q}_0} \subset \mathcal{O} \}$, where $\overline{\mathcal{Q}_0}$ denotes the closure of $\mathcal{Q}_0$ in $\mathbb{R}^4$.

(3) Poincaré invariance: the group $U(\mathscr{P}_+^\uparrow)$ implements the natural action on the net $\{ \mathscr{F}(\mathcal{O}) \}$: for any $\lambda \in \mathscr{P}_+^\uparrow$ and any $\mathcal{O} \subset \mathbb{R}^4$,

$$U(\lambda)\mathscr{F}(\mathcal{O})U(\lambda)^{-1} = \mathscr{F}(\mathcal{O}_\lambda),$$

where $\mathcal{O}_\lambda$ is the image of $\mathcal{O}$ under λ.

(4) Locality: there exists a unitary involution Z such that $[Z, U(\lambda)] = 0$ for any $\lambda \in \mathscr{P}_+^\uparrow$ and such that, setting $F_\pm = \frac{1}{2}(F \pm ZFZ^*)$, one has:

$$\begin{aligned} &F_+G_+ - G_+F_+ = 0 \\ &F_+G_- - G_-F_+ = 0, \quad \forall F \in \mathscr{F}(\mathcal{O}_1), \quad G \in \mathscr{F}(\mathcal{O}_2), \quad \mathcal{O}_1 \subset \mathcal{O}_2', \quad \mathcal{O}_2 \subset \mathbb{R}^4, \\ &F_-G_- + G_-F_- = 0 \end{aligned}$$

where primes on space-time regions signify the interior of their causal complements and primes on algebras their commutants. If

$$\mathscr{F}(\mathcal{O})^Z \equiv \{ F^Z \equiv Z_1FZ_1^* \mid F \in \mathscr{F}(\mathcal{O}) \}, \quad \text{where} \quad Z_1 \equiv \frac{1-i}{2}I + \frac{1+i}{2}Z,$$

then one verifies that locality means precisely $\mathscr{F}(\mathcal{O}_1) \subset \mathscr{F}(\mathcal{O}_2)^{Z'}$, for any $\mathcal{O}_1 \subseteq \mathcal{O}_2'$, $\mathcal{O}_2 \subset \mathbb{R}^4$ (note that $\mathscr{F}(\mathcal{O})^{Z'} = \mathscr{F}(\mathcal{O})'^{Z}$).

Remark. — The physical significance of the existence of such a Z is that we are admitting both Bose and Fermi statistics [*2*].

(5) There is an $\Omega \in P_0\mathscr{H}$ that is cyclic for $\mathscr{F} \equiv \left(\bigcup_{\mathscr{O}\in\mathscr{K}} \mathscr{F}(\mathscr{O}) \right)''$ in $\mathscr{H}$ and $Z\Omega = \Omega$ (see Remark 4 after Prop. 3.1). Note that we do not require that Ω be cyclic for $\mathscr{F}(\mathscr{O})$, $\mathscr{O} \in \mathscr{K}$.

The quintuple $\{ \mathscr{H}, U(\mathscr{P}_+^\uparrow), \Omega, \{ \mathscr{F}(\mathscr{O}) \}, Z \}$ will be called a Poincaré-invariant field system, and when $\mathscr{H}, U(\mathscr{P}_+^\uparrow)$, Ω and Z are understood, $\{ \mathscr{F}(\mathscr{O}) \}$ will be referred to as a Poincaré-invariant net of local field algebras. If only the translation subgroup of $\mathscr{P}_+^\uparrow$ is unitarily implemented, « Poincaré-invariant » is replaced by « translation-invariant ». If $\mathscr{F}$ is irreducible in $\mathscr{H}$, the word « irreducible » will be prefixed to the relevant name. Further structure, that we do not use here, is commonly added (see e. g. [*2*]).

III. STRUCTURE RESULTS

We next prove some preliminary propositions giving structure results for Poincaré-invariant field systems. Portions of the first proposition are only generalizations to the field algebra case of results known for observable algebras (see Remarks below). In the following, for any von Neumann algebra $\mathscr{M}$, $\mathscr{Z}(\mathscr{M}) \equiv \mathscr{M} \cap \mathscr{M}'$ is the center of $\mathscr{M}$, and an overbar on a set of vectors signifies the strong closure of the set in $\mathscr{H}$. Moreover, if $\mathscr{M}$ and $\mathscr{N}$ are von Neumann algebras $\mathscr{N} \vee \mathscr{M}$ will denote the smallest von Neumann algebra containing both.

PROPOSITION 3.1. — If $\{ \mathscr{H}, U(\mathscr{P}_+^\uparrow), \Omega, \{ \mathscr{F}(\mathscr{O}) \}, Z \}$ is a Poincaré-invariant field system, then

(1) $T(a) \in \mathscr{F}$, $\forall a \in \mathbb{R}^4$,

(2) $Z\Omega_1 = \Omega_1$, $\forall \Omega_1 \in P_0\mathscr{H}$,

(3) $\overline{\mathscr{Z}(\mathscr{F})\Omega} = P_0\mathscr{H}$,

(4) $\mathscr{F}' = \mathscr{Z}(\mathscr{F})$,

(5) $Z \in \mathscr{F}$,

(6) Ω is cyclic for $\mathscr{F}(W)$, $\forall W \in \mathscr{W}$,

(7) $\mathscr{Z}(\mathscr{F}) = \mathscr{Z}(\mathscr{F}(W))$, $\forall W \in \mathscr{W}$.

Proof. — For observable algebras the counterpart to (1) was proven in Prop. 2 of [*1*] using only the spectrum property. The argument thus carries over to the present situation without change.

Let $F \in \mathscr{F}(\mathscr{O})$, $\mathscr{O} \in \mathscr{K}$, and $a_n = (0, \underline{a}_n)$ with $| \underline{a}_n | \to \infty$ as $n \to \infty$. Since $\mathscr{H}$ is separable and $\{ F(a_n) \}_{n\in\mathbb{N}}(F(a_n) \equiv T(a_n)FT(a_n)^{-1})$ is uniformly bounded in norm, there exists a subsequence $\{ F(a_{n_k}) \}_{k\in\mathbb{N}}$ that is weakly convergent. By locality, the weak limit

$$w - \lim_{k\to\infty} F(a_{n_k}) \equiv F_\infty \in \mathscr{F}^{Z'} . \tag{3.1}$$

Since $Z \in U(\mathscr{P}_+^\uparrow)'$, it follows that $w - \lim_{k\to\infty} F_-(a_{n_k}) = F_{\infty,-}$. By locality, for any $i \in \mathbb{N}$ there exists an $N(i)$ such that for all $k \geq N(i)$,

$$F_-(a_{n_i})F_-(a_{n_k}) = -F_-(a_{n_k})F_-(a_{n_i}).$$

Thus, first taking $k \to \infty$ and then $i \to \infty$, one obtains $(F_{\infty,-})^2 = -(F_{\infty,-})^2$. If F is symmetric or antisymmetric, the same is true of $F_{\infty,-}$. Therefore, decomposing F into its symmetric and antisymmetric parts and performing the above argument on each part separately, one may conclude that $F_{\infty,-} = 0$, for any $F \in \mathscr{F}(\mathscr{O})$, $\mathscr{O} \in \mathscr{K}$. So $F_\infty^Z = F_\infty$, which with (3.1) implies $F_\infty \in \mathscr{Z}(\mathscr{F})$, since $\mathscr{F}$ is weakly closed.

Note for later use that F_∞ in (3.1) is independent of the choice of sequence $\{(0, \underline{a}_n)\}_{n\in\mathbb{N}}$ with $|\underline{a}_n| \to \infty$ such that $\{F(a_n)\}_{n\in\mathbb{N}}$ is weakly Cauchy. This follows because $F_\infty\Omega = w - \lim_{n\to\infty} F(a_n)\Omega = P_0F\Omega$ [12] is independent of the choice of $\{a_n\}$ as specified and Ω is separating for $\mathscr{Z}(\mathscr{F}) \subset \mathscr{F}'$. Define now $\mathscr{F}_\infty \equiv \left\{ F_\infty \mid F \in \bigcup_{\mathscr{O}\in\mathscr{K}} \mathscr{F}(\mathscr{O}) \right\}$.

Let $\Omega_1 \in P_0\mathscr{H}$. Since $Z\Omega_1 \in P_0\mathscr{H}$, for any

$$F \in \bigcup_{\mathscr{O}\in\mathscr{K}} \mathscr{F}(\mathscr{O}) \qquad \langle F\Omega, (Z-I)\Omega_1 \rangle = \langle F(a)\Omega, (Z-I)\Omega_1 \rangle, \qquad \forall a \in \mathbb{R}^4.$$

Thus,

$$\langle F\Omega, (Z - I)\Omega_1 \rangle = \langle F_\infty\Omega, (Z - I)\Omega_1 \rangle = 0,$$

since $F_\infty^Z = F_\infty$. But Ω is cyclic for $\bigcup_{\mathscr{O}\in\mathscr{K}} \mathscr{F}(\mathscr{O})$, proving (2).

Now let $\Phi \in P_0\mathscr{H}$ and choose an increasing sequence $\{\mathscr{O}_n\}_{n\in\mathbb{N}} \subset \mathscr{K}$ with $\cup\, \mathscr{O}_n = \mathbb{R}^4$, and for each $n \in \mathbb{N}$ choose an $F_n \in \mathscr{F}(\mathscr{O}_n)$ so that the strong limit $s-\lim_{n\to\infty} F_n\Omega = \Phi$ (this is possible since $\cup\, \mathscr{F}(\mathscr{O}_n)\Omega$ is dense in $\mathscr{H}$). By the obvious diagonalization procedure find a sequence

$$\{a_e\}_{e\in\mathbb{N}} = \{(0, \underline{a}_e)\}_{e\in\mathbb{N}} (|\underline{a}_e| \to \infty) \quad \text{such that} \quad \{F_n(a_e)\}_{e\in\mathbb{N}}$$

converges weakly for each $n \in \mathbb{N}$ and let $(F_n)_\infty \in \mathscr{F}_\infty$ be the weak limit. Since $\Phi \in P_0\mathscr{H}$, it follows easily that $(F_n)_\infty\Omega \xrightarrow{s} \Phi$, so that $\overline{\mathscr{F}_\infty\Omega} \supseteq P_0\mathscr{H}$. On the other hand, (1) implies that $\mathscr{F}_\infty\Omega \subseteq \mathscr{Z}(\mathscr{F})\Omega \subseteq P_0\mathscr{H}$. Therefore, $\overline{\mathscr{F}_\infty\Omega} = \overline{\mathscr{Z}(\mathscr{F})\Omega} = P_0\mathscr{H}$, proving (3). (1) also implies that $\mathscr{F}'\Omega \subseteq P_0\mathscr{H}$, which with the above yields $\overline{\mathscr{F}'\Omega} = P_0\mathscr{H}$. Thus, $P_0 \in \mathscr{F}$, so that

$$P_0\mathscr{F}'P_0 = (P_0\mathscr{F}P_0)' \qquad \text{on} \qquad P_0\mathscr{H}.$$

It has already been seen that Ω is cyclic for $P_0\mathscr{F}_\infty P_0$ in $P_0\mathscr{H}$, so $P_0\mathscr{F}_\infty''P_0$ is maximally Abelian and must equal $P_0\mathscr{Z}(\mathscr{F})P_0$. But $P_0\mathscr{Z}(\mathscr{F})P_0 = \mathscr{Z}(P_0\mathscr{F}P_0)$ [13, Corollaire, p. 18]. Thus,

$$P_0\mathscr{F}P_0 \cap P_0\mathscr{F}'P_0 = \mathscr{Z}(P_0\mathscr{F}P_0) = \mathscr{Z}(P_0\mathscr{F}P_0)' = P_0\mathscr{F}P_0 \vee P_0\mathscr{F}'P_0,$$

so that $$P_0\mathscr{Z}(\mathscr{F})P_0 = P_0\mathscr{F}P_0 = P_0\mathscr{F}'P_0.$$

In consequence, $\mathscr{Z}(\mathscr{F})\Omega = \mathscr{F}'\Omega$, implying $\mathscr{Z}(\mathscr{F}) = \mathscr{F}'$, since Ω is separating for both algebras. This proves (4). Similarly, $\mathscr{F}''_\infty = \mathscr{Z}(\mathscr{F})$, and since $Z \in \mathscr{F}'_\infty$, assertion (5) also follows.

(6) will be proven next. Ω is cyclic for $\bigcup_{\mathcal{O}\in\mathscr{K}} \mathscr{F}(\mathcal{O})$. But for each $\mathcal{O} \in \mathscr{K}$ and any $W \in \mathscr{W}$ there exists an open subset $\mathscr{N}(\mathcal{O}, W) \subset \mathbb{R}^4$ such that $\mathcal{O}_a \subset W$, $\forall a \in \mathscr{N}(\mathcal{O}, W)$. But for any $\Phi \in \mathscr{H}$, using the spectrum condition,

$$\overline{\text{span}\,\{\, T(a)\Phi \mid a \in \mathbb{R}^4 \,\}} = \overline{\text{span}\,\{\, T(a)\Phi \mid a \in \mathscr{N}(\mathcal{O}, W) \,\}}$$

(Reeh-Schlieder principle). It follows easily that $\overline{\bigcup_{\mathcal{O}\in\mathscr{K}} \mathscr{F}(\mathcal{O})\Omega} = \overline{\mathscr{F}(W)\Omega}$, for any $W \in \mathscr{W}$.

Finally, note that for any $F \in \bigcup_{\mathcal{O}\in\mathscr{K}} \mathscr{F}(\mathcal{O})$ and any $W \in \mathscr{W}$, a sequence $\{\, a_n \,\}_{n\in\mathbb{N}} = \{\, (0, \underline{a}_n) \,\}_{n\in\mathbb{N}}$ with $|\, \underline{a}_n \,| \to \infty$ can be chosen such that $F(a_n) \in \mathscr{F}(W)$, for all $n \in \mathbb{N}$. By the aforementioned independence of F_∞ of such a choice, this implies that $\mathscr{F}_\infty \subseteq \mathscr{Z}(\mathscr{F}(W))$. Thus, $\mathscr{Z}(\mathscr{F}) \subseteq \mathscr{Z}(\mathscr{F}(W))$, for any $W \in \mathscr{W}$. To see the containment $\mathscr{Z}(\mathscr{F}(W) \subseteq \mathscr{Z}(\mathscr{F}))$, let $A = A^* \in \mathscr{Z}(\mathscr{F}(W))$ and $\{\, W(\tau) \,\}_{\tau\in\mathbb{R}} \equiv \{\, e^{iP_{||}\tau} \,\}_{\tau\in\mathbb{R}}$ be the strongly continuous, unitary group of translations along the lightlike direction given by the intersection of the closure of the given wedge W with the closed forward light cone. Then as in part (1) of the proof of Theorem 2.2 in [*14*], which uses only the lightlike monotonicity of the wedge regions, one can easily show that $\langle A\Omega, W(\tau)A\Omega \rangle = \langle A\Omega, W(-\tau)A\Omega \rangle$, $\forall \tau \in \mathbb{R}$. Since by the spectrum condition $P_{||} \geq 0$ in $\mathscr{H}$, one can analytically continue these scalar products and obtain a bounded entire function. Thus, $\langle A\Omega, W(\tau)A\Omega \rangle$ is constant in τ. Using arguments in [*12*], one sees that $w\!-\!\lim_{\tau\to\infty} W(\tau) = P_0$, so that $\|\, A\Omega \,\|^2 = \|\, P_0 A\Omega \,\|^2$, implying $A\Omega \in P_0\mathscr{H}$. Since by (6) Ω is separating for $\mathscr{F}(W)$, it follows that $A \in T(\mathbb{R}^4)'$, so that $A = A_\infty \in \mathscr{Z}(\mathscr{F})$. (7) follows at once. □

Remarks. (1). — Note that the only role played by the unitary implementability of the (homogeneous) Lorentz group in this proof is to assure that the spectrum of the generators of the translations $T(\mathbb{R}^4)$ is absolutely continuous in $(I - P_0)\mathscr{H}$, so that $w\!-\!\lim_{n\to\infty} T((0, \underline{a}_n)) = P_0$ [*12*].

(2). — We sketch an alternative proof for the new observation (7) below (see Remark following proof of Theorem 4.1).

(3). — Assertion (1) was shown for observable algebras in [*1*]. (3) and (4) can be concluded for observable algebras using [*1*] and [*15*, Corollary, p. 179].

(4). — As seen above, without assuming $Z\Omega = \Omega$, we have $\langle \Omega, F_-\Omega \rangle = 0$,

$\forall F \in \mathscr{F}$. Thus, $\langle \Omega, F\Omega \rangle = \langle Z\Omega, FZ\Omega \rangle$, $\forall F \in \mathscr{F}$, and $Z\Omega = \Omega$ can always be attained by multiplying Z by a unitary in $\mathscr{Z}(\mathscr{F})$. This observation is related to a result in [*16*].

The following lemma, which is also properly a structure result and which will be used in the proof of Proposition 3.3, will be proven in the next section (after the proof of Theorem 4.1).

LEMMA 3.2. — If $\{ \mathscr{H}, U(\mathscr{P}_+^\uparrow), \Omega, \{ \mathscr{F}(\mathscr{O}) \}, Z \}$ is a Poincaré-invariant field system, then $\mathscr{F}(W)$ is a type III von Neumann algebra, for all $W \in \mathscr{W}$.

PROPOSITION 3.3. — If $\{ \mathscr{H}, U(\mathscr{P}_+^\uparrow), \Omega, \{ \mathscr{F}(\mathscr{O}) \}, Z \}$ is a Poincaré-invariant field system, then $\mathscr{F}$ is homogeneous of type I_∞. Thus, there exists an Abelian von Neumann algebra $\mathscr{Z}$, *-isomorphic to $\mathscr{Z}(\mathscr{F})$, and a separable Hilbert space $\mathscr{H}_1$ such that $\mathscr{Z} \otimes \mathscr{B}(\mathscr{H}_1)$ is unitarily equivalent to $\mathscr{F}$.

Proof. — Suppose $Q \in \mathscr{Z}(\mathscr{F})$ is a finite central projection in $\mathscr{F}$. Then $Q\mathscr{F}Q$ is finite, so that its subalgebra $Q\mathscr{F}(W)Q$ must also be finite. But by Prop. 3.1 (7), Q is also in $\mathscr{Z}(\mathscr{F}(W))$. Lemma 3.2 entails, however, that $\mathscr{F}(W)$ is purely infinite. Thus, $Q = 0$. Since, in addition, $\mathscr{F}'$ is Abelian (Prop. 3.1 (4)), $\mathscr{F}$ is type I_∞ using [*13*, Section III.3.1] and the separability of $\mathscr{H}$. The rest of the proposition follows from [*13*, Section III.3.1]. □

IV. CENTRAL DECOMPOSITION

With this information we can proceed to the central decomposition into irreducible representations of a general Poincaré-invariant net of local field algebras, which corresponds to the decomposition into pure phases of the given physical system. The following theorem is a generalization to field algebras of the main result of [*1*], shown for observable algebras.

THEOREM 4.1. — Let $\Lambda = \{ \mathscr{H}, U(\mathscr{P}_+^\uparrow), \Omega, \{ \mathscr{F}(\mathscr{O}) \}, Z \}$ be a Poincaré-invariant field system. The central decomposition of $\mathscr{F}$ leads to a unique integral decomposition of Λ into irreducible, translation-invariant field systems. Precisely, there exists a measure ν on the spectrum S of $\mathscr{Z}(\mathscr{F})$ and measurable families of Hilbert spaces $\zeta \to \mathscr{H}(\zeta)$, von Neumann algebras $\zeta \to \mathscr{F}(\zeta) \subseteq \mathscr{B}(\mathscr{H}(\zeta))$, and strongly continuous, unitary representations of the translations $\zeta \to T(\mathbb{R}^4)(\zeta)$, such that

$$\mathscr{H} = \int_S^\oplus \mathscr{H}(\zeta) d\nu(\zeta), \quad \mathscr{F} = \int_S^\oplus \mathscr{F}(\zeta) d\nu(\zeta), \quad T(a) = \int_S^\oplus T(a)(\zeta) d\nu(\zeta), \quad \forall a \in \mathbb{R}^4.$$

For ν-almost all ζ, $T(\mathbb{R}^4)(\zeta)$ satisfies the spectrum condition, $\mathscr{F}(\zeta)$ is irreducible in $\mathscr{H}(\zeta)$, $P_0(\zeta)\mathscr{H}(\zeta) = (P_0\mathscr{H})(\zeta)$ is one-dimensional (unique vacuum in $\mathscr{H}(\zeta)$), and $\{ \mathscr{H}(\zeta), T(\mathbb{R}^4)(\zeta), \Omega(\zeta), \{ \mathscr{F}(\mathcal{O})(\zeta) \}, Z(\zeta) \}$ is an irreducible, translation-invariant field system, where $\Omega = \int^{\oplus} \Omega(\zeta)d\nu(\zeta)$ and $Z = \int^{\oplus} Z(\zeta)d\nu(\zeta)$. Moreover, if $F \in \mathscr{F}(\mathcal{O})$ and $F = \int^{\oplus} F(\zeta)d\nu(\zeta)$, then $F(\zeta) \in \mathscr{F}(\mathcal{O})(\zeta)$ ν-almost everywhere.

Proof. — For details concerning the decomposition of $\mathscr{H}$ and $\mathscr{F}$ with respect to the center of $\mathscr{F}$, see [*13*]. By Prop. 3.1 (1), $T(a)$ decomposes, $\forall a \in \mathbb{R}^4$, and since for any $\Psi = \int^{\oplus} \Psi(\zeta)d\nu(\zeta) \in \mathscr{H}$

$$\| T(a)\Psi \|^2 = \int_S \| T(a)(\zeta)\Psi(\zeta) \|^2 d\nu(\zeta),$$

the strong continuity of $T(a)(\zeta)$ for ν-almost all ζ easily follows. If

$$T(a) = \int e^{ia.p}E(dp), \qquad a \in \mathbb{R}^4, \qquad a.p = a^0p^0 - \underline{a}.\underline{p},$$

is the spectral decomposition of the translation group $T(\mathbb{R}^4)$, for any Borel set $\Delta \subseteq \mathbb{R}^4$, $E(\Delta) \in \mathscr{F}$ also decomposes, and one has, by the uniqueness of the spectral decomposition of $T(\mathbb{R}^4)(\zeta)$,

$$E(\Delta) = \int_S^{\oplus} E(\Delta)(\zeta)d\nu(\zeta), \qquad T(a)(\zeta) = \int e^{ia.p}E(dp)(\zeta), \quad \nu\text{-almost all } \zeta. \tag{4.1}$$

The spectrum condition, $E(\Delta) = 0$ for any Borel set Δ with empty intersection with the forward light cone, yields at once $E(\Delta)(\zeta) = 0$ for any such Δ and therewith the spectrum condition for $T(\mathbb{R}^4)(\zeta)$, ν-almost everywhere. That all of these claims are true, and in particular that (4.1) holds for all $\zeta \in N_1 \subseteq S$ with $\nu(S \backslash N_1) = 0$ and N_1 independent of a, has been shown carefully in [*17*].

Although it is clear from [*13*] that all $F \in \mathscr{F}$ decompose, it is necessary that one finds a set $N \subseteq S$ with $\nu(S\backslash N) = 0$, for which one can construct a translation-invariant field system $\Lambda(\zeta)$ for every $\zeta \in N$. This, however, involves *prima facie* uncountably many conditions, which could lead to a zero-set catastrophe. For that reason it is necessary to take some pains with the construction.

Let $\mathscr{K}_r$ be the set of all double cones with apexes at rational points in $\mathbb{R}^4$ (with respect to some given axis system). For each $\mathcal{O} \in \mathscr{K}_r$ pick a countable, weakly dense set in $\mathscr{F}(\mathcal{O})$ (this is possible since $\mathscr{H}$ is separable) and take the union over all $\mathcal{O} \in \mathscr{K}_r$ of such sets. Let $\mathscr{R}$ be the set consisting of the operators in this union conjugated with all $T(a_i)$, $a_i \in \mathbb{R}^4$ rational, and with Z.

Then $\mathcal{R}$ is countable. Thus it is possible to pick a set $N_2 \subseteq S$ with $\nu(S \backslash N_2)=0$ such that for all $\zeta \in N_2$ and all $\mathcal{R} \ni F = \int^{\oplus} F(\zeta)d\nu(\zeta)$, one has $F(\zeta) \in \mathcal{B}(\mathcal{H}(\zeta))$. By construction and [*13*], for every $F \in \mathcal{R}$, every rational $a_i \in \mathbb{R}^4$ and every $\zeta \in N_3 \subseteq N_1 \cap N_2$, $(T(a_i)FT(a_i)^{-1})(\zeta) = T(a_i)(\zeta)F(\zeta)T(a_i)(\zeta)^{-1}$, where, of course, $\nu(S \backslash N_3)=0$.

By Prop. 3.1 (5), Z also decomposes into a direct integral of unitarities: $Z = \int^{\oplus} Z(\zeta)d\nu(\zeta)$. In view of the fact that for any $\Phi = \int^{\oplus} \Phi(\zeta)d\nu(\zeta)$, $\Psi = \int^{\oplus} \Psi(\zeta)d\nu(\zeta) \in \mathcal{H}$, and any $F \in \mathcal{F}$,

$$\langle \Phi, (F \pm ZFZ^*)\Psi \rangle = \int_S \langle \Phi(\zeta), (F(\zeta) \pm Z(\zeta)F(\zeta)Z(\zeta)^*) \Psi(\zeta) \rangle d\nu(\zeta),$$

one can find an $N_4 \subseteq N_3$ with $\nu(S \backslash N_4) = 0$ such that the locality of $\{ \mathcal{F}(\mathcal{O}) \}$ determined by Z carries over into each $\mathcal{H}(\zeta)$, $\zeta \in N_4$, to give the locality of $\mathcal{R}(\zeta)$ determined by $Z(\zeta)$.

Define, for $\zeta \in N_4$ and $\mathcal{O} \in \mathcal{K}_r$, $\mathcal{F}(\mathcal{O})(\zeta)$ to be the von Neumann algebra generated by all $F(\zeta)$ such that $F \in \mathcal{F}(\mathcal{O}) \cap \mathcal{R}$. By construction,

$$T(a_i)(\zeta)\mathcal{F}(\mathcal{O})(\zeta)T(a_i)(\zeta)^{-1} = \mathcal{F}(\mathcal{O}_{a_i})(\zeta)$$

for all rational $a_i \in \mathbb{R}^4$, $\mathcal{O} \in \mathcal{K}_r$. Define further for any $\mathcal{O} \subset \mathbb{R}^4$,

$$\mathcal{F}(\mathcal{O})(\zeta) = \mathrm{V} \{ \mathcal{F}(\mathcal{O}_0)(\zeta) \mid \mathcal{O}_0 \in \mathcal{K}_r, \mathcal{O}_0 \subseteq \mathcal{O} \}.$$

It is clear that $\{ \{ \mathcal{F}(\mathcal{O})(\zeta) \}, Z(\zeta) \}$ defines a local net. And using the condition of local generation assumed for $\mathcal{F}$ and the arguments of [*13*, Théorème II.3.1], the last claim of the theorem follows easily.

It was seen in the proof of Prop. 3.1 that $P_0\mathcal{F}P_0$ is Abelian. Thus, $(P_0\mathcal{F}P_0)(\zeta) = P_0(\zeta)\mathcal{F}(\zeta)P_0(\zeta)$ is Abelian for ν-almost all ζ. Since $\mathcal{F}' = \mathcal{Z}(\mathcal{F})$, Corollaire II.3.1 (*i*) and Prop. 2.1 (2) of [*13*] entail $\mathcal{F}(\zeta) = \mathcal{B}(\mathcal{H}(\zeta))$ (ν-almost everywhere), implying that $P_0(\zeta)\mathcal{F}(\zeta)P_0(\zeta)$ must be irreducible on $P_0(\zeta)\mathcal{H}(\zeta)$. Thus, $P_0(\zeta)\mathcal{H}(\zeta)$ is one-dimensional. Indeed, one has from Prop. 3.1 (3) that $\mathbb{C}\Omega(\zeta) = P_0(\zeta)\mathcal{H}(\zeta)$, since $\mathcal{Z}(\mathcal{F})(\zeta) = \mathbb{C}I(\zeta)$, ν-almost everywhere. Therefore, $\Omega(\zeta)$ is the unique translation-invariant state in $\mathcal{H}(\zeta)$. Since the isotony and the translation covariance of each net $\{ \mathcal{F}(\mathcal{O})(\zeta) \}$ is obvious, the theorem is proved, up to uniqueness. The strong continuity of $T(\mathbb{R}^4)$ and the condition of local generation on $\mathcal{F}$ yield the uniqueness of this decomposition in the sense of [*18*, Theorem 8.23]. □

Remarks. (1). — We note that Prop. 3.1 (7) was not used in the proof of Theorem 4.1. The containment $\mathcal{Z}(\mathcal{F}(W)) \subseteq \mathcal{Z}(\mathcal{F})$ is, in fact, a consequence of Theorem 4.1. In each irreducible translation-invariant field system $\Lambda(\zeta)$, one can apply a result from [*14*] as extended by Longo [*19*, Theorem 3] to conclude that $\mathscr{F}(W)(\zeta)$ is a type III_1 factor for each $W \in \mathscr{W}$ (ν-almost everywhere); here one must use Prop. 3.1 (2) and (6). Therefore, both $\mathscr{F}(\zeta)$ and $\mathscr{F}(W)(\zeta) \vee \mathscr{F}(W)(\zeta)'$ are irreducible in $\mathscr{H}(\zeta)$. From the proof of Prop. 3.1, $\mathscr{Z}(\mathscr{F}) \subseteq \mathscr{Z}(\mathscr{F}(W))$; thus $\mathscr{F}(W) = \int^{\oplus} \mathscr{F}(W)(\zeta) d\nu(\zeta)$ [*13*, Section II.3]. One can then conclude $\mathscr{Z}(\mathscr{F}) = \mathscr{Z}(\mathscr{F}(W))$ from [*13*, Cor. II.3.1 (*ii*)].

(2). — $F \to F(\zeta)$ does not yield a representation of $\mathscr{F}$ on $\mathscr{H}(\zeta)$ unless there is a projection $P \in \mathscr{Z}(\mathscr{F})$ such that $\mathscr{H}(\zeta) = P\mathscr{H}$, i. e. unless $\mathscr{F}(\zeta)$ is a direct summand.

We can now give a quick proof of Lemma 3.2.

Proof of Lemma 3.2. — As already noted in the preceding Remark, $\mathscr{F}(W)(\zeta)$ is a type III_1 factor for all $W \in \mathscr{W}$ and ν-almost all ζ. One can conclude from [*13*, Corollaire II.5.2] that $\mathscr{F}(W) = \int^{\oplus} \mathscr{F}(W)(\zeta) d\nu(\zeta)$ is itself type III, for any $W \in \mathscr{W}$. □

V. SPONTANEOUS BREAKING OF LORENTZ GROUP SYMMETRY

In [*1*] Araki gives an example of a Poincaré-invariant field system $\{ \mathscr{H}, U(\mathscr{P}_+^\uparrow), \Omega, \{ \mathscr{F}(\mathscr{O}) \}, Z \}$ such that *no* vector in $P_0\mathscr{H}$ is invariant under the Lorentz subgroup $U(\mathscr{L}_+^\uparrow)$ of $U(\mathscr{P}_+^\uparrow)$. His construction was carried out for a net of observable algebras $\{ \mathscr{A}(\mathscr{O}) \}$, but it works as well for a net of field algebras. Therefore, in the decomposition given in Theorem 4.1, the Lorentz group symmetry is spontaneously broken in each $\mathscr{H}(\zeta)$, i. e. the representation $U(\mathscr{L}_+^\uparrow)$ decomposes into unitary representations $U(\mathscr{L}_+^\uparrow)(\zeta)$ in $\mathscr{H}(\zeta)$ only for a collection of ζ's contained in a set of ν-measure 0, as can be seen by examining his example. And in the Appendix we present an example of a Poincaré-invariant field system wherein the special vector Ω *is* $U(\mathscr{P}_+^\uparrow)$-invariant (and is the only such vector in $P_0\mathscr{H}$), but $\dim (P_0\mathscr{H}) > 1$. Once again, in this new example the Lorentz group symmetry is spontaneously broken in each $\mathscr{H}(\zeta)$. We now show that the group $U(\mathscr{P}_+^\uparrow)$ decomposes properly if and only if every vector in $P_0\mathscr{H}$ is $U(\mathscr{P}_+^\uparrow)$-invariant.

PROPOSITION 5.1. — Let $\{ \mathscr{H}, U(\mathscr{P}_+^\uparrow), \Omega, \{ \mathscr{F}(\mathscr{O}) \}, Z \}$ be a Poincaré-invariant field system. The following are equivalent.

(1) Every vector in $P_0\mathscr{H}$ is invariant under $U(\mathscr{P}_+^\uparrow)$.

(2) $U(\mathscr{P}_+^\uparrow) \subset \mathscr{Z}(\mathscr{F})' (= \mathscr{F})$.

(3) Under the central decomposition of $\mathscr{F}$, $U(\mathscr{P}_+^\uparrow)$ decomposes into a direct integral of strongly continuous unitary representations of (the universal covering group of) the Poincaré group $\mathscr{P}_+^\uparrow$.

Remark. — In two space-time dimensions the implication (2) → (1) is not true. The implication (1) → (3) was shown in [*1*], using another argument.

Proof. — It is clear from the proof of Theorem 4.1 and [*13*, Section II.3] that (2) and (3) are equivalent. Assume now (2) or (3). Since the translations form an invariant subgroup of $\mathscr{P}_+^\uparrow$, it is easy to see that $U(\lambda)P_0\mathscr{H} = P_0\mathscr{H}$, $\forall\lambda \in \mathscr{L}_+^\uparrow$. Thus, $P_0 \in U(\mathscr{L}_+^\uparrow)'$. It therefore follows that $V_0(\mathscr{L}_+^\uparrow) \equiv P_0U(\mathscr{L}_+^\uparrow)P_0$ defines a strongly continuous unitary representation of $\mathscr{L}_+^\uparrow$. But it was seen in the proof of Prop. 3.1 that $P_0\mathscr{F}P_0$ is Abelian. By (2), $V_0(\mathscr{L}_+^\uparrow)$ must be Abelian, which is only possible, in three or four space-time dimensions, if the representation is trivial. Thus, $U(\mathscr{L}_+^\uparrow)\Phi = V_0(\mathscr{L}_+^\uparrow)\Phi = \Phi$, $\forall\Phi \in P_0\mathscr{H}$, proving (1).

If, on the other hand, one assumes (1), then for any $H \in \mathscr{Z}(\mathscr{F})$ and $\lambda \in \mathscr{P}_+^\uparrow$, $H\Omega = U(\lambda)H\Omega = U(\lambda)HU(\lambda)^{-1}\Omega$. But $U(\lambda)\mathscr{Z}(\mathscr{F})U(\lambda)^{-1} = \mathscr{Z}(\mathscr{F})$, so $H = U(\lambda)HU(\lambda)^{-1}$, since Ω is separating for $\mathscr{Z}(\mathscr{F})$. Thus, (2) follows. □

We next show that any $\mathscr{H}(\zeta)$ in the central decomposition of $\mathscr{H}$ that actually occurs as a direct summand must reduce the entire Poincaré group representation $U(\mathscr{P}_+^\uparrow)$, not just the translations. Thus, the spontaneous breaking of the Lorentz group symmetry can only occur when the (up to measure isomorphism unique) measure space (S, ν) given in Theorem 4.1 is not purely atomic.

PROPOSITION 5.2. — Let $\{\mathscr{H}, U(\mathscr{P}_+^\uparrow), \Omega, \{\mathscr{F}(\mathscr{O})\}, Z\}$ be a Poincaré-invariant field system, and let (S, ν) be the measure space given by Theorem 4.1. Then for any $\zeta \in S$ such that $\nu(\zeta) > 0$, $U(\mathscr{P}_+^\uparrow)(\zeta) \equiv U(\mathscr{P}_+^\uparrow) \restriction \mathscr{H}(\zeta)$ is a strongly continuous unitary representation of the Poincaré group, and $\{\mathscr{H}(\zeta), U(\mathscr{P}_+^\uparrow)(\zeta), \Omega(\zeta), \{\mathscr{F}(\mathscr{O})(\zeta)\}, Z(\zeta)\}$ is an irreducible, Poincaré-invariant field system.

Proof. — If $\nu(\zeta) > 0$, then there exists a projection $P_\zeta \in \mathscr{Z}(\mathscr{F})$ (corresponding to the characteristic function of ζ in $L_\infty(S, d\nu)$) such that $P_\zeta\mathscr{H} = \mathscr{H}(\zeta)$ and $\mathscr{F}(\zeta) = P_\zeta\mathscr{F}P_\zeta$. In light of Theorem 4.1, it suffices to show that $P_\zeta \in U(\mathscr{P}_+^\uparrow)'$.

Assume there is a $\lambda \in \mathscr{P}_+^\uparrow$ with $U(\lambda)P_\zeta U(\lambda)^{-1} \neq P_\zeta$. By the strong continuity of $U(\mathscr{P}_+^\uparrow)$, there exists a $\lambda_0 \in \mathscr{P}_+^\uparrow$ and an open neighbourhood $\mathscr{N} \subseteq \mathscr{P}_+^\uparrow$ of λ_0 such that $U(\lambda)P_{\zeta,\lambda_0}U(\lambda)^{-1} \neq P_{\zeta,\lambda_0}$ for all $\lambda \in \mathscr{N}$, where

$$P_{\zeta,\lambda_0} \equiv U(\lambda_0)P_\zeta U(\lambda_0)^{-1}.$$

Moreover, there exists a $\lambda_1 \in \mathscr{N}$ such that the projection

$$P \equiv P_{\zeta,\lambda_0}U(\lambda_1)P_{\zeta,\lambda_0}U(\lambda_1)^{-1}$$

is nonzero. Of course, $P \leq P_{\zeta,\lambda_0}$ and both are contained in $\mathscr{Z}(\mathscr{F})$. But $\mathscr{Z}(\mathscr{F}) \restriction P_{\zeta,\lambda_0}\mathscr{H}$ is trivial by Theorem 4.1, so that $P = P_{\zeta,\lambda_0}$, entailing

$P_{\zeta,\lambda_0} \leq U(\lambda_1)P_{\zeta,\lambda_0}U(\lambda_1)^{-1}$. Hence, once again $P_{\zeta,\lambda_0} = U(\lambda_1)P_{\zeta,\lambda_0}U(\lambda_1)^{-1}$, a contradiction. □

There is, however, an interesting property in the context of Poincaré-invariant field systems that insures that the representation of the Poincaré group in a field system Λ has the proper action on $\mathscr{Z}(\mathscr{F})$ in order to be able to decompose. This property is probably not a necessary condition, but we discuss it here in some detail because it has other interesting consequences and because it is very likely to be closely related to the implicit presence of quantum fields. As we shall see in Section 6, if $U(\mathscr{P}_+^\uparrow)$ implements the Poincaré transformations on a relativistic quantum field $\varphi(x)$ satisfying the Wightman axioms, then every vector in $P_0\mathscr{H}$ is $U(\mathscr{P}_+^\uparrow)$-invariant. This is due to additional analyticity properties inherent in quantum fields [*11*], that are not at hand in the general algebraic framework. The condition to be discussed provides at *least* enough additional analyticity that the properties (1)-(3) in Prop. 5.1 hold: this is the lesson presented here. To formulate this condition we must recall some facts from the Tomita-Takesati theory [*20*].

If Φ is a cyclic and separating vector for the von Neumann algebra $\mathscr{M}$, then there exists an antiunitary involution J and a positive, selfadjoint (generally unbounded) operator Δ such that $\mathscr{M}\Phi \subset D(\Delta^{1/2})$, the domain of definition of $\Delta^{1/2}$, $J\Phi = \Phi$, and

$$J\Delta^{1/2}A\Phi = A^*\Phi, \qquad \Delta^{it}A\Delta^{-it} \equiv \sigma_t(A) \in \mathscr{M}, \qquad \forall A \in \mathscr{M}, \quad \forall t \in \mathbb{R}. \quad (5.1)$$

Also $J\mathscr{M}J = \mathscr{M}'$. $\{\sigma_t\}_{t\in\mathbb{R}}$ is called the modular automorphism group of $\{\mathscr{M}, \Phi\}$ and J the modular conjugation.

Given a Poincaré-invariant field system Λ, we consider $\mathscr{M} = \mathscr{F}(W_j)$, where $W_j \equiv \{x \in \mathbb{R}^4 \mid |x^0| < x^j\}$, $j = 1, 2, 3$ (thus $W_1 = W_R$), and let $V_j(t) \equiv U(v_j(t), 0)$ be the Abelian group of unitaries implementing the Lorentz velocity transformations in the x_j-direction, $j = 1, 2, 3$. Then by the Lorentz invariance of the net, $V_j(t)\mathscr{F}(W_j)V_j(t)^{-1} = \mathscr{F}(W_j)$, $j = 1, 2, 3$, for any $t \in \mathbb{R}$. In other words, the restriction of the action of the appropriate Lorentz velocity transformation to the appropriate wedge algebra yields a group of automorphisms of this algebra. Although Prop. 3.1 assures us that the Tomita-Takesaki theory can be applied to $\{\mathscr{F}(W), \Omega\}$, $W \in \mathscr{W}$, the above mentioned automorphism groups need not coincide with the modular automorphism groups. Indeed, the examples in the Appendix and in [*1*] are incidences of this noncoincidence. This can be seen as follows. The coincidence we are speaking of means precisely

$$V_j(t)FV_j(t)^{-1} = \Delta_j^{it}F\Delta_j^{-it}, \qquad \forall t \in \mathbb{R}, \qquad F \in \mathscr{F}(W_j), \qquad j = 1, 2, 3, \quad (5.2)$$

where Δ_j is the modular operator of $\{\mathscr{F}(W_j), \Omega\}$. (Note that we do not specify *which* parametrization of the velocity transformations leads to the equality (5.2) – any would do.) It has already been mentioned that in Araki's example *no* vector in $P_0\mathscr{H}$ and that in the example in the Appendix only Ω is invariant under $\{ V_j(t) \}_{t\in\mathbb{R}}, j = 1, 2, 3$. If, however, (5.2) is *assumed*, every vector in $P_0\mathscr{H}$ is $U(\mathscr{P}_+^\uparrow)$-invariant. This is shown in the following proof.

PROPOSITION 5.3. — If $\{ \mathscr{H}, U(\mathscr{P}_+^\uparrow), \Omega, \{ \mathscr{F}(\mathcal{O}) \}, Z \}$ is a Poincaré-invariant field system and (5.2) holds, then $U(\mathscr{P}_+^\uparrow) \subset \mathscr{F}$ and, at least in more than two space-time dimensions, $\Phi \in P_0\mathscr{H}$ implies $U(\lambda)\Phi = \Phi$, all $\lambda \in \mathscr{P}_+^\uparrow$.

Proof. — From Prop. 3.1 (7) one has $\mathscr{Z}(\mathscr{F}) = \mathscr{Z}(\mathscr{F}(W)) \subseteq \mathscr{F}(W)_\Omega$, the centralizer of Ω in $\mathscr{F}(W)$, defined by

$$\mathscr{F}(W)_\Omega \equiv \{ F \in \mathscr{F}(W) \mid \langle \Omega, FG\Omega \rangle = \langle \Omega, GF\Omega \rangle, \qquad \forall G \in \mathscr{F}(W) \}.$$

However, for any $F \in \mathscr{F}(W)_\Omega$, $\Delta^{it} F \Delta^{-it} = F$, $\forall t \in \mathbb{R}$, where Δ is the modular operator for $\{ \mathscr{F}(W), \Omega \}$ [*20*, Lemma 15.8]. Thus, $\mathscr{Z}(\mathscr{F})$ is elementwise invariant under $\{ V_j(t) \}_{t\in\mathbb{R}}$, $j = 1, 2, 3$; these operators are therefore contained in $\mathscr{F}$. Thus, arguing as in the proof of Prop. 5.1, the velocity transformations (in three orthogonal directions) in $P_0 U(\mathscr{L}_+^\uparrow) P_0$ must generate together an Abelian group. Hence, the generators of $\{ V_j(t) \}_{t\in\mathbb{R}}$, $j = 1, 2, 3$, must commute on $P_0\mathscr{H}$. By the algebraic relations of the Lie algebra of the Lorentz group, this implies once again (in three or more space-time dimensions) that $P_0 U(\mathscr{P}_+^\uparrow) P_0$ must be trivial. Thus, $U(\mathscr{P}_+^\uparrow)\Phi = \Phi$ $\forall \Phi \in P_0\mathscr{H}$. Since in two space-time dimensions the Lorentz group consists solely of velocity transformations in one direction, the proposition is proven. □

Remark. — Note that Prop. 5.3 implies that (5.2) is equivalent to $V_j(t) = \Delta_j^{it}$, all $t \in \mathbb{R}$, $j = 1, 2, 3$, at least in more than two space-time dimensions.

Propositions 5.1 and 5.3 entail that (5.2) is a sufficient condition to exclude spontaneous symmetry breaking of the Lorentz group in the pure phases of the theory. Yet another consequence of (5.2) is that the wedge algebras $\{ \mathscr{F}(W) \mid W \in \mathscr{W} \}$ satisfy (twisted) duality.

PROPOSITION 5.4. — Given the hypothesis of Prop. 5.3, $\mathscr{F}(W)' = \mathscr{F}(W')^Z$, for any $W \in \mathscr{W}$.

Proof. — By Poincaré invariance it suffices to show $\mathscr{F}(W_R)' = \mathscr{F}(W_R')^Z$. Since $\mathscr{F}(W_R')^Z \subseteq \mathscr{F}(W_R)'$, one has from the Tomita-Takesaki theory [*20*] $\mathscr{F}(W_R')^Z\Omega \subseteq D(\Delta_1^{-1/2})$ and $J\Delta_1^{-1/2}A\Omega = A^*\Omega$, for any $A \in \mathscr{F}(W_R')^Z$. And by Poincaré invariance and the equality $\Delta_1^{it} = V_1(t)$, for any $t \in \mathbb{R}$, one has $\Delta_1^{it}\mathscr{F}(W_R')^Z\Delta_1^{-it} = F(W_R')^Z$, for all $t \in \mathbb{R}$. Since the same is true of $\mathscr{F}(W_R)'$, which has Ω as a separating vector, the proof of Theorem 2 (*e*) in [*6*] gives the desired equality $\mathscr{F}(W_R')^Z = \mathscr{F}(W_R)'$. □

In the following theorem we summarize results relevant to the central decomposition of $\mathscr{F}$, and we show that the equality of the modular automorphism groups of the wedge algebras with the appropriate Lorentz velocity transformations is *conserved* in the decomposition.

THEOREM 5.5. — Let $\Lambda = \{ \mathscr{H}, U(\mathscr{P}_+^\uparrow), \Omega \{ \mathscr{F}(\mathscr{O}) \}, Z \}$ be a Poincaré-invariant field system such that the modular automorphism group $\{ \sigma_{t,j} \}_{t\in\mathbb{R}}$ of $\{ \mathscr{F}(W_j), \Omega \}$ coincides with the automorphism group of $\mathscr{F}(W_j)$ implemented by $\{ V_j(t) \}_{t\in\mathbb{R}}$ for each $j = 1, 2, 3$ (i. e. (5.2) holds). Then the central decomposition of $\mathscr{F}$ leads to a unique integral decomposition of Λ into irreducible, *Poincaré*-invariant field systems. Precisely, one has the conclusions of Theorem 4.1 and a measurable family of strongly continuous, unitary representations of (the universal covering group of) the Poincaré group, $\zeta \to U(\mathscr{P}_+^\uparrow)(\zeta)$, satisfying the spectrum condition for ν-almost all ζ, such that

$$U(\lambda) = \int_S^\oplus U(\lambda)(\zeta) d\nu(\zeta), \qquad \forall \lambda \in \mathscr{P}_+^\uparrow,$$

and for ν-almost all ζ, $\{ \mathscr{H}(\zeta), U(\mathscr{P}_+^\uparrow)(\zeta), \Omega(\zeta), \{ \mathscr{F}(\mathscr{O})(\zeta) \}, Z(\zeta) \}$ is an irreducible, Poincaré-invariant field system. In addition, the modular automorphism group $\{ \sigma_{t,j}(\zeta) \}_{t\in\mathbb{R}}$ of $\{ \mathscr{F}(W_j)(\zeta), \Omega(\zeta) \}$ coincides with the automorphism group on $\mathscr{F}(W_j)(\zeta)$ implemented by $\{ V_j(t)(\zeta) \}_{t\in\mathbb{R}}$, for $j = 1, 2, 3$, and for ν-almost all ζ. Moreover, $\mathscr{F}(W)(\zeta)' = \mathscr{F}(W')(\zeta)^{Z(\zeta)}$, for any $W \in \mathscr{W}$.

Proof. — One proceeds as in the proof of Theorem 4.1, except that the operators in $\mathscr{R}$ are conjugated in addition by the operators $U(\lambda_i)$, where λ_i is an element of the subgroup $\mathscr{L}_r$ of the Lorentz group $\mathscr{L}_+^\uparrow$ that leaves the set $\mathscr{K}_r$ invariant. $\mathscr{L}_r$ is countable, so $\mathscr{R}$ remains countable. Since by Prop. 5.3 $U(\mathscr{P}_+^\uparrow) \subset \mathscr{F}$, the operators $U(\mathscr{P}_+^\uparrow)$ are all decomposable. Once again there exists a set $N \subseteq S$ with $\nu(S \backslash N) = 0$ such that for all $F \in \mathscr{R}$, all $\lambda_i \in \mathscr{L}_r$, and all $\zeta \in N$,

$$(U(\lambda_i) F U(\lambda_i)^{-1})(\zeta) = U(\lambda_i)(\zeta) F(\zeta) U(\lambda_i)(\zeta)^{-1}.$$

If $\mathscr{L}_r$ is dense in $\mathscr{L}_+^\uparrow$, then it is clear that one can construct a Poincaré-invariant field system $\Lambda(\zeta)$ for each ζ in a set with complement of ν-measure zero.

To see that $\mathscr{L}_r$ is dense in $\mathscr{L}_+^\uparrow$, consider as an example the velocity transformations in the 1-direction:

$$\lambda(t) = \begin{pmatrix} \cosh t & \sinh t & 0 & 0 \\ \sinh t & \cosh t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad t \in \mathbb{R}.$$

To be in $\mathscr{L}_r$, $\cosh t$ and $\sinh t$ must be rational. Since $\mathscr{L}_r$ is a group, it suffices to show that every neighbourhood of the identity $\lambda(0)$ contains an element $\lambda(t) \in \mathscr{L}_r$ (excluding $\lambda(0)$, of course). But this can be done by the velocity transformations of the form

$$\begin{pmatrix} \frac{n+1}{n} & \frac{m}{n} & 0 & 0 \\ \frac{m}{n} & \frac{n+1}{n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad m \text{ odd}, \quad n = \frac{m^2-1}{2}.$$

A similar argument clearly functions for the other subgroups of $\mathscr{L}_+^\uparrow$. Thus the only claim in the theorem that is not now obviously true is the identification of the modular automorphism group $\{ \sigma_{t,j}(\zeta) \}_{t\in\mathbb{R}}$ (this is *prima facie* an abuse of notation, but it will be shown not to be) of $\{ \mathscr{F}(W_j)(\zeta), \Omega(\zeta) \}$ with the automorphism group on $\mathscr{F}(W_j)(\zeta)$ implemented by $\{ V_j(t)(\zeta) \}_{t\in\mathbb{R}}$.

In [*17*, Theorem III.3] it is shown that if $\mathscr{A} = \int^\oplus \mathscr{A}(\zeta)d\mu(\zeta)$ is a decomposable von Neumann algebra in a separable Hilbert space, $\phi = \int^\oplus \phi(\zeta)d\mu(\zeta)$ a normal, faithful, strictly semifinite weight on $\int^\oplus \mathscr{A}(\zeta)d\mu(\zeta)$, where $\zeta \to \phi(\zeta)$ is a measurable field of normal, faithful, strictly semifinite weights on $\{ \mathscr{A}(\zeta) \}$, and σ_t (resp. $\sigma_t(\zeta)$) is the modular automorphism group of $\{ \mathscr{A}, \phi \}$ (resp. $\{ \mathscr{A}(\zeta), \phi(\zeta) \}$), then $\sigma_t = \int^\oplus \sigma_t(\zeta)d\mu(\zeta)$, for every $t \in \mathbb{R}$.

Of course, here $\mathscr{A} = \mathscr{F}(W)$, any $W \in \mathscr{W}$, and the decomposition of $\mathscr{A}$ is that given by the center of $\mathscr{F}$ (Theorem 4.1). The vector Ω (resp. $\Omega(\zeta)$) determines a normal, positive, linear form on $\mathscr{F}(W)$ (resp. $\mathscr{F}(W)(\zeta)$), which is by Prop. 4.2 of [*21*] a normal, strictly semifinite weight on $\mathscr{F}(W)$ (resp. $\mathscr{F}(W)(\zeta)$). Since, moreover, Ω (resp. $\Omega(\zeta)$) is separating for $\mathscr{F}(W)$ (resp. $\mathscr{F}(W)(\zeta)$), the weight it determines is faithful on $\mathscr{F}(W)$ (resp. $\mathscr{F}(W)(\zeta)$). Thus, the result from [*17*] may be applied to the present case.

Since the equation (5.2) also decomposes, i. e. the equality $V_j(t) = \Delta_j^{it}$, $\forall t \in \mathbb{R}$, $j = 1, 2, 3$, carries over to $V_j(t)(\zeta) = \Delta_j^{it}(\zeta)$, for ν-almost all ζ, the proof of the claim is now clear. (At the cost of possibly another set of ν-measure zero, this equality is established for all $\zeta \in N$ and all rational t; strong continuity completes the argument.) □

Before closing this section, we would like to point out a further interesting consequence of the equality $\mathscr{Z}(\mathscr{F}) = \mathscr{Z}(\mathscr{F}(W))$, any $W \in \mathscr{W}$.

PROPOSITION 5.6. — Let $\{ \mathscr{H}, U(\mathscr{P}_+^\uparrow), \Omega, \{ \mathscr{F}(\mathcal{O}) \}, Z \}$ be a Poincaré-invariant field system. Then for any $W \in \mathscr{W}$, if Δ is the modular operator for $\{ \mathscr{F}(W), \Omega \}$, Δ^{it} implements an inner automorphism of $\mathscr{F}$, i. e. $\Delta^{it}\mathscr{F}\Delta^{-it} = \mathscr{F}$, $\forall t \in \mathbb{R}$, and $\{ \Delta^{it} \}_{t\in\mathbb{R}} \subset \mathscr{F}$.

Proof. — By the Tomita-Takesaki theory,

$$\Delta^{it}(\mathscr{F}(W) \vee \mathscr{F}(W)')\Delta^{-it} = \mathscr{F}(W) \vee \mathscr{F}(W)', \qquad \forall t \in \mathbb{R}.$$

And by Prop. 3.1 (4 and 7), one has $\mathscr{F} = \mathscr{F}(W) \vee \mathscr{F}(W)'$. As has already been seen in the proof of Prop. 5.3, $\mathscr{Z}(\mathscr{F}) = \mathscr{Z}(\mathscr{F}(W)) \subset (\{ \Delta^{it} \}_{t\in\mathbb{R}})'$. Since $\mathscr{Z}(\mathscr{F}) = \mathscr{F}'$, it follows that $\Delta^{it} \in \mathscr{F}$, $\forall t \in \mathbb{R}$. □

Thus, if (5.2) does not hold, the theory has many additional symmetries that would have to be understood. It could happen that in a concrete case one could exclude the existence of such additional symmetries, thereby demonstrating that (5.2) must obtain, along with the consequences we have shown.

VI. IN THE PRESENCE OF A QUANTUM FIELD

As mentioned in the Introduction, if we are given a Poincaré-invariant field system $\Lambda = \{ \mathscr{H}, U(\mathscr{P}_+^\uparrow), \Omega, \{ \mathscr{F}(\mathcal{O}) \}, Z \}$ *and* a relativistic quantized field $\varphi(x)$ satisfying the usual axioms [*22*] [*23*], which transforms under $U(\mathscr{P}_+^\uparrow)$ and which has Ω as a cyclic vector, then all vectors in $P_0\mathscr{H}$ are $U(\mathscr{P}_+^\uparrow)$-invariant [*11*, Theorem 3]. (Note, however, that Ω must itself already be $U(\mathscr{P}_+^\uparrow)$-invariant in order to use [*11*]. This assumption is a component of the axioms in [*22*] [*23*].) Thus, as we have seen in the proof of Prop. 5.1 using Prop. 3.1, $U(\mathscr{P}_+^\uparrow) \subset \mathscr{F}$, and we can conclude that the central decomposition of $\mathscr{F}$ leads to a direct integral of Poincaré-invariant field systems. Therefore, if there is a quantum field about, which is weakly related to Λ as above, then the Lorentz symmetry cannot be spontaneously broken in the decomposition into pure phases.

What is more, if there is such a quantum field $\varphi(x)$, which satisfies, in addition, the following equation (we state it for simplicity's sake for the case of a Hermitian scalar field—the more general case can be found in [*7*]):

$$\langle F\Omega, X\Omega \rangle = \langle X^*\Omega, F^*\Omega \rangle, \qquad \forall F \in \mathscr{F}(W_R'), \tag{6.1}$$

for all $X \in \mathscr{P}_0(W_R)$, the set of all polynomials of field operators $\varphi(f)$ averaged with test functions f with support in W_R, then $\{ \mathscr{F}(\mathcal{O}) \}$ satisfies a special condition of duality (see [*9*] [*10*] for details), which itself implies the equation (5.2).

Either the condition (5.2) holds only when a quantum field satisfying the properties above, along with (6.1), exists, in which case (5.2) would be a beautiful necessary and sufficient condition, formulated solely in the framework of the algebraic relativistic quantum theory, for the implicit presence of such a field, or it holds more generally. Even then its combination of the analyticity properties related to the KMS boundary condition of the modular automorphism group (see e. g. Chapter 13 of [*20*]) and the geometric properties of the Lorentz group, which is implicit in the proof of Prop. 5.3, is likely to have more far-reaching consequences than those we have discussed here—particularly in the direction of bringing the extensive analyticity properties of field theories [*22*] [*23*] [*11*] into the algebraic relativistic quantum theory.

Whatever the situation, and we would not be surprised if (5.2) along with technicalities really does signify the presence of some field in the sense of the second paragraph, this matter is worth further examination.

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APPENDIX

In this appendix we construct a class of examples of Poincaré-invariant field systems $\{ \mathcal{H}, U(\mathcal{P}_+^\uparrow), \Omega, \{ \mathcal{F}(\mathcal{O}) \}, Z \}$, wherein the special vector Ω is the unique (up to a phase factor) $U(\mathcal{P}_+^\uparrow)$-invariant vector in $\mathcal{H}$ and the subspace $P_0\mathcal{H}$ is not one-dimensional. The groups of modular automorphisms of the wedge algebras do not, therefore, coincide with the action of the Lorentz velocity transformations on these algebras (Prop. 5.3). We produce examples of the above situation where duality holds and examples for which it does not hold.

Let $\mathcal{C} \equiv \{ f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^d) \mid f(-x) = f(x) \}$ ($\mathcal{S}_{\mathbb{R}}(\mathbb{R}^d)$ is the set of real-valued functions in $\mathcal{S}(\mathbb{R}^d)$) and let $\varphi(x)$ be the free scalar Hermitian field of mass m on the Fock space $\mathcal{F}_m$ [22]. For any $f, g \in \mathcal{C}$, the commutator $[\varphi(f), \varphi(g)]$ is an antisymmetric operator, so that $[\varphi(f), \varphi(g)] \in \{ irI \}_{r\in\mathbb{R}}$, since it must be proportional to the identity operator I on $\mathcal{F}_m$. On the other hand, if Θ is the TCP-operator for the field $\varphi(x)$, we have, since $\Theta\varphi(f)\Theta = \varphi(f)$ for any $f \in \mathcal{C}$,

$$[\varphi(f), \varphi(g)] = \Theta[\varphi(f), \varphi(g)]\Theta = -[\varphi(f), \varphi(g)],$$

since Θ is antiunitary. Therefore, the Weyl groups generated by the (selfadjoint) closures of $\{ \varphi(f) \mid f \in \mathcal{C} \}$ on D_0, the usual domain of definition [22], commute pairwise and generate an Abelian von Neumann algebra $\mathcal{A}$. Let $U_m(\mathcal{P}_+^\uparrow)$ be the usual unitary representation of the Poincaré group on $\mathcal{F}_m$. Since the set $\mathcal{C}$ is invariant under the induced action of the Lorentz group $\mathcal{L}_+^\uparrow$, the corresponding subgroup $U_m(\mathcal{L}_+^\uparrow) \subset U_m(\mathcal{P}_+^\uparrow)$ acts automorphically on $\mathcal{A}$.

Now let Ω_m be the (up to a phase factor unique) $U_m(\mathcal{P}_+^\uparrow)$-invariant unit vector in $\mathcal{F}_m$ and define $\mathcal{H} \equiv \mathcal{F}_m \otimes \overline{A\Omega_m}$ (the closure taken in $\mathcal{F}_m$). Also let $\Omega = \Omega_m \otimes \Omega_m$. Define $U(\mathcal{L}_+^\uparrow)$ to act on $\mathcal{H}$ as $U_m(\mathcal{L}_+^\uparrow) \otimes U_m(\mathcal{L}_+^\uparrow)$ in the obvious manner and $T(\mathbb{R}^d)$ to act on $\mathcal{F}_m$ as $T_m(\mathbb{R}^d)$ and as the identity on $\overline{\mathcal{A}\Omega_m}$. For each $\mathcal{O} \subseteq \mathbb{R}^d$, set $\mathcal{F}(\mathcal{O}) \equiv \mathcal{F}_m(\mathcal{O}) \otimes \mathcal{A}$, where $\mathcal{F}_m(\mathcal{O})$ is the von Neumann algebra generated by the Weyl groups generated by the closures of $\{ \varphi(f) \mid f \in \mathcal{S}_{\mathbb{R}}(\mathcal{O}) \}$ on D_0. Z is taken to be the identity on $\mathcal{H}$. The thusly defined quintuple $\{ \mathcal{H}, U(\mathcal{P}_+^\uparrow), \Omega, \{ \mathcal{F}(\mathcal{O}) \}, Z \}$ is easily seen to be a Poincaré-invariant field system. Moreover, Ω is the only $U(\mathcal{P}_+^\uparrow)$-invariant unit vector in $\mathcal{H}$, while the subspace $\mathbb{C}\Omega_m \otimes \overline{\mathcal{A}\Omega_m}$ is $T(\mathbb{R}^d)$-invariant. Thus, we may use Prop. 5.3 to conclude that the modular automorphism groups of the wedge algebras do not coincide with the action of the appropriate Lorentz velocity transformations.

Since $\mathcal{A}$ is maximally Abelian in $\mathcal{B}(\overline{\mathcal{A}\Omega_m})$, we see by the known duality of the free field [24] that for any $\mathcal{O} \in \mathcal{K} \cup \mathcal{W}$, $\mathcal{F}(\mathcal{O}') = \mathcal{F}_m(\mathcal{O})' \otimes \mathcal{A}' = \mathcal{F}_m(\mathcal{O}') \otimes A = \mathcal{F}(\mathcal{O}')$. Thus, duality holds for this net of local algebras. If we define $\mathcal{A}(\mathcal{O})$ to be the von Neumann algebra generated by the field operator $\varphi(f)$ as above but taking only those $f \in \mathcal{C}$ with support in $\mathcal{O} \cup -\mathcal{O}$, then with $\mathcal{F}_0(\mathcal{O}) \equiv \mathcal{F}_m(\mathcal{O}) \otimes \mathcal{A}(\mathcal{O})$, we once again can define a Poincaré-invariant field system. $\{ \mathcal{H}, U(\mathcal{P}_+^\uparrow), \Omega, \{ \mathcal{F}_0(\mathcal{O}) \}, Z \}$ with $\dim(P_0\mathcal{H}) > 1$ and Ω the only $U(\mathcal{P}_+^\uparrow)$-invariant vector, where, however, duality is not satisfied (since $\mathcal{A}(\mathcal{O}) \neq \mathcal{A}$, for any $\mathcal{O} \in \mathcal{K} \cup \mathcal{W}$).

We remark that in these examples any known net of local field algebras can replace $\{ \mathcal{F}_m(\mathcal{O}) \}$ in the first factor of $\{ \mathcal{F}(\mathcal{O}) \}$. If this new set satisfies (twisted) duality, all results above are unchanged, and if (twisted) duality fails, only the assertions concerning duality must be dropped.

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