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Legendre transforms and *r*-particle irreducibility in quantum field theory: the formalism for fermions (*)

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ABSTRACT. — For Euclidean field theories involving both a boson field ϕ and fermi field ψ, let $G^M\{J\}$ be the generating functional of connected Green's functions with source terms $\int J_m \psi^{m_f} \phi^{m_b} dx$, $m = (m_f, m_b) \in M$. We analyze the Legendre transform $\Gamma^M\{A\}$ of $G^M\{J\}$ in the framework of formal power series on a Grassmann algebra. By using Spencer's *t*-derivatives, we establish, independently of perturbation theory, that Γ^M is cluster-irreducible for various choices of second order M. For certain simple M's we extend these results up to fourth order. We show that Γ^M generates M-field projectors and Bethe-Salpeter kernels with appropriate irreducibility properties.

RÉSUMÉ. — Soit $G^M\{J\}$ la fonction génératrice des fonctions de Green connexes dans une théorie des champs euclidiens contenant un champ ψ de fermions et un champ ϕ de bosons. Les termes de source sont $\int J_m \psi^{m_f} \phi^{m_b} dx$, $m = (m_f, m_b) \in M$. On analyse $\Gamma^M\{A\}$, la transformée de Legendre de $G^M\{J\}$, dans le contexte des séries formelles de puissances sur une algèbre de Grassmann. Grâce aux *t*-dérivées de Spencer, on établit, indépendamment de la théorie des perturbations, les propriétés d'irréduc-

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tibilité de Γ^M pour M de deuxième ordre. Pour M assez simple on étend ces résultats jusqu'au quatrième ordre. On montre que Γ^M engendre les projections sur les champs de M et les noyaux de Bethe-Salpeter avec l'irréductibilité appropriée.

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§ I. INTRODUCTION

This series of papers [*1*]-[*4*] is dedicated to the proposition that the rth Legendre transform $\Gamma^{(r)}$ provides a unifying framework for the concept of « r-particle-irreducibility » in quantum field theory. In our previous papers (which we shall refer to as I-IV), we gave elementary non-perturbative proofs of the irreducibility properties of $\Gamma^{(r)}$ $(r \leq 4)$ and we explained the role played by $\Gamma^{(r)}$ as the generator of (channel-) irreducible objects such as Bethe-Salpeter kernels. In these previous papers we considered the case of (Euclidean) quantum field theories involving a single scalar boson field ϕ. The purpose of the present paper is to extend our analysis to field theories that involve a fermi field ψ. This will enable us (see Ref. [*5*]) to apply the machinery of the Legendre transform to the study of the energy-momentum spectrum of the Yukawa$_2$ model with interaction $\phi\bar{\psi}\psi$. The extension to general multi-field theories or to more general fields is fairly obvious, and except for a few remarks at the end of this section, we shall not pursue this generalization any further here.

To define the higher Legendre transform $\Gamma^{(N)}$ for boson theories (see I) one introduces higher degree source terms,

$$\phi^n . J_n = \int \phi(x_1) \dots \phi(x_n) J_n(x_1, \dots, x_n) dx_1 \dots dx_n ,$$

J_n being a symmetric function of n arguments $x_i \in \mathbb{R}^d$, into the generating functional for connected Green's functions

$$G^{(N)}\{ J_1, \dots, J_N \} = \ln \left\langle \exp \sum_{n=1}^{N} \phi^n . J_n \right\rangle ;$$

and one then makes a (functional) Legendre transform in the « variables » $(J_1, \dots, J_N)$:

$$G^{(N)}\{ J_1, \dots, J_N \} \rightarrow \Gamma^{(N)}\{ A_1, \dots, A_N \} .$$

Now consider a theory involving anticommuting fermi fields. (We shall use the notation $\psi = (\psi_-, \psi_+)$ for the (independent) Euclidean fields in place of the relativistic notation ψ, $\bar{\psi}$; spinor indices will be suppressed.)

It is clear that source terms such as $\psi\alpha + \phi f = \psi_-\alpha_- + \psi_+\alpha_+ + \phi f$, with $\alpha = (\alpha_-, \alpha_+)$ being ordinary functions, are inadequate. In fact,

$$e^{\psi\alpha+\phi f} = (1 + \psi\alpha)e^{\phi f}$$

so that it is impossible to generate a Green's function containing more than one fermion.

The formal solution to this problem goes back to Schwinger [6]: one requires that the fermion source « functions » $\alpha_\pm$ anticommute with themselves and with the fermion fields

$$\{ \psi_\varepsilon, \psi_{\varepsilon'} \} = \{ \psi_\varepsilon, \alpha_{\varepsilon'} \} = \{ \alpha_\varepsilon, \alpha_{\varepsilon'} \} = 0, \qquad \varepsilon, \varepsilon' = \pm . \tag{1.1}$$

Of course, the bosonic objects ϕ and f commute with $\psi_\pm$, $\alpha_\pm$ and among themselves. The partition function for a theory of interacting (Euclidean) bosons and fermions

$$Z\{ \alpha, f \} = \langle e^{\psi\alpha+\phi f} \rangle \tag{1.2}$$

then generates the Schwinger functions of the theory provided one interprets the functional derivatives $\frac{\delta}{\delta\alpha_\pm}$ of Z in a suitable non-commutative sense (basically as derivatives on a Grassmann algebra [7]). The expectation $\langle\ .\ \rangle$ in (1.2) corresponds to the formal measure

$$\text{const. } e^{-V}(e^{-\frac{1}{2}\phi C^{-1}\phi}\delta\phi)(e^{-\psi_- S_0^{-1}\psi_+}\delta\psi), \tag{1.3}$$

where C and S_0 are the free propagators or two-point functions for the boson and fermion fields

$$C = (-\Delta + m_b)^{-1} \qquad S_0 = (-i\partial\!\!\!/ + m_f)^{-1} \tag{1.4}$$

and V is the (local) interaction. For example, for the (unrenormalized) Yukawa model

$$V = \lambda \int \phi\psi_-\psi_+ dx$$

and we have

$$Z\{ \alpha, f \} = \text{const.} \int d\mu(\phi)e^{\phi f} \int e^{\psi\alpha}e^{-\psi_- S^{-1}\psi_+}\delta\psi \tag{1.5}$$

where $d\mu(\phi) = \text{const. } e^{-\frac{1}{2}\phi C^{-1}\phi}\delta\phi$ is the well-defined boson Gaussian measure and

$$S^{-1} = S_0^{-1} + \lambda\phi = -i\partial\!\!\!/ + m_f + \lambda\phi \tag{1.6}$$

is the Euclidean Dirac operator with external field $\lambda\phi$. The fermion integral over anticommuting variables in (1.5) is interpreted according to the prescription [8]

$$\int e^{-\psi_- A\psi_+}\delta\psi = \det A . \tag{1.7}$$

It is worth noting that the formulation (1.5) of Z in terms of anticommuting sources is consistent with the Matthews-Salam-Seiler [9] formulation of the Yukawa model. To see this we complete the square in (1.5)

$$-\psi_-S^{-1}\psi_+ + \psi_-\alpha_- + \psi_+\alpha_+ = -(\psi_- + S^*\alpha_+)S^{-1}(\psi_+ - S\alpha_-) - \alpha_+S\alpha_-,$$

and, since we may regard $(\psi_- + S^*\alpha_+)$ and $(\psi_+ - S\alpha_-)$ as new *anticommuting* integration variables, we apply (1.7) to obtain

$$\begin{aligned} Z\{\alpha, f\} &= \text{const.} \int d\mu(\phi) e^{\phi f} e^{-\alpha_+ S\alpha_-} \det S^{-1} \\ &= \text{const.} \int d\mu(\phi) e^{\phi f - \alpha_+ S\alpha_-} \det(1 + \lambda S_0 \phi), \end{aligned} \tag{1.8}$$

(with a different constant). Taking formal functional derivatives of Z (in the anticommuting sense) yields the usual (unrenormalized) MSS formulae for the Schwinger functions (see (2.35)).

The procedure for obtaining the Legendre transform in a theory with fermions may now be obvious to the reader. For instance, to define the first transform $\Gamma^{(1)}$, we would set $G^{(1)}\{\alpha, f\} = \ln Z\{\alpha, f\}$, introduce variables $\{\beta, A\}$ conjugate to $\{\alpha, f\}$ by taking appropriate derivatives $\frac{\delta}{\delta\alpha}G^{(1)}, \frac{\delta}{\delta f}G^{(1)}$, and then define $\Gamma^{(1)}\{\beta, A\} = G^{(1)}\{\alpha, f\} - \alpha\frac{\delta G}{\delta\alpha} - f\frac{\delta G}{\delta f}$ in the usual way, always respecting the non-commutative nature of the variables. However it is very awkward to make rigorous the notions of (complex-valued?) « functionals » Z, $G^{(1)}$, ... on a space of non-commuting variables and of the desired operations on such « functionals ». Fortunately for the purposes of our analysis it is not necessary to do so. Instead we shall interpret such a « functional » as the sequence of its formal moments, each moment being itself a well-defined multilinear form on a space of ordinary functions. The desired structure (1.1) of anticommuting source theory will be realized in the way we define various operations on the sequences of moments. Now the natural setting for this point of view is the framework of formal power series (fps) that we introduced in II for the special case of pure boson theories and that we develop for the case of theories involving fermions in § II of this paper. For example, consider the partition function Z of (1.2). By a formal computation based on (1.1)

$$\begin{aligned} Z\{\alpha, f\} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{m!}\frac{1}{n!}\langle (\psi\alpha)^m(\phi f)^n \rangle \\ &= \sum_{m,n} \frac{\varepsilon_m}{m!\,n!} \int dxdy \langle \psi(x_1) \dots \psi(x_m)\phi(y_1) \dots \phi(y_n) \rangle \alpha(x_1) \\ &\qquad \dots \alpha(x_m) f(y_1) \dots f(y_n) \end{aligned} \tag{1.9 a}$$

where

$$\varepsilon_m = (-1)^{m(m-1)/2}. \tag{1.9 b}$$

Here the y's are integrated over $\mathbb{R}^d$ and the x's are integrated/summed over $\mathbb{R}^d \times \{-,+\} \times \{1, 2, \ldots, 2^{[d/2]}\}$ where the first index set provides for the two independent fermi fields $\psi_\pm$, and the second set for the $2^{[d/2]}$ spinor components of each field. We thus regard Z as the sequence of multilinear forms

$$Z_{m,n}(a_1, \ldots, a_m; f_1, \ldots, f_n) = \varepsilon_m \langle \psi(a_1) \ldots \psi(a_m)\phi(f_1) \ldots \phi(f_n) \rangle \tag{1.9 c}$$

where now each a_j is an element of the (ordinary) function space

$$\mathscr{H}_f = \mathscr{H}_{-1/2}(\mathbb{R}^d) \times \mathbb{C}^{2[d/2].2} \tag{1.10 a}$$

where

$$\mathscr{H}_{-\lambda}(\mathbb{R}^d) = \{ f : \mathbb{R}^d \to \mathbb{C} \mid (f, (-\Delta + 1)^{-\lambda} f) < \infty \}$$

and each f_j is an element of

$$\mathscr{H}_b = \mathscr{H}_{-1}(\mathbb{R}^d). \tag{1.10 b}$$

As is known [9]-[*11*] for εY_2 (the weakly coupled renormalized Yukawa model in $d = 2$ dimensions), $Z_{m,n} \in \mathscr{M}_{m,n}(\mathscr{H}_f, \mathscr{H}_b)$, the space of jointly continuous multilinear forms on $\mathscr{H}_f^m \times \mathscr{H}_b^n$ that are antisymmetric under permutation of the first m arguments and symmetric under permutation of the last n arguments. $Z = \{ Z_{m,n} \}$ is thus an element of the space of $\mathbb{C}$-valued fps on $(\mathscr{H}_f, \mathscr{H}_b)$:

$$Z \in \mathscr{F}_Y \equiv \mathscr{F}(\mathscr{H}_f, \mathscr{H}_b; \mathbb{C}) \equiv \bigtimes_{m,n=0}^{\infty} \mathscr{M}_{m,n}(\mathscr{H}_f, \mathscr{H}_b). \tag{1.11}$$

Besides allowing us to interpret the notion of a « functional of anticommuting sources » in terms of functionals on ordinary spaces, the fps framework has the same advantage that we have already exploited in II for boson theories, i. e., it enables us to avoid the difficult analytic question of convergence of the series: Operations on the « functionals » ($Z \to Z'$) are interpreted as corresponding operations on the coefficients ($\{ Z_{m,n} \} \to \{ Z'_{m,n} \}$). For most standard operations, $Z'_{m,n}$ is given by a *finite* expression in the $Z_{m,n}$'s. Hence our results, while stated in terms of « functionals », are actually shorthand for well-defined statements about the coefficients. In this way we can draw rigorous conclusions about the coefficients while retaining the conceptual and notational advantages of the « functional » point of view.

In the next section we shall carefully define the fps calculus that underlies this reinterpretation. However, for the remainder of this introduction we shall revert to the language of anticommuting source theory in order to describe the Legendre transform in an intuitively direct way. We introduce general source terms of the form

$$U^M\{J\} = \sum_{m \in M} \Phi^m(\vec{x}_m) J_m(\vec{x}_m) \tag{1.12}$$

where

a) $M \subset \mathscr{M} = \{ m = (m_f, m_b) \mid m_f, m_b = 0, 1, 2, \ldots ; (m_f, m_b) \neq (0, 0) \}$, m_f being the degree of the fermion source and m_b the degree of the boson source;

b) $\Phi^m(\vec{x}_m)$ is a product of m_f fermion fields and m_b boson fields,

$$\Phi^m(\vec{x}_m) = \psi(x_1) \ldots \psi(x_{m_f}) \phi(x_{m_f+1}) \ldots \phi(x_{m_f+m_b}) ;$$

c) for notational compactness, each fermion argument

$$x_i \in \mathbb{R}^d \times \{ -, + \} \times \{ 1, 2, \ldots, 2^{[d/2]} \}$$

carries a space-time point, an index ($\pm$) which determines whether the field is $\psi_\pm$, and a spinor index; each boson argument x_i is simply a point in $\mathbb{R}^d$;

d) the source « functions » $J_m(\vec{x}_m)$ are antisymmetric in their fermion arguments and symmetric in their boson arguments;

e)

$$\begin{aligned} J_m J_n &= (-1)^{m_f n_f} J_n J_m \\ J_m \Phi^n &= (-1)^{m_f n_f} \Phi^n J_m \end{aligned} \tag{1.13}$$

so that sources and fields of odd fermion degree anticommute with each other and commute with sources and fields of even fermion degree.

f) repeated arguments are integrated/summed over.

Remarks 1. — In the body of this paper we shall concern ourselves with the case of « second order » M's, i. e. the case of source terms with

$$|m| = m_f + m_b \leq 2$$

(however the Appendix treats the case of fourth order M's).

2. In order to concentrate on the formalism we shall ignore questions of rigour such as what spaces the J_m's belong to (for a discussion of such points, which entail Wick-ordering the sources, see Ref. [5].

3. As the need arises, it will be appropriate to place further restrictions on M (see Definitions III.5 and III.10 for the notions of « gapless » and « triangular » M).

4. We shall often write $f = (1, 0)$, $b = (0, 1)$, $fb = (1, 1)$, etc.

The generating functional of Schwinger functions is

$$Z^M\{J\} = \langle e^{U^M\{J\}} \rangle \tag{1.14}$$

and the generating functional of connected Green's functions is

$$G^M\{J\} = \ln \langle e^{U^M\{J\}} \rangle .$$

The Schwinger functions and Green's functions are generated by taking functional derivatives of Z^M and G^M. For a complete discussion of derivatives on a Grassmann algebra and in particular of distinction between left and right differentiation, see Berezin's text [7] or § II; as examples of the product rule we have for the left derivative ($\alpha \equiv J_{(1,0)}$)

$$\frac{\delta}{\delta\alpha(x)}\alpha(y_1)\alpha(y_2) = \delta(x, y_1)\alpha(y_2) - \alpha(y_1)\delta(x, y_2) \qquad (1.15\,a)$$

and for the right derivative

$$\alpha(y_1)\alpha(y_2)\frac{\delta}{\delta\alpha(x)} = -\delta(x, y_1)\alpha(y_2) + \alpha(y_1)\delta(x, y_2) \qquad (1.15\,b)$$

where $\delta(x, y)$ is the appropriate product of δ-functions in the continuous variables and Kronecker-δ's in the discrete variables. We shall denote the left derivative of G with respect to J_m by both $\frac{\delta}{\delta J_m}G$ and ${}_{J_m}G$, and the right derivative of G with respect to J_m by both $G\frac{\delta}{\delta J_m}$ and G_{J_m}. One part of our attempt to organize chaos is the (arbitrarily chosen) convention that whenever possible we shall apply only *right* derivatives to $G^M\{J\}$. Consequently we define the variable A_m conjugate to J_m by

$$A_m(\vec{x}_m) + A_m^0(\vec{x}_m) = G^M_{J_f(x_1)\dots J_f(x_{m_f})J_b(x_{m_f+1})\dots J_b(x_{m_f+m_b})}\{J\} \qquad (1.16)$$

where $A_m^0(\vec{x}_m)$ is chosen independent of J so that $A_m = 0$ when $J = 0$.

Note that as in I-IV we use the « connected variables » as the conjugate variables rather than the « Schwinger variables » $G^M_{J_m(\vec{x}_m)}$ (see Lemma III.4 for the relation between the two). The Legendre transform Γ^M of G^M is then defined by

$$\Gamma^M\{A\} = G^M\{J\} - \sum_{m\in M} G^M_{J_m}J_m \,|_{J=J^M\{A\}} \qquad (1.17)$$

where $J^M\{A\}$ refers to the inverse of the map $J \to A$ defined in (1.16). For derivatives of Γ^M we adopt the opposite convention to that of G^M, i. e. we use only left derivatives. (This convention eliminates certain factors of ± 1 from our formulas). In § III we interpret the definition (1.17) in the context of fps and we work out the conjugate relations (expressing J in terms of ${}_{A_m}\Gamma^M$) and the Jacobian relation for Γ^M.

In § IV we establish the irreducibility properties of Γ^M (i. e. of the generalized vertex functions generated by Γ^M) for various choices of « second order » M. As in I-IV we employ an analytic notion of irreducibility due to Spencer [12] which does not rely on perturbation theory. The basic idea is as follows (for more details see § II of I). Let σ be a $(d-1)$-dimensional hyperplane separating $\mathbb{R}^d$ into two half-spaces $\mathbb{R}^d_+$ and $\mathbb{R}^d_-$, and let $S_{0,\sigma}$ and C_σ be the fermion [5] [10] and boson propagators (1.4) but with Dirichlet B.C. on σ. For $t = (t_f, t_b)$ in $[0,1] \times [0,1]$ we define the interpolating propagators

$$S_0(t_f) = t_f S_0 + (1 - t_f) S_{0,\sigma} \tag{1.18 a}$$

and

$$C(t_b) = t_b C + (1 - t_b) C_\sigma . \tag{1.18 b}$$

Note that at $t = 0$, $S_0(t_f) = S_{0,\sigma}$ and $C(t_b) = C_\sigma$.

We then introduce this t-dependence into the basic expectation $\langle \, . \, \rangle$. For a model defined by (1.3) this is accomplished by replacing S_0^{-1} and C^{-1} in the Gaussian part of the measure by $S_0(t_f)^{-1}$ and $C(t_b)^{-1}$.

The effect of this replacement on the graphs of perturbation theory is to replace fermion lines S_0 by $S_0(t_f)$ and boson lines C by $C(t_b)$. At $t=0$ (Dirichlet B.C.)

$$S_0(t_f)(x, y) = 0 \qquad \text{and} \qquad C(t_b)(x, y) = 0 \text{ if } x\sigma y \tag{1.19 a}$$

where by $x\sigma y$ we mean that x and y be on opposite sides of σ, i. e. the fermion or boson line crosses σ. On the other hand,

$$\partial_{t_f} S_0(t_f)(x, y) \neq 0 \qquad \text{and} \qquad \partial_{t_b} C(t_b)(x, y) \neq 0 \text{ if } x\sigma y \tag{1.19 b}$$

since e. g. $\partial_{t_b} C(t_b) = C - C_\sigma$ and C does not vanish across σ. Now consider a perturbation theory graph $\mathcal{G}(t)$ with external vertices at x and y with $x\sigma y$. By (1.19) $\mathcal{G}(0) = 0$ if x and y are connected in $\mathcal{G}$, and $\partial_{t_f}^{r_f} \partial_{t_b}^{r_b} \mathcal{G}(0) = 0$ if x and y are connected by at least $r_f + 1$ fermion or $r_b + 1$ boson lines in $\mathcal{G}$. Thus the vanishing of t-derivatives of $\mathcal{G}(t)$ measures its degree of irreducibility. This idea applies directly to more general objects than graphs to yield an analytic definition of irreducibility independent of but consistent with perturbation theory. For instance the cluster-irreducibility of Γ^M is expressed as the cluster-connectedness of t-derivatives $\partial_t^r \Gamma^M$ at $t = 0$ (we write $\partial_t^r \Gamma^M \sim 0$; see (1.23) and Definition III.3). A practical advantage of this analytic definition is that the formulas for t-derivatives of G^M and Γ^M have a fairly simple structure (see Theorem IV.2).

Naively one would expect the irreducibility of Γ^M to be simply that $\partial_t^r \Gamma^M \sim 0$ for exactly those r's in M. However, for M's that are not « triangular », this expectation is dashed by the presence of more than one basic field (this has nothing to do with fermions per se as we illustrate at the end of this section by looking at a Gaussian, purely bosonic, multi-field model). On the other hand, the presence of fermions enhances irreducibility because of « fermion symmetry » (the expectation of an odd power of fermi fields vanishes). As a result, the irreducibility results of Theorems IV.2 and IV.4, are a little more complicated than originally anticipated.

Besides generating vertex functions (via degree one derivatives $\dfrac{\delta}{\delta A_f}$ and

$\left.\frac{\delta}{\delta A_b}\right)$, Γ^M generates (via its higher degree derivatives) « M-field projectors » P_M, M-truncated expectations $E^M_{m,n} \equiv \langle \Phi^m(1 - P_M)\Phi^n \rangle$, and Bethe-Salpeter kernels

$$K^M_{m,n} = -\text{ connected part of } \frac{\delta}{\delta A_m}\frac{\delta}{\delta A_n}\Gamma^M\{0\}.$$

We describe this generating role in § V and in so doing derive Bethe-Salpeter equations relating E^M's and K^M's. Given the irreducibility results of § IV, we shall thus have completed one of the main objectives of this paper, namely, to establish the analytic irreducibility of the various Bethe-Salpeter kernels needed for the analysis of the spectrum of the Y_2 model [5].

The irreducibility results of § IV can be extended to the case of certain higher order M's (see Theorem A.1), but the discussion of these extensions has been banished to an appendix.

In our previous study of the transformation from $G\{J\}$ to $\Gamma\{A\}$ for a single boson field we found that transforming more source terms led to greater irreducibility of Γ. This same pattern more or less holds for the more general transform considered here, but, as remarked above, some of the irreducibility properties of Γ^M turn out to be rather counterintuitive. To illustrate this point we consider a free (i. e. Gaussian) multi-field model where the fields are all bosonic. Define the partition function with linear and quadratic sources by

$$Z\{J, L; t\} = \int e^{\phi J + L\phi^2} e^{-\frac{1}{2}\phi C^{-1}(t)\phi}\delta\phi \Big/ \int e^{-\frac{1}{2}\phi C^{-1}(t)\phi}\delta\phi$$

where

$\phi = (\phi_1, \ldots, \phi_n)$ consists of n boson fields,
$J = (J_1, \ldots, J_n)$ is the set of linear source functions,
$L = (L_{ij})$, $i, j = 1, \ldots, n$ is the set of quadratic source functions $L_{ij}(x, y)$, each symmetric under $i \leftrightarrow j$, $x \leftrightarrow y$,
$t = (t_1, \ldots, t_n) \in [0, 1]^n$,
$C^{-1}(t)_{ij} = C_i^{-1}(t_i)\delta_{i,j} + V_{i,j}$ where $C_i(t_i)$ is a boson covariance as in (1.18 b) and where the « interaction » $V_{i,j}(x_i, x_j) = \lambda_{ij}\delta(x_i - x_j)$.

Here $\{\lambda_{ij}\}$ is the set of coupling constants, and we set $\lambda_{ii} = 0$.
By a straightforward Gaussian integration (we suppress the t)

$$G\{J, L\} = \ln Z\{J, L\} = \frac{1}{2}J(C^{-1} - 2L)^{-1}J - \frac{1}{2}\ln\det(1 - 2LC). \quad (1.20)$$

Let P be a subset of the set $\mathscr{P}$ of unordered pairs (ij). We transform all the J's to conjugate variables

$$A_i \equiv G_{J_i} = ((C^{-1} - 2L)^{-1}J)_i \quad (1.21\,a)$$

but only those L_{ij}'s with $(ij) \in P$ to the conjugate variables

$$B_{ij} \equiv G_{J_i J_j} - C_{ij} = (C^{-1} - 2L)^{-1}_{ij} - C_{ij}. \qquad (1.21\,b)$$

For $(ij) \in P^c \equiv \mathscr{P} \backslash P$, we set $L_{ij} = 0$. The Legendre transform Γ^P is then a functional of A and $B_P = \{ B_{ij} \mid (ij) \in P \}$ and it is easy to compute that

$$\Gamma^P \{ A, B_P \} \equiv G - G_J J - G_L L$$
$$= -\frac{1}{2} A C^{-1} A + \frac{1}{2} \operatorname{tr} [\ln (1 + C^{-1} B) - C^{-1} B] \qquad (1.22)$$

where in (1.22) we use the symbol B_{ij}, $(ij) \in P^c$, to mean the functional $B_{ij} \{ B_p \}$ determined by (1.21 *b*) (we shall not attempt an explicit determination of $B_{P^c} \{ B_P \}$!).

To illustrate the paradoxical nature of the irreducibility of Γ^P, suppose first that $P = \{ (11) \}$. Then, as we show below, $\Gamma^P \{ A, B_{(11)} ; t \}$ is 2-cluster-irreducible (2-CI) in t_1. However if we *further* transform G, say by taking $P = \{ (11), (22) \}$, then Γ^P is no longer 2-CI in t_1 ! By 2-CI in t_1 we mean that

$$\partial^r_{t_1} \Gamma^P \sim 0, \qquad r = 0, 1, 2, \qquad (1.23)$$

where we write $F \sim 0$ to mean that the functional $F \{ A, B ; t \}$ is *cluster connected*, i. e.,

$$F_{f_1(x) f_2(y)} \{ A, B ; t \} = 0 \quad \text{at} \quad 0 \ \text{if} \ x \sigma y$$

where $f_i(x) = A(x)$ (in which case $x \in \mathbb{R}^d$) or $f_i(x) = B(x)$ (in which case $x \in \mathbb{R}^{2d}$) ; « at 0 » means that $t = 0$ and

$$\not{B} = 0 \quad \text{i. e.,} \qquad B(x_1, x_2) = 0 \quad \text{if} \quad x_1 \sigma x_2 ;$$

and A and B are in $\mathcal{N}_\sigma$, the class of C_0^∞ functions that vanish in a neighbourhood of σ. The functions in $\mathcal{N}_\sigma$ are insensitive to changes in B. C. in $C^{-1}(t)$ so that (see Lemma V.2 of II)

$$\partial_{t_i} A C^{-1} A = \partial_{t_i} C^{-1} B = 0 \qquad \text{for} \qquad A, B \in \mathcal{N}_\sigma. \qquad (1.24)$$

For general P, we have by (1.22) and (1.24) for $A, B_P \in \mathcal{N}_\sigma$,

$$\partial_{t_i} \Gamma^P = \frac{1}{2} \partial_{t_i} \operatorname{tr} [\ln (1 + C^{-1} B) - C^{-1} B]$$
$$= \frac{1}{2} \operatorname{tr} \{ (\partial_{t_i} C^{-1} B) [(1 + C^{-1} B)^{-1} - 1] \}$$
$$= - \operatorname{tr} [C (\partial_{t_i} C^{-1} B) L] \qquad \text{(by (1.21 } b\text{))}$$
$$= - \operatorname{tr} \sum_{\substack{j,k \\ (ij) \in P^c,\ (jk) \in P}} C_{ki} (\partial_{t_i} C^{-1}_{ii} B_{ij}) L_{jk}$$

where the restriction $(ij) \in P^c$ arises because of (1.24) (for $(ij) \in P$, B_{ij} is an independent variable in $\mathcal{N}_\sigma$) and the restriction $(jk) \in P$ arises because $L_{jk} = 0$ for $(jk) \in P^c$.

Clearly if there are no j and k with $(ij) \in P^c$ and $(jk) \in P$, then $\partial_t \Gamma^P = 0$, i.e. Γ^P is independent of t_i. In particular, this verifies (1.23) for $P = \{(11)\}$ and all $r \geq 1$. On the other hand if there are j and k with $(ij) \in P^c$ and $(jk) \in P$, then Γ^P is not 2-CI in t_i in general. In particular, for $P = \{(11), (22)\}$ and for $\lambda_{12} \neq 0$ the perturbation series for $\Gamma^P_{B_{(22)}(x)B_{(22)}(y)}$ contains terms of the form

$$(\lambda_{12})^4 [C_1(x_1, y_1)C_1(x_2, y_2) + C_1(x_1, y_2)C_1(x_2, y_1)],$$

where $C_i(x, y) = [C_{ii}^{-1}(x, y; t_i)]^{-1}$. These terms are manifestly not 2-CI in t_1. Of course, by enlarging P to $\{(11), (12), (22)\}$ we would recover the irreducibility (1.23). In fact in the Gaussian model with $n = 2$ the expected irreducibility statement

$$\partial_{t_i}\partial_{t_j}\Gamma^P \sim 0 \qquad \text{for all} \quad (ij) \in P$$

fails only for $P = \{(11), (22)\}$ and $P = \{(12)\}$.

The conclusions of the above paragraph for the Gaussian model motivate the results we obtain for the Yukawa model (see Theorem IV.1) and serve as a guide for irreducibility results we expect for a general multi-field model.

§ II. THE FORMAL POWER SERIES FRAMEWORK

The purpose of this section is to develop the framework of formal power series (fps) for « functionals » whose moments are multilinear forms on $\mathcal{L}_0^m \times \mathcal{L}_e^n$ that are antisymmetric on $\mathcal{L}_0^m$ and symmetric on $\mathcal{L}_e^n$. This framework will enable us to define « functionals » like $Z^M\{J\}$ of (1.14) rigorously as fps and to realize the structure (1.13) of anticommuting source theory in terms of a rigorous fps calculus. In order to consider an object like

$$Z^M\{J\}\frac{\delta}{\delta J_k(\vec{x}_k)} = \langle e^{U^M\{J\}}\psi(x_1)\dots\psi(x_{k_1})\phi(x_{k+1})\dots\phi(x_{k_1+k_2})\rangle \quad (2.1)$$

which takes values not in $\mathbb{C}$ but rather in a space (see (1.10)-(1.11)) of multilinear forms we need to generalize the definition (1.11) to allow our fps to take values in a general vector space $\mathcal{L}$:

DEFINITION II.1 (*Formal power series*). — Given topological vector spaces $\mathcal{L}_0$, $\mathcal{L}_e$ and $\mathcal{L}$ we define the *space of $\mathcal{L}$-valued fps on* $(\mathcal{L}_0, \mathcal{L}_e)$ by

$$\mathcal{F}(\mathcal{L}_0, \mathcal{L}_e; \mathcal{L}) = \bigtimes_m \mathcal{M}_m(\mathcal{L}_0, \mathcal{L}_e; \mathcal{L}) \qquad (2.2)$$

where $m = (m_0, m_e)$ runs through the index set $\mathbb{N}^2 = \{0, 1, 2, \ldots\}^2$ and $\mathcal{M}_m(\mathcal{L}_0, \mathcal{L}_e; \mathcal{L})$ is the space of $\mathcal{L}$-valued jointly continuous multilinear forms on $\mathcal{L}_0^{m_0} \times \mathcal{L}_e^{m_e}$ which are antisymmetric (symmetric) under permutation of the first m_0 (last m_e) arguments.

Remarks 1. — As indicated by the symmetry conventions, the space $\mathcal{L}_0$ (resp. $\mathcal{L}_e$) will consist of functions which are associated with an odd (resp. even) number of fermions and which we shall usually denote by Greek letters $\alpha, \beta, \ldots$ (resp. Latin letters $f, g, \ldots$). For example, to define the right side of (2.1) as an element of $\mathcal{F}(\mathcal{L}_0, \mathcal{L}_e; \mathcal{M}_k(\mathcal{H}_f, \mathcal{H}_b))$ when $\mathrm{M} = \{(1,0), (0,1), (1,1), (2,0)\} = \{f, b, fb, ff\}$ we would choose

$$\alpha \equiv (\mathrm{J}_f, \mathrm{J}_{fb}) \in \mathcal{L}_0 \equiv \mathcal{L}_{1,0} + \mathcal{L}_{2,0} \equiv \mathcal{H}_f \oplus (\mathcal{H}_f \otimes \mathcal{H}_b)$$

and

$$f \equiv (\mathrm{J}_b, \mathrm{J}_{ff}) \in \mathcal{L}_e \equiv \mathcal{L}_{1,e} \oplus \mathcal{L}_{2,e} \equiv \mathcal{H}_b \oplus (\mathcal{H}_f \underset{a}{\otimes} \mathcal{H}_f)$$

2. We may occasionally split the test function space in different ways. For instance instead of regarding the test function space in the above example as $\mathcal{L}_0 + \mathcal{L}_e$ with $\mathrm{J} = (\mathrm{J}_0, \mathrm{J}_e) \equiv (\alpha, f)$ we may write it as $\mathcal{L}_1 + \mathcal{L}_2$ with $\mathrm{J} = (\mathrm{J}_1, \mathrm{J}_2)$ where $\mathcal{L}_1 = \mathcal{L}_{1,0} \oplus \mathcal{L}_{1,e}, \mathcal{L}_2 = \mathcal{L}_{2,0} \oplus \mathcal{L}_{2,e}, \mathrm{J}_1 = (\mathrm{J}_f, \mathrm{J}_b)$ and $\mathrm{J}_2 = (\mathrm{J}_{fb}, \mathrm{J}_{ff})$. In general an element of a subspace of will be labelled with the same subscripts or superscripts used to label the subspace.

The space $\mathcal{F}(\mathcal{L}_0, \mathcal{L}_e; \mathcal{L})$ is a topological vector space with the obvious definitions of vector addition and scalar multiplication and with the product topology. In the important case $\mathcal{L} = \mathbb{C}$ we can make $\mathcal{F}$ into an algebra with multiplication defined as follows.

DEFINITION II.2 (*Products*). — If $u = \{u_m\}$, $v = \{v_m\} \in \mathcal{F}(\mathcal{L}_0, \mathcal{L}_e; \mathbb{C})$ then uv is defined by

$$(uv)_m \equiv \sum_{m'=0}^{m} \binom{m}{m'} u_{m'} \odot v_{m-m'} \tag{2.3 a}$$

where m' runs over the set $\{(m'_0, m'_e) \mid 0 \leq m'_0 \leq m_0,\ 0 \leq m'_e \leq m_e\}$, $\binom{m}{m'} = \binom{m_0}{m'_0}\binom{m_e}{m'_e}$ and $\odot$ denotes the tensor product of multilinear forms antisymmetrized in the $\mathcal{L}_0$-arguments and symmetrized in the $\mathcal{L}_e$-arguments. Explicitly,

$$u_{m'} \odot v_{m-m'}(\alpha_1, \ldots, \alpha_{m_0}; f_1, \ldots, f_{m_e})$$
$$= \mathrm{A}_\alpha \mathrm{S}_f u_{m'}(\alpha_1, \ldots, \alpha_{m'_0}; f_1, \ldots, f_{m'_e}) v_{m-m'}(\alpha_{m'_0+1}, \ldots, \alpha_{m_0}; f_{m'_e+1}, \ldots, f_{m_e})$$
$$= \frac{1}{m!} \sum_{\pi_0 \in \mathrm{S}_{m_0}} \sum_{\pi_e \in \mathrm{S}_{m_e}} \operatorname{sgn} \pi_0 \; u_{m'}(\alpha_{\pi_0(1)}, \ldots; f_{\pi_e(1)}, \ldots) v_{m-m'}(\ldots \alpha_{\pi_0(m_0)}; \ldots f_{\pi_e(m_e)})$$

where A_α antisymmetrizes in the α's, S_f symmetrizes in the f's, and $\frac{1}{m!} = \frac{1}{m_0!\, m_e!}$.

Remark. — More generally when $u \in \mathscr{F}(\mathscr{L}_0, \mathscr{L}_e; \mathscr{L}_1)$ and $v \in \mathscr{F}(\mathscr{L}_0, \mathscr{L}_e; \mathscr{L}_2)$ and a jointly continuous product is defined on $\mathscr{L}_1 \times \mathscr{L}_2$ then we can define uv and vu as in (2.3) (see for example (2.39) below where the product is an operator product). Whenever we write a product uv we shall be assuming this additional structure, and whenever we write formulas like (2.8) below we shall be implicitly assuming that the product on $\mathscr{L}_1 \times \mathscr{L}_2$ is commutative.

Although we have cautioned the reader about the mathematical impropriety of « functionals » of « anticommuting sources », the motivation for Defin II.2 and for most of the definitions to follow comes from thinking of $u \in \mathscr{F}(\mathscr{L}_0, \mathscr{L}_e; \mathbb{C})$ as the formal series

$$u\{\alpha, f\} = \sum_m \frac{1}{m!} \int dxdy\, u_m(x_1, \ldots, x_{m_0}; y_1, \ldots, y_{m_e}) \alpha(x_1) \ldots \alpha(x_{m_0}) f(y_1) \ldots f(y_{m_e}) \quad (2.4)$$

where the α's satisfy the multiplication rule

$$\alpha(x_1)\alpha(x_2) = -\alpha(x_2)\alpha(x_1) \quad (2.5)$$

and differentiation rules based on (1.15), and the f's are ordinary functions. Note the order of the α's in (2.4). Multiplying two series of the form (2.4) together using (2.5) leads at once to the definition (2.3) (the binomial coefficients in (2.3 *a*) arise from our convention of placing factorials in (2.4)). In fact we shall generally write down a formal expression like (2.4)-(2.5) in order to *define* a fps u, i. e. the formal series is to be interpreted simply as a specification of each component $u_m \in \mathscr{M}_m$.

We pause to identify the above multiplicative structure in terms of Grassmann algebras. The Grassmann algebra [7] associated with the finite set of generators $\{ \alpha_k \mid 1 \leq k \leq p \}$ is the vector space of dimension 2^p having as a basis $\{1\} \cup \{ \alpha_{k_1} \ldots \alpha_{k_m} \mid 1 \leq m \leq p,\ k_1 < k_2 < \ldots < k_m \}$. The multiplication law in this algebra is the natural extension of $\alpha_i\alpha_j = -\alpha_j\alpha_i$. Hence there is a $1-1$ correspondence between elements

$$v_0 + \sum_{m=1}^{p} \sum_{k_1 \ldots k_m = 1}^{p} \frac{1}{m!} v_m(k_1, \ldots, k_m) \alpha_{k_1} \ldots \alpha_{k_m}$$

and sets $v = \{ v_m \}$ of coefficients provided we impose the condition that each function $v_m(k_1, \ldots, k_m)$ be antisymmetric under permutation of its arguments. In terms of these sets the multiplication law is

$$(uv)_m(k_1, \ldots, k_m) = \sum_{m'=0}^{m} \binom{m}{m'} A u_{m'}(k_1, \ldots, k_{m'}) v_{m-m'}(k_{m'+1}, \ldots, k_m)$$

where A is the antisymmetrization operator. When the Grassmann algebra has infinitely many generators the situation is similar. An element is a set $\{ v_m \mid m = 0, 1, 2, 3, \ldots \}$ with $v_0 \in \mathbb{C}$ and for $m \geq 1$ v_m is an antisymmetric « function ». But now there are topological considerations. One choice of topology is the product topology:

$$\mathscr{G}(\mathscr{L}_0) \equiv \bigtimes_{m_0=0}^{\infty} \mathscr{M}_{(m_0,0)}(\mathscr{L}_0, \mathscr{L}_e; \mathbb{C}). \tag{2.6 a}$$

By rewriting $\{ u_m \mid m \}$ as $\{ \{ u_{(m_0,m_e)} \mid m_0 \geq 0 \} \mid m_e \geq 0 \}$ we may view any element u of $\mathscr{F} \{ \mathscr{L}_0, \mathscr{L}_e; \mathbb{C} \}$ as an fps (in the sense of II) on $\mathscr{L}_e$ taking values in $\mathscr{F}(\mathscr{L}_0)$. In other words

$$\mathscr{F}(\mathscr{L}_0, \mathscr{L}_e; \mathbb{C}) = \mathscr{F}(\mathscr{L}_e, \mathscr{G}(\mathscr{L}_0)). \tag{2.6 b}$$

(2.3) is nothing more than the natural multiplication law in $\mathscr{F}(\mathscr{L}_e, \mathscr{G}(\mathscr{L}_0))$. More generally when the values occur in a space $\mathscr{L}$ we write

$$\mathscr{G}(\mathscr{L}_0 ; \mathscr{L}) = \bigtimes_{m_0=0}^{\infty} \mathscr{M}_{(m_0,0)}(\mathscr{L}_0, \mathscr{L}_e ; \mathscr{L}). \tag{2.6 c}$$

In general, $\mathscr{G}(\mathscr{L}_0 ; \mathscr{L})$ is not an algebra but it is a module over the ring $\mathscr{G}(\mathscr{L}_0)$ since multiplication between an element of $\mathscr{G}(\mathscr{L}_0 ; \mathscr{L})$ and an element of $\mathscr{G}(\mathscr{L}_0)$ is defined as in (2.3). As a generalization of (2.6 b) we have

$$\mathscr{F}(\mathscr{L}_0, \mathscr{L}_e ; \mathscr{L}) = \mathscr{F}(\mathscr{L}_e, \mathscr{G}(\mathscr{L}_0 ; \mathscr{L})). \tag{2.6 d}$$

The point of view implied by the right side of (2.6 d) is convenient for multiplication of fps and for carrying over results from II.

We also introduce the following notation to record the degree in the $\mathscr{L}_0$-argument :

$$\mathscr{F}^i(\mathscr{L}_0, \mathscr{L}_e ; \mathscr{L}) \equiv \{ u \in \mathscr{F}(\mathscr{L}_0, \mathscr{L}_e ; \mathscr{L}) \mid u_m = 0 \text{ unless } m_0 = i \}, \tag{2.7 a}$$

$$\mathscr{F}_e \equiv \bigoplus_{i \text{ even}} \mathscr{F}^i, \qquad \mathscr{F}_0 \equiv \bigoplus_{i \text{ odd}} \mathscr{F}^i, \tag{2.7 b}$$

and similarly for $\mathscr{G}^i$, $\mathscr{G}_0$ and $\mathscr{G}_e$. It is easy to see that

$$u \in \mathscr{F}^i, \qquad v \in \mathscr{F}^j \Rightarrow uv = (-1)^{ij} vu \tag{2.8 a}$$

and that if at least one of u or v is even (i. e. u or $v \in \mathscr{F}_e$) then

$$uv = vu. \tag{2.8 b}$$

Having a definition of product we also have a definition of the power u^k of an element $u \in \mathscr{F}(\mathscr{L}_0, \mathscr{L}_e ; \mathbb{C})$, and, given a function $\mathrm{F}(z) = \sum_{k=0}^{\infty} a_k z^k$ analytic at $z = 0$, a definition of $\mathrm{F}(u) = \sum_{k=0}^{\infty} a_k u^k$. If $u_{0,0} = 0$, as we shall usually assume, then we need never worry about the convergence of the sum over k because the sum defining $F(u)_m$ will have only finitely many nonzero terms. (Indeed this will be the case even if $\Sigma a_k z^k$ has radius of convergence zero.) A certain amount of care must be exercised when computing with $F(u)$ when u is not even. For example, by (2.8 a)

$$u \text{ odd} \Rightarrow F(u) = a_0 + a_1 u$$

so that in general

$$\exp(u + v) \neq \pm \exp u \exp v .$$

On the other hand, if at least one of u or v is even then (2.8 b) entitles us to use such familiar tools as the binomial theorem

$$(u + v)^k = \sum_{r=0}^{k} \binom{k}{r} u^r v^{k-r} \tag{2.9}$$

and the product rules for exponentials and logarithms

$$\exp(u + v) = \exp u \exp v \tag{2.10}$$

$$\log(1 + u)(1 + v) = \log(1 + u) + \log(1 + v) \tag{2.11}$$

The existence of multiplicative inverses is guaranteed by the next theorem whose proof follows that of Lemma II.6 of II. Here the identity 1 in $\mathcal{F}(\mathcal{L}_0, \mathcal{L}_e; \mathbb{C})$ is defined by $1_0 = 1$ with $1_m = 0$ otherwise.

THEOREM II.3 (*Multiplicative inverse*).

a) If $u \in \mathcal{F}(\mathcal{L}_0, \mathcal{L}_e; \mathbb{C})$ has $u_0 \neq 0$ there exists a unique $u^{-1} \in \mathcal{F}(\mathcal{L}_0, \mathcal{L}_e; \mathbb{C})$ with $u^{-1}u = uu^{-1} = I$.

b) If u is even, then so is u^{-1}.

EXAMPLE II.4. — Consider the partition function for the *renormalized* εY_2 model [9]-[11] (corresponding to the unrenormalized form (1.8))

$$Z\{J\} = Z\{\alpha, f\} = \int dv(\phi) e^{u\{\alpha, f; \phi\}} \tag{2.12}$$

where

$$J = (J_f, J_b) = (\alpha, f) = ((\alpha_-, \alpha_+), f),$$

$$dv(\phi) = \det{}_{\text{ren}}(1 + \lambda S_0 \phi) d\mu(\phi) \Big/ \int \det{}_{\text{ren}} d\mu , \tag{2.13}$$

($\det_{ren}$ is obtained from det by Wick ordering and mass renormalization factors [9]), and

$$u\{\alpha, f; \phi\} = \hat{\phi} f - \alpha_+ S a_- \tag{2.14 a}$$

where we have replaced $\phi(x)$ by $\hat{\phi}(x) \equiv \phi(x) - \int \phi(x) dv$ and where

$S = (1 + \lambda S_0 \phi)^{-1} S_0$ acts on $\mathscr{H}_{-1/2} \otimes \mathbb{C}^2$. As we remarked after (2.4)-(2.5), the relation (2.14 *a*) defines a fps $u = u_{(0,1)} + \frac{1}{2\,!} u_{(2,0)}$, where

$$u_{(2,0)}(\alpha, \beta) = -\,\alpha \vec{S} \beta = -\,(\alpha_-, \alpha_+) \begin{pmatrix} 0 & -S^t \\ S & 0 \end{pmatrix} \begin{pmatrix} \beta_- \\ \beta_+ \end{pmatrix}. \qquad (2.14\,b)$$

Here S^t denotes the transpose of S with respect to both spatial and spinor variables so that $\alpha_- . S^t \beta_+ = S\alpha_- . \beta_+$ and (2.14 *b*) defines an antisymmetric bilinear form on $\mathscr{H}_f = \mathscr{H}_{-1/2} \otimes \mathbb{C}^4$.

The expression (2.14 *b*) is still only formal even for smooth ϕ's since $\vec{S}$ need not exist as a bounded operator on $\mathscr{H}_f$ and so $u \{ \alpha, f; \phi \}$ need not be an element of $\mathscr{F}_Y \equiv \mathscr{F}_e(\mathscr{H}_f, \mathscr{H}_b; \mathbb{C})$. Nevertheless, we shall pretend u is a well-defined element of $\mathscr{F}_Y$ for all ϕ, interpret e^u as explained above, and then carry out the ϕ-integration. The justification for this formal procedure is that the integration over ϕ effects a cancellation [*9*] between the « poles » of S and « zeros » of $\det_{\text{ren}}$, and so the resulting formulas for Z_m correctly define $Z \{ J \}$ as an element of $\mathscr{F}_Y$. The advantage of this formal procedure is that certain algebraic properties of $Z \{ J \}$ become transparent.

Continuing with the example, we have $Z = 1 + v$ where v is an even element of $\mathscr{F}_Y$ with $v_0 = 0$. Hence

$$G \{ J \} \equiv \ln Z \{ J \} \qquad (2.15)$$

is a well-defined element of $\mathscr{F}_Y$. Suppose we introduce a t-dependence into Z and G as described in the introduction:

$$\begin{aligned} Z \{ J ; t \} &= \int e^{u\{J;\phi;t\}} d\nu_t(\phi), \quad G \{ J ; t \} = \ln Z \{ J ; t \} \\ d\nu_t(\phi) &= \det_{\text{ren}} (1 + \lambda S_0(t_f)\phi) d\mu_{C(t_b)}(\phi) \Big/ \int \det_{\text{ren}} d\mu_{C(t_b)} \\ u \{ J ; \phi ; t \} &= \hat{\phi} f - \alpha_+ S(t_f) \alpha_- \left(\hat{\phi} \equiv \phi - \int \phi d\nu_t \right) \\ S(t_f) &= (1 + \lambda S_0(t_f)\phi)^{-1} S_0(t_f). \end{aligned} \qquad (2.16)$$

Here $d\mu_{C(t_b)} = d\mu_{t_b}$ is the boson Gaussian measure with covariance $C(t_b)$ (see (1.18)). At $t = 0$ the measure $d\nu_t$ decouples across σ [*10*]; so does $S(t_f)$ so that

$$u \{ J; \phi; 0 \} = u \{ \chi_+ J; \phi; 0 \} + u \{ \chi_- J; \phi; 0 \} \qquad (2.17)$$

where $\chi_\pm(x)$ is the characteristic function of the half-space $\mathbb{R}^d_\pm$. From (2.17), (2.10) and (2.11) we then obtain the crucial decoupling relations

$$\begin{aligned} Z \{ J ; 0 \} &= Z \{ \chi_+ J ; 0 \} Z \{ \chi_- J ; 0 \} \\ G \{ J ; 0 \} &= G \{ \chi_+ J ; 0 \} + G \{ \chi_- J ; 0 \}. \end{aligned} \qquad (2.18)$$

Functional derivatives are defined as in II except that for an $\mathscr{L}_0$-functional derivative it is natural to define both a right and left derivative. If we formally apply the rule (1.15) to the series (2.4) we obtain

$$\frac{\delta}{\delta\alpha(x)} u\{\alpha, f\} = \sum_m \frac{1}{(m_0-1)!\,m_e!} \int u_m(x, x_2, \ldots, x_{m_0}; y_1, \ldots, y_{m_e})\alpha(x_2) \ldots \alpha(x_{m_0}) f(y_1) \ldots f(y_{m_e}) dx dy \quad (2.19\,a)$$

and

$$u\{\alpha, f\} \frac{\delta}{\delta\alpha(x)} = \sum_m \frac{1}{(m_0-1)!\,m_e!} \int u_m(x_1, \ldots, x_{m_0-1}, x; y_1, \ldots, y_{m_e})\alpha(x_1) \ldots \alpha(x_{m_0-1}) f(y_1) \ldots f(y_{m_e}) dx dy. \quad (2.19\,b)$$

These expressions motivate the following definitions:

DEFINITION II.5 (*Functional derivatives*). — Given $u \in \mathscr{F}(\mathscr{L}_0, \mathscr{L}_e; \mathscr{L})$ the *left $\mathscr{L}_0$-functional derivative* $\frac{\delta}{\delta\alpha} u \equiv \delta_\alpha u \equiv {}_\alpha u$ is defined as a fps in $\mathscr{F}(\mathscr{L}_0, \mathscr{L}_e; \mathscr{M}_{(1,0)}(\mathscr{L}_0, \mathscr{L}_e; \mathscr{L}))$ by

$$\left(\frac{\delta}{\delta\alpha} u\right)_m (\alpha_1, \ldots, \alpha_{m_0}; f_1, \ldots, f_{m_e}) : \alpha_0 \to u_{(m_0+1, m_e)}(\alpha_0, \alpha_1, \ldots, \alpha_{m_0}; f_1, \ldots, f_{m_e}), \quad (2.20\,a)$$

the *right $\mathscr{L}_0$-functional derivative* $u \frac{\delta}{\delta\alpha} \equiv u\delta_\alpha \equiv u_\alpha$ is defined as a fps in $\mathscr{F}(\mathscr{L}_0, \mathscr{L}_e; \mathscr{M}_{(1,0)}(\mathscr{L}_0, \mathscr{L}_e; \mathscr{L}))$ by

$$\left(u \frac{\delta}{\delta\alpha}\right)_m (\alpha_1, \ldots, \alpha_{m_0}; f_1, \ldots, f_{m_e}) : \alpha_0 \to u_{(m_0+1, m_e)}(\alpha_1, \ldots, \alpha_{m_0}, \alpha_0; f_1, \ldots, f_{m_e}) \quad (2.20\,b)$$

and the *$\mathscr{L}_e$-functional derivative* $\frac{\delta u}{\delta f} \equiv \delta_f u \equiv u_f \equiv {}_f u$ is defined as a fps in $\mathscr{F}(\mathscr{L}_0, \mathscr{L}_e; \mathscr{M}_{(0,1)}(\mathscr{L}_0, \mathscr{L}_e; \mathscr{L}))$ by

$$\left(\frac{\delta u}{\delta f}\right)_m (\alpha_1, \ldots, \alpha_{m_0}; f_1, \ldots, f_{m_e}) : f_0 \to u_{(m_0, m_e+1)}(\alpha_1, \ldots, \alpha_{m_0}; f_0, f_1, \ldots, f_{m_e}). \quad (2.20\,c)$$

In the case where $\mathscr{L}_0$ or $\mathscr{L}_e$ is a space of (generalized) functions $\{\alpha(x)\}$ or $\{f(x)\}$ we write $\frac{\delta}{\delta\alpha(x)} u$, $u \frac{\delta}{\delta\alpha(x)}$, $\frac{\delta u}{\delta f(x)}$ for the kernels of the functionals (2.20).

Note that a multiple functional derivative

$$\frac{\delta}{\delta f(y_1)} \cdots \frac{\delta}{\delta f(y_k)} \frac{\delta}{\delta\alpha(x_i)} \cdots \frac{\delta}{\delta\alpha(x_1)} u \frac{\delta}{\delta\alpha(x_{i+1})} \cdots \frac{\delta}{\delta\alpha(x_{i+j})} \quad (2.21)$$

is an element of $\mathscr{F}(\mathscr{L}_0, \mathscr{L}_e ; \mathscr{M}_{i+j,k}(\mathscr{L}_0, \mathscr{L}_e ; \mathscr{L}))$ where, on the basis of (2.19), we adopt the following sign convention for the form (2.21):

$$\int dxdy\left(\frac{\delta}{\delta f(y_1)} \cdots \frac{\delta}{\delta f(y_k)} \frac{\delta}{\delta\alpha(x_i)} \cdots \frac{\delta}{\delta\alpha(x_1)} u \frac{\delta}{\delta\alpha(x_{i+1})} \cdots \frac{\delta}{\delta\alpha(x_{i+j})}\right)_m$$
$$(\alpha_1, \ldots, \alpha_{m_0} ; f_1, \ldots, f_{m_e})\beta_1(x_1) \ldots \beta_{i+j}(x_{i+j})g(y_1) \ldots g(y_k)$$
$$= u_{m+i+j,n+k}(\beta_1, \ldots, \beta_i, \alpha_1, \ldots, \alpha_{m_0}, \beta_{i+j}, \ldots, \beta_{i+1} ; f_1, \ldots, f_{m_e}, g_1, \ldots, g_k). \tag{2.22}$$

Note also that if $u \in \mathscr{F}^l$ then the above derivative is in $\mathscr{F}^{l-i-j}$.

Another convenient way of defining a fps $u\{J\}$ is thus to specify all its derivatives $\left(\frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_N)} u\right)\{0\}$.

As examples of (2.22) we have for the fps $Z\{\alpha, f\}$ of (1.9) ($V\{0,0\}$ means $V_{(0,0)}$)

$$\frac{\delta}{\delta f(y_1)} \cdots \frac{\delta}{\delta f(y_k)} \frac{\delta}{\delta\alpha(x_i)} \cdots \frac{\delta}{\delta\alpha(x_1)} Z \frac{\delta}{\delta\alpha(x_{i+1})} \cdots \frac{\delta}{\delta\alpha(x_{i+j})} \{0, 0\}$$
$$= Z_{(i+j,k)}(x_1, \ldots, x_i, x_{i+j}, \ldots, x_{i+1} ; y_1, \ldots, y_k)$$
$$= \varepsilon_{i+j} \langle \psi(x_1) \ldots \psi(x_i)\psi(x_{i+j}) \ldots \psi(x_{i+1})\phi(y_1) \ldots \phi(y_k) \rangle$$
$$= (-1)^{ij} \langle \phi(y_1) \ldots \phi(y_k)\psi(x_i) \ldots \psi(x_1)\psi(x_{i+1}) \ldots \psi(x_{i+j}) \rangle, \tag{2.23}$$

since $\varepsilon_{i+j}\varepsilon_i\varepsilon_j = (-1)^{ij}$, and for the fps $Z^M\{J\}$ of (1.14)

$$Z^M\{J\} \frac{\delta}{\delta J_f(x_1)} \cdots \frac{\delta}{\delta J_f(x_{k_1})} \frac{\delta}{\delta J_b(x_{k_1+1})} \cdots \frac{\delta}{\delta J_b(x_{k_1+k_2})}$$
$$= \langle e^{U^M\{J\}}\psi(x_1) \ldots \psi(x_{k_1})\phi(x_{k_1+1}) \ldots \phi(x_{k_1+k_2}) \rangle = Z^M\{J\} \frac{\delta}{\delta J_k(\vec{x}_k)}. \tag{2.24}$$

The following rules for computing with functional derivatives follow readily from the definitions (2.20) and (2.22):

TABLE II.6. *Fps calculus.*

Commutativity and associativity

$$\frac{\delta}{\delta f(x)} \frac{\delta}{\delta f(y)} = \frac{\delta}{\delta f(y)} \frac{\delta}{\delta f(x)}, \quad \frac{\delta}{\delta f(x)} \frac{\delta}{\delta\alpha(y)} = \frac{\delta}{\delta\alpha(y)} \frac{\delta}{\delta f(x)} \tag{2.25 a}$$

$$\frac{\delta}{\delta\alpha(x)} \frac{\delta}{\delta\alpha(y)} = -\frac{\delta}{\delta\alpha(y)} \frac{\delta}{\delta\alpha(x)} \tag{2.25 b}$$

$$\left(\frac{\delta}{\delta\alpha(x)} u\right) \frac{\delta}{\delta\alpha(y)} = \frac{\delta}{\delta\alpha(x)} \left(u \frac{\delta}{\delta\alpha(y)}\right) \tag{2.25 c}$$

Relationship between left and right derivatives

$$\frac{\delta}{\delta\alpha} u = (-1)^{i-1} u \frac{\delta}{\delta\alpha} \quad \text{for} \quad u \in \mathscr{F}^i \tag{2.26 a}$$

$$\frac{\delta}{\delta\alpha} u = - u \frac{\delta}{\delta\alpha} \quad \text{for} \quad u \in \mathscr{F}_e \tag{2.26 b}$$

$$\frac{\delta}{\delta\alpha} u = u \frac{\delta}{\delta\alpha} \quad \text{for} \quad u \in \mathscr{F}_0 \tag{2.26 c}$$

Product rules

$$\frac{\delta}{\delta f}(uv) = \frac{\delta u}{\delta f} v + u \frac{\delta v}{\delta f} \tag{2.27 a}$$

$$\frac{\delta}{\delta\alpha}(uv) = \frac{\delta u}{\delta\alpha} v + (-1)^i u \frac{\delta v}{\delta\alpha} \quad \text{for} \quad u \in \mathscr{F}^i \tag{2.27 b}$$

$$(uv)\frac{\delta}{\delta\alpha} = (-1)^j \left(u \frac{\delta}{\delta\alpha} \right) v + u \left(v \frac{\delta}{\delta\alpha} \right) \quad \text{for} \quad v \in \mathscr{F}^j \tag{2.27 c}$$

$$\frac{\delta}{\delta\alpha(x_1)} \cdots \frac{\delta}{\delta\alpha(x_m)} uv = \mathrm{A}_x \sum_{r=0}^{m} \binom{m}{r} \left(\frac{\delta}{\delta\alpha(x_1)} \cdots \frac{\delta}{\delta\alpha(x_r)} u \right) \left(\frac{\delta}{\delta\alpha(x_{r+1})} \cdots \frac{\delta}{\delta\alpha(x_m)} v \right) \quad \text{for} \quad u \in \mathscr{F}_e \tag{2.27 d}$$

$$uv \frac{\delta}{\delta\alpha(x_1)} \cdots \frac{\delta}{\delta\alpha(x_m)} = \mathrm{A}_x \sum_{r=0}^{m} \binom{m}{r} \left(u \frac{\delta}{\delta\alpha(x_1)} \cdots \frac{\delta}{\delta\alpha(x_r)} \right) \left(v \frac{\delta}{\delta\alpha(x_{r+1})} \cdots \frac{\delta}{\delta\alpha(x_m)} \right) \quad \text{for} \quad v \in \mathscr{F}_e \tag{2.27 e}$$

Derivatives of multiplicative inverse

$$\begin{aligned} \frac{\delta}{\delta f} u^{-1} &= - u^{-1} \frac{\delta u}{\delta f} u^{-1} \\ \frac{\delta}{\delta\alpha} u^{-1} &= - u^{-1} \left(\frac{\delta}{\delta\alpha} u \right) u^{-1} \quad \text{for} \quad u \in \mathscr{F}_e \\ u^{-1} \frac{\delta}{\delta\alpha} &= - u^{-1} \left(u \frac{\delta}{\delta\alpha} \right) u^{-1} \quad \text{for} \quad u \in \mathscr{F}_e \end{aligned} \tag{2.28}$$

Chain rule

For $u \in \mathscr{F}_e$

$$\frac{\delta}{\delta f}\mathrm{F}(u) = \mathrm{F}'(u)\frac{\delta u}{\delta f}$$
$$\frac{\delta}{\delta \alpha}\mathrm{F}(u) = \mathrm{F}'(u)\frac{\delta}{\delta \alpha}u \tag{2.29}$$
$$\mathrm{F}(u)\frac{\delta}{\delta \alpha} = \mathrm{F}'(u)\left(u\frac{\delta}{\delta \alpha}\right)$$

As an illustration of the above manipulations consider the partition function Z for the Y_2 model in the MSS form $\int dve^u$ (see (2.12)). By (2.22) and (2.26 *a*)

$$u\frac{\delta}{\delta\alpha(x)}\frac{\delta}{\delta\alpha(y)} = -\frac{\delta}{\delta\alpha(x)}u\frac{\delta}{\delta\alpha(y)} = \vec{\mathrm{S}}(x, y; \phi). \tag{2.30}$$

Let $\pi_\pm$ denote the restriction operators from $\mathscr{H}_f$ to $\mathscr{H}_{-1/2} \otimes \mathbb{C}^2$ defined by $\pi_-(\alpha_-, \alpha_+) = \alpha_-$ and $\pi_+(\alpha_-, \alpha_+) = \alpha_+$, and let

$$\frac{\delta}{\delta\alpha_\pm(x)} = \pi_\pm^* \frac{\delta}{\delta\alpha(x)} \tag{2.31}$$

so that multiple derivatives $\left(\frac{\delta}{\delta\alpha_+}\right)^i\left(\frac{\delta}{\delta\alpha_-}\right)^j$ yield (antisymmetric) functionals on $(\mathscr{H}_{-1/2} \otimes \mathbb{C}^2)^i \times (\mathscr{H}_{-1/2} \otimes \mathbb{C}^2)^j$. Then by (2.30) and (2.14 *b*)

$$u\frac{\delta}{\delta\alpha_-}\frac{\delta}{\delta\alpha_-} = u\frac{\delta}{\delta\alpha_+}\frac{\delta}{\delta\alpha_+} = 0$$

and

$$\frac{\delta}{\delta\alpha_+(x)}u\frac{\delta}{\delta\alpha_-(y)} = -\mathrm{S}(x, y; \phi). \tag{2.32}$$

Hence by (2.29)

$$\frac{\delta}{\delta\alpha_+(x_n)}\cdots\frac{\delta}{\delta\alpha_+(x_1)}e^u\frac{\delta}{\delta\alpha_-(y_1)}\cdots\frac{\delta}{\delta\alpha_-(y_n)}\{0, 0\}$$
$$= \left(\frac{\delta}{\delta\alpha_+(x_n)}u\right)\cdots\left(\frac{\delta}{\delta\alpha_+(x_1)}u\right)\frac{\delta}{\delta\alpha_-(y_1)}\cdots\frac{\delta}{\delta\alpha_-(y_n)}$$
$$= \sum_{\sigma\in \mathrm{S}_n}\operatorname{sgn}\sigma\prod_{i=1}^{n}\frac{\delta}{\delta\alpha_+(x_i)}u\frac{\delta}{\delta\alpha_-(y_{\sigma_i})} \qquad \text{(by (2.27 } c\text{))}$$
$$= (-1)^n \det \mathrm{S}(x_i, y_j; \phi). \tag{2.33}$$

In this way we obtain

$$\frac{\delta}{\delta f(z_1)} \cdots \frac{\delta}{\delta f(z_k)} \frac{\delta}{\delta\alpha_+(x_n)} \cdots \frac{\delta}{\delta\alpha_+(x_1)} Z \frac{\delta}{\delta\alpha_-(y_1)} \cdots \frac{\delta}{\delta\alpha_-(y_n)} \{ 0, 0 \}$$
$$= (-1)^n \int dv(\phi) \det S(x_i, y_j; \phi)\phi(z_1) \ldots \phi(z_k). \quad (2.34)$$

Identifying (2.23) and (2.34) yields the MSS formula for Schwinger functions,

$$\langle \psi_+(x_n) \ldots \psi_+(x_1)\psi_-(y_1) \ldots \psi_-(y_n)\phi(z_1) \ldots \phi(z_k) \rangle$$
$$= \int dv(\phi) \det S(x_i, y_j; \phi)\phi(z_1) \ldots \phi(z_k). \quad (2.35)$$

(As the computation (2.33) shows, derivatives $\delta_f^k \delta_{\alpha_+}^i Z \delta_{\alpha_-}^j \{ 0, 0 \}$ vanish if $i \neq j$.)

DEFINITION II.7 (*Composition of fps*). — Suppose that $\beta \in \mathscr{F}_0(\mathscr{L}_0, \mathscr{L}_e; \mathscr{V}_0)$, $g \in \mathscr{F}_e(\mathscr{L}_0, \mathscr{L}_e; \mathscr{V}_e)$ with $g_{(0,0)} = 0$, and $u \in \mathscr{F}(\mathscr{V}_0, \mathscr{V}_e; \mathscr{L})$. Then the composition $u\{ \beta\{ .,. \}, g\{ .,. \} \} = u \circ (\beta, g) \in \mathscr{F}(\mathscr{L}_0, \mathscr{L}_e; \mathscr{L})$ is defined by

$$(u \circ \{ \beta, g \})_m(\alpha_1, \ldots, \alpha_{m_0}; f_1, \ldots, f_{m_e})$$
$$= S_f A_\alpha \sum_{m'} \frac{1}{m'!} \sum_{j=(j^1,\ldots,j^{m'_0+m'_e})} \binom{m}{j} u_{m'}(\beta_{j^1}, \ldots, \beta_{j^{m'_0}}; g_{j^{m'_0+1}}, \ldots, g_{j^{m'_0+m'_e}})$$

where $m' = (m'_0, m'_e)$, $j^k = (j_0^k, j_e^k)$ are pairs of non-negative integers with

$$j_0^1 + \ldots + j_0^{m'_0+m'_e} = m_0, \qquad j_e^1 + \ldots + j_e^{m'_0+m'_e} = m_e,$$
$$\binom{m}{j} = \binom{m_0}{j_0^1, \ldots, j_0^{m'_0+m'_e}} \binom{m_e}{j_e^1, \ldots, j_e^{m'_0+m'_e}},$$

and the arguments of β_{j^k} and g_{j^k} are $(\alpha_{j_0^1+\ldots+j_0^{k-1}+1}, \ldots, \alpha_{j_0^1+\ldots+j_0^k}; f_{j_e^1+\ldots+j_e^{k-1}+1}, \ldots, f_{j_e^1+\ldots+j_e^k})$. Note that because $\beta_{0,0} = g_{0,0} = 0$ we must have $j_0^k + j_e^k \geq 1$ for all k; consequently $m'_0 + m'_e \leq m_0 + m_e$ and all sums in (2.36) are finite.

A simple example of composition of fps is that of composition with a linear operator. If $u \in \mathscr{F}(\mathscr{L}_0, \mathscr{L}_e; \mathscr{L})$ and T_i is a continuous linear operator on $\mathscr{L}_i$, then

$$(Tu\{ T_0\alpha, T_e f \})_m(\alpha_1, \ldots, f_{m_e}) = Tu_m(T_0\alpha_1, \ldots, T_e f_{m_e}). \quad (2.37)$$

Just as in II we have:

THEOREM II.8 (*Chain rule*). — Let $\beta \in \mathscr{F}_0(\mathscr{L}_0, \mathscr{L}_e; \mathscr{V}_0)$, $g \in \mathscr{F}_e(\mathscr{L}_0, \mathscr{L}_e; \mathscr{V}_e)$ with $g_0 = 0$, and $u \in \mathscr{F}(\mathscr{V}_0, \mathscr{V}_e; \mathscr{L})$

$$a)\ \frac{\delta}{\delta f} u\{\beta\{\alpha, f\}, g\{\alpha, f\}\} = \frac{\delta \beta}{\delta f}\{\alpha, f\} \frac{\delta u}{\delta \beta}\{\beta\{\alpha, f\}, g\{\alpha, f\}\}$$

$$+ \frac{\delta u}{\delta g}\{\beta\{\alpha, f\}, g\{\alpha, f\}\} \frac{\delta g}{\delta f}\{\alpha, f\} = \left(u \frac{\delta}{\delta \beta}\right) \frac{\delta \beta}{\delta f} + \frac{\delta u}{\delta g} \frac{\delta g}{\delta f}$$

$$b)\quad \frac{\delta}{\delta \alpha} u = \left(\frac{\delta}{\delta \alpha} \beta\right) \frac{\delta}{\delta \beta} u + \left(\frac{\delta}{\delta \alpha} g\right) \frac{\delta u}{\delta g}$$

$$c)\quad u \frac{\delta}{\delta \alpha} = \left(u \frac{\delta}{\delta \beta}\right)\left(\beta \frac{\delta}{\delta \alpha}\right) + \frac{\delta u}{\delta g}\left(g \frac{\delta}{\delta \alpha}\right)$$

d) If β, g, and u depend differentiably on a parameter t, then

$$\frac{d}{dt} u\{\beta, g\} = \frac{\partial u}{\partial t} + \frac{\partial \beta}{\partial t} \frac{\delta}{\delta \beta} u + \frac{\partial g}{\partial t} \frac{\delta u}{\delta g}$$

$$= \frac{\partial u}{\partial t} + \left(u \frac{\delta}{\delta \beta}\right) \frac{\partial \beta}{\partial t} + \frac{\delta u}{\delta g} \frac{\partial g}{\partial t}.$$

Remark. — The suppressed arguments in *b*), *c*), *d*) and the second result of *a*) are determined by the pattern of arguments in the first result *a*). An expression like $\frac{\delta u}{\delta g} \frac{\delta g}{\delta f}$ is to be interpreted as the value of $\frac{\delta u}{\delta g}$ (regarded as a linear functional on $\mathscr{V}_e$) at $\frac{\delta u}{\delta g}$ (which takes values in $\mathscr{V}_e$ because g does).

By imitating the proof of Lemma II.10 of II we have:

THEOREM II.9 (*Composition inverse*). — Let $\beta \in \mathscr{F}_0(\mathscr{L}_0, \mathscr{L}_e; \mathscr{V}_0)$ and $g \in \mathscr{F}_e(\mathscr{L}_0, \mathscr{L}_e, \mathscr{V}_e)$. If $\beta_{(1,0)}^{-1} : \mathscr{V}_0 \to \mathscr{L}_0$ and $g_{(0,1)}^{-1} : \mathscr{V}_e \to \mathscr{L}_e$ exist as continuous linear maps, then there exists a unique composition inverse (γ, h) to (β, g); i.e., $\gamma \in \mathscr{F}_0(\mathscr{V}_0, \mathscr{V}_e; \mathscr{L}_0)$ and $h \in \mathscr{F}_e(\mathscr{V}_0, \mathscr{V}_e; \mathscr{L}_e)$ with

$$\gamma\{\beta\{\alpha, f\}, \quad g\{\alpha, f\}\} = \alpha, \qquad h\{\beta\{\alpha, f\}, \quad g\{\alpha, f\}\} = f \tag{2.38 a}$$

and

$$\beta\{\gamma\{\kappa, k\}, \quad h\{\kappa, k\}\} = \kappa, \qquad g\{\gamma\{\kappa, k\}, \quad h\{\kappa, k\} = k. \tag{2.38 b}$$

Here the right side α (and similarly for f, κ, k) represents the fps in $\mathscr{F}(\mathscr{L}_0, \mathscr{L}_e; \mathscr{L}_0)$ whose components are all zero with the exception of the (1, 0)-component which is the identity operator on $\mathscr{L}_0$.

The following theorem is a direct consequence of the uniqueness of the composition inverse.

THEOREM II.10 (*Inversion of symmetry*). — Under the hypotheses of Theorem II.9, if there exist continuous linear operators T_i on $\mathscr{L}_i$ and U_i on $\mathscr{V}_i$ with continuous linear inverses such that

$$U_0\beta\{T_0\alpha, T_e f\} = \beta\{\alpha, f\}$$
$$U_e g\{T_0\alpha, T_e f\} = g\{\alpha, f\},$$

then the composition inverse (γ, h) to (β, g) satisfies

$$T_0^{-1}\gamma\{U_0^{-1}\beta, U_e^{-1}g\} = \gamma\{\beta, g\}$$
$$T_e^{-1}h\{U_0^{-1}\beta, U_e^{-1}g\} = h\{\beta, g\}.$$

If we right-differentiate the two equations in (2.38 *a*) with respect to α and f using the chain rule, we obtain 4 equations which can be written in matrix form as

$$\begin{bmatrix} \gamma\dfrac{\delta}{\delta\beta} & \dfrac{\delta\gamma}{\delta g} \\ h\dfrac{\delta}{\delta\beta} & \dfrac{\delta h}{\delta g} \end{bmatrix} \begin{bmatrix} \beta\dfrac{\delta}{\delta\alpha} & \dfrac{\delta\beta}{\delta\alpha} \\ g\dfrac{\delta}{\delta\alpha} & \dfrac{\delta g}{\delta f} \end{bmatrix} = \begin{bmatrix} I^{\mathscr{L}_0} & 0 \\ 0 & I^{\mathscr{L}_e} \end{bmatrix} = I \qquad (2.39)$$

Here $I^{\mathscr{L}_0}$, $I^{\mathscr{L}_e}$ and I are the identity operators on $\mathscr{L}_0$, $\mathscr{L}_e$ and $\mathscr{L}_0 \oplus \mathscr{L}_e$. (2.39) is to be interpreted as an equality of fps in $\mathscr{F}(\mathscr{L}_0, \mathscr{L}_e; \mathscr{B}(\mathscr{L}_0 \oplus \mathscr{L}_e))$ where $\mathscr{B}(\mathscr{L}_0 \oplus \mathscr{L}_e)$ is the space of continuous linear operators on $\mathscr{L}_0 \oplus \mathscr{L}_e$. The first factor on the left side of (2.39) takes values in the space $\mathscr{B}(\mathscr{V}_0 \oplus \mathscr{V}_e, \mathscr{L}_0 \oplus \mathscr{L}_e)$ of continuous linear operators from $\mathscr{V}_0 \oplus \mathscr{V}_e$ to $\mathscr{L}_0 \oplus \mathscr{L}_e$, and the second factor takes values in the space $\mathscr{B}(\mathscr{L}_0 \oplus \mathscr{L}_e, \mathscr{V}_0 \oplus \mathscr{V}_e)$. Similarly we can left-differentiate (2.38 *a*) or differentiate with respect to a parameter:

THEOREM II.11 (*Jacobian relations*). — Let $\gamma(h)$ be the composition inverse to (β, g) (assuming the hypotheses of Theorem II.9).

a) As an equality in $\mathscr{F}(\mathscr{L}_0, \mathscr{L}_e; \mathscr{B}(\mathscr{L}_0 \oplus \mathscr{L}_e, \mathscr{V}_0 \oplus \mathscr{V}_e))$

$$\begin{bmatrix} \gamma\dfrac{\delta}{\delta\beta} & \dfrac{\delta\gamma}{\delta g} \\ h\dfrac{\delta}{\delta\beta} & \dfrac{\delta h}{\delta g} \end{bmatrix}^{-1} = \begin{bmatrix} \beta\dfrac{\delta}{\delta\alpha} & \dfrac{\delta\beta}{\delta f} \\ g\dfrac{\delta}{\delta\alpha} & \dfrac{\delta g}{\delta f} \end{bmatrix}. \qquad (2.40\,a)$$

b) As an equality in $\mathscr{F}(\mathscr{L}_0, \mathscr{L}_e; \mathscr{B}(\mathscr{L}_0 \oplus \mathscr{L}_e, \mathscr{V}_0 \oplus \mathscr{V}_e))$

$$\begin{bmatrix} \dfrac{\delta}{\delta\beta}\gamma & \dfrac{\delta}{\delta\beta}h \\ \dfrac{\delta\gamma}{\delta g} & \dfrac{\delta h}{\delta g} \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{\delta}{\delta\alpha}\beta & \dfrac{\delta}{\delta\alpha}g \\ \dfrac{\delta\beta}{\delta f} & \dfrac{\delta g}{\delta f} \end{bmatrix}. \qquad (2.40\,b)$$

c) If in addition β and g depend differentiably on a parameter t then so do γ and h and

$$\begin{bmatrix} \dfrac{\partial\gamma}{\partial t} \\ \dfrac{\partial h}{\partial t} \end{bmatrix} = - \begin{bmatrix} \beta\dfrac{\delta}{\delta\gamma} & \dfrac{\delta\beta}{\delta h} \\ g\dfrac{\delta}{\delta\gamma} & \dfrac{\delta g}{\delta h} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\partial\beta}{\partial t} \\ \dfrac{\partial g}{\partial t} \end{bmatrix}. \tag{2.40 c}$$

As an example of the application of Defn. II.7 and Theorems II.8-II.11, we consider the construction of the first order Legendre transform $\Gamma^{(1)} = \Gamma^{M}$ where $M = \{(1,0), (0,1)\}$. Instead of using the notation $J = (J_{(1,0)}, J_{(1,0)})$ and $A = (A_{(1,0)}, A_{(0,1)})$ for the basic conjugate variables (as in (1.17)) we shall use $J = (\alpha, f)$ and $A = (\beta, g)$. From Example II.4 (we suppress the t-dependence) we know that

$$G\{\alpha, f\} \equiv \ln Z\{\alpha, f\} \in \mathscr{F}_Y \equiv \mathscr{F}_e(\mathscr{H}_f, \mathscr{H}_b; \mathbb{C}).$$

It follows that

$$\beta(x) \equiv G\{\alpha, f\}\frac{\delta}{\delta\alpha(x)} \in \mathscr{F}_0(\mathscr{H}_f, \mathscr{H}_b; \mathscr{H}_f^*) \tag{2.41 a}$$

and

$$g(x) \equiv \frac{\delta G\{\alpha, f\}}{\delta f(x)} - \frac{\delta G\{0,0\}}{\delta f(x)} \in \mathscr{F}_e(\mathscr{H}_f, \mathscr{H}_b; \mathscr{H}_b^*) \tag{2.41 b}$$

where (see (1.10))

$$\mathscr{H}_f^* = \mathscr{H}_{1/2}(\mathbb{R}^2) \otimes \mathbb{C}^4, \qquad \mathscr{H}_b^* = \mathscr{H}_1(\mathbb{R}^2). \tag{2.42}$$

In order to apply Theorem II.9 to invert the map $(\alpha, f) \to (\beta, g)$ of (2.41) we first compute (using (2.27)-(2.29)) that

$$\begin{aligned} \beta_{(1,0)}(x, y) &= \beta(x)\frac{\delta}{\delta\alpha(y)}\{0,0\} \\ &= G\frac{\delta}{\delta\alpha(x)}\frac{\delta}{\delta\alpha(y)}\{0,0\} \\ &= \left[Z^{-1}\left(Z\frac{\delta}{\delta\alpha(x)}\frac{\delta}{\delta\alpha(y)}\right) + Z^{-2}\left(Z\frac{\delta}{\delta\alpha(y)}\right)\left(Z\frac{\delta}{\delta\alpha(x)}\right)\right]\{0,0\} \\ &= \langle\, \psi(x)\psi(y)\,\rangle \qquad \text{(by (2.23))} \end{aligned} \tag{2.43 a}$$

and

$$\begin{aligned} g_{(0,1)}(x, y) &= \frac{\delta g(x)}{\delta f(y)}\{0,0\} \\ &= \langle\, \hat{\phi}(x)\hat{\phi}(y)\,\rangle. \end{aligned} \tag{2.43 b}$$

Assuming, as is true for εY_2 (see Ref. 5), that $\beta_{(1,0)}^{-1}$ is a bounded operator from $\mathcal{H}_f^*$ to H_f and that $g_{(0,1)}^{-1}$ is a bounded operator from $\mathcal{H}_b^*$ to $\mathcal{H}_b$, we may invert (2.41) to get

$$\alpha\{\beta, g\} \in \mathcal{F}_0(\mathcal{H}_f^*, \mathcal{H}_b^*; \mathcal{H}_f)\,.$$
$$f\{\beta, g\} \in \mathcal{F}_e(\mathcal{H}_f^*, \mathcal{H}_b^*; \mathcal{H}_b)\,. \tag{2.44}$$

The first order Legendre transform is then defined by

$$\Gamma^{(1)}\{\beta,g\} \equiv G\{\alpha, f\} - G_\alpha . \alpha - G_f . f \,|_{\substack{\alpha=\alpha\{\beta,g\}\\ f=f\{\beta,g\}}} \tag{2.45}$$

as an element of $\mathcal{F}_Y^* \equiv \mathcal{F}_e(\mathcal{H}_f^*, \mathcal{H}_b^*; \mathbb{C})$. The chain rule (Theor. II.8) together with the product rule (2.27 b) give the conjugate relations:

$$\frac{\delta}{\delta\beta}\Gamma^{(1)} = \left(\frac{\delta}{\delta\beta}\alpha\right)\left(\frac{\delta}{\delta\alpha}G\right) + \left(\frac{\delta}{\delta\beta}f\right)\frac{\delta G}{\delta f} + \beta\frac{\delta\alpha}{\delta\beta} - \alpha - G_f\frac{\delta}{\delta\beta}f = -\alpha \tag{2.46a}$$

$$\frac{\delta\Gamma^{(1)}}{\delta g} = \left(G\frac{\delta}{\delta\alpha}\right)\frac{\delta\alpha}{\delta g} + \frac{\delta G}{\delta f}\frac{\delta f}{\delta g} - \beta\frac{\delta\alpha}{\delta g} - f - G_f\frac{\delta f}{\delta g} = -f \tag{2.46b}$$

(in each case the first and third terms cancel as do the second and fifth). Theorem II.11 b) gives the Jacobian relation:

$$-\begin{bmatrix} \frac{\delta}{\delta\beta}\frac{\delta}{\delta\beta}\Gamma^{(1)} & \frac{\delta}{\delta\beta}\frac{\delta}{\delta g}\Gamma^{(1)} \\ \frac{\delta}{\delta g}\frac{\delta}{\delta\beta}\Gamma^{(1)} & \frac{\delta^2}{\delta g\delta g}\Gamma^{(1)} \end{bmatrix} = \begin{bmatrix} \frac{\delta}{\delta\beta}\alpha & \frac{\delta}{\delta\beta}f \\ \frac{\delta\alpha}{\delta g} & \frac{\delta f}{\delta g} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\delta}{\delta\alpha}\beta & \frac{\delta}{\delta\alpha}g \\ \frac{\delta\beta}{\delta f} & \frac{\delta g}{\delta f} \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} \frac{\delta}{\delta\alpha}G\frac{\delta}{\delta\alpha} & \frac{\delta}{\delta\alpha}\frac{\delta G}{\delta f} \\ \frac{\delta}{\delta f}G\frac{\delta}{\delta\alpha} & \frac{\delta^2 G}{\delta f^2} \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} G\frac{\delta}{\delta\alpha}\frac{\delta}{\delta\alpha} & G\frac{\delta}{\delta\alpha}\frac{\delta}{\delta f} \\ G\frac{\delta}{\delta\alpha}\frac{\delta}{\delta f} & G\frac{\delta}{\delta f}\frac{\delta}{\delta f} \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.47}$$

In the next section we generalize the above construction of Γ^M to the case of general M.

§ III. DEFINITIONS AND BASIC PROPERTIES OF THE LEGENDRE TRANSFORMS

We now set out to interpret the formal definition of Γ^M given in (1.12)-(1.17) in terms of the fps calculus of § II. We shall soon place some restrictions on M, but for now let it be any subset of

$\mathscr{M}=\{ m=(m_f, m_b) \mid m_i \in \mathbb{N} \text{ for } i=f \text{ or } b, \text{ and } m_f+m_b>0 \} = \mathbb{N}^2 \backslash \{0,0\}$.

Since the components of $m \in \mathscr{M}$ represent numbers of fermi and bose arguments, we find it convenient to denote $m = (m_f, m_b)$ alternatively by $m=m_f f m_b f$ or by a string of m_f $f's$ and m_b b's eg. $(2, 3)=2f3b=ffbbb$. We shall order $\mathscr{M}$ in the obvious way, i. e. $r \leq s$ in $\mathscr{M}$ iff $r_b \leq s_b$ and $r_f \leq s_f$ and norm it by $|m| = m_b + m_f$. We shall also adopt the familiar multi-index conventions such as $mn = (m_f n_f, m_b n_b)$, $m! = m_b!\, m_f!$, etc.

In particular, as in (1.12), we shall consider the « direct sum » formal field $\Phi = (\Phi_f, \Phi_b) = (\psi, \phi)$, and for $\vec{x}_m = (x_1 \ldots x_{m_f}, g_1 \ldots g_{m_b})$ (with $y_i \in \mathbb{R}^d$ and $x_i \in \mathbb{R}^d \times \{+, -\} \times 2^{[d/2]}$) we let $\Phi^m(\vec{x}_m)=\psi(x_1)\ldots\psi(x_{m_f})\phi(y_1)\ldots\phi(y_{m_b})$, and we shall let $\sigma=(-1, +1)$ so that $\sigma^m=(-1)^{m_f}$.

Now consider two « single field test function » spaces $\mathscr{L}_f$ and $\mathscr{L}_b$ along with their direct sum $\mathscr{L}_1 = \mathscr{L}_f \oplus \mathscr{L}_b$. With each point $m \in$ M we associate a « source function » space

$$\mathscr{L}_m = \mathscr{L}_f^{@m f} \otimes \mathscr{L}_b^{\circledS m b}. \tag{3.1}$$

Here the symmetric and antisymmetric tensor products $\mathscr{L}_b^{@m b}$ and $\mathscr{L}_f^{\circledS m f}$ are defined in the usual way as subspaces of the corresponding tensor product spaces and we shall also refer to the « appropriately symmetrized » tensor product of (2.3 b) in terms of which we have

$$\mathscr{L}_{m_1} \odot \mathscr{L}_{m_2} = \mathscr{L}_{m_1+m_2}$$

and for $\mathrm{N} \in \mathbb{N}$

$$(\mathscr{L}_f \oplus \mathscr{L}_b)^{\odot \mathrm{N}} = \sum_{\substack{m \in \mathscr{M} \\ |m|=\mathrm{N}}} \mathscr{L}_m .$$

For $\mathrm{M} \subset \mathscr{M}$ consider

$$\mathscr{L}^{\mathrm{M}} \equiv \bigoplus_{m \in \mathrm{M}} \mathscr{L}_m, \qquad \mathscr{L}_e^{\mathrm{M}} \equiv \bigoplus_{\substack{m \in \mathrm{M} \\ m_f \text{ even}}} \mathscr{L}_m \quad \text{and} \quad \mathscr{L}_0^{\mathrm{M}} = \bigoplus_{\substack{m \in \mathrm{M} \\ m_f \text{ odd}}} \mathscr{L}_m \tag{3.2}$$

as subspaces of $\mathscr{L}^{\mathscr{M}}$ and define P^{M} to be the projector in $\mathscr{L}^{\mathscr{M}}$ on $\mathscr{L}^{\mathrm{M}}$. (We shall use the same notation, P^{M}, regardless of the base spaces $\mathscr{L}_f$ and $\mathscr{L}_b$, with any necessary distinctions being inferred from the context.) Whenever the superscript M is omitted, we are referring by convention to the case $\mathrm{M} = \mathscr{M}$.

As in § I, for $J_m \in \mathscr{L}_m$ we let

$$\Phi^m(J_m) = \int \Phi^m(\vec{x}_m) J_m(\vec{x}_m) d\vec{x}_m$$

and

$$U^M\{J\} = \sum_{m \in M} \Phi^m(J_m).$$

Our definition of the partition function $Z\{J\} = Z^{\mathscr{M}}\{J\}$ is then motivated by (1.14) and the following formal calculation.

$$Z\{J\} = \langle e^{U\{J\}} \rangle = \sum_{N=0}^{\infty} \frac{1}{N!} \langle U\{J\}^N \rangle = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{m_1 \ldots m_N} \left\langle \prod_{i=1}^{N} \Phi^{m_i}(J_{m_i}) \right\rangle$$
$$= \sum_{n=0}^{\infty} \frac{1}{N!} \sum_{m_1 \ldots m_N} \int \langle \Phi^{m_1}(\vec{x}_{m_1}) \ldots \Phi^{m_N}(\vec{x}_{m_N}) \rangle J_{m_N}(\vec{x}_{m_N}) \ldots J_{m_1}(\vec{x}_{m_1}) d\vec{x}_{m_1} \ldots d\vec{x}_{m_N} \quad (3.3)$$

(The products $\Phi^m(J_m)$ all commute with everything, and if in the last step we extract J's starting from the right, then each is commuted past an even total power of anticommuting objects on the way to its final destination.)

DEFINITION III.1. — The partition function, Z, is the unique element of $\mathscr{F}(\mathscr{L}_0, \mathscr{L}_e; \mathbb{C})$ obeying

$$Z \frac{\delta}{\delta J_{m_1}(\vec{x}_{m_1})} \cdots \frac{\delta}{\delta J_{m_N}(\vec{x}_{m_N})} \{0\} = \langle \Phi^{m_1}(\vec{x}_{m_1}) \ldots \Phi^{m_N}(\vec{x}_{m_N}) \rangle \quad (3.4)$$

for all N, $m_1, \ldots, m_N, \vec{x}_{m_1}, \ldots, \vec{x}_{m_N}$.

Remarks 1. — Here $\langle \Phi^{m_1}(.) \ldots \Phi^{m_N}(.) \rangle$ denotes an appropriate Schwinger function. For Y_2 without cutoffs the Schwinger functions are constructed in refs. [*10*] and [*11*]. To introduce cutoffs and/or t-dependence (as described in § 1) in Z we just repeat the above definition with the appropriately modified expectation.

2. The choices of $\mathscr{L}_b$ and $\mathscr{L}_f$ are dictated by the requirement that $\langle \, . \, \rangle$ be sufficiently regular in t.

3. The vanishing of expectations involving net odd powers of ψ implies that Z is always even.

Using a straightforward fps composition we now have

DEFINITION III.2. $\quad G = \ln\{Z\}. \quad (3.5)$

DEFINITION III.3.

Let
$$A^M_m\{J\} \equiv \begin{cases} G^M_{J^m_1}\{J\} - A^0_m & \text{for } m \in M \\ J_m \quad \text{for} \quad m \in M^c = \mathscr{M}\backslash M \end{cases} \tag{3.6}$$
and
$$J^M \equiv A^{M^{-1}}. \tag{3.7}$$
Then
$$\Gamma^M\{A\} \equiv G\{J^M\{A\}\} - G_J\{J^M\{A\}\}P^M J^M\{A\}. \tag{3.8}$$
(Here, in the same spirit as our previous notational conventions, by $G_{J^m_1}(\vec{x}_m)$ we mean $G\dfrac{\delta}{\delta J_f(x_1)}\cdots\dfrac{\delta}{\delta J_f(x_{m_f})}\dfrac{\delta}{\delta J_b(y_1)}\cdots\dfrac{\delta}{\delta J_b(y_{m_b})}$.)

Remarks 1. — $A^0_m \equiv G_{J^m_1}\{0\}$ is subtracted so that $A = 0$ at $J = 0$.

2. For Definition III.3 to make sense as it stands it is necessary for A^M to have a fps composition inverse taking values in the domain of $\theta^M \equiv G - G_J P^M$. In [ref. *5*] we shall establish that, for εY_2, Γ^M is a well defined element of $\mathscr{F}(\mathscr{L}_0'^M, \mathscr{L}_e'^M; \mathbb{C})$ and C^∞ in t for suitable test function spaces $\mathscr{L}_0'^M$ and $\mathscr{L}_e'^M$.

3. Unless otherwise specified we restrict our attention to evaluation of Γ^M at arguments, A, with $A_m = 0$ for $m \in M^c$.

We now explain how to interpret in terms of fps the discussion of irreducibility given in § I.

To start with we let $P_\pm$ denote the projectors defined by

$$(P_\pm J)_m(\vec{x}_m) = \begin{cases} J_m(\vec{x}_m) & \text{if all components of } \vec{x}_m \text{ have } \mathbb{R}^d \text{ component on the } \pm \text{ side of } \sigma \\ 0 & \text{otherwise} \end{cases} \tag{3.9}$$

and $P_0 = 1 - P_+ - P_-$. We also let $\mathscr{L}^\pm_0 = P^\pm_0 \mathscr{L}$.

The phrase « at 0 » and symbol $|_0$ then denote restriction to both $t=(0,0)$ and to smooth text functions J supported away from σ with $P_0 J = 0$. So $F = G$ at 0, means $(F - G)|_0 = 0$. (It follows directly from the definition of fps derivatives and equality that if $F = G$ at 0, then also $P_\pm \dfrac{\delta}{\delta J}F = P_\pm \dfrac{\delta}{\delta J}G$ at 0, i.e. $\dfrac{\delta}{\delta J_m(\vec{x}_m)}F = \dfrac{\delta}{\delta J_m(\vec{x}_m)}G$ at 0, whenever the arguments of $\vec{x}_m$ are all on the same side of σ.)

The same decoupling properties used to establish (2.18) now easily generalize to yield the results that
$$Z = Z\{P_+\}.Z\{P_-\} \quad \text{at} \quad 0 \tag{3.10}$$
and so
$$G = G\{P_+\} + G\{P_-\} \quad \text{at} \quad 0 \tag{3.11}$$

By $\vec{x}_m \sigma \vec{x}'_{m'}$ we shall mean that (the $\mathbb{R}^d$ component of) each component x_i of $\vec{x}_m = (x_1 \dots x_m)$ is separated by σ from all components of $\vec{x}'_{m'}$. It follows immediately that

$$G_{J_m(\vec{x}_m)J_{m'}(\vec{x}'_{m'})}|_0 = 0 \quad \text{whenever} \quad \vec{x}_m \sigma \vec{x}'_{m'} \tag{3.12}$$

This is what we mean by the statement « G is cluster connected » which from now on we abbreviate as « $G \sim 0$ » (we shall also write $F \sim G$ whenever $F - G$ is cluster connected). More generally, we have the following definition.

DEFINITION III.4. — For $r = (r_f, r_b) \in \mathbb{N}^2$ we say that $\Gamma\{A, t\}$ is r — CI (« r cluster irreducible ») if and only if for all $s \leq r$ we have

$$\partial_t^s \Gamma \sim 0 \tag{3.13}$$

(i. e. for all $m, m' \in \mathcal{M}$, $\partial^s_{tA_m(\vec{x}_m)A_{m'}(\vec{x}_{m'})}\Gamma|_0 = 0$ whenever $\vec{x}_m \sigma \vec{x}'_m$). The property of being (0, 0)-CI is the same as « cluster connectedness ».

Remark 1. — Here, since the argument of Γ is A, $|_0$ denotes restriction to $t = 0$ and $P_0 A = 0$. Fortunately, there is no confusion here with the restriction to $P_0 J = 0$ since, formally speaking, « at $t = 0$, $P_0 J = 0 \Leftrightarrow P_0 A = 0$ », i.e. at $t=0$, $P_0 \circ A \circ (1 - P_0) = 0$ and $P_0 \circ J \circ (1 - P_0) = 0$, (see (4.17) of I and Lemma II.11 of II).

2. The cluster connectedness of all Γ^M follows easily from the above remark together with (3.11) and (3.8).

In order to apply this definition and to investigate the other irreducibility properties discussed in § I we must compute various t-derivatives. A simple application of the chain rule (Theorem II.8 d)) yields, for $i = f$ or b

$$\begin{aligned}\partial_{t_i}\Gamma^M = & [(\partial_{t_i}G)\{J^M\} + G_J\{J^M\}.\partial_{t_i}J^M] \\ & \qquad - [G_J\{J^M\}.\partial_{t_i}J^M + \partial_{t_i}(G_J\{J^M\}).J^M] \\ = & (\partial_{t_i}G)\{J^M\} - \partial_{t_i}(G_J\{J^M\}).J^M\end{aligned} \tag{3.14 a}$$

For the first term on the right, the explicit formula

$$\partial_{t_i}G = -\frac{1}{2}\dot{C}_i^{-1}[G_{J_iJ_i} + G_{J_i}G_{J_i}|_0^J = -\frac{1}{2}\dot{C}_i^{-1}[A_{ii} + A_iA_i + 2A_i^0A_i] \text{ if } ii \in M \tag{3.14 b}$$

with $\dot{C}_i^{-1} = C_i^{-1}(\partial_{t_i}C_i)C_i^{-1}$, will be derived in Theorem III.16.

As far as the second term is concerned, if Γ^M had been defined using the Schwinger functions $A_m^S \equiv G_{J_m}$ for $m \in M$ as dual variables instead of the connected functions $A_m \equiv G_{J_1^m} - A_m^0$, then for $k \in M$ we would have $G_{J_k}\{J^{S,M}\{A^S\}\} = A_k^S\{J^{S,M}\{A\}\} + A_k^S$, independent of t, and so the second term would drop out. But our « connected variables » A_m are *not* the same as the « Schwinger variables » G_{J_m} though they *are* closely related, and the relationship bears looking at.

In fact, by (3.5) and (2.24), for $m \in \mathscr{M}$ we have

$$G_{J_m}\{J\} = \frac{1}{Z\{J\}} Z_{J_m}\{J\} = \frac{1}{Z\{J\}} Z_{J_1^m}\{J\}. \tag{3.15}$$

But if

$$Z\{J+g\} \in \mathscr{F}(\mathscr{L}_0 \oplus \mathscr{L}_f, \mathscr{L}_e \oplus \mathscr{L}_b; \mathbb{C})$$

is now defined by the obvious rule

$$Z\{J+g\}_{J_1^m g^n}|_{g=0} \equiv Z\{J\}_{J_1^{m+n}}$$

then from (3.15) we have, for $m \in \mathscr{M}$,

$$G_{J_m}\{J\} = \frac{1}{Z\{J\}} Z\{J+g\}_{g^m}|_{g=0} = (\exp \circ H\{J,g\})_{g^m}|_{g=0} \tag{3.16}$$

where

$$H\{J,g\} = G\{J+g\} - G\{J\} \tag{3.17}$$

is even. Hence by (2.27 e)

$$\begin{aligned}(\exp \circ H\{J,g\})_{g^m} &= \sum_N \frac{1}{N!}(H\{J,g\}^N)_{g^m} \\ &= \sum_N \frac{1}{N!} \sum_{\substack{m_1,\dots,m_N \in \mathscr{M} \\ m_1+\dots+m_N=m}} \binom{m}{m_1,\dots,m_N} \bigodot_{i=1}^{N} H\{J,g\}_{g^{m_i}}\end{aligned} \tag{3.18}$$

But

$$H\{J\{A\},g\}_{g^k}|_{g=0} = G_{J_1^k}\{J\{A\}\} = (A^0 + A)_k \tag{3.19}$$

so that by (3.16)-(3.19) we obtain:

Lemma III.5 (*Relation between Schwinger and connected variables*).

$$G_{J_m}\{J^{\mathscr{M}}\{A\}\} = F_m\{A^0 + A\} \tag{3.20}$$

where

$$F_m\{a\} = \sum_{N=1}^{|m|} \frac{1}{N!} \sum_{\substack{m_1,\dots,m_N \\ m_1+\dots+m_N=m}} \binom{m}{m_1,\dots,m_N} \bigodot_{i=1}^{N} a_{m_i} \quad \text{for} \quad m \in \mathscr{M} \tag{3.21}$$

and

$$F_{(0,0)}\{\underline{a}\} = 1 \tag{3.22}$$

Remark. — $F_m\{a\}$ depends only on a_k for $k \leq m$. So, as long as $k \leq m$ implies $k \in M$, we have $F_m = F_m \circ P_M$ and so

$$\begin{aligned}G_{J_m}\{J^M\{A^m\}\} &= F_m\{A^0 + A\{J^M\{A^M\}\}\} \\ &= F_m\{A^0 + A^M\{J^M\{A^M\}\}\} = F_m\{A^0 + A^M\}\end{aligned} \tag{3.23}$$

(independent of M).

We can now obtain the relation between the J-variables and derivatives of Γ^M that is « conjugate » to the definition of A in terms of derivatives of G.

In view of the last remark, our task is considerably simplified if we restrict to the case of « gapless » M, where we say

DEFINITION III.6. — A subset M of $\mathscr{M}$ is said to be gapless if and only if for $m \in M$,

$$l \leq m \Rightarrow l \in M .$$

In fact we have

THEOREM III.7 (*Conjugate relation*). — If M is gapless, and $m \in M$ then

$$_{A_m}\Gamma^M \{ A \} = - \sum_{k \in M} {}_{A_m}F_k \{ A^0 + A \} J_k^M \{ A \} \tag{3.24 a}$$

$$= - \sum_{\substack{k \in M \\ k \geq m}} \binom{k}{m} (I_m \odot F_{k-m} \{ A^0 + A \}) J_k^M \{ A \} \tag{3.24 b}$$

where

$$I_m(x_1, \ldots, x_m, y_1, \ldots, y_m) \equiv \prod_{i=1}^{m} \delta(x_i - y_i)$$

and the $\odot$ appropriately symmetrizes the m y-arguments of I_m with all k-m arguments of F_{k-m}.

Proof:

$$_{A_m}\Gamma^M = \frac{\delta}{\delta A_m} \left[G \{ J^M \} - \sum_{k \in M} G_{J_k} \{ J^M \} J_k^M \right]$$

$$= \sum_{k \in M} {}_{A_m}J_k^M ({}_{J_k}G) \{ J^M \}$$

$$- \sum_{k \in M} ({}_{A_m}[G_{J_k} \{ J^M \}] J_k^M + \sigma^{mk} G_{J_k} \{ J^M \} {}_{A_m}J_k^M)$$

by the product rule (2.27). If $m \in M$, then in the first sum only terms with $k \in M$ survive (since $_{A_m}J_k^M = 0$ for $m \in M$, $k \notin M$). The first and last terms now cancel because changing G_{J_k} to $_{J_k}G$ introduces a factor of σ^k (see (2.26)) and commuting G_{J_k} with $_{A_m}J_k^M$ (by (2.8 a)) introduces $\sigma^{k(k-m)}$, so that the total power of σ adds up to $mk + k + k^2 - mk \equiv (0, 0)$ mod evens.

So $$_{A_m}\Gamma^M = - \sum_{k \in M} {}_{A_m}[G_{J_k} \{ J^M \}] J_k^M .$$

But by the remark following Lemma III.5, we see that for $k \in \mathrm{M}$ (assumed gapless)

$$ {}_{\mathrm{A}_m}\mathrm{G}_{\mathrm{J}_k}\{\mathrm{J}^{\mathrm{M}}\{\mathrm{A}\}\} = {}_{\mathrm{A}_m}\mathrm{F}_k\{\mathrm{A}^0 + \mathrm{A}\} \tag{3.25} $$

giving (3.24 *a*).

To obtain (3.24 *b*) we compute

$$ {}_{\mathrm{A}_m}\mathrm{F}_k\{\mathrm{A}^0+\mathrm{A}\} = \sum_{\mathrm{N}=1}^{\infty}\frac{1}{\mathrm{N}!}\sum_{\substack{k_i\in\mathcal{M}\\ \sum_{i=1}^{\mathrm{N}} k_i=k}}\binom{k}{k_1\ldots k_{\mathrm{N}}}_{\mathrm{A}_m}\Big[\bigodot_{i=1}^{\mathrm{N}}(\mathrm{A}^0+\mathrm{A})_{k_i}\Big] \quad \text{(by (3.21)} $$

$$ = \sum_{\mathrm{N}=0}^{\infty}\frac{1}{\mathrm{N}!}\sum_{\Sigma k_i=k}\sum_{j=1}^{\mathrm{N}}\binom{k}{k_1\ldots k_{\mathrm{N}}}\delta_{m,k_j}\mathrm{I}_{k_j}\odot\bigodot_{i\neq j}(\mathrm{A}^0+\mathrm{A})_{k_i} $$

$$ = \mathrm{I}_m\odot\sum_{\mathrm{N}=1}\frac{1}{\mathrm{N}!}\sum_{j=1}^{\mathrm{N}}\sum_{\substack{\sum_{i=1}^{\mathrm{N}} k_i=k\\ k_j=m}}\begin{bmatrix}k\\ k_1\ldots k_{\mathrm{N}}\end{bmatrix}\bigodot_{i\neq j}(\mathrm{A}^0+\mathrm{A})_{k_i} $$

Now the inside sum is empty if $m \nleq k$. When $m \leq k$ we can combine the combinatoric factors, noting that N choices of j (for k_j to equal m) raises $\frac{1}{\mathrm{N}!}$ to $\frac{1}{(\mathrm{N}-1)!}$ and $\binom{k}{k_1\ldots k_{\mathrm{N}-1}} = \binom{k-m}{k_1\ldots k_{\mathrm{N}-1}}\binom{k}{m}$, to see that with $n = \mathrm{N} - 1$, the second factor on the right becomes

$$ \binom{k}{m}\sum_{n=0}\frac{1}{n!}\sum_{\sum_{i=1}^{n} k_i=k-m}\binom{k-m}{k_1\ldots k_n}\bigodot_{i=1}^{n}(\mathrm{A}^0+\mathrm{A})_{k_i} = \begin{cases}\binom{k}{m}\mathrm{F}_{k-m}\{\mathrm{A}^0+\mathrm{A}\} & \text{for } k<m\\ 1 & k=m\\ 0 & k\leq m\end{cases} $$

So

$$ {}_{\mathrm{A}_m}\mathrm{F}_k\{\mathrm{A}^0+\mathrm{A}\} = \binom{k}{m}\mathrm{I}_m\,.\,\mathrm{F}_{k-m}\{\mathrm{A}^0+\mathrm{A}\} \tag{3.26} $$

where $\mathrm{F}_{(0,0)} \equiv 1$ and $\mathrm{F}_r \equiv 0$ if $r \ngeq (0,0)$ giving (3.24 *b*). ■

If we now combine (3.14) and (3.23) we find

$$ \begin{aligned} \partial_{t_i}\Gamma^{\mathrm{M}}\{\mathrm{A}\} &= (\partial_{t_i}\mathrm{G})\{\mathrm{J}^{\mathrm{M}}\{\mathrm{A}\}\} - [\partial_{t_i}\mathrm{F}_k\{\mathrm{A}^0+\mathrm{A}\}]\mathrm{J}_k^{\mathrm{M}}\{\mathrm{A}\} \\ &= (\partial_{t_i}\mathrm{G})\{\mathrm{J}^{\mathrm{M}}\{\mathrm{A}\}\} - \partial_{t_i}\mathrm{A}_m^0{}_{\mathrm{A}_m}\mathrm{F}_k\{\mathrm{A}^0+\mathrm{A}\}\mathrm{J}_k^{\mathrm{M}}\{\mathrm{A}\} \\ &= -\frac{1}{2}\dot{\mathrm{C}}_i^{-1}[\mathrm{A}_{ii}+\mathrm{A}_i\mathrm{A}_i+2\mathrm{A}_i^0\mathrm{A}_i] + (\partial_{t_i}\mathrm{A}_m^0)_{\mathrm{A}_m}\Gamma\{\mathrm{A}\} \end{aligned} \tag{3.27} $$

by (3.14 *b*) and (3.24 *a*).

Iteration of this formula yields formulae for higher derivatives $\partial_t^s\Gamma^M\{A\}$ in terms of derivatives $\dot{C}_i^{-1}, \ddot{C}_i^{-1}, \ldots, \partial_t^q A_m^0$, and ${}_{A^r}\Gamma^M$, for various $p, r \leq |s|$ and $q \leq s$. Thus the study of general t-derivatives of Γ^M is reduced to the study of t-derivatives of A_m^0 and of the moments ${}_{A^n}\Gamma^M$.

In particular, to study the second moments ${}_{AA}\Gamma^M$, we shall combine the conjugate relation expressing ${}_A\Gamma^M$ in terms of J^M with the inverse fps Jacobian relation ${}_AJ^M = ({}_JA^M)^{-1}$ (Theorem II.11). To prepare for this we shall first express the A_J's in terms of a generating functional.

DEFINITION III.8. — For $J \in \mathscr{L}$, $f \in \mathscr{L}_1$, $g \in \mathscr{L}_1$, we define

$$\mathscr{S}\{J, f, g\} \equiv \exp\{G\{J+f+g\} - G\{J+f\} - G\{J+g\} + G\{J\}\} \in \mathscr{F}(\mathscr{L}_0 \oplus \mathscr{L}_f \oplus \mathscr{L}_f, \mathscr{L}_e \oplus \mathscr{L}_b \oplus \mathscr{L}_b; \mathbb{C}) \quad (3.28)$$

and

$$S_{mn} \equiv (\mathscr{S}_{f^m g^n}|_{f=g=0}) \in \mathscr{F}(\mathscr{L}_0, \mathscr{L}_e; \mathscr{M}_{m+n}(\mathscr{L}_1, \mathbb{C})) \quad (3.29)$$

LEMMA III.9.

$$A_{mJ_n} = \sum_{\substack{n' \in \mathscr{M} \\ n' \leq n}} \binom{n}{n'} S_{mn'} \odot F_{n-n'}\{A^0 + A\} \quad (3.30\,a)$$

$$= \sum_{n' \in \mathscr{M}} S_{mn'}({}_{A_{n'}}F_n)\{A^0 + A\} \quad (3.30\,b)$$

(Here $\odot$ appropriately symmetrizes the n' rightmost arguments of $S_{mn'}$ with all $n - n'$ arguments of $F_{n-n'}$.)

Proof:

$$\begin{aligned}
A_{mJ_n}\{J\} &= G_{J_1^m J_n}\{J\} = \sigma^{nm} G_{J_n J_1^m}\{J\} \\
&= \sigma^{nm}(\exp\{G\{J+g\} - G\{J\}\})_{g^n J_1^m}|_{f=0} && \text{(by (3.16))} \\
&= \sigma^{nm}(\exp\{G\{J+f+g\} - G\{J+f\}\})_{g^n f^m}|_{f=g=0} \\
&= (\exp\{G\{J+f+g\} - G\{J+f\}\})_{f^m g^n}|_{f=g=0} \\
&= (\mathscr{S}\{J, f, g\} . \exp H\{J, g\})_{f^m g^n}|_{f=g=0} && \text{(by (3.28) and (3.20))} \\
&= \sum_{\substack{n' \\ n' \leq n}} \binom{n}{n'} S_{mn'}\{J\} . F_{n-n'}\{A^0 + A\{J\}\} && \text{(by (2.27) since } f \text{ is even)}
\end{aligned}$$

establishing (3.30 *a*). (3.30 *b*) then follows by (3.26). ■

Remark. — It is natural to interpret the S_{mn}'s as operator valued fps in J from $\mathscr{L}_n$ to $\mathscr{L}_m^*$ and to consider these operators as matrix elements of a single operator valued fps S (in just the same sense that A_{mJ_n} are matrix elements of the Jacobian operator valued fps A_J).

In the same spirit we also let

$$S^{M} \equiv (P^{M}SP^{M} \mid \mathscr{L}^{M}),$$

and
$$K = K^{\mathscr{M}} \equiv {}_{A}F\{A^0 + A^{\mathscr{M}}\}$$
$$K^{M} = (P^{M}KP^{M} \mid \mathscr{L}^{M})$$

where

$$K_{nm} = {}_{A_n}F_m\{A^0 + A^{\mathscr{M}}\}$$
$$= \begin{cases} \binom{m}{n} I_n \odot F_{m-n}\{A^0 + A^{\mathscr{M}}\} & \text{if } n \leq m \quad \text{(by (3.26))} \\ 0 & \text{otherwise} \end{cases}$$

So, since K is « triangular » with identity operators on the « diagonal » it is formally invertible. Also, if M is gapless, then (by the remark following Lemma III.5)

$$KP^{M} = P^{M}KP^{M}$$

and $\quad F_m\{A^0+A^{\mathscr{M}}\}=F_m\{A^0+A^{M'}\}$ for all $m \in M \subset M'$.

So for any $M' \supset M$ we have

$$K^{M}\{J^{M'}\{A\}\} = {}_{A}F\{A^0 + A\} \quad \text{on } \mathscr{L}^{M}$$

In terms of this matrix notation we have the following immediate consequences of Lemma III.9.

COROLLARY III.10.

a) $$A_J = SK \quad \text{so} \quad K = S^{-1}A_J$$

b) $$A_J^{M} = \begin{bmatrix} P^{M}SKP^{M} & P^{M}SKP^{M^c} \\ 0 & P^{M^c} \end{bmatrix} = P^{M}SK + P^{M^c}. \qquad (3.31\,a)$$

If M is gapless, then

$$A_J^{M}P^{M} = S^{M}K^{M} \quad \text{and} \quad K^{M} = S^{M^{-1}}A_J^{M}P^{M} \qquad (3.31\,b)$$

c) $$(A_J^{M})^{-1} = (P^{M}SKP^{M} \mid \mathscr{L}^{M})^{-1}(P^{M} - P^{M}SKP^{M^c}) + P^{M^c}.$$

If M is gapless, then

$$(J_A^{M})P^{M} = P^{M}[A_J^{M}\{J^{M}\}]^{-1}P^{M} = K^{M}\{J^{M}\}^{-1}S^{M}\{J^{M}\}^{-1}$$

Remarks 1. — Here S^{-1} denotes the operator inverse of S interpreted for S an operator valued fps by the reciprocal rule (Theorem II.3).

2. Similar formulae hold with right and left derivatives interchanged provided all products are also written in reverse order. In particular, if

$$\rtimes_{mn} \equiv ({}_{J_m}G \circ J)_{A_n} \circ A = \sigma^{m+n+mn}K_{mn}$$

then part (a) has the « adjoint version »

$$_{\mathrm{J}}\mathrm{A} = \rtimes \mathrm{S}, \text{ etc}.$$

We now compute $_{\mathrm{AA}}\Gamma^{\mathrm{M}}$.

THEOREM III.11. — If M is gapless, then for n, $m \in \mathrm{M}$ we have

$$_{\mathrm{A}_n\mathrm{A}_m}\Gamma^{\mathrm{M}} = \in^{\mathrm{M}}_{n+m}\binom{m+n}{m}\sigma^{nm}{}_{\mathrm{A}_{n+m}}\Gamma^{\mathrm{M}} - \sigma^{m}[\mathrm{S}^{\mathrm{M}} \circ \mathrm{J}^{\mathrm{M}}]^{-1}_{nm} \qquad (3.32)$$

where $\in^{\mathrm{M}}_{k} = \begin{cases} 1 & \text{if } k \in \mathrm{M} \\ 0 & \text{if } k \notin \mathrm{M} \end{cases}$ (Here $[\mathrm{S}^{\mathrm{M}} \circ \mathrm{J}^{\mathrm{M}}]^{-1}_{nm}$ represents the kernel of the nm component of the operator inverse to $\mathrm{S}^{\mathrm{M}} \circ \mathrm{J}^{\mathrm{M}}$ considered as an operator valued fps in the variable A.)

Proof. — By (3.24 a)

$$_{\mathrm{A}_m(\vec{y}_m)\mathrm{A}_n(\vec{x}_n)}\Gamma^{\mathrm{M}}\{\mathrm{A}\} = \frac{\delta}{\delta \mathrm{A}_m(\vec{y}_m)}\left[-{}_{\mathrm{A}_n(\vec{x}_n)}\mathrm{F}_k\{\mathrm{A}^0+\mathrm{A}\}\,\mathrm{P}^{\mathrm{M}}\mathrm{J}^{\mathrm{M}}_k\{\mathrm{A}\}\right]$$

$$= -\frac{\delta}{\delta \mathrm{A}_m(\vec{y}_n)}\left[\binom{k}{n}\mathrm{I}_n(\vec{x}_n, \vec{u}_n) \odot \mathrm{F}_{k-n}(\vec{z}_{k-n})\{\mathrm{A}^0+\mathrm{A}\}\right]\mathrm{P}^{\mathrm{M}}\mathrm{J}^{\mathrm{M}}_k\{\mathrm{A}\}(\vec{u}_n, \vec{z}_{k-n})$$

$$- \sigma^{m(k-n)}{}_{\mathrm{A}_n(\vec{x}_n)}\mathrm{F}_k\{\mathrm{A}^0 + \mathrm{A}\}_{\mathrm{A}_m(\vec{y}_m)}\mathrm{P}^{\mathrm{M}}\mathrm{J}^{\mathrm{M}}_k\{\mathrm{A}\} \quad \text{((by (2.27) and (3.26))}$$

where, since $\mathrm{J}^{\mathrm{M}}_k \in \mathscr{L}_k$, we do not need to make explicit the symmetrization in the first term and so have dropped the $\odot$. For the first term we now apply (3.26) again, and in the second term we use (3.23) to write $_{\mathrm{A}_n}\mathrm{F}_k\{\mathrm{A}^0+\mathrm{A}\}\mathrm{P}^{\mathrm{M}}$ as $\mathrm{K}^{\mathrm{M}}_{nk}\{\mathrm{J}^{\mathrm{M}}\{\mathrm{A}\}\}$ and then apply Corollary III.10 (b) to get

$$\mathrm{A}_m(\vec{y}_m)\mathrm{A}_n(\vec{x}_n)\Gamma^{\mathrm{M}}\{\mathrm{A}\} =$$

$$-\binom{k}{n}\mathrm{I}_n(\vec{x}_n, \vec{u}_n)\left[\binom{k-n}{m}\mathrm{I}_m(\vec{y}_m, \vec{v}_m)\mathrm{F}_{k-m-n}(\vec{w}_{k-m-n})\{\mathrm{A}^0+\mathrm{A}\}\right]$$

$$\mathrm{P}^{\mathrm{M}}\mathrm{J}^{\mathrm{M}}_k\{\mathrm{A}\}(\vec{u}_n, \vec{v}_n, \vec{w}_{k-m-n})$$

$$-\sigma^{m(k-n)}[\mathrm{S}^{\mathrm{M}}\{\mathrm{J}^{\mathrm{M}}\{\mathrm{A}\}\}]^{-1}_{np}(\vec{x}_n, \vec{\alpha}_p)(\mathrm{A}_p(\vec{\alpha}_p)_{\mathrm{J}_k(\beta_k)}\{\mathrm{J}^{\mathrm{M}}\{\mathrm{A}\}\})$$

$$({}_{\mathrm{A}_m(\vec{y}_m)}\mathrm{P}^{\mathrm{M}}\mathrm{J}^{\mathrm{M}}_k(\vec{\beta}_k)\{\mathrm{A}\}).$$

Now

$$\binom{k}{n}\binom{k-n}{m} = \binom{k}{m+n}\binom{m+n}{m}$$

and

$$\mathrm{I}_n(\vec{x}_n, \vec{u}_n) \odot \mathrm{I}_m(\vec{y}_m, \vec{v}_m) = \mathrm{I}_{n+m}(\vec{x}_n\vec{y}_m, \vec{u}_n\vec{v}_m)$$

So if $n + m \in \mathrm{M}$ the first term becomes

$$-\binom{m+n}{m}\left[\binom{k}{m+n}\mathrm{I}_{n+m}(\vec{x}_n\vec{y}_m, \vec{u}_m\vec{v}_m) \odot \mathrm{F}_{k-n-m}(\vec{w}_{k-n-m})\{\mathrm{A}^0+\mathrm{A}\}\right]$$

$$\mathrm{P}^{\mathrm{M}}\mathrm{J}^{\mathrm{M}}_k\{\mathrm{A}\}(\vec{u}_n\vec{v}_m\vec{w}_{k-n-m}) = \binom{m+n}{m}{}_{\mathrm{A}_{m+n}(\vec{x}_n\vec{y}_m)}\Gamma^{\mathrm{M}}\{\mathrm{A}\}.$$

If $n + m \notin \mathrm{M}$ the first term is zero.

Now ${}_{A_m}J_k^M = J_k^M {}_{A_m}\sigma^{m(k-m)}$ (by (2.26))

and $$m_f(k_f - n_f) + m_f(k_f - m_f) \equiv m_f n_f + m_f \pmod 2$$

so the Jacobian identity (2.39) converts the second term to

$$- \sigma^{nm+m}[S^M\{J^M\{A\}\}]_{nm}^{-1}(\vec{x}_n, \vec{y}_m)$$

But $${}_{A_nA_m}\Gamma^M = \sigma^{nm}{}_{A_mA_n}\Gamma^M$$

and so we are done. ■

Remarks 1. — At A = J = 0, (3.32) gives

$$P^M{}_{AA}\Gamma P^M = -\sigma^m[S^M]^{-1} \tag{3.33}$$

2. If we define the reversal operator R applied to

$$\vec{x}_m = (x_1^f \dots x_{m_f}^f, x_1^b \dots x_{m_b}^b) \quad \text{by} \quad R\vec{x}_m \equiv (x_{m_f}^f \dots x_1^f, x_{m_b}^b \dots x_1^b)$$

and to any (appropriately symmetrized) function $f_m \in \mathscr{L}_m$ by

$$(Rf_m)(\vec{x}_m) \equiv f_m(R\vec{x}_m) = \sigma^{m(m-1)/2} f_m(\vec{x}_m),$$

then, since $\dfrac{n(n-1)}{2} + \dfrac{m(m-1)}{2} \equiv \dfrac{(n+m)(n+m-1)}{2} + nm \pmod 2$, we have

$$f_{n+m}(R(\vec{x}_n, \vec{x}_m)) = \sigma^{nm} f(R\vec{x}_n, R\vec{x}_m).$$

The sign factor σ^{nm} in the first term of (3.32) could thus be eliminated by reversing the arguments of all A derivatives to write

$${}_{RA_nRA_m}\Gamma^M = \in_{n+m}^M \binom{m+n}{m} {}_{RA_{n+m}}\Gamma^M - \sigma^m(R[S^M \circ J^M]^{-1}R)_{nm}$$

3. We can also absorb the σ^m factor in the second term of (3.32) if we write our result in terms of the matrix .S. with entries

$${}_n S_m = {}_{f^n}\mathscr{S}_{g^m}|_{f=g=0} = \sigma^n S_{nm}$$

In fact, if $\underline{\sigma} = P_{\text{even}} - P_{\text{odd}}$ (which is the linear operator on $\mathscr{L}$ with eigenvalue σ^m on $\mathscr{L}_m$) then .S. $= \underline{\sigma}S$, and so

$$[.S.]^{-1} = [\underline{\sigma}S]^{-1} = S^{-1}\underline{\sigma}^{-1} = S^{-1}\underline{\sigma}$$

so the matrix elements ${}_n[.S.^{-1}]_m$ of $.S.^{-1}$ satisfy

$${}_n[.S.^{-1}]_m = [S^{-1}]_{mn}\sigma^m.$$

So (3.32) becomes

$${}_{RA_nRA_m}\Gamma^M = \in_{n+m}^M \binom{m+n}{m} {}_{RA_{n+m}}\Gamma^M - R_n[.S.^M \circ J^M]_m^{-1} R \tag{3.32'}$$

The virtue of these results is that by expressing ${}_{AA}\Gamma$ in terms of S (and so in terms of connected derivatives of G) they provide us with a tool for the study of decoupling properties of ${}_{AA}\Gamma$ (over and above its obvious connectedness between the two A variables).

For the purposes of such a study we find it convenient to abuse the notation a bit and to think (as we did in III) of a subscript $r \in \mathscr{M}$ as representing also a value of the corresponding argument $\vec{x}_r$. In fact, at any point in our discussion, r may be being considered in any of several ways, either just as the pair of integers (r_f, r_b) (as we have done up to now) or as an ordered « r-tuple » $\vec{x}_r = (x_1 \ldots x_{r_f}, y_1 \ldots y_{r_b})$ or as the (unordered) set of points $\{ x_i \} \cup \{ y_j \}$ (e. g. we write $r \subset r'$ to mean $\{ x_1 \ldots x_{r_f} y_1 \ldots y_{r_b} \}$ is a subset of $\{ x'_1 \ldots x'_{r_f} y'_1 \ldots y'_{r_b} \}$).

In terms of this convention it follows from Definition III.8 that each matrix element S_{mn} is a polynomial in the variables

$$\{ G_k \mid k \subset m \cup n,\ k \cap m \neq \emptyset,\ k \cap n \neq \emptyset \}$$

i. e. every point in m is connected to at least one point in n and *vice versa*. (For example since $G_{fb} = 0$ at $J = 0$ we have

$$S_{fb,fb}(xz\,;x'z') = G_{ffbb}(x, x', z, z') + G_{ff}(x, x')G_{bb}(z, z')\,.)$$

This remark has the consequence that, at 0,

$$P_{\pm}SP_{\mp} = P_{\pm}SP_0 = P_0SP_{\pm} = 0\,. \tag{3.34}$$

In fact we can say more. By (3.11) we see that, at 0

$$G\{ J + f \} = G\{ J + P_+ f \} + G\{ J + P_- f \} - G\{ J \} \tag{3.35}$$

and so

$$\mathscr{S}\{ J, f, g \} = \mathscr{S}\{ J, P_+ f, P_+ g \}\,\mathscr{S}\{ J, P_- f, P_- g \} \tag{3.36}$$

So by (2.27) if we let $\overline{M} = M \cup \{ (0,0) \}$ and $\underset{(\pm)}{S} = \underset{(\pm)}{P}S\underset{(\pm)}{P}$ with

$$\underset{\pm}{S}_{m,(0,0)} = \underset{\pm}{S}_{(0,0),m} = 0 \qquad \text{for} \qquad m \neq (0,0)$$

and $\underset{\pm}{S}_{(0,0)} = 1$ then, at 0

$$S_{ij} = \sum_{\substack{i' \in \mathscr{M} \\ i' \subset i}} \sum_{\substack{j' \in \mathscr{M} \\ j' \subset j}} \underset{+}{S}_{i',j'} \underset{-}{S}_{i\sim i', j\sim j'} \sigma^{j'(i-i')} \operatorname{sign}(i', i) \operatorname{sign}(j', j) \tag{3.37 a}$$

where sign (i', i) is the sign of the permutation $\begin{pmatrix} & i_f & \\ i'_f & & i_f \sim i'_f \end{pmatrix}$ which places the primed arguments first but otherwise preserves the order of i_f. But here only one term contributes (namely that with i' being *exactly* the set, i_+, of arguments on the $+$ side of σ, and similarly for $j' = j_+$) and so we are left with

$$S_{ij} = \underset{+}{S}_{i_+ j_+} \underset{-}{S}_{i_- j_-} \sigma^{j_+ i_-}, \quad \text{at } 0. \tag{3.37 b}$$

Thus (apart from the σ factor) S looks rather like a tensor product of $\underset{+}{S}$

and $\underline{S}$. In fact, if we interpret operator valued fps as operators on vector valued fps, then a natural definition of operator tensor product is given by

$$[(A_1 \otimes A_2)(f_1 \otimes f_2)](x_1, x_2) \equiv [A_1 f_1 \otimes A_2 f_2](x_1, x_2)$$
$$= \int A_1(x_1 y_1) f(y_1) dy_1 \int A_2(x_2, y_2) f(y_2) dy_2$$
$$= \int A_1(x_1 y_1) A_2(x_2 y_2) \sigma^{\deg A_2 \deg f_1} f_1(y_1) f_2(y_2) dy_1 dy_2$$

Keeping in mind the overall appropriate symmetrization of each group of arguments in S, and the fact that for us the parity of a function valued fps is always given by the total number of fermi arguments, we are lead to define, for kernel functions with appropriate symmetry and parity, the fps operator tensor product, $\circledast$, by

$$(A_1)_{i_1 j_1} \circledast (A_2)_{i_2 j_2} \equiv P_{i_1 + i_2} (A_1)_{i_1 j_1} (A_2)_{i_2 j_2} \sigma^{j_1 (i_2 + j_2)}$$

If we now recall Remark 2 following Theorem III.11, we see that at 0,

$$SR_{i,j} = \underset{+}{S}R_{i_+ j_+} \underset{-}{S}R_{i_- j_-} \sigma^{j_+ (i_- + j_-)} \tag{3.37 c}$$
$$= (\underset{+}{S}R)_{i_+ j_+} \circledast (\underset{-}{S}R)_{i_- j_-}$$

or (keeping in mind the survival of only one term in (3.37 *a*)) at 0,

$$SR_{i,j} = \sum_{\substack{i' \in \mathcal{M} \\ i' \leq i}} \sum_{\substack{j' \in \mathcal{M} \\ j' \leq j}} \binom{i}{i'} \binom{j}{j'} (\underset{+}{S}R)_{i'j'} \circledast (\underset{-}{S}R)_{i-i', j-j'} \tag{3.37 d}$$

Let $\overline{S} \equiv (S_{nm})_{n,m \in \overline{\mathcal{M}}} = 1 \oplus S$ on $\mathbb{C} \oplus \mathscr{L}$, and $\underset{\pm}{\overline{S}} = 1 + \underset{\pm}{S}$.

Following § 3 of III we rewrite (3.37) in matrix form so as to represent $\overline{S}$ as a « tensor » product of $\underset{+}{\overline{S}}$ and $\underset{-}{\overline{S}}$. If π denotes the map from $\mathscr{L}_+ \otimes \mathscr{L}_-$ to $\mathscr{L}$ which simply extends each function using the appropriate symmetry it is natural to define, for $\underset{\pm}{A} : \mathscr{L}_\pm \to \mathscr{L}_\pm$,

$$\underset{+}{A} \times \underset{-}{A} = \pi \underset{+}{A} \circledast \underset{-}{A} \pi^{-1}.$$

The relationship between the kernels of $\underset{+}{A}$, $\underset{-}{A}$ and $\underset{+}{A} \times \underset{-}{A}$ is

$$(\underset{+}{A} \times \underset{-}{A})_{mn} = \sum_{\substack{m' \leq m \\ n' \leq n}} \binom{m}{m'} \underset{+}{A}_{m'n'} \circledast \underset{-}{A}_{m-m', n-n'}. \tag{3.38}$$

Hence we may rewrite (3.37) as

$$\overline{S} = (\underset{+}{\overline{S}} F^{-1} \times \underset{-}{\overline{S}} F^{-1}) F \quad \text{at} \quad 0 \tag{3.39}$$

where F is the operator on $\mathscr{L}$ (or $\mathscr{L}_\pm$) with $F_{ij} = i!\,\delta_{ij}$. Consequently

$$\bar{S}^{-1} = F^{-1}(F\underset{+}{S}^{-1} \times F\underset{-}{S}^{-1}) \tag{3.40 a}$$

$$= \sum_{\substack{m' \leq m \\ n' \leq n}} \underset{+}{S}^{-1}_{m',n'} \circledS \underset{-}{S}^{-1}_{m-m',n-n'} \tag{3.40 b}$$

by (3.38). However since the projections that restrict S to $S^{(M)}$ do not commute with taking either tensor products or inverses (3.40) is of little value. But for certain M's we do have a decomposition formula analogous to (3.15) of III.

DEFINITION III.12. — $M \subset \mathscr{M}$ is said to be decomposable if there is a (finite) strictly increasing sequence $M_0, \ldots, M_N$ of subsets of M with $M_0 = \emptyset$ and $M_N = M$ such that for $i = 1, \ldots, N$,

$$m \in M_i \backslash M_{i-1} \Rightarrow C_M(m) \equiv \{ m' \in \mathscr{M} \mid m + m' \in M \} = M_{N-i}.$$

(So if $m \in M_i$ and $m' \in M_{i'}$ with $i + i' \leq N$, then $m + m' \in M$.)

For example, $M = \{ m \in \mathscr{M} \mid |m| \leq N \}$ is decomposable with

$$M_i = \{ m \in \mathscr{M} \mid |m| \leq i \}.$$

LEMMA III.13. — *a*) At 0,

$$\bar{S}^{-1} = F^{-1}(F \times F)(\underset{+}{S}^{-1} \times \underset{-}{S}^{-1}) \tag{3.41 a}$$

b) If M is decomposable, then, at 0

$$S^{M^{-1}} = F^{-1}(F \times F)\left[\sum_{l=0}^{N} (\underset{+}{S}^{M_l})^{-1} \times (\underset{-}{S}^{M_{N-l}})^{-1} - \sum_{l=1}^{N-2} (S_{+}^{M_l})^{-1} \times (\underset{-}{S}^{M_{N-l-1}})^{-1}\right] \tag{3.41 b}$$

Here, by another of our many abuses of notation we view $(S_{\pm}^{M_i})^{-1}$ as being defined to be zero off $\mathscr{L}_{\pm}^{M_i} \equiv P_{\pm}\mathscr{L}^{M_i}$ (or $\mathbb{C}$ when $i = 0$).

Proof. — Part *a*) has already been proven. To prove part *b*) we compute that the product of S^M times the right hand side of (3.43 *b*) is P^M. Let $T_i = (S^{M_i})^{-1}$. By (3.39) and our condition on M this product is

$$[P^M(\underset{+}{S} \times \underset{-}{S})(F^{-1} \times F^{-1})FP^M]F^{-1}(F \times F)\left[\sum_{i=0}^{N} \underset{+}{T}_i \times \underset{-}{T}_{N-i} - \sum_{i=1}^{N-2} \underset{+}{T}_i \times \underset{-}{T}_{N-i-1}\right]$$

$$= P^M \cdot \left[\sum_{i=0}^{N} \underset{+}{S}\underset{+}{T}_i \times \underset{-}{S}\underset{-}{T}_{N-i} - \sum_{i=1}^{N-2} \underset{+}{S}\underset{+}{T}_i \times \underset{-}{S}\underset{-}{T}_{N-i-1}\right]$$

The proof now continues as in Theorem III.4 of III using the fact that

$$\underset{\pm}{\bar{S}}\underset{\pm}{T}_i = \underset{\pm}{P}^{M_i} + \underset{\pm}{E}_i$$

with

$$\underset{\pm}{P}^{M_i}\underset{\pm}{E}_i = \underset{\pm}{P}^{M_i}(\underset{\pm}{\bar{S}} - \underset{\pm}{S}^{M_i})\underset{\pm}{T}_i = 0 . \qquad \blacksquare$$

As we remarked after Definition III.12, any M defined by an upper bound on the « particle number » $|m|$ is decomposable. More generally, so is any « triangular » M, where

DEFINITION III.14. — M is said to be triangular if there exist positive numbers a_f, a_b, and a_0 such that

$$M = \{ M \in \mathcal{M} \mid a \cdot m \equiv a_f m_f + a_b m_b \leq a_0 \}$$

Remark. — There are precisely two gapless subsets of $\{ m \mid m_f + m_b \leq 2 \}$ that fail to be triangular, namely $\{ f, b, fb \}$ and $\{ f, b, ff, bb \}$, and these are precisely the two gapless second order M's with anomalously poor irreducibility properties. (See the end of § I and beginning of § IV.)

LEMMA III.15. — If M is triangular, then M is decomposable.

Proof. — For $\mu \in [0, a_0]$ define

$$M(\mu) = \{ m \in \mathcal{M} \mid a.m \leq \mu \}.$$

Clearly $M(\mu)$ increases with μ, $M(0) = \phi$ and $M(a_0) = M$. Let $\mu_1 < \mu_2 < \ldots < \mu_p$ be the points of increase of $M(\mu)$ and set $D(\mu_j) = M(\mu_j)M(\mu_{j}-)$. It is obvious that the complements $C_M(m)$ decrease with μ in the sense that

$$m \in D(\mu_j), \quad m' \in D(\mu_{j'}), \quad j \leq j' \Rightarrow C_M(m) \supset C_M(m') .$$

Let $\lambda_1 < \lambda_2 < \ldots < \lambda_{N-1}$ be those μ_j's for which the inclusion above is proper if $j < j'$ and let $\lambda_N = a_0$. We then define $M_0 = \emptyset$,

$$M_i = M(\lambda_i) \qquad \text{and} \qquad D_i = M_i \backslash M_{i-1}$$

for $i = 1, \ldots, N$.

It remains to show that for $k = 1, \ldots, N$

$$m \in D_k \Rightarrow C_M(m) = M_{N-k}.$$

Since there are exactly N distinct C_M's and M_i's $(0 \leq i \leq N - 1)$ it suffices to show that for any $m \in M$

$$C_M(m) = M_i \qquad \text{for some } i, \qquad 0 \leq i \leq N - 1 .$$

Since $C_M(m) \subsetneq M$ this amounts to the claim that if $n \in D_i$ is in $C_M(m)$ and $n' \in D_{i'}$ for some $i' \leq i$ then $n' \in C_M(m)$ too. If $i' < i$ it is obvious that $n' \in C_M(m)$ because $n'_f a_f + n'_b a_b < n_f a_f + n_b a_b$.

If $i' = i$

$$\begin{aligned} n \in C_M(m) &\Rightarrow m \in C_M(n) \\ &\Rightarrow m \in C_M(n') = C_M(n) \quad \text{by construction} \\ &\Rightarrow n' \in C_M(m). \end{aligned}$$

The properties of ${}_A\Gamma^M$ and ${}_{AA}\Gamma^M$ established so far will suffice for our $r - p$I results with $|r| \leq 2$. For higher irreducibility properties we shall need to study ${}_{A^N}\Gamma^M$ for $N > 2$ but since the majority of our results involve just $|r| \leq 2$ we relegate the study of higher order derivatives to the Appendix.

We now turn to the calculation of t-derivatives. Formal differentiation of the (unrenormalized) expression

$$Z\{J\} = \langle e^{U\{J\}} \rangle = \text{const.} \int e^{U\{J\} + V - \frac{1}{2}(\phi C_b^{-1}\phi + \psi C_f^{-1}\psi)} \delta\phi\delta\psi ,$$

with $C_b = C$ and $C_f = \begin{pmatrix} 0 & -S_0^+ \\ S_0 & 0 \end{pmatrix}$, yields for $i = b$ or f

$$\begin{aligned} \frac{d}{dt_i} Z\{J\} &= -\frac{1}{2} \dot{C}_i^{-1} [\langle \Phi^{2i} e^{U\{J\}} \rangle - \langle \Phi^{2i} \rangle \langle e^{U\{J\}} \rangle] \\ &\equiv -\frac{1}{2} \dot{C}_i^{-1} \langle \Phi^{2i} ; e^{U\{J\}} \rangle \end{aligned} \tag{3.42}$$

where

$$\dot{C}_i^{-1} \equiv - C_i^{-1} \frac{d}{dt_i} C_i C_i^{-1}$$

is formally the derivative with respect to t_i of C_i^{-1} (but in fact must be interpreted as a quadratic form).

Of course, for Y_2, in order to make sense of (1.3) in the absence of cutoffs, it is necessary to renormalize V with counterterms that depend on t.

$$V_{\text{ren}} = V_{\text{ren}}(t) = V + \int \omega(t, x)\phi(x)dx + \int \frac{\delta m^2(t, x)}{2} : \phi^2 : (x)dx .$$

The t-dependence of V leads to additional terms in our derivative formula.

$$\frac{d}{dt_i} Z_{\text{ren}}\{J\} = \left\langle -\frac{1}{2} \dot{C}_i^{-1} \Phi^{2i} - \frac{\partial}{\partial t_i} V_{\text{ren}} ; e^{U\{J\}} \right\rangle \tag{3.43}$$

This result will be derived rigorously for a suitable cutoff version of (2.13) in a separate paper. But alas, the individual terms on the right diverge as the cutoffs are removed. In fact, to write down a well defined expression for the t-derivatives in terms of uncutoff quantities requires a rearrangement of terms which destroys the locality properties and so is less suitable for proving irreducibility results. Thus all of our irreducibility results are proved first with cutoffs in place and then carried to the limit. For this approach to work we must of course introduce the cutoffs in such a way as not to interfere with the decoupling properties that we aim to establish.

We may, for example, cutoff the fermion covariance through the introduction of a small second order derivative

$$S_\varepsilon(1) = (-\varepsilon\Delta - \not\nabla + m_p)^{-1}. \tag{3.44}$$

The cutoff covariance with Dirichlet boundary conditions on σ, $S_\varepsilon(0)$, and interpolating covariances $S_\varepsilon(t)$ are then defined in the usual way. $S_\varepsilon(0)$ is the inverse of the Friedrichs extension of

$$(-\varepsilon\Delta - \not\nabla + m_f) \mid C_c^\infty(\mathbb{R}^d\backslash\sigma) \times \mathbb{C}^2 \times \mathbb{C}^{[d/2]}.$$

It is a simple exercise in abstract nonsense to verify that $S_\varepsilon(0)$ is indeed a bounded everywhere defined (on L^2) operator that decouples across σ.

In the course of the rigorous argument it will be apparent that what really occurs where we have written $\dot{C}_i^{-1}$ is the quadratic form defined by

$$\dot{C}_i^{-1}(t)(f, g) \equiv -\left(C_i(t)^{-1}f, \frac{d}{dt_i} C_i C_i(t)^{-1} g\right). \tag{3.45}$$

Our previous warnings [*1*]-[*2*] about the singular nature of $\dot{C}_b^{-1}$ apply equally well to $\dot{S}_\varepsilon^{-1}$. In particular, although $\dot{S}_\varepsilon^{-1}(f, g)$ is equal to zero if either f or g is in $C_c^\infty(\mathbb{R}^d\backslash\sigma) \times \mathbb{C}^2 \times \mathbb{C}^{[d/2]}$ (and so is zero on a *dense* domain), it is also defined, and *not* identically zero (see e. g. Remark (2), p. 168 of I) on $\mathscr{D}(S_\varepsilon(0)^{*-1} \times \mathscr{D}(S_\varepsilon(0)^{-1})$.

(This despite the fact that the quadratic form closures of $S_\varepsilon(t)^{-1}$ are all defined and independent of t on $\mathscr{D}(C_b(0)^{-\frac{1}{2}}) \times \mathscr{D}(C_b(0)^{-\frac{1}{2}})$ which includes $\mathscr{D}(S_\varepsilon(0)^{*-1}) \times \mathscr{D}(S_\varepsilon(0)^{-1})$!). We shall not in the rest of this paper indicate the presence of cutoffs explicitly in our formulas. We shall also drop the extra $\frac{d}{dt_i}V_{ren}$ terms so recently introduced, remarking only that their locality renders them harmless in so far as our irreducibility arguments are concerned.

We now apply (3.42) to compute derivatives of G and Γ.

THEOREM III.16 (*t-derivatives*). — For $i = b$ or f (and $\sim i = f$ or b resp.) we have

a)
$$\partial_{t_i} G = -\frac{1}{2}\dot{C}_i^{-1}[G_{J_iJ_i} + G_{J_i}G_{J_i}|_0^J]$$
$$= -\frac{1}{2}\dot{C}_i^{-1}[A_{ii} + A_iA_i + 2A_i^0A_i] \quad (\text{if } ii \in M) \tag{3.46 a}$$

b)
$$\partial_{t_i}\Gamma^M\{A\} = -\frac{1}{2}\dot{C}_i^{-1}[G_{J_iJ_i} + G_{J_i}G_{J_i}]_0^{J\{A\}} + (\partial_{t_i}A_m^0)_{A_m}\Gamma^M\{A\}$$
$$= -\frac{1}{2}\dot{C}_i^{-1}[A_{ii} + A_i^2 + 2A_i^0A_i] + \partial_{t_i}A_{mA_m}^0\Gamma^M\{A\} \quad (\text{if } ii \in M) \tag{3.46 b}$$

c)
$$\partial_{t_i} A^0_m = -\frac{1}{2}\dot{C}_i^{-1}[S_{ii,m}\{0\} + 2S_{i,m}\{0\}A^0_i] = -[\vec{b}_i S(0)]_m \quad (3.46\,c)$$

d) Furthermore, if M is gapless and if $m \in M \Rightarrow m + i - (\sim i) \in M$, then for $m \in M$ and A supported away from σ with $P_0 A = 0$ we find that

$$\partial_{t_i} A^0_m |_{t=0} P_0 = -\frac{1}{2}\dot{C}_i^{-1} S_{ii,m}\{J^M\{A\}\}P^0|_{t=0} \quad (3.46\,d)$$

Remarks 1. — In fact $A^0_i = 0$ for all t if i_f is odd (and also if i_b is odd for psY_2 by Furry's Theorem).

2. The condition $m \in M \Rightarrow m + i - (\sim i) \in M$ in (*d*) requires that M be a union of sets of the form $\{ m \in \mathcal{M} \mid |m| \leq \beta, m_{\sim i} \leq \alpha \}$.

Proof. — *a*) From (3.42)
$$\partial_{t_i} G = \frac{1}{2}\partial_{t_i} Z = -\frac{1}{2}\dot{C}_i^{-1}[G_{J_{ii}}|^J_0$$
and by Lemma III.5,
$$[G_{J_{ii}}|^J_0 = [G_{J_iJ_i} + G_{J_i}G_{J_i}|^J_0 = [(A_{2i} + A^0_{2i}) + (A_i + A^0_i)^2] - [A^0_{2i} + (A^0_i)^2] \quad \text{if} \quad ii \in M$$

b) Follows directly by combining part (*a*) with (3.27)

c) From part *a*) we have

$$\begin{aligned}
\partial_{t_i} A^0_m &= -\frac{1}{2}\dot{C}_i^{-1}[G_{J_iJ_i} + G_{J_i}G_{J_i}]_{J^M_1}|_{J=0}. \\
&= -\frac{1}{2}\dot{C}_i^{-1}\,[\exp(G\{f+g\} - G\{g\})]_{f^{2i}g^m}|_{f=g=0} \\
&= -\frac{1}{2}\dot{C}_i^{-1}[\mathcal{S}\{f,g\}e^{G\{f\}}]_{f^{2i}g^m}|_{f=g=J=0} \qquad \text{(by (3.28))} \\
&= -\frac{1}{2}\dot{C}_i^{-1}\sum_{\mathcal{M}\ni j\leq 2i}\binom{2i}{j}S_{2i-j,m}Z_{J_j}|_{J=0} \\
&= -\frac{1}{2}\dot{C}_i^{-1}[S_{2i,m}\{0\} + 2S_{i,m}\{0\}A^0_i].
\end{aligned}$$

d) It follows from *c*) and (3.37) that, at 0

$$\partial_{t_i} A^0_m P_0 = -\frac{1}{2}\dot{C}_i^{-1}P_0 S_{2i,m}\{0\}P_0 = -\frac{1}{2}\dot{C}_i^{-1}\,\text{sign}\,\pi \underset{+}{S}_{i_+m_+}\{0\}\underset{-}{S}_{i_-m_-}\{0\} \quad (3.47)$$

where π is the permutation which reorders the fermion arguments of im_+im_-

to $2im$. (Here we are letting m, and $m_\pm$ denote the actual sets of arguments as well as the numbers thereof.)

Now if M is gapless and includes $i + m_\pm$ then both of $S_{\underset{+}{i+m+}}\{J^M\}$ are polynomials in $G_k\{J^M\} = A_k + A_k^0$ $(k \in M)$. So if we restrict to arguments A with $P_0A = 0$ and supported away from σ, then we find by virtue of the discussion following (3.45) that $\dot{C}_i^{-1}A_k = 0$, so

$$-\frac{1}{2}\dot{C}_i^{-1}S_{\underset{+}{i+m+}}\{J^M\{A\}\}S_{\underset{-}{i-m-}}\{J^M\{A\}\} = -\frac{1}{2}\dot{C}_i^{-1}S_{\underset{+}{i+m+}}\{0\}S_{\underset{-}{i-m-}}\{0\}.$$

Finally, we check that $i + m_\pm \in M$. Since $m_\pm$ are both nonempty both are strictly less than m. So either $m_\pm < m - i$ or $m_\pm \leq m - (\sim i)$ and so $i + m_\pm$ is less than or equal to either m or $m + i - (\sim i)$. But then our assumption on M forces $i + m_\pm \in M$ and we are done. ■

§ IV. IRREDUCIBILITY OF Γ^M

We now study the irreducibility properties of first order and second order gapless Legendre transforms Γ^M. By n^{th} order we mean that $\max_{m\in M} |m| = n$. (« Gapless » is defined in Def. III.5). The irreducibility properties we shall prove fall into two categories. The first consists of general irreducibility statements

$$\partial_t^\alpha \Gamma^M|_0 \sim 0 \tag{4.1}$$

that apply to the *generating functional* Γ^M. The second consists of special irreducibility statements about *specific moments* of Γ^M (evaluated at $A = 0$) of the form,

$$_{xy}\Gamma^M\{0, t\} \quad \text{is } \beta\text{-irreducible between } x \text{ and } y$$

where

$$x = (x_1, \ldots, x_m),\ y = (y_1, \ldots, y_n), \tag{4.2}$$

$$_{xy}\Gamma^M\{0, t\} \equiv \left(\prod_{j=1}^{M} \delta_{A_{\mu_j}(x_j)}\right)\left(\prod_{k=1}^{n} \delta_{A_{\nu_k}(y_k)}\right)\Gamma^M\{A, t\}|_{A=0},$$

and

DEFINITION IV.1. — A function $\gamma(x, y; t)$ is r-irreducible between the variables x and y ($r-$I between x and y) if

$$\partial_t^{r'}\gamma(x, y; 0) = 0 \qquad \text{for all} \quad r' \leq r$$

whenever x and y are separated by σ.

These special irreducibility properties arise from the fermion symmetry « every graph must have an even number of external fermion lines ».

One might expect that (4.1) applies whenever $\alpha \in M$. If so one should reread the part of § I dealing with the Gaussian model. On the basis of the Gaussian model one expects that (4.1) holds for all $\alpha \in M$ whenever M is triangular and only for $|\alpha| = 1$ when M is not triangular (i. e. if $M = \{f, b, ff, bb\}$ or $\{f, b, fb\}$). However the Gaussian model does not take into account the above-stated fermion symmetry. This symmetry is responsible for the applicability of (4.1) in the following cases:

i) $M = \{f, b, fb\}$ and $\alpha = fb$
ii) $M = \{b\}$ and $\alpha = f, fb$
iii) $M = \{b, bb\}$ and $\alpha = f, fb, fbb$.

We shall see that (4.1) holds in case (*i*) because $P_0 A^0 = 0$ for all t. In the latter two cases M is purely bosonic so that no graph from Γ^M can have external fermion lines. Consequently the number of internal lines cut in disconnecting such a graph must be even.

THEOREM IV.2 (*Irreducibility of* Γ^M). — Let M be first order or second order and gapless. Then

$$\partial_t^\alpha \Gamma^M |_0 \sim 0$$

for the set of α's listed below.

Case	M	Allowed α's	Remarks
a)	$\{f\}$	M	
b)	$\{\ b\}$	$M \cup \{f, fb\}$	fermion symmetry increases irreducibility
c)	$\{f, b\}$	M	
d)	$\{f,\ ff\}$	M	
e)	$\{f, b, ff\}$	M	
f)	$\{\ b,\ bb\}$	$M \cup \{f, fb, fbb\}$	fermion symmetry increases irreducibility
g)	$\{f, b,\ bb\}$	M	
h)	$\{f, b,\ fb\}$	M	M non-triangular but fermion symmetry helps
i)	$\{f, b, ff, fb\}$	M	
j)	$\{f, b,\ fb, bb\}$	M	
k)	$\{f, b, ff,\ bb\}$	$M \backslash \{ff, bb\}$	M non-triangular
l)	$\{f, b, ff, fb, bb\}$	M	

Remark. — A nonzero multiple of $C_f(x_1, x_2)C_f(x_3, x_4)C_b(x_1, x_3)C_b(x_2, x_4)$ occurs in the perturbation theory expansion of $\left(\prod_{i=1}^{4} \frac{\delta}{\delta A_f(x_i)}\right)\Gamma^M\{0\}$ for $M = \{f, b, ff, bb\}$ so the restriction $\alpha \neq bb, ff$ is indeed necessary for this M.

Proof. — The tools we will be using here are similar to those we used in the case of a single scalar field [1]-[4]. Before getting into the details of the proof we present a brief résumé of our principal tools.

RESUME. — Our goal is to prove the connectedness of various t-derivatives of Γ^M. The t-derivatives are evaluated by (possibly repeated) application of

$$\partial_{t_i}\Gamma^M = -\frac{1}{2}\dot{C}_i^{-1}[G_{J_iJ_i} + G_{J_i}G_{J_i}]_0^{J^M\{A\}} + (\partial_{t_i}A_m^0)_{A_m}\Gamma^M \quad (4.3)$$

with $i = f, b$. (See Theorem III.16). There are three main tools used in handling J-derivatives of G. Firstly the connectedness (2.18) of G implies that for a product of J-derivatives each localized on one side of σ

$$\Pi\left(P_\pm \frac{\delta}{\delta J}\right)G\,|_{J^M\{A\}} \sim 0\,. \quad (4.4)$$

In fact we have equality in (4.4) if both $P_+\frac{\delta}{\delta J}$ and $P_-\frac{\delta}{\delta J}$ appear in the left-hand side. Secondly, since $C_i^{-1}(t_i)$ and $C_i^{-1}(t_i)^*$ are independent of t when applied to functions supported away from σ,

$$f\dot{C}_i^{-1}(G_{J^m}\,|_0^{J^M\{A\}}) = (G_{J^m}\,|_0^{J^M\{A\}})\dot{C}_i^{-1}f = 0 \quad (4.5)$$

for all f whenever $m \in M$ and $A_m \in \mathcal{N}_\sigma$. Thirdly when M is purely bosonic (cases (b) and (f))

$$\left(G\prod_k \frac{\delta}{\delta J_{i_k}}\right)(J^M\{A\}) = 0 \qquad \text{if} \qquad \left(\sum_k i_k\right)_f \text{ is odd}\,. \quad (4.6)$$

The reason for this is that $J^M\{A\}$ is purely bosonic so that no matter how many additional derivatives are applied to the left-hand side of (4.6) every term must contain a moment of G having odd fermion number. Hence every moment of the left-hand side of (4.6) is zero when evaluated at $A = 0$.

There are three principal tools used in handling A-derivatives of Γ^M. Just as for G the connectedness of Γ^M (see remarks following Definition III.4) implies that

$$\Pi\left(P_\pm \frac{\delta}{\delta A}\right)\Gamma^M \sim 0\,. \quad (4.7)$$

Consequently we may insert projections P_0:

$$_A\Gamma^M \sim P_{0A}\Gamma^M \quad (4.8\,a)$$

$$_{AA}\Gamma^M \sim (P_0 \otimes (I-P_0) + (I-P_0)\otimes P_0 + P_0 \otimes P_0)_{AA}\Gamma^M\,. \quad (4.8\,b)$$

Secondly the conjugate relation (Theorem III.7) expresses ${}_{A}\Gamma^{M}$ as a « linear combination » of J's so that

$$P_{0A}\Gamma^{M} = 0 \qquad \text{at zero}. \tag{4.9}$$

Thirdly the Jacobian relation (Lemma III.9) is used to express ${}_{AA}\Gamma^{M}$ (in Theorem III.11) and ${}_{AAA}\Gamma^{M}$ (in the Appendix, see (A.8)) in terms of $G_{J^M}\{J^M\{A\}\}$ via the matrix S_{ij} of Definition III.8. It is also used to express $\frac{\delta}{\delta A_m} G_{J^n}\{J^M\{A\}\}$ in terms of S via

$$\frac{\delta}{\delta A_m} G_{J^n}\{J^M\{A\}\} = S^{M^{-1}}_{mn'} S_{n'n}\,. \tag{4.10}$$

First order derivatives. Let $i \in \{f, b\}$ and consider the formula (4.3) for $\partial_{t_i}\Gamma^M$. We always have

$$G_{J_iJ_i}\big|_0^{J^M\{A\}} \sim 0 \qquad \text{(by (4.4))}$$
$$(P_+ \otimes P_+ + P_- \otimes P_-)[G_{J_i}G_{J_i}]_0^{J^M\{A\}} \sim 0 \qquad \text{(by (2.8))}$$
$$P_{\pm A_m}\Gamma^M \sim 0 \qquad \text{(by (4.7))}$$

and

$$P_{0A_m}\Gamma^M \sim 0 \qquad \text{(by (4.9))}$$

Consequently

$$\begin{aligned}\partial_{t_i}\Gamma^M &\sim P_+G_{J_i}\{J^M\{A\}\}\dot{C}_i^{-1}P_-G_{J_i}\{J^M\{A\}\}\\ &\sim P_+G_{J_i}\big|_0^{J^M\{A\}}\dot{C}_i^{-1}P_-G_{J_i}\big|_0^{J^M\{A\}}.\end{aligned}$$

If $i \in M$ this is zero for all $A \in \mathcal{N}_\sigma$ by (4.5). The case $i \notin M$ occurs only when $i = f$ and M is purely bosonic (cases b) and f)). But then $G_{J_f}\big|_0^{J^M\{A\}} = 0$ by (4.6).

Second order derivatives. The general second order derivative of Γ^M is

$$\begin{aligned}\partial_{t_j}\partial_{t_i}\Gamma^M = &-\frac{1}{2}(\partial_{t_j}\dot{C}_i^{-1})[G_{J_iJ_i} + G_{J_i}G_{J_i}]_0^{J^M\{A\}}\\ &-\frac{1}{2}\dot{C}_i^{-1}\partial_{t_j}[G_{J_iJ_i} + G_{J_i}G_{J_i}]_0^{J^M\{A\}}\\ &+(\partial_{t_j}\partial_{t_i}A_m^0)_{A_m}\Gamma^M\\ &+(\partial_{t_i}A_m^0)\frac{\delta}{\delta A_m}[G_{J_jJ_j} + G_{J_j}G_{J_j}]\,\big|_{J^M\{A\}}\left(-\frac{1}{2}\dot{C}_j^{-1}\right)\\ &+(\partial_{t_i}A_m^0)(\partial_{t_j}A_n^0)_{A_mA_n}\Gamma^M\end{aligned} \tag{4.11}$$

for any $i, j \in \{f, b\}$.

We first consider those cases for which we can choose $2i \in M$ (namely all cases except $\alpha = fb$, $M = \{ b \}$ or $\{ f, b, fb \}$). Then by (4.5), (4.11) simplifies to

$$\begin{aligned} \partial_{t_j}\partial_{t_i}\Gamma^M &= (\partial_{t_j}\partial_{t_i}A_m^0)_{A_m}\Gamma^M \\ &+ (\partial_{t_i}A_m^0)\frac{\delta}{\delta A_m}[G_{J_jJ_j} + G_{J_j}G_{J_j}]|_{J^M\{A\}}\left(-\frac{1}{2}\dot{C}_j^{-1}\right) \\ &+ (\partial_{t_i}A_m^0)(\partial_{t_j}A_n^0)_{A_mA_n}\Gamma^M \\ &\sim (P_0\partial_{t_i}A_m^0)\frac{\delta}{\delta A_m}G_{J_jJ_j}\{J^M\{A\}\}\left(-\frac{1}{2}\dot{C}_j^{-1}\right) \\ &+ (P_0\partial_{t_i}A_m^0)(P_0\partial_{t_j}A_n^0)_{A_mA_n}\Gamma^M . \end{aligned} \tag{4.12}$$

To get rid of the $G_{J_j}G_{J_j}$ term we have observed that either $j \in M$ or (in the case $i=b$, $j=f$, $M = \{ b, bb \}$) $G_{J_j}\{ J^M\{ A \} \} = 0$ by fermion symmetry. Now $(P_0\partial_{t_i}A_m^0)\frac{\delta}{\delta A_m}G_{J_jJ_j}\{J^M\{A\}\}\left(-\frac{1}{2}\dot{C}_j^{-1}\right)$ is independent of A and hence OCI unless $2j \notin M$. This happens only when $i \neq j$ and $M = \{ f, b, fb, 2i \}$ or $M = \{ b, bb \}$. Then if $\bar{i}$ denotes $b(f)$ when $i = f(b)$

$$\begin{aligned} &(P_0\partial_{t_i}A_m^0)\frac{\delta}{\delta A_m}G_{J_{\bar{i}}J_{\bar{i}}}\{J^M\{A\}\}\left(-\frac{1}{2}\dot{C}_{\bar{i}}^{-1}\right) \\ &=\left(-\frac{1}{2}\dot{C}_i^{-1}\right)S_{(2i)m}\{J^M\{A\}\}P_0)S_{mm'}^{(M)^{-1}}\{J^M\{A\}\}S_{m'(2\bar{i})}\{J^M\{A\}\}\left(-\frac{1}{2}\dot{C}_{\bar{i}}^{-1}\right) \end{aligned}$$

(by Theorem III.16 c) and (4.10))

$$\begin{aligned} &= \left(-\frac{1}{2}\dot{C}_i^{-1}\right)P_0S_{(2i)(2\bar{i})}\{J^M\{A\}\}\left(-\frac{1}{2}\dot{C}_{\bar{i}}^{-1}\right) \qquad \text{(since } (2i) \in M) \\ &= 0 \end{aligned}$$

since $P_0S_{(2i)(2\bar{i})}\{J^M\{A\}\}$ contains a factor $G_{J_fJ_b}\{J^M\{A\}\}$ which is in $\mathcal{N}_\sigma$ for $M = \{ f, b, fb, 2i \}$ and is zero for the purely bosonic $M = \{ b, bb \}$. That leaves the second term in (4.12):

$$\begin{aligned} &(P_0\partial_{t_i}A_m^0)(P_0\partial_{t_j}A_n^0)_{A_mA_n}\Gamma^M \\ &\quad = \left(-\frac{1}{2}\dot{C}_i^{-1}S_{(2i)m}\{J^M\{A\}\}\right)(-\sigma^n S_{mn}^{M-1}\{J^M\{A\}\}P_0)(\partial_{t_j}A_n^0) \end{aligned}$$

by Theorems III.11 and III.16,

$$\sim 0 .$$

We now return to those cases for which it is not possible to choose i so that $2i \in M$, namely $\partial_{t_f}\partial_{t_b}\Gamma^M$ with $M = \{ b \}$ or $\{ f, b, fb \}$. In both these cases we have $P_0 A^0_m = 0$ so that (4.11) leads to

$$\begin{aligned}
\partial_{t_f}\partial_{t_b}\Gamma^M &\sim -\frac{1}{2}\dot{C}_b^{-1}\partial_{t_f}(G_{J_bJ_b}\{ J^M\{ A \} \}) \\
&+ (\partial_{t_b}A^0_m)\frac{\delta}{\delta A_m}(G_{J_f}\{ J^M\{ A \} \} G_{J_f}\{ J^M\{ A \} \})\left(-\frac{1}{2}\dot{C}_f^{-1}\right) \\
&\sim -\frac{1}{2}\dot{C}_b^{-1}\partial_{t_f}(G_{J_bJ_b}\{ J^M\{ A \} \}) \qquad \text{((by (4.5), (4.6))}
\end{aligned}$$

The ∂_{t_f} may act either on the $G_{J_b^2}$ or on the $J^M\{ A \}$. In the former case we get, by Theorem III.16 *a*)

$$\begin{aligned}
(\partial_{t_f}G_{J_b^2})\{ J^M\{ A \} \} &= -\frac{1}{2}\dot{C}_f^{-1}[G_{J_f^2J_b^2} + 2G_{J_f}G_{J_fJ_b^2} + 2G^2_{J_fJ_b}] \\
&\sim 0 . \qquad \text{((by (4.4), (4.5), (4.6))}
\end{aligned}$$

In the latter case we use

$$0 = \partial_{t_f}A^M\{J^M\{ . \}\} = \left(\partial_{t_f}A^M\{J^M\{ . \}\} + \left(A^M\frac{\delta}{\delta J}\right)\{ J^M\{ . \}\}(\partial_{t_f}J^M\{ . \}\right)$$

to get

$$\partial_{t_f}J^M\{ A \} = -\left(J^M\frac{\delta}{\delta A_n}\right)\partial_{t_f}A^M_n\{ J^M\{ A \} \}$$

and hence

$$\begin{aligned}
&G_{J_b^2J_m}\{ J^M\{ A \} \}\partial_{t_f}J^M_m\{ A \} \\
&= -G_{J_b^2J_m}\{ J^M\{ A \} \}\left(J^M_m\frac{\delta}{\delta A_n}\right)(\partial_{t_f}A^M_n)\{ J^M\{ A \} \} \\
&= -\left(G_{J_b^2}\{ J^M\{ A \} \}\frac{\delta}{\delta A_n}\right)(\partial_{t_f}A^M_n)\{ J^M\{ A \} \} \\
&= -\sigma^n\left(\frac{\delta}{\delta A_n}G_{J_b^2}\{ J^M\{ A \} \}(\partial_{t_f}A^M_n)\{ J^M\{ A \} \} \qquad \text{((by (2.26))} \\
&= -\sigma^n S^{M-1}_{nn'}S_{n'bb}(\partial_{t_f}A^M_n)\{ J^M\{ A \} \} \qquad \text{((by (4.10), (4.13))}
\end{aligned}$$

Now

$$\begin{aligned}
(\partial_{t_f}A^M_n)\{ J^M\{ A \} \} &= -\frac{1}{2}\dot{C}_f^{-1}[G_{J_f^2} + G_{J_f}G_{J_f}]\frac{\delta^n}{\delta J^n}\Big|_0^{J^M\{A\}} \\
&= -\frac{1}{2}\dot{C}_f^{-1}G_{J_f^2J^n}\Big|_0^{J^M\{A\}} \qquad \text{((by (4.5), (4.6))}
\end{aligned}$$

for any $n \in M$. Hence $P_0(\partial_{t_f}A^M_n)\{ J^M\{ A \} \} = 0$ and $P_\pm(\partial_{t_f}A^M_n)\{ J^M\{ A \} \}$ depends only on $P_\pm A$ so that $G_{J_b^2J^m}\{ J^M\{ A \} \}\partial_{t_f}J^M_m\{ A \} \sim 0$ too.

Third order derivatives. There is only one third order derivative to consider, $\partial_t^{(1,2)}\Gamma^M\{A\}$ for $M = \{b, bb\}$. By (4.11).

$$\partial_{t_b}^2\Gamma^M\{A\} = (\partial_{t_b}^2 A_m^0)_{A_m}\Gamma^M + (\partial_{t_b}A_m^0)(\partial_{t_b}A_n^0)_{A_mA_n}\Gamma^M + \text{const.}$$

and by (4.3) and the fact that $G_{J_f}\{J^M\{A\}\} = 0$

$$\partial_{t_f}\Gamma^M = \frac{1}{2}\dot{C}_f^{-1}[G_{J_fJ_f}]_0^{J^M\{A\}} + (\partial_{t_f}A_p^0)_{A_p}\Gamma^M$$

so that

$$\begin{aligned}\partial_t^{(1,2)}\Gamma^M = {} & (\partial_t^{(1,2)}A_m^0)_{A_m}\Gamma^M - \frac{1}{2}(\partial_{t_b}^2 A_m^0)\frac{\delta}{\delta A_m}G_{J_f^2}\{J^M\{A\}\}\dot{C}_f^{-1} \\ & + (\partial_{t_b}^2 A_m^0)(\partial_{t_f}A_p^0)_{A_mA_p}\Gamma^M + 2(\partial_t^{(1,1)}A_m^0)(\partial_{t_b}A_n^0)_{A_mA_n}\Gamma^M \\ & - \frac{1}{2}(\partial_{t_b}A_n^0)(\partial_{t_b}A_m^0)\frac{\delta}{\partial A_n}\frac{\delta}{\delta A_m}G_{J_f^2}\{J^M\{A\}\}\dot{C}_f^{-1} \\ & + (\partial_{t_b}A_m^0)(\partial_{t_b}A_n^0)(\partial_{t_f}A_p^0)_{A_mA_nA_p}\Gamma^M + \text{const.} \\ \sim {} & -\frac{1}{2}(\partial_{t_b}^2 A_m^0)\frac{\delta}{\delta A_m}G_{J_f^2}\{J^M\{A\}\}\dot{C}_f^{-1} \\ & - \frac{1}{2}(\partial_{t_b}^2 A_n^0)(\partial_{t_b}A_m^0)\frac{\delta}{\delta A_n}\frac{\delta}{\delta A_m}G_{J_f^2}\{J^M\{A\}\}\dot{C}_f^{-1} \\ & + (\partial_{t_b}A_m^0)(\partial_{t_b}A_n^0)(\partial_{t_f}A_p^0)_{A_mA_nA_p}\Gamma^M .\end{aligned}$$

We have used (4.8 *a*) and (4.9) to get rid of the $_{A_m}\Gamma^M$ term and (by Theorem III.16 *c*))

$$\begin{aligned}\partial_{t_f}A_{bb}^0P_0 &= -\frac{1}{2}\dot{C}_f^{-1}S_{ff,bb}\{0\}P_0 \\ &= -\frac{1}{2}\dot{C}_f^{-1}(2G_{J_fJ_b}\{0\}G_{J_fJ_b}\{0\}) \\ &= 0 \end{aligned} \tag{4.14}$$

and

$$\partial_{t_b}A_n^0P_0 = -\frac{1}{2}\dot{C}_b^{-1}S_{bb,n}\{J^M\{A\}\}P_0$$

to get rid of the $_{A_mA_p}\Gamma^M$ and $_{A_mA_n}\Gamma^M$ terms respectively. We now consider each of the remaining terms in turn,

$$\begin{aligned}P_0\frac{\delta}{\delta A_m}G_{J_f^2}\{J^M\{A\}\} &= S_{mm'}^{M^{-1}}S_{m',ff}P_0 \qquad \text{(by (4.10))} \\ &= P_0S_{m,bb}^{M^{-1}}2G_{J_fJ_b}\{J^M\{A\}\}G_{J_fJ_b}\{J^M\{A\}\} \\ &= 0 .\end{aligned}$$

$$\begin{aligned}\frac{\delta}{\delta A_n}\frac{\delta}{\delta A_m}G_{J_f^2}\{J^M\{A\}\} &= \frac{\delta}{\delta A_n}S_{mm'}^{M^{-1}}S_{m',ff} \\ &= -S_{mm'}^{M^{-1}}S_{nn'}^{M^{-1}}T_{n'm'k}S_{km''}^{M^{-1}}S_{m'',ff} \\ &\quad + S_{mm'}^{M^{-1}}S_{nn'}^{M^{-1}}T_{n',m',ff}\end{aligned}$$

by Lemma A.3. (Note that n is purely bosonic.) Suppose firstly that at least one of $\frac{\delta}{\delta A_m}$ and $\frac{\delta}{\delta A_n}$ is not hit by a P_0. We may assume without loss of generality that the $\frac{\delta}{\delta A_n}$ is hit by a P_+. But then it follows from the definition (A.7 (c)) of T that

$$P_+ \otimes I \otimes IT_{n'm'k} = P_+ \otimes P_+ \otimes P_+T_{n'm'k} + P_+ \otimes P_0 \otimes P_0T_{n'm'k}.$$

The $P_+ \otimes P_+ \otimes P_+$ term gives a cluster connected contribution to $\frac{\delta}{\delta A_n}\frac{\delta}{\delta A_m} G_{J_f^2}\{J^M\{A\}\}$. All contributions to $\frac{\delta}{\delta A_n}\frac{\delta}{\delta A_m} G_{J_f^2}\{J^M\{A\}\}$ from $P_+ \otimes P_0 \otimes P_0$ term contain a factor of $G_{J_f J_b}\{J^M\{A\}\}$ (either in $S_{m'',ff}$ or in $T_{n',m',ff}$) and hence are zero. Suppose now that both $\frac{\delta}{\delta A_n}$ and $\frac{\delta}{\delta A_m}$ are hit by P_0's. Then by Theorem III.16 d) we obtain a cancellation of the « live » factors $S^{M^{-1}}_{mm'}$ and $S^{M^{-1}}_{nn'}$

$$-\frac{1}{2}(\partial_{t_b}A^0_nP_0)(\partial_{t_b}A^0_mP_0)\frac{\delta}{\delta A_n}\frac{\delta}{\delta A_m}G_{J_f^2}\{J^M\{A\}\}\dot{C}_f^{-1}$$
$$= -\frac{1}{8}(\dot{C}_b^{-1}P_0)(\dot{C}_b^{-1}P_0)(-T_{bb,bb,k}S^M_{km''}S_{m'',ff} + T_{bb,bb,ff})\dot{C}_f^{-1}.$$

If the $\dot{C}_f^{-1}$ is hit with a P_0 we always get a factor of $G_{J_f J_b^n}\{J^M\{A\}\} = 0$. If the $\dot{C}_f^{-1}$ is hit with a P_+ (e. g.) then P_-A dependence may enter only through $G_{J_b^2}\{J^M\{A\}\} = A_{bb} + A^0_{bb}$ all of whose A dependence is destroyed by a $\dot{C}_b^{-1}$. Hence

$$-\frac{1}{2}(\partial_{t_b}A^0_n)(\partial_{t_b}A^0_m)\frac{\delta}{\delta A_n}\frac{\delta}{\delta A_m}G_{J_f^2}\{J^M\{A\}\}\dot{C}_f^{-1} \sim 0.$$

This leaves only

$$\begin{aligned}(\partial_{t_b}A^0_m)(\partial_{t_b}A^0_n)(\partial_{t_f}A^0_p)_{A_mA_nA_p}\Gamma^M \\ &\sim (\partial_{t_b}A^0_mP_0)(\partial_{t_b}A^0_nP_0)[\partial_{t_f}A^0_p(1-P_0)]_{A_mA_nA_p}\Gamma^M \quad \text{(by (4.14))} \\ &= \frac{1}{4}(\dot{C}_b^{-1}P_0)(\dot{C}_b^{-1}P_0)[\partial_{t_f}A^0_p(1-P_0)]S^{M^{-1}}_{pk}T_{bb,bb,k} \\ &\sim 0\end{aligned}$$

as in the $T_{bb,bb,k}$ discussion above. ■

The remaining irreducibility properties we shall consider (cf. (4.2)) are motivated by the fact that in perturbation theory if no fermion line is allowed to cross σ (i.e. if $t_f = 0$) then every graph must have an even number of external fermion lines in $\mathbb{R}^d_-$ and $\mathbb{R}^d_+$. We find it convenient to express this fact in terms of the symmetry operators

$$(\mathrm{E}_\pm \mathrm{J})_m(\vec{x}) = (-1)^{m_\pm} \mathrm{J}_m(\vec{x}) \tag{4.15}$$

where $m_+(m_-)$ is the number of the m_f fermion arguments $x_1, \ldots, x_{m_f}$ that are in $\mathbb{R}^d_+$ ($\mathbb{R}^d_-$).

LEMMA IV.3. — If $t_f = 0$

$$\mathrm{Z^M}\{\mathrm{E_\pm J}\} = \mathrm{Z^M}\{\mathrm{J}\}$$
$$\mathrm{G^M}\{\mathrm{E_\pm J}\} = \mathrm{G^M}\{\mathrm{J}\}$$
$$\Gamma^{\mathrm{M}}\{\mathrm{E_\pm A}\} = \Gamma^{\mathrm{M}}\{\mathrm{A}\}$$

Proof. — To get $\mathrm{Z^M}\{\mathrm{E_\pm J}\} = \mathrm{Z^M}\{\mathrm{J}\}$, and hence $\mathrm{G^M}\{\mathrm{E_\pm J}\} = \mathrm{G^M}\{\mathrm{J}\}$ we need only observe that, when $t_f = 0$,

$$\begin{aligned}
&\langle f(\phi)(-1)^{m_\pm}\psi(x_1)\ \ldots\ \psi(x_m)\rangle \\
&\qquad = \text{const.} \int d\mu(\phi) f(\phi) \int (-1)^{m_\pm}\psi(x_1)\ \ldots\ \psi(x_m) e^{-\mathrm{V}-\bar\psi + \mathrm{S}_\sigma^{-1}\psi +} \delta\psi \\
&\qquad = \langle f(\phi)\psi(x_1)\ \ldots\ \psi(x_m)\rangle
\end{aligned}$$

because the fermion integral is zero if m_+ or m_- is odd.

From $\mathrm{G^M}\{\mathrm{E_\pm J}\} = \mathrm{G^M}\{\mathrm{J}\}$ we get

$$\mathrm{E_\pm A^M}\{\mathrm{E_\pm J}\} = \mathrm{A^M}\{\mathrm{J}\},$$

then

$$\mathrm{E_\pm J^M}\{\mathrm{E_\pm A}\} = \mathrm{J^M}\{\mathrm{A}\} \quad \text{(by Theorem II.10)}$$

and finally

$$\begin{aligned}
\Gamma^{\mathrm{M}}\{\mathrm{E_\pm A}\} &= \mathrm{G^M}\{\mathrm{J^M}\{\mathrm{E_\pm A}\}\} - \mathrm{G^M_J}\{\mathrm{J^M}\{\mathrm{E_\pm A}\}\}.\mathrm{J}\{\mathrm{E_\pm A}\} \\
&= \mathrm{G^M}\{\mathrm{J^M}\{\mathrm{A}\}\} - \mathrm{G^M_J}\{\mathrm{J^M}\{\mathrm{A}\}\}.\mathrm{J^M}\{\mathrm{A}\} = \Gamma^{\mathrm{M}}\{\mathrm{A}\}
\end{aligned}$$

Remarks 1. — If we were being completely rigorous, even to the extent of worrying about which spaces things lived in, A and J would live in different spaces each with its own projectors $\mathrm{E}_\pm$.

2. It follows immediately from $\mathrm{F}\{\mathrm{E_\pm A}\} = \mathrm{F}\{\mathrm{A}\}$ that

$$\prod_m \left(\mathrm{E}_\pm \frac{\delta}{\delta \mathrm{A}_m}\right) \mathrm{F}\{\mathrm{E_\pm A}\} = \left(\prod_m \frac{\delta}{\delta \mathrm{A}_m}\right) \mathrm{F}\{\mathrm{A}\}$$

and hence that $\left(\prod_m \frac{\delta}{\delta \mathrm{A}_m}\right) \mathrm{F}\{0\}$ is zero unless m_+ and m_- are even.

3. The Yukawa model actually obeys a much stronger symmetry than this. Any graph with no fermion lines crossing σ must have the same number

of ψ_- (fermion) and ψ_+ (antifermion) external vertices in $\mathbb{R}^d_\pm$. This symmetry may be expressed in terms of the operators

$$(F_\pm J)_m(\vec{x}) = 2^{f_\pm - \bar{f}_\pm} J_m(\vec{x}) \tag{4.16}$$

where $f_\pm$ ($\bar{f}_\pm$) is the number of fermion (antifermion) arguments in $\mathbb{R}^d_\pm$. We will not discuss, in this paper, the extra irreducibility that results from this symmetry.

Consider a graph contributing to be perturbation theory expansion of

$$\partial_t^\beta P_+ \otimes P_{-\,xy}\Gamma^M |_{A=t=0} = \partial_t^\beta \left(\prod_{j=1}^m P_+ \delta_{A_{\mu_j}(x_j)} \right) \left(\prod_{k=1}^n P_- \delta_{A_{\nu_k}(y_k)} \right) \Gamma^M |_{A=t=0} . \tag{4.17}$$

Define x_f to be $\sum_{j=1}^m (\mu_j)_f$, the number of fermion arguments in x. Suppose $\beta_f + x_f$ ($\beta_f + y_f$) is odd. Then no matter where in $\mathbb{R}^d$ we place its vertices our graph must have strictly fewer than β_f fermion propagators crossing σ. For otherwise the subgraph consisting of everything in $\mathbb{R}^d_+$ ($\mathbb{R}^d_-$) would have an odd number of external fermion lines. Hence if $\partial_t^{\beta'}\Gamma^M \sim 0$ for all β' obeying $\beta'_f < \beta_f$ and $\beta'_b \le \beta_b$ we would expect (4.17) to be zero. The intuition, combined with Theorem IV.2, suggests

THEOREM IV.4. — Let M be first order or second order and gapless.

a) If at least one of x_f and y_f is odd then the following hold.

i) For all M $G^M_{xy}\{0, t\}$ and ${}_{xy}\Gamma^M\{0, t\}$ are ∞b-I between x and y (i. e. m-I for all m with $m_f = 0$).

ii) If $f \in M$, then ${}_{xy}\Gamma^M\{0, t\}$ is $2f$-I between x and y.

iii) If $\{f, b, fb\} \subset M$, then ${}_{xy}\Gamma^M\{0, t\}$ is ffb-I between x and y.

b) If at least one of x_f and y_f is even then the following hold.

i) For all M, $G^M_{xy}\{0, t\}$ and ${}_{xy}\Gamma^M\{0, t\}$ are f-I between x and y.

ii) If $b \in M$, then ${}_{xy}\Gamma^M\{0, t\}$ is fb-I between x and y.

iii) If $\{b, bb\} \subset M \neq \{f, b, ff, bb\}$, then ${}_{xy}\Gamma^M\{0, t\}$ is fbb-I between x and y.

iv) If $\{f, ff\} \subset M \neq \{f, b, ff, bb\}$, then ${}_{xy}\Gamma^M\{0, t\}$ is $3f$-I between x and y.

Proof. — We may immediately dispense with *a*) *i*) thanks to Lemma IV.3. (Note that t_b is not set to zero in Lemma IV.3.) Furthermore we need not consider $\partial_t^{m'}$ derivatives with $m'_f < m_f$ in our proofs of m-irreducibility because these have all been taken care of in Theorem IV.2. We now consider the remaining first, second and third order derivatives in turn.

First order derivatives (part *b*) *i*), *b*) *ii*))

$$\partial_{t_f} G = -\frac{1}{2}\dot{C}_f^{-1}[G_{J_f^2} + G_{J_f}G_{J_f}]_0^J$$
$$\sim -\dot{C}_f^{-1}G_{J_f}\{P_+J\}G_{J_f}\{P_-J\}$$

Now apply Lemma IV.3. From now on applying Lemma IV.3 (and in particular remark 2 following it) will be referred to as fermion counting.

$$\partial_{t_f}\Gamma^M = -\frac{1}{2}\dot{C}_f^{-1}[G_{J_f^2} + G_{J_f}G_{J_f}]_0^{J^M\{A\}} + (\partial_{t_f}A_m^0)_{A_m}\Gamma^M\{A\}$$
$$\sim -\dot{C}_f^{-1}G_{J_f}\{J^M\{P_+A\}\}G_{J_f}\{J^M\{P_-A\}\}.$$

Count fermions.

Second order derivatives (parts *a*) (*ii*), *b*) (*ii*), *a*) (*iii*) and *b*) (*iii*)).

Let $j \in \{b, f\}$ with $j \in M$. Then from (4.11)

$$(\partial_{t_j}\partial_{t_f}\Gamma^M \sim -\frac{1}{2}\dot{C}_f^{-1}\partial_{t_j}[G_{J_f^2}\{J^M\{A\}\}]$$
$$+ (\partial_{t_f}A_m^0P_0)\frac{\delta}{\delta A_m}[G_{J_j^2}\{J^M\{A\}\}]\left(-\frac{1}{2}\dot{C}_j^{-1}\right)$$
$$+ (\partial_{t_f}A_m^0P_0)(\partial_{t_j}A_n^0P_0)_{A_mA_n}\Gamma^M\{A\}.$$

We consider each of these terms in turn, starting with the last one. By part *b*) (*i*) $\partial_{t_f}A_m^0P_0$ is zero unless $m_+ = m_- = f$ and by parts *a*) *i*) and *b*) *i*)

$$\partial_{t_j}A_n^0P_0 = 0 \quad \text{unless} \quad n_+ = n_- = j. \tag{4.18}$$

Hence we need only count fermions in ${}_{A_mA_n}\Gamma^M\{A\}$.

To get a nonzero answer from the second term $(\partial_{t_f}A_m^0P_0)$ must be nonzero, which implies $m_+ = m_- = f$ which implies $ff \in M$. Either $j = f$ in which case $\frac{\delta}{\delta A_m}[G_{J_j^2}\{J^M\{A\}\}]$ is independent of A or $j = b$ in which case we need only count fermions in $\frac{\delta}{\delta A_m}[G_{J_f^2}\{J^M\{A\}\}]$.

$$-\frac{1}{2}\dot{C}_f^{-1}\partial_{t_j}[G_{J_f^2}\{J^M\{A\}\}]$$
$$= \frac{1}{4}\dot{C}_j^{-1}[G_{J_j^2} + G_{J_j}G_{J_j}]_{J_f^2}\{J^M\{A\}\}\dot{C}_f^{-1}$$
$$+ \sigma^n S_{nn'}^{M^{-1}}S_{n',ff}\left\{\left(-\frac{1}{2}\dot{C}_j^{-1}\right)[G_{J_j^2} + G_{J_j}^2]\frac{\delta^n}{\delta J^n}\Big|_0^{J^M\{A\}}\right\}\frac{1}{2}\dot{C}_f^{-1} \quad \text{(as in (4.13))}$$
$$\sim \sigma^j\dot{C}_j^{-1}G_{J_jJ_f}\{J^M\{P_+A\}\}G_{J_jJ_f}\{J^M\{P_-A\}\}\dot{C}_f^{-1}$$
$$-\frac{1}{4}(\dot{C}_j^{-1})\sigma^n S_{nn'}^{M^{-1}}S_{n'ff}P_0[G_{J_j}G_{J_j}]\left(P_0\frac{\delta^n}{\delta J^n}\right)\Big|_0^{J^M\{A\}}\dot{C}_f^{-1}$$

since $j \in \mathrm{M}$ and P_0 commutes with $\mathrm{S}^{\mathrm{M}^{-1}}_{nn'}$ and $\mathrm{S}_{n',ff}$ at zero. Because there must be one J_j in $\mathbb{R}^d_+$ and one in $\mathbb{R}^d_-$ we may now count fermions. For example let μ^a_+ (μ^b_+, μ^c_+) be the number of fermions in those of the $\prod_{j=1}^{m} \mathrm{P}_+ \delta_{\mathrm{A}_{\mu_j(x_j)}}$ derivatives that are applied to $\mathrm{S}^{\mathrm{M}^{-1}}_{nn'}\left(\mathrm{S}_{n',ff}\mathrm{P}_0, [\mathrm{G}_{\mathrm{J}_j}\mathrm{G}_{\mathrm{J}_j}]\left(\mathrm{P}_0 \frac{\delta^n}{\delta \mathrm{J}^n}\right)\Big|_0^{\mathrm{J}^{\mathrm{M}}\{\mathrm{A}\}}\right)$.

Then to get a nonzero answer we must have all of $n_{f,+} + n'_{f,+} + \mu^a_+$, $n'_{f+} + 1 + \mu^b_+$ and $(j)_{f,+} + n_{f,+} + \mu^c_+$ even and hence

$$1 + (j)_{f,+} + \mu^a_+ + \mu^b_+ + \mu^c_+ = 1 + (j)_{f,+} + \mu_+$$

even.

Third order derivatives (cases *a) iii*), *b) iii*), and *b) iv*)). We merely sketch the argument. First we consider those cases involving only derivatives of the form $\partial_t^{2i+j}\Gamma$ with $\{ i, j, 2i \} \subset \mathrm{M}$ (possibly with $i = j$). (This covers all but those cases of part (*a*) for which $2f \notin \mathrm{M}$.) Iterating (4.3) and throwing out all those terms which either vanish for $\mathrm{A} \in \mathcal{N}_\sigma$ by virtue of a $\dot{\mathrm{C}}^{-1}$ hitting a factor of $\mathrm{G}_{\mathrm{J}^m_1}\{ \mathrm{J}^{\mathrm{M}} \{ \mathrm{A} \} \}$ for $m \in \mathrm{M}$ (see (4.5)) or which are manifestly connected (such as $(\partial_t^{j+2i}\mathrm{A}^0_m)_{\mathrm{A}_m}\Gamma^{\mathrm{M}}$), we find that for $\mathrm{A} \in \mathcal{N}_\sigma$

$$\begin{aligned}
\partial_t^{j+2i}\Gamma^{\mathrm{M}} \sim & -\frac{1}{2}(\partial_t^{2i}\mathrm{A}^0_m\mathrm{P}_0)(\dot{\mathrm{C}}_j^{-1})\frac{\delta}{\delta \mathrm{A}_m}\mathrm{G}_{\mathrm{J}_j\mathrm{J}_j}\{ \mathrm{J}^{\mathrm{M}}\{ \mathrm{A} \} \} \\
& + (\partial_t^{2i}\mathrm{A}^0_m\mathrm{P}_0)(\partial_{t_j}\mathrm{A}^0_n\mathrm{P}_0)_{\mathrm{A}_m\mathrm{A}_n}\Gamma^{\mathrm{M}} \\
& + 2(\partial_t^{i+j}\mathrm{A}^0_m\mathrm{P}_0)(\partial_{t_i}\mathrm{A}^0_n\mathrm{P}_0)_{\mathrm{A}_m\mathrm{A}_n}\Gamma^{\mathrm{M}} \\
& - \frac{1}{2}(\partial_{t_i}\mathrm{A}^0_m\mathrm{P}_0)(\partial_{t_i}\mathrm{A}^0_n\mathrm{P}_0)(\dot{\mathrm{C}}_j^{-1})\frac{\delta}{\delta \mathrm{A}_m}\frac{\delta}{\partial \mathrm{A}_n}\mathrm{G}_{\mathrm{J}_j\mathrm{J}_j}\{ \mathrm{J}^{\mathrm{M}}\{ \mathrm{A} \} \} \\
& - (\partial_{t_i}\mathrm{A}^0_m(1-\mathrm{P}_0))(\partial_{t_i}\mathrm{A}^0_n\mathrm{P}_0)(\dot{\mathrm{C}}_j^{-1})\frac{\delta}{\delta \mathrm{A}_m}\frac{\delta}{\delta \mathrm{A}_n}\mathrm{G}_{\mathrm{J}_j\mathrm{J}_j}\{ \mathrm{J}^{\mathrm{M}}\{ \mathrm{A}\} \} \\
& + (\partial_{t_i}\mathrm{A}^0_m)(\partial_{t_i}\mathrm{A}^0_n)(\partial_{t_j}\mathrm{A}^0_p)_{\mathrm{A}_m\mathrm{A}_n\mathrm{A}_p}\Gamma^{\mathrm{M}}
\end{aligned}$$

It turns out that, when the external (x and y) derivatives are applied and the result evaluated at 0, all of these terms vanish for the relevant choices of i, j and M. This involves essentially a case by case analysis with fermion counting taking care of all cases not covered by the tools of Theorem IV.2. We illustrate this by considering just the first term.

$$\begin{aligned}
& -\frac{1}{2}(\partial_t^{2i}\mathrm{A}^0_m\mathrm{P}_0)(\dot{\mathrm{C}}_j^{-1})\frac{\delta}{\delta \mathrm{A}_m}\mathrm{G}_{\mathrm{J}_j\mathrm{J}_j}\{ \mathrm{J}^{\mathrm{M}}\{ \mathrm{A} \} \} \\
= & -\frac{1}{2}(\partial_t^{2i}\mathrm{A}^0_m\mathrm{P}_0)(\dot{\mathrm{C}}_j^{-1})\mathrm{S}^{\mathrm{M}^{-1}}_{mm'}\mathrm{S}_{m',2j}\mathrm{P}_0
\end{aligned}$$

If $2j \in \mathrm{M}$ this is independent of A, so we may restrict to $2j \notin \mathrm{M}$. Now m

is either $2b$ or $2f$ (by fermion counting applied to $\partial_t^{2i}A_m^0$ on each side of σ) and the $2b$ case is okay by fermion counting. But if $m = 2f$ then, since $m \in M$, we must have $j = b$. The only such case is $i = f, j = b$, $M = \{ f, b, fb, 2f \}$. But then for $m' \in M$, $S_{m',2j}\{ J^M\{ A \} \} P_0$ contains a factor of $A_{fb} \in \mathcal{N}_\sigma$ which is annihilated by the $\dot{C}_j^{-1}$.

For the cases of part (*a*) (involving $2f + b$ derivatives) with $2f \notin M$, we get a different pattern of terms dropping out by virtue of (4.5). In particular, if we compute $\partial_t^{2f+b}\Gamma$ by using (4.11) for $\partial_t^{f+b}\Gamma$ and then apply the second ∂_{t_f}, using (4.3) when it hits a Γ factor, we find after the usual reductions

$$\begin{aligned}
\partial_t^{2f+b}\Gamma^M \sim & -\frac{1}{2}(\dot{C}_b^{-1})\partial_t^{2f}[G_{J_bJ_b}\{ J^M\{ A \} \}] \\
& - (\partial_t^{fb}A_m^0P_0)(\dot{C}_f^{-1})\frac{\delta}{\delta A_m}[G_{J_fJ_f}\{ J^M\{ A \} \}] \\
& - \frac{1}{2}(\partial_{t_b}A_m^0P_0)(\ddot{C}_f^{-1})\frac{\delta}{\delta A_m}\partial_{t_f}[G_{J_fJ_f}\{ J^M\{ A \} \}] \\
& - \frac{1}{2}(\partial_{t_b}A_m^0)(\dot{C}_f^{-1})\frac{\delta}{\delta A_m}\partial_{t_f}[G_{J_fJ_f}\{ J^M\{ A \} \}] \\
& + (\partial_{t_b}A_m^0P_0)(\partial_{t_f}^2A_n^0P_0)_{A_mA_n}\Gamma^M \\
& - \frac{1}{2}(\partial_{t_f}A_n^0(1-P_0))(\partial_{t_b}A_m^0P_0)(\dot{C}_f^{-1})\frac{\delta}{\delta A_n}\frac{\delta}{\delta A_m}[G_{J_fJ_f}\{ J^M\{ A \} \}].
\end{aligned}$$

For the case $M = \{ f, b, fb, 2b \}$ all of these terms can be handled along the lines we have already discussed. But for the case $M = \{ f, b, fb \}$ the first term (alone) requires a more subtle analysis: after expanding out $\partial_{t_f}^2 G_{J_bJ_b}$ it is necessary to effect cancellations between some of the resulting terms in addition to invoking all of our standard tools including fermion counting. The computation proceeds as in Theorem IV.2 to get

$$\begin{aligned}
\left(-\frac{1}{2}\dot{C}_b^{-1}\right)\partial_{t_f}G_{J_b^2}\{ J^M\{ A \} \} = & \frac{1}{4}(\dot{C}_b^{-1})(\dot{C}_f^{-1})G_{J_f^2J_b^2}\{ J^M\{ A \} \} \\
& - \frac{1}{4}\sigma^n(\dot{C}_b^{-1})\left(\frac{\delta}{\delta A_n}G_{J_b^2}\{ J^M\{ A \} \} \right)(\dot{C}_f^{-1})(G_{J_f^2J^n}|_0^{J^M\{A\}}).
\end{aligned}$$

Applying the second ∂_{t_f} derivative then results in an expansion which includes in particular the terms

$$\frac{1}{4}(\dot{C}_b^{-1})(\dot{C}_f^{-1})(\partial_{t_f}G_{J_f^2J_b^2})\{ J^M\{ A \} \}$$

and

$$-\frac{1}{4}\sigma^n(\dot{C}_b^{-1})\left(\frac{\delta}{\delta A_n}G_{J_b^2}\{ J^M\{ A \} \} \right)(\dot{C}_f^{-1})(\partial_{t_f}G_{J_f^2J^n}|_0^{J^M\{A\}}).$$

These can be shown by our standard techniques to give the same connected contribution to $\partial_t^{2f+b}{}_{xy}\Gamma^M$ (for x, y not both even) as

$$\mp \frac{1}{8}(\dot{C}_b^{-1})G_{J_b^2 J_f}(\dot{C}_f^{-1})G_{J_f^3}(\dot{C}_f^{-1})$$

and so these contributions cancel. All other terms can be shown to give zero contribution individually. ◢

§ V. THE M-FIELD PROJECTORS AND BETHE-SALPETER KERNELS

In this section we develop the relationship between the Legendre transform Γ^M and the M-field projectors P_M and Bethe-Salpeter kernels K^M in analogy to the results for boson models in I. One major difference for fermion models is that the Euclidean expection $\langle \,.\, \rangle$ is not associated with a positive measure and a related Hilbert space in which the P_M's are self-adjoint operators. Nevertheless we can obtain sufficient algebraic structure to define P_M, K^M, etc., by equipping the formal vector space $\mathscr{V}$ of all polynomials in the fields with the bilinear form

$$\langle f, g \rangle = \langle fg \rangle . \tag{5.1}$$

Our manipulations on $\mathscr{V}$ will be purely formal; for instance, we write, for an element $f \in \mathscr{V}$, $f = \sum_{m \in \mathscr{M}} \int f_m(\vec{y}_m)\Phi^m(\vec{y}_m)d\vec{y}_m$ for suitable family of (distribution) kernels $f_m(y_m)$ (with only the appropriately symmetric parts of f_m having a nonzero contribution). One way of attaching rigorous meaning to such expressions is to identify each $f \in \mathscr{V}$ with its family of kernels $(f_m)_{m \in \overline{\mathscr{M}}}$ with $f_m \in \mathscr{L}_m$ (so that $\mathscr{V}$ might be thought of as a suitable completion of $\mathscr{L}^0 \equiv \bigcup_{\substack{M \in \overline{\mathscr{M}} \\ M \text{ finite}}} \mathscr{L}^{\mathscr{M}}$ in $\mathscr{L}^{\overline{\mathscr{M}}}$).

Even after restriction to kernels of appropriate symmetry, the bilinear form (5.1) is still not positive definite or even symmetric. But writing $f = P_0 f + P_e f$ where $P_0 f (P_e f)$ is the part of f of odd (even) fermion degree, we have (since $\langle P_0 f \rangle = 0$)

$$\begin{aligned} \langle f, g \rangle &= \langle P_0 f, P_0 g \rangle + \langle P_e f, P_e g \rangle \\ &= - \langle P_0 g, P_0 f \rangle + \langle P_e g, P_e f \rangle \end{aligned} \tag{5.2}$$

It follows from (5.2) that $\langle f, g \rangle = 0$ for all $g \in \mathscr{V}$ if and only if $\langle g, f \rangle = 0$ for all $g \in \mathscr{V}$. Hence it is natural to define the orthogonal complement of a subspace $\mathscr{U} \subset \mathscr{V}$ as

$$\mathscr{U}^\perp = \{ g \in \mathscr{V} \mid \langle f, g \rangle = 0 \quad \text{for all } f \in \mathscr{U} \} .$$

Throughout this section we assume that M is gapless (i. e. $m \in \mathrm{M}$, $0 < n \leq m \Rightarrow n \in \mathrm{M}$) and we let $\mathscr{V}_{\mathrm{M}}$ be the subspace of $\mathscr{V}$ generated by $\{ \Phi^m(\vec{x}_m) \mid m \in \overline{\mathrm{M}} = \mathrm{M} \cup \{0, 0\} \}$. The physically Wick ordered powers are defined by

$$\vdots \Phi^m \vdots = \frac{e^{\Phi f}}{\langle e^{\Phi f} \rangle} \frac{\delta}{\delta f^m} \bigg|_{f=0} . \tag{5.3 a}$$

It is easy to see that for all m

$$\langle \vdots \Phi^m \vdots \rangle = 0 \tag{5.3 b}$$

and

$$\vdots \Phi^m \vdots = \Phi^m + \sum_{n \subsetneq m} c_{mn} \Phi^n \tag{5.3 c}$$

with coefficients c_{mn} depending on the variables in mn. Hence $\vdots \Phi^m \vdots \in \mathscr{V}_{\mathrm{M}}$ and $\{ \Phi^m \mid m \in \overline{\mathrm{M}} \}$ also generates $\mathscr{V}_{\mathrm{M}}$. Note that the matrix $\mathrm{S}_{m,n}\{ \mathrm{J} \}$ of Defin. III.8 is related to the Wick powers by

$$\begin{aligned} \mathrm{S}_{mn}\{0\} &= \exp\{ \mathrm{G}\{f+g\} - \mathrm{G}\{f\} - \mathrm{G}\{g\} \} \frac{\delta}{\delta f^m} \frac{\delta}{\delta g^n} \bigg|_{f=g=0} \\ &= \left\langle \frac{e^{\Phi f}}{\langle e^{\Phi f} \rangle} \frac{e^{\Phi g}}{\langle e^{\Phi g} \rangle} \right\rangle \frac{\delta}{\delta f^m} \frac{\delta}{\delta g^n} \bigg|_{f=g=0} \\ &= \left\langle \frac{e^{\Phi f}}{\langle e^{\Phi f} \rangle} \frac{\delta}{\delta f^m} \bigg|_{f=0} \frac{e^{\Phi g}}{\langle e^{\Phi g} \rangle} \frac{\delta}{\delta g^n} \bigg|_{g=0} \right\rangle \\ &= \langle \vdots \Phi^m \vdots, \vdots \Phi^n \vdots \rangle \end{aligned} \tag{5.4}$$

Since $\langle . , . \rangle$ is not positive definite there is always the danger that $\mathscr{V}_{\mathrm{M}} \cap \mathscr{V}_{\mathrm{M}}^{\perp} \neq \{0\}$. However as we are making the basic assumption that Γ^{M} exists, $\mathrm{S}^{\mathrm{M}}_{mn}$ must be invertible (see (3.32)). It follows that there cannot be a polynomial $\sum_{n \in \mathrm{M}} c_n \Phi^n$ in $\mathscr{V}_{\mathrm{M}}$ which is orthogonal to all $\vdots \Phi^m \vdots$, $m \in \mathrm{M}$. Hence $\mathscr{V}_{\mathrm{M}} \cap \mathscr{V}_{\mathrm{M}}^{\perp} = \{0\}$, $\langle . , . \rangle$ is non-degenerate, and the decomposition $f = f_{\mathrm{M}} + f_{\mathrm{M}^{\perp}}$ where $f \in \mathscr{V}$, $f_{\mathrm{M}} \in \mathscr{V}_{\mathrm{M}}$, $f_{\mathrm{M}^{\perp}} \in \mathscr{V}_{\mathrm{M}}^{\perp}$ must be unique (if it exists). We now establish existence by showing that f_{M} can be explicitly given in terms of the Legendre transform as $f_{\mathrm{M}} = \mathrm{P}_{\mathrm{M}} f$ where:

DEFINITION V.1. — For each $f \in \mathscr{V}$ define

$$\mathrm{P}_{\mathrm{M}} f = \langle f \rangle - \sum_{m,n \in \mathrm{M}} \sigma^m \vdots \Phi^m \vdots_{\mathrm{A}_m \mathrm{A}_n} \Gamma^{\mathrm{M}}\{0\} \langle \vdots \Phi^n \vdots, f \rangle \tag{5.5}$$

$$\mathrm{P}_{\phi} f = \mathrm{P}_{\{0,0\}} f = \langle f \rangle$$

THEOREM V.2. — *a*) The range of P_M is $\mathscr{V}_M$.

b) $$P_M^2 = P_M.$$

c) $$\langle P_M f, g \rangle = \langle f, P_M g \rangle.$$

Proof. — *a*), *b*) Since $P_M \mathscr{V} \subset \mathscr{V}_M$ it suffices to show that $P_M \upharpoonright \mathscr{V}_M$ is the identity, i. e. that $P_M \vdots \Phi^r \vdots = \vdots \Phi^r \vdots$ for all $r \in \overline{M}$. The case $r = (0, 0)$ is obvious from (5.3 *b*). When $r \in M$ we have from (5.4) and the Jacobian relation (3.33)

$$\begin{aligned} P_M \vdots \Phi^r \vdots &= - \sum_{m,n \in M} \sigma^m \vdots \Phi^m \vdots_{A_m A_n} \Gamma^M \{ 0 \} S_{nr} \{ 0 \} \\ &= \vdots \Phi^r \vdots . \end{aligned}$$

c) From the definition (5.5) with $T_{mn} = {}_{A_m A_n}\Gamma^M \{ 0 \}$

$$\langle P_M f, g \rangle = \langle f \rangle \langle g \rangle - \sum_{m,n \in M} \sigma^m \langle \vdots \Phi^m \vdots g \rangle T_{mn} \langle \vdots \Phi^n \vdots f \rangle \quad (5.6\,a)$$

$$\langle f, P_M g \rangle = \langle f \rangle \langle g \rangle - \sum_{m,n \in M} \sigma^n \langle f \vdots \Phi^n \vdots \rangle T_{nm} \langle \vdots \Phi^m \vdots g \rangle \quad (5.6\,b)$$

The equality of the two sums follows from the observation that T_{mn} satisfies

$$T_{mn} = 0 \qquad \text{if} \qquad m_f + n_f \text{ is odd} \quad (5.7\,a)$$

$$T_{mn} = \sigma^{mn} T_{nm} = (-1)^{m_f n_f} T_{nm}. \quad (5.7\,b)$$

For suppose that f is of pure fermi degree d. Then n_f and m_f must have the same parity as d and the sum in (5.6 *b*) equals

$$\Sigma \langle \Phi^n f \rangle \sigma^{mn} T_{mn} \langle \Phi^m g \rangle$$

which agrees with the sum in (5.6 *a*) since $\sigma^{mn} = \sigma^m$. ■

Properties *a*)-*c*) of Theorem V.2 ensure that both $P_M f \in \mathscr{V}_M$, and $(1 - P_M) f \in \mathscr{V}_M^\perp$ thus giving our decomposition. By virtue of the uniqueness of such a decomposition, these properties completely characterize P_M.

LEMMA V.3. — If $Q : \mathscr{V} \to \mathscr{V}$ satisfies

a) the range of Q is $\mathscr{V}_M$,

b) $Q^2 = Q$,

c) $\langle Qf, g \rangle = \langle f, Qg \rangle$,

then $Q = P_M$.

Just as in the case of a single scalar field (see Theorem IV.1 of III) we can characterize the matrix elements of $(1 - P_M)$ in terms of the partial Legendre transforms Γ^M viewed as a functional of $\{ A_\alpha \mid \alpha \in M \}$ and $\{ J_i \mid i \notin M \}$ (here we use the convention that Greek letters run over M and Latin letters over M^c).

THEOREM V.4. — $\langle \Phi^i, (1-P_M)\Phi^k \rangle = \Gamma^M_{J_iJ_k}\{0\} = {}_{J_iJ_k}\Gamma^M\{0\} \, \forall i, k \notin M$. When $M = \phi$ replace Γ^M by G.

Remark. — Since the conclusions of Theorems IV.2 and IV.4 apply to the partial Legendre transforms, we can read off the irreducibility properties of $\langle \phi^i, (1-P)\phi^k \rangle$ from those theorems.

Proof. — The case $M = \phi$ is the well-known identity

$$G_{J_iJ_k}\{\phi\} = \langle \Phi^i\Phi^k \rangle - \langle \Phi^i \rangle \langle \Phi^k \rangle$$

So consider $M \neq \phi$. Since $G_{J_\alpha}\{J\{A\}\} = F_\alpha\{A^0+A\}$ is independent of J_i, we have $\Gamma^M_{J_i} = G_{J_\alpha}J_{\alpha J_i} + G_{J_i} - G_{J_\alpha}J_{\alpha J_i} = G_{J_i}$ and

$$\Gamma^M_{J_iJ_k} = G_{J_iJ_\alpha}J_{\alpha J_k} + G_{J_iJ_k}.$$

But

$$0 = G_{J_\gamma}\{J\{A\}\}\frac{\delta}{\delta J_k} = G_{J_\gamma J_\beta}J_{\beta J_k} + G_{J_\gamma J_k}$$

so

$$J_\alpha\frac{\delta}{\delta J_k} = -[G^M_{\cdot\cdot}]^{-1}_{\alpha\gamma}G_{J_\gamma J_k}$$

and

$$\begin{aligned}\Gamma^M_{J_iJ_k}\{0\} &= G_{J_iJ_k} - G_{J_iJ_\alpha}[G^M_{\cdot\cdot}]^{-1}_{\alpha\gamma}G_{J_\gamma J_k} \\ &= \langle \Phi^i\Phi^k \rangle - \langle \Phi^i \rangle \langle \Phi^k \rangle - \langle \Phi^i\Sigma^\alpha \rangle [\langle \Sigma^\cdot\Sigma^\cdot \rangle]^{-1}_{\alpha\gamma} \langle \Sigma^\alpha\Sigma^k \rangle\end{aligned}$$

where Σ^i refers to the polynomial $\Phi^i - \langle \Phi^i \rangle$. Hence it suffices to prove that

$$P_M\Phi^k = \langle \Phi^k \rangle + \Sigma^\alpha[\langle \Sigma^\cdot\Sigma^\cdot \rangle]^{-1}_{\alpha\gamma} \langle \Sigma^\gamma\Phi^k \rangle. \tag{5.8}$$

Define $Q\Phi^k$ to be the right hand side of (5.8). It is straightforward to verify that Q obeys the hypotheses (*a*) and (*b*) of Lemma V.3. As for hypothesis *c*) we note that the matrix $T_{\alpha\gamma} = \langle \Sigma^\alpha\Sigma^\gamma \rangle$ satisfies (5.7), and hence by Lemma V.5 below so does its inverse. Thus by the argument of Theorem V.2 *c*) Q satisfies hypothesis *c*) of Lemma V.3 and thus $Q = P_M$. ■

LEMMA V.5. — If $\{T_{mn} \mid m, n \in M\}$ is an ordinary (i. e. not Grassmann algebra valued) invertible matrix/operator satisfying (5.7) then so is T^{-1}.

Proof. — The condition (5.7 *a*) can be restated as $T_{mn} = \sigma^{m+n}T_{mn}$ or as $T = \underline{\sigma}T\underline{\sigma}$, where the matrix $\underline{\sigma}$ has elements $\underline{\sigma}_{mn} = \delta_{mn}\sigma^m$. Inverting we have $T^{-1} = \underline{\sigma}^{-1}T^{-1}\underline{\sigma}^{-1} = \underline{\sigma}T^{-1}\underline{\sigma}$, as desired.

We now establish (5.7 *b*) for T^{-1} by computing that the matrix

$$U_{mn} = \sigma^{mn}(T^{-1})_{mn}$$

inverts T. By (5.7 *b*)

$$U_{mn}T_{nr} = \sigma^{mn}(T^{-1})_{mn}\sigma^{nr}T_{rn}.$$

But by (5.7 *a*) applied to T and T^{-1}, *m* and *r* must have the same fermion parity and so $\sigma^{mn+nr} = 1$. Hence $U_{mn}T_{nr} = T_{rn}(T^{-1})_{mn} = \delta_{rm}$. ■

We turn next to the construction of projectors P_m onto single powers Φ^m. The natural definition of P_m given P_M is

$$P_m = P_{[m]} - P_{(m)} \tag{5.9}$$

where $[m]$ and (m) consist of those n's in $\mathscr{M}$ that are, respectively, « smaller than or equal to » and « strictly smaller than » m. In fact we have a choice in interpreting the notion « smaller » at this stage; it can refer to the natural partial ordering on $\mathscr{M}$ ($n \leq m \Leftrightarrow n_f \leq m_f$ and $n_b \leq m_b$; $n < m \Leftrightarrow n \leq m$ and $n \neq m$) or to a total ordering by some « weight »:

DEFINITION V.6. — Let $\mu_f > 0$, $\mu_b > 0$ with μ_f/μ_b irrational. Then n is *less than m in ordering by weight*, denoted $n \underset{w}{<} m$, if $\mu_f n_f + \mu_b b_b < \mu_f m_f + \mu_b m_b$ and $n \underset{w}{\leq} m$ if $\mu_f n_f + \mu_b n_b \leq u_f m_f + \mu_b m_b$.

Remarks 1. — We require μ_f/μ_b irrational to prevent ties. Hence exactly one of $n \underset{w}{<} m, n = m$ or $n \underset{w}{>} m$ holds.

2. Ordering by weight is consistent with the natural partial ordering and with addition (i. e. $n \underset{w}{\leq} m,\ r \in \mathscr{M} \Rightarrow n + r \underset{w}{\leq} m + r$).

3. Given any total ordering on $\mathscr{M}$ consistent with the natural partial ordering and addition and given any finite subset M of $\mathscr{M}$ it can be shown that there exists an ordering by weight that agrees with the given ordering on M.

Since $P_M P_N = P_N P_M = P_N$ if $N \subset M$ it follows that whichever definition of $[m]$ and (m) we choose $P_m^2 = P_m$ and $P_m P_n = 0$ if $n < m$. However, while $P_m \Phi^n = 0$ if $n < m$, it is not the case that $P_m \Phi = 0$ if $m < n$. Moreover it is not true as in the single field case that

$$P_M = \sum_{m \in \bar{M}} P_m \tag{5.10}$$

for all gapless M. Of course (5.10) does hold (with the ordering by weight definition) for any M of the form $\{ n \in \mathscr{M} \mid n \underset{w}{\leq} m \}$. To see that (5.10) cannot hold in general for all gapless M we note that the consistency condition

$$P_{M \cup \{m,n\}} = P_{M \cup \{m\}} + P_{M \cup \{n\}} - P_M$$

is violated. For choose $M = \{f\}$, $m = ff$, $n = b$. Then none of $P_{M \cup \{m,n\}} \Phi^n = \Phi^n$, $P_{M \cup \{n\}} \Phi^n = \Phi^n$ or $P_M \Phi^n = \langle \Phi^n \rangle$ contains a Φ^m contribution whereas

$$\begin{aligned} P_{M \cup \{m\}} \Phi^n &= \langle \Phi^n \rangle - \vdots \Phi^m \vdots_{A_m A_m} \Gamma^{M \cup \{m\}} \{ 0 \} \langle : \Phi^m : \Phi^n \rangle \\ &= \langle \Phi^n \rangle + \vdots \Phi^m \vdots \langle \vdots \Phi^m \vdots \vdots \Phi^m \vdots \rangle^{-1} \langle \vdots \Phi^m \vdots \Phi^n \rangle \end{aligned}$$

does.

Explicit expressions for P_m can be given in terms of the Legendre transform.

THEOREM V.7. — *a*) Let $P_m = P_{[m]} - P_{(m)}$ where the sets $[m]$ and (m) are defined either in terms of the natural partial ordering on $\mathcal{M}$ or by an ordering by weight. Then

$$P_m f = -\sigma^m(1 - P_{(m)})\Phi^m{}_{A_m A_m}\Gamma^{[m]}\{0\}\langle \Phi^m, (1 - P_{(m)})f\rangle. \quad (5.11)$$

b) Define $R_m = \langle \Phi^m, P_m\Phi^m\rangle$. Then

$$R_m^{-1} = -\sigma^m{}_{A_m A_m}\Gamma^{[m]}\{0\} = (S^{[m]}\{0\})^{-1}_{mm}.$$

Proof. — *a*) By the definition (5.5) of $P_{[m]}$

$$\begin{aligned} P_m f &= (1 - P_{(m)})P_{[m]}(1 - P_{(m)})f \\ &= -(1 - P_{(m)})\sum_{i,j[m]} \sigma^i \vdots \Phi^i \vdots_{A_i A_j}\Gamma^{[m]}\{0\}\langle (1 - P_{(m)}) \vdots \Phi^j \vdots, f\rangle \end{aligned}$$

We obtain (5.11) by noting that $(1 - P_{(m)})\mathcal{V}_{(m)} = 0$, $\Phi^i \in \mathcal{V}_{(m)}$ if $i \in (m)$ and $\vdots \Phi^m \vdots - \Phi^m \in \mathcal{V}_{(m)}$.

b) By part *a*)

$$R_m = -\sigma^m\langle \Phi^m, (1 - P_{(m)})\Phi^m\rangle_{A_m A_m}\Gamma^{[m]}\{0\}\langle \Phi^m, (1 - P_{(m)})\Phi^m\rangle.$$

But $(1 - P_{(m)})\Phi^m = P_m\Phi^m$ so that

$$R_m = -\sigma^m R_{m A_m A_m}\Gamma^{[m]}\{0\} R_m$$

and the theorem follows once we prove that R_m is invertible. Now

$$P_i\Phi^i = (P_{[i]} - P_{(i)}) \vdots \Phi^i \vdots = \sum_{j[i]} v_{ij} \vdots \Phi^j \vdots$$

where $V_{ii} = 1$ and $V_{ij} = 0$ unless $j \in [i]$. Then letting all indices run over $[m]$

$$\begin{aligned} V_{ik}S^{[m]}_{kl}V^t_{lj} &= \langle P_i \vdots \Phi^i \vdots, P_j \vdots \Phi^j \vdots \rangle \\ &= \begin{cases} R_m & \text{if } i = j = m \\ 0 & \text{if } i \in (j) \text{ or } j \in (i) \end{cases} \end{aligned} \quad (5.12)$$

V and V^t are invertible by « triangularity » and $S^{[m]}$ is invertible by the hypothesis that $\Gamma^{[m]}$ exists. Hence the left side of (5.12) is invertible. It follows that R_m is invertible. ■

We now consider the Bethe-Salpeter kernels. In any model involving only one field, the « n to n Euclidean Bethe-Salpeter kernel », $K^{(n)}(x_n, y_n)$, consists, in perturbation theory, of all graphs from the $2n$-point Schwinger function $-S_{2n}(x_n, y_n)$ that are connected and n-irreducible between the clusters x_n and y_n.

In models involving more than one field there is no one obvious desired degree of cluster irreducibility. For example in a 2-fermion to 2-fermion kernel one can imagine wanting 0, 1, 2 or even more boson irreducibility in addition to two fermion irreducibility. Hence there are many « m to m Euclidean Bethe-Salpeter » kernels. However one would want any such kernel to be connected and to be n-cluster irreducible for any n « smaller » than or equal to m. The question is: what is the most general reasonable definition of « smaller » ? As the following proofs show it is desirable to define « small » in terms of a total ordering on $\mathcal{M}$. On the basis of Remark 3 after Defin. V.6 we shall determine such an ordering by a fixed ordering by weight. Such a determination is physically reasonable; for example μ_f and μ_b might be chosen as the (bare) fermi and boson masses.

As in the bosonic case (§ V of III) we can now define Bethe-Salpeter kernels K^m inductively by

$$K^m = (S^{[m]})^{-1}_{mm} \qquad |m| = 1$$

$$K^m = (S^{[m]})^{-1}_{mm} - \sum_{p \in \mathcal{D}_2^m} n_p^{-1} K^p \qquad |m| \geq 2 \tag{5.13}$$

where

$\mathcal{D}^m$ = set of decompositions of m into summands from $\mathcal{M}$ without regard to order

$$= \{ \{ m_1, \ldots, m_s \} \mid s \geq 1, m_i \in \mathcal{M}, m_1 + \ldots + m_s = m \}$$

$$\mathcal{D}_2^m = \mathcal{D}^m \backslash \{ \{ m \} \}$$

$$= \{ \{ m_1, \ldots, m_s \} \mid s \geq 2, \ m_i \in \mathcal{M}, \ m_1 + \ldots + m_s = m \}$$

K^p = appropriate symmetrization of $\bigotimes_{n \in p} K^n$

$$n_p = \prod_{n \in P} n_p(n)\,!$$

$n_p(n)$ = number of times n occurs in p.

Remarks.

1. The induction is well-defined since $n \in p \in \mathcal{D}_2^m \Rightarrow n \underset{w}{<} m$.

2. The order of the factors in $\bigotimes_{n \in p} K^n$ is irrelevant since each K^n must contain the same number of outgoing fermion arguments as it does incoming fermion arguments.

3. When $J = 0$ (as in the conventional definition of K^m) K^m may be expressed directly in terms of the Legendre transform as

$$K^m \{ 0 \} = - {}_{A_m A_m}\Gamma^{[m]} \{ 0 \} - \sum_{p \in \mathcal{D}_2^m} n_p^{-1} K^p \{ 0 \} \tag{5.14}$$

4. The subtractions in (5.13) have no effect on the cluster irreducibility properties of K^m: if K^m is n-irreducible for each $n \underset{w}{<} m$ then K^p is m-irreducible. The role of the subtractions is to ensure connectedness (not just cluster connectedness) of the K^m's i. e. to eliminate « lower body scattering processes ».

5. Different choices of ordering by weight (i. e. different μ_b/μ_f's) will give different K^m's. The principal differences will be in cluster irreducibility properties that are not comparable in the natural partial ordering (see the discussion after Theor. V.10).

THEOREM V.8. — K^m is connected. In other words $K^m = 0$ whenever there are arguments on both sides of σ.

Proof. — We use induction on m in the order given by our ordering by weight. The result is trivial for the smallest m, which must be one of f or b, thanks to the connectedness of $S^{[m]}$. Assume the result for all $n \underset{w}{<} m$. $S^{[m]}$ commutes with P_+, P_- and P_0 as do (by the induction hypotheses) all K^p's with $p \in \mathscr{D}_2^m$. So it suffices to prove that for any decomposition $m_1 + m_2$ of m with $m_i \neq (0, 0)$

$$\underset{+}{\pi}^{m_1} \times \underset{-}{\pi}^{m_2} \,|\, K^m = \underset{+}{\pi}^{m_1} \times \underset{-}{\pi}^{2} \left\{ (S^{[m]})^{-1}_{mm} - \sum_{p \in \mathscr{D}_2^m} n_p^{-1} K^p \right\} = 0 .$$

But there is precisely one term in the decomposition formula (3.41 b) that can have m_1 + 's and m_2 − 's. Hence

$$\underset{+}{\pi}^{m_1} \times \underset{-}{\pi}^{m_2} S^{[m]^{-1}}_{mm} = \underset{+}{\pi}^{m_1} \times \underset{-}{\pi}^{m_2} \binom{m}{m_1}^{-1} \underset{+}{S}^{[m_1]^{-1}}_{m_1 m_1} \times \underset{-}{S}^{[m_2]^{-1}}_{m_2 m_2} . \tag{5.15}$$

By the induction hypothesis each nonzero factor in K^p with $p \in \mathscr{D}_2^m$ must have arguments of pure sign. So $\underset{+}{\pi}^{m_1} \times \underset{-}{\pi}^{m_2} n_p^{-1} K^p$ is zero unless $p = p_1 \cup p_2$ with $p_i \in \mathscr{D}^m$.

$$\underset{+}{\pi}^{m_1} \times \underset{-}{\pi}^{m_2} \sum_{p \in \mathscr{D}_2^m} n_p^{-1} K^p$$
$$= \underset{+}{\pi}^{m_1} \times \underset{-}{\pi}^{m_2} \sum_{\substack{p \in \mathscr{D}^m \\ \text{s.t. } p = p_1 \cup p_2 \\ \text{with } p_i \in \mathscr{D}^{m_i}}} n_p^{-1} K^p$$
$$= \underset{+}{\pi}^{m_1} \times \underset{-}{\pi}^{m_2} \sum_{P_i \in \mathscr{D}^{m_i}} n_{p_1}^{-1} n_{p_2}^{-1} \underset{+}{K}^{p_1} \odot \underset{-}{K}^{p_2}$$

$\left(\text{since there are } \prod_r \frac{n_p(r)\,!}{n_{p_1}(r)\,!\, n_{p_2}(r)\,!} \text{ ways of assigning signs to the factors of } K^p \text{ to get } \underset{+}{K}^{P_1} \odot \underset{-}{K}^{P_2}\right)$

$$= (\underset{+}{\pi}^{m_1} \times \underset{-}{\pi}^{m_2})(\underset{+}{S}^{[m_1]^{-1}}_{m_1 m_1} \odot \underset{-}{S}^{[m_2]^{-1}}_{m_2 m_2}) \qquad \text{(by (5.13))}$$

$$= (\underset{+}{\pi}^{m_1} \times \underset{-}{\pi}^{m_2})\binom{m}{m_1}^{-1} (\underset{+}{S}^{[m_1]^{-1}}_{m_1 m_1} \times \underset{-}{S}^{[m_2]^{-1}}_{m_2 m_2}) \qquad (5.16)$$

So (5.15) and (5.16) cancel precisely. ■

From (5.14) and the known irreducibility properties Γ^M (Theorems IV.2 and IV.4) we can now read off the irreducibility properties of the second order BS kernels $K^m(\,|m| = 2, J = 0)$. The second column in the following table gives the (triangular) set $[m]$ induced by a choice of ordering by weight and the third column gives the allowed set of t-derivatives in the relation

$$\partial_t^\alpha P_\pm K^m P_\mp \,|_{t=0} = 0$$

as authorized by Theorems IV.2 and IV.4 (referenced in columns 4 and 5)

TABLE V.9. *Irreducibility of* K^m.

Case	m	$[m]$	Allowed α's	Theor. IV.2	Theor. IV.4
i)	ff	$\{f, ff\}$	$[m] \cup \{fff\}$	*d)*	*b) iv)*
ii)		$\{f, b, ff\}$	$[m] \cup \{fff, fb\}$	*e)*	*b) ii), iv)*
iii)		$\{f, b, ff, bf, bb\}$	$[m] \cup \{fff, fbb\}$	*l)*	*b) iii), iv)*
iv)	bb	$\{b, bb\}$	$[m] \cup \{f, fb, fbb\}$	*f)*	
v)		$\{f, b, bb, fbb\}$	$[m] \cup \{fb\}$	*g)*	*b) ii), iii)*
vi)		$\{f, b, ff, fb, bb\}$	$[m] \cup \{fff, fbb\}$	*l)*	*b) iii), iv)*
vii)	fb	$\{f, b, ff, fb\}$	$[m] \cup \{ffb, \infty b\}$	*i)*	*a) i), iii)*
viii)		$\{f, b, fb, bb\}$	$[m] \cup \{ff, ffb, \infty b\}$	*j)*	*a)*
ix)		$\{f, b, ff, fb, bb\}$	$[m] \cup \{ffb, \infty b\}$	*l)*	*a) i), iii)*

For an alternate and possibly more concrete description of the kernels K^m we next consider the « Bethe-Salpeter equations » they satisfy. These equations are simply Jacobian relations for partial Legendre transforms, as we now explain. Given two gapless subsets $M \subset M'$ of $\mathscr{M}$ we may regard the transform $\Gamma^{M'}\{A^{M'}\}$ as arising from first transforming $G\{J^{M'}\}$ to $\Gamma^M\{A^M, J^{M'\setminus M}\}$ and then transforming the remaining J's. (We adopt the convention that unprimed (primed) indices belong to M ($M'\setminus M$) and that $A^M = \{A_m \mid m \in M\}$, etc.) More precisely, define

$$A_m + A_m^0 = G_{J_1^m}\{J^{M'}\} \qquad m \in M$$

$$\Gamma^M\{A^M, J^{M'\setminus M}\} = G\{J^{M'}\} - \sum_{m \in M} G_{J_m} . J_m \,|_{J^M = J^M\{A^M\}} \qquad (5.17)$$

Note that A^M and $J^{M'\backslash M}$ are independent variables. As in the proof of Theorem V.4, $S_m \equiv G_{J_m}\{ J^M\{A^M\}\} = F_m\{A^M + (A^M)^0\}$ is independent of $J_{m'}$ so that

$$S_{m'} \equiv \Gamma^M_{J_{m'}}\{A^M, J^{M'\backslash M}\} = G_{J_{m'}}\{J^{M'}\}|_{J^M = J^M\{A^M\}} \tag{5.18}$$

Now transform the sources $J^{M'\backslash M}$ to connected variables $A^{M'\backslash M}$ defined in terms of the Schwinger variables $S^{M'}$:

$$A'_m + A^0_{m'} = F^{-1}_{m'}\{S^{M'}\}, \qquad m' \in M'\backslash M. \tag{5.19}$$

Here F^{-1} is the inverse of the mapping F of (3.22); like F_m

$$F^{-1}_m\{S^M\} = S_m + \text{terms involving } S_n\text{'s with } n < m.$$

The definition (5.19) of $A^{M'\backslash M}$ agrees with the direct definition in terms of G, i. e.,

$$A_{m'}\{A^M, J^{M'\backslash M}\} + A^0_{m'} = G_{J_1^{m'}}\{J^M\{A^M\}, J^{M'\backslash M}\}, \tag{5.20}$$

and so by (5.17) and (5.18) the iterated transform agrees with the direct one:

$$\left[\Gamma^M\{A^M, J^{M'\backslash M}\} - \sum_{n' \in M'\backslash M} \Gamma^M_{J'_n}\{A^M, J^{M'\backslash M}\} J_{n'}\right]\Bigg|_{J^{M'\backslash M} = J^{M'\backslash M}\{A^{M'\backslash M}\}}$$
$$= \Gamma^{M'}\{A^{M'}\}.$$

The conjugate relations for the iterated transform are (by the usual cancellation)

$$\frac{\delta}{\delta A_{m'}}\Gamma^{M'} = -\left(\frac{\delta}{\delta A_{m'}}\Gamma^M_{J_{n'}}\right)J_{n'} \tag{5.21}$$

Now suppose that no two different indices $m', n' \in M'\backslash M$ are comparable with respect to the partial ordering on $\mathcal{M}$, i. e. neither $m' < n'$ or $m' > n'$. From (5.18) and (5.19) and the definition (3.22) of F we see that

$$\Gamma^M_{J_{n'}}\{A^M, J^{M'\backslash M}\{A^{M'\backslash M}\}\} = A_{n'} + A^0_{n'} + \text{terms involving } A_\alpha\text{'s with } \alpha < n'.$$

Hence $\dfrac{\delta}{\delta A_{m'}}\Gamma^M_{J_{n'}} = \delta_{m',n'}$ and so from (5.21)

$$J_{n'} = -\frac{\delta}{\delta A_{n'}}\Gamma^{M'}. \tag{5.22}$$

Moreover, from (5.19) and (5.18),

$$A_{m'} + A^0_{m'} = \sigma^{m'}{}_{J_{m'}}\Gamma^M + \text{terms involving } S_n\text{'s with } n < m'.$$

Since S_n with $n \in M$ is independent of $J_{n'}$ we conclude that

$$\frac{\delta}{\delta J_{n'}} A_{m'} = \sigma^{m'}_{J_{n'}, J_{m'}}\Gamma^M.$$

Evaluating at $A = J = 0$ we obtain from Theorem V.4

$$\frac{\delta}{\delta J_{n'}} A_{m'} = E^{M}_{n'm'}\sigma^{m'} \tag{5.23}$$

where

$$E^{M}_{n'm'} \equiv \langle \Phi^{n'}, (1 - P_M)\Phi^{m'} \rangle$$

is the « M-truncated expectation between $\Phi^{n'}$ and $\Phi^{m'}$.

From (5.22), (5.23) and the Jacobian relation

$$\left(\frac{\delta}{\delta J_{m'}} A_{r'}\right)\left(\frac{\delta}{\delta A_{r'}} J_{n'}\right) = \delta_{m',n'}$$

we arrive at the following result:

THEOREM V.10. — Suppose that $M \subset M'$ are two gapless subsets of $\mathscr{M}$ such that no two indices in $M' \backslash M$ are comparable (with respect to the partial ordering on $\mathscr{M}$). Then

$$E^{M}_{m'r'}\sigma^{r'}{}_{A_{r'}A_{n'}}\Gamma^{M'}\{0\} = -\delta_{m'n'} \tag{5.24}$$

where $m', n', r' \in M' \backslash M$.

Equations (5.24) constitute the Bethe-Salpeter equations. Consider for example the ff kernel of case *ii*) in Table V.9 which we denote by $K^{M'}$, where $M' = [m] = \{f, b, ff\}$. For $i = f$ or b, $S^{[i]}$ is diagonal at $A = J = 0$ whatever $[i]$ is, and so

$$K^{i} = G^{-1}_{J_iJ_i} \qquad i = f \quad \text{or} \quad b\,. \tag{5.25}$$

Hence by (5.14)

$$K^{M'} = -{}_{A_{ff}A_{ff}}\Gamma^{M'} - \frac{1}{2} G^{-1}_{J_fJ_f} \otimes G^{-1}_{J_fJ_f}$$

Taking $M = \{f, b\}$ and letting $E_0 = 2G_{J_fJ_f} \otimes G_{J_fJ_f}$ we can write (5.24) as

$$K^{M'} = (E^{M})^{-1} = E_0^{-1} \tag{5.26 a}$$

(as operators on the two-fermion space), or as

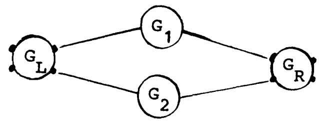
$$E^{M} = E_0 - E_0K^{M'}E^{M}\,. \tag{5.26 b}$$

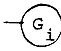
The BS equation (5.26 *b*) has a simple interpretation in perturbation theory which is best seen by writing it in amputated form. With $E^{M}_{amp} \equiv E_0^{-1}E^{M}E_0^{-1}$ the equation becomes

$$E^{M}_{amp} = E_0^{-1} - K^{M'}E_0E^{M}_{amp} \tag{5.26 c}$$

All graphs contributing to E^{M}_{amp} have at least two lines (f or b) joining the left and right ff clusters. The first term on the right of (5.26 *c*) consists of the disconnected graphs contributing to E^{M}_{amp}. The second term consists of the connected graphs in E^{M}_{amp} visualized in factored form as follows. Either the graph can be channel-disconnected by cutting two f lines or not.

If not, it is a graph in $-K^{M'}$. If so, consider the first place starting from the left where it can be channel disconnected by cutting two f lines. Such a graph will have the form



where  $(i = 1, 2)$ is a graph contributing to $G_{J_f J_f}$,  is a graph contributing to $-K^{M'}$, and  a graph with only M-irreducibility which contributes to E^{M}_{amp}. Summing up all these graphs gives $-K^{M'}E_0E^{M}_{amp}$ (the $2f$-channel irreducible graphs are captured by the singular contribution E_0^{-1} to E^{M}_{amp}).

Consider next case *iii*) in Table V.9 where $m = ff$ and $M' = [m] = \{ f, b, ff, bf, bb \}$. Choosing $M = M' \backslash \{ m \} = \{ f, b, bf, bb \}$ we obtain an equation of exactly the same form as (5.26) except that now the M-trucated expectation E^M enjoys more irreducibility (since $bf, bb \in M$) as does $K^{M'}$. The interpretation in perturbation theory is exactly the same with the factorization $K^{M'}E_0E^{M}_{amp}$ displaying the place at which a contributing graph may be channel-disconnected by cutting two f lines (no graph can be disconnected by cutting two b lines or one b and one f line).

In this second example there are other possibilities for, M such as $M = \{ f, b \}$. With this choice $M' \backslash M = \{ ff, fb, bb \}$ and (5.24) yields a set of coupled BS equations for kernels

$$\begin{aligned} K^{M'}_{ff,ff} &= -\, {}_{A_{ff}A_{ff}}\Gamma^{M'}\{0\} - E^{-1}_{0,ff} \\ K^{M'}_{bb,bb} &= -\, {}_{A_{bb}A_{bb}}\Gamma^{M'}\{0\} - E^{-1}_{0,bb} \\ K^{M'}_{bb,ff} &= -\, {}_{A_{bb}A_{ff}}\Gamma^{M'}\{0\} = K^{M'}_{ff,bb} \end{aligned}$$

and

$$K^{M'}_{fb,fb} = -\, {}_{A_{fb}A_{fb}}\Gamma^{M'}\{0\} - E^{-1}_{0,fb}$$

where $E_{0,ff} = 2G_{J_fJ_f}G_{J_fJ_f}$, $E_{0,bb} = 2G_{J_bJ_b}G_{J_bJ_b}$ and $E_{0,fb} = G_{J_fJ_f}G_{J_bJ_b}$. In matrix from (5.24) becomes (the 0 entries result from fermion symmetry)

$$\begin{pmatrix} E^M_{ff,ff} & E^M_{ff,bb} & 0 \\ E^M_{bb,ff} & E^M_{bb,bb} & 0 \\ 0 & 0 & -E^M_{fb,fb} \end{pmatrix}$$

$$\begin{pmatrix} K^{M'}_{ff,ff} + E^{-1}_{0,ff} & K^{M'}_{ff,bb} & 0 \\ K^{M'}_{bb,ff} & K^{M'}_{bb,bb} + E^{-1}_{0,bb} & 0 \\ 0 & 0 & K^{M'}_{fb,fb} + E^{-1}_{0,fb} \end{pmatrix} = I\,.$$

The fb, fb entry decouples from the others to give a BS equation of the form (5.26) whereas the other kernels satisfying the coupled BS equations

$$\begin{aligned} E^{M}_{ff,ff} &= E_{0,ff} - E_{0,ff}K^{M'}_{ff,ff}E^{M}_{ff,ff} - E_{0,ff}K^{M'}_{ff,bb}E^{M}_{bb,ff} \\ E^{M}_{ff,bb} &= -E_{0,ff}K^{M'}_{ff,bb}E^{M}_{bb,bb} - E_{0,ff}K^{M'}_{ff,ff}E^{M}_{ff,bb} \\ E^{M}_{bb,bb} &= E_{0,bb} - E_{0,bb}K^{M'}_{bb,ff}E^{M}_{ff,bb} - E_{0,bb}K^{M'}_{bb,bb}E^{M}_{bb,bb} \end{aligned} \tag{5.27}$$

These equations have an obvious interpretation in perturbation theory. For example the second and third terms on the right of the first equation display the connected graphs in $E^{M}_{ff,ff}$ which can be channel disconnected by cutting two f and two b lines, respectively.

We remark that by developing tree graph expansions (e. g. for $G_{J_{i_1}\dots J_{i_r}}$ in terms of vertex functions ${}_{A_{i_1}\dots A_{i_s}}\Gamma^{\{f,b\}}$) in analogy to the bosonic case (see I, § VI), it is possible to characterize the vertex functions as appropriate irreducible parts of $G^{amp}_{J_{i_1}\dots J_{1_r}}$. The analysis is complicated by the sign factors arising from fermi statistics and by the occurrence of lines of two types, and so we choose to omit it.

APPENDIX A :

HIGHER DERIVATIVES

In this appendix we develop and apply the tools needed to prove irreducibility results involving more than two t-derivatives. Our main result deals with M's determined by « total power of the field ». In fact, if for $N \in \mathbb{Z}$ we let $M_N \equiv \{ m \in \mathscr{M} \mid |m| \leq N \}$ and $\Gamma^{(N)} \equiv \Gamma^{M_N}$, then we have

THEOREM A.1. — For all $r \in M_4$ if $N \geq |r|$, then the generalized vertex functions ${}_{A_{i_1} \ldots A_{i_n}}\Gamma^{M_N}$ are r-CI for all i_k if $|r| < 4$, and provided $|i_k| \neq 2$ if $|r| = 4$.

The proof of this theorem is similar to that which we gave in IV for the case of a theory with just one scalar boson field. In that case the proofs proceeded in three main steps: first expanding t-derivatives of Γ in terms of A-derivatives of Γ and t-derivatives of A^0, secondly using « tree graph expansions » to express the A-derivatives of Γ in terms of S^{-1}'s and connected Greens functions and the t-derivatives of A^0 in terms of these and $\dot{C}^{-1}$'s, and finally using the factorization properties of S and S^{-1} to obtain expressions in which the applied $\dot{C}^{-1}$'s could be seen to « kill » the « live » (i. e. A-dependent) part on at least one side of the hyperplane σ. In fact, in the present situation, for M of the form M_N, the decomposition property (3.17) and derivative formulae of Theorem III.16, are so closely analogous to the corresponding results of IV that we shall not follow through all of the argument in detail. The only new point on which we might need to elaborate is the control of σ factors. For $N \leq 3$ this is of little significance as the terms arising in our expansion of $\partial_t^r \Gamma$ are all controlled individually and an overall σ factor does not affect connectedness. But for $N = 4$ our argument involved a partial cancellation between $(\ddot{A}^0)^2 \Gamma_{AA}$ and $(\dot{A}^0)^4 \Gamma_{AAAA}$. It is therefore important to keep track of the σ factors that are associated with the various terms in our « tree-graph » expansions.

We first recall the previously stated results for $n = 1$ and 2, namely (3.24)

$$
{}_{A_i}\Gamma^M \{ A \} = - \sum_{M \ni k \geq i} \binom{k}{i} I_i \odot F_{k-i} \{ \underline{A}^0 + \underline{A} \} J_k^M \{ \underline{A} \} \tag{A.1}
$$

and (3.32′)

$$
{}_{A^R A^R}\Gamma^M = \in^M_{n+m} \binom{m+n}{n} {}_{A^R}\Gamma^M - [R(.S^M_{.} \circ J^M)^{-1} R] \tag{A.2}
$$

Note that we prefer to work with the σ factors of (3.32) suppressed by an appropriate rearrangement of arguments as in (3.32′). Also we shall augment the notation of (3.32′) by defining (in the obvious way) operators ..S with matrix elements

$$
{}_{nm}S = [{}_{f^n g^m} \mathscr{S} \restriction_{f=g=0}, \text{ etc.}
$$

To compute ${}_{A^R A^R A^R}\Gamma^M$ we start by iterating (A.2) to get

$$
\begin{aligned}
{}_{A_i^R A_j^R A_k^R}\Gamma^M &= {}_{A_i^R}\left({}_{A_j^R A_k^R}\Gamma^M\right) = {}_{A_i^R}\left\{ \binom{j+k}{j} {}_{A^R_{j+k}}\Gamma^M - {}_j[RS^M \{ J^M \}^{-1} R]_k \right\} \\
&= \binom{j+k}{j} \left\{ \binom{i+j+k}{i} {}_{A^R_{i+j+k}}\Gamma^M - {}_i[RS^M \{ J^M \}^{-1} R]_{j+k} \right\} - {}_{A_i^R} {}_j[RS^M \{ J^M \}^{-1} R]_k \\
&= \binom{i+j+k}{i,\ j,\ k} {}_{A^R_{i+j+k}}\Gamma^M - {}_i[RS^M \{ J^M \}^{-1} R]_{j+k} \binom{j+k}{j,\ k} - {}_{A_i^R} {}_j[RS^M \{ J^M \}^{-1} R]_k
\end{aligned} \tag{A.3}
$$

Next, motivated by the « tree-graph » expansions of III we extract « external » S^{-1} factors by means of the following two lemmas (analogous to Theorem III.7 *a*), and Lemma III.6 in III).

LEMMA A.2. — If M is gapless, then

$$ {}_i[\mathrm{R.S}^{\mathrm{M}^{-1}}_{\cdot}\mathrm{R}]_{j_1+\dots+j_n}\binom{j_1+\dots+j_n}{j_1,\ \dots,\ j_n} = -\ {}_{i,j_1,\dots,j_n}([\mathrm{R.S}^{\mathrm{M}^{-1}}_{\cdot}]^{\circledast(n+1)}\dots \mathrm{U}) \qquad (\mathrm{A}.4\,a) $$

with

$$ {}_{i',j'_1,\dots,j'_n}\mathrm{U} = -\sum_{\substack{i'_1\dots i'_n\\ i'=i'_1+\dots+i'_n}} \binom{i'}{i'_1,\dots,i'_n}\sigma^{\sum_{l=1}^{n-1}(\sum_{m<l} j'_m)}\ {}_{i'_1j'_1}\mathrm{S}\dots {}_{i'_nj'_n}\mathrm{S} \qquad (\mathrm{A}.4\,b) $$

Remarks 1. — Here the left hand side is interpreted as the kernel of the operator as in (3.32), and as usual we include arguments in subscripts with $j = j_1 + \dots + j_n$ representing the j-tuple consisting of j_1 up to j_n listed in the order written in the sum.

2. The « tensor product » $\circledast$ on the right is the fps operator tensor product introduced in (3.37).

3. The sign factor in each term in U is exactly that required to reorder the arguments from their positions as occurring on the right side of (A.4 *b*) to the ordering on the left side.

4. Aside from the R's in (A.4 *a*) and σ's in (A.4 *b*) and the distinction between left and right derivatives in the definitions of .S. and ..S, this Lemma is of exactly the same form as the result used to prove Theorem III.7 *a*) of III.

Proof. — Following our convention of regarding the variables $\vec{x}_m$ as included in a subscript m, and in particular considering variables corresponding to a sum as occurring in the order in which the sum is written (so that $f_{m+n}=f_{m+n}(\vec{x}_m, \vec{y}_n)$) we have

$$ \begin{aligned} {}_i[.\mathrm{S}^{\mathrm{M}^{-1}}_{\cdot}]_{j_n+\dots+j_1} &= {}_i[.\mathrm{S}^{\mathrm{M}^{-1}}_{\cdot}]_{j_1+\dots+j_n}\sigma^{[\sum_{l=1}^{n} j_l \sum_{m<l} j_m]} \\ &= {}_i[.\mathrm{S}^{\mathrm{M}^{-1}}_{\cdot}]_{i'_1+\dots+i'_n}\delta_{i'_1 j_1}\dots\delta_{i'_n j_n}\sigma^{[\sum_{l=1}^{n} i'_l \sum_{m<l} j_m]} \\ &= {}_i[.\mathrm{S}^{\mathrm{M}^{-1}}_{\cdot}]_{i'_1+\dots+i'_n}({}_{i'_1}\mathrm{S}_{j'_1}\,{}_{j'_1}[.\mathrm{S}^{\mathrm{M}^{-1}}_{\cdot}]_{j_1})\dots({}_{i'_n}\mathrm{S}_{j'_n}[.\mathrm{S}^{\mathrm{M}^{-1}}_{\cdot}]_{j_n})\sigma^{[\sum_{l=1}^{n} i'_l \sum_{m>l} j_m]} \\ &= {}_i[.\mathrm{S}^{\mathrm{M}^{-1}}_{\cdot}]_{i'j_1}[.\mathrm{S}^{\mathrm{M}^{-1}}_{\cdot}]_{j'_1\dots j_n}[.\mathrm{S}^{\mathrm{M}^{-1}}_{\cdot}]_{j'_n}\mathrm{P}_{i'}\sum_{\Sigma i'_l=i'}\delta_{i',i'_1+\dots+i'_n i'_1 j'_1}\mathrm{S}\dots {}_{i'_n j'_n}\mathrm{S} \\ &\quad \sigma^{[\sum_{l=1}^{n} i'_l(\sum_{m<l} j_m)+\sum_l j_l j'_l+\sum_l (j_l+j'_l)(\sum_{m\le l}(i'_m+j'_m))+\sum_l j'_l]} \\ &= {}_{i+j_1+\dots+j_n}([.\mathrm{S}^{\mathrm{M}^{-1}}_{\cdot}]^{(n+1)})_{i'+j'_1+\dots+j'_n}\mathrm{P}_{i'}\sum_{\Sigma i'_l=i'}\delta_{i',i'_1+\dots i'_n i'_1 j'_1}\mathrm{S}\dots {}_{i'_n j'_n}\mathrm{S} \\ &\quad \sigma^{[\sum_{l=1}^{n} i'_l(\sum_{m<l} j_m)+\sum_l j_l j'_l+\sum_l (j_l+j'_l)(\sum_{m\leq l}(i'_m+j'_m))+\sum_l j'_l+i'\sum_l (j_l+j'_l)+\sum_l j'_l(\sum_{m>l}(j_m+j'_m))]} \end{aligned} $$

Here in the last step we have used the fact that

$$ \ker_{j_0+\dots+j_n}[.\mathrm{S}^{\mathrm{M}^{-1}}_{\cdot}]^{\circledast(n+1)}_{j'_0+\dots+j_n} = \prod_{l=0}^{n} {}_{j_l}[.\mathrm{S}^{\mathrm{M}^{-1}}_{\cdot}]_{j'_l}\sigma^{\sum_l j'_l\sum_{m<l}(j_m+j'_m)} $$

But now, since

$$\sum_{l=1}^{n} i'_l\Big(\sum_{m<l} j_m\Big) + \sum_l j_l j'_l + \sum_l (j_l + j'_l)\Big(\sum_{m\leq l}(i'_m + j'_m)\Big) + \sum_l j'_l + i'\sum_l (j_l + j'_l) + \sum_l j'_l \sum_{m>l}(j_m + j'_m) \equiv \sum_{l=1}^{n} i'_l\Big(\sum_{m<l} j'_m\Big) \text{ (mod evens)}$$

we find that

$${}_i[.\mathrm{S}^{\mathrm{M}-1}_{\cdot}]_{j_n+\ldots+j_1} = {}_{i+j_1+\ldots+j_n}([.\mathrm{S}^{\mathrm{M}-1}_{\cdot}]^{\circledast(n+1)}[\ldots\ldots - u])$$

with

$${}_{i',j'_1,\ldots,j'_n} - u \equiv \mathrm{P}_{i'} \sum_{\sum_{l=1}^{n} i'_l = i'} \delta_{i',i'_1+\ldots+i'_n}\, {}_{i'_1 j'_1}\mathrm{S} \ldots {}_{i'_n j'_n}\mathrm{S}\,\sigma^{\Sigma i'_l(\sum_{m<l} j'_m)}$$

But for each term in this sum, the j' integrals yield factors of $\delta_{i'_l j'_l}$. So for any function $f(j_1, \ldots, j_n)$ we have

$${}_i[.\mathrm{S}^{\mathrm{M}-1}_{\cdot}]_{j_n+\ldots j_1} f(j_1, \ldots, j_n) = [.\mathrm{S}^{\mathrm{M}-1}_{\cdot}]^{\circledast(n+1)} \sum_{\sum_{l=1}^{n} i'_l = i} f(i'_1, \ldots, i'_n)\delta_{i',i'_1+\ldots+i'_n}\, {}_{i'_1 j'_1}\mathrm{S} \ldots {}_{i'_n j'_n}\mathrm{S}\,\sigma^{\Sigma i'_l(\sum_{m<l} j'_m)}$$

In particular, with $f(j_1, \ldots, j_n) = \binom{j_1 + \ldots + j_n}{j_1, \ \ldots, \ j_n}$ we get

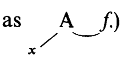
$${}_i[.\mathrm{S}^{\mathrm{M}-1}_{\cdot}]_{j_n+\ldots+j_1}\binom{j_1 + \ldots + j_n}{j_1, \ \ldots, \ j_n} = {}_{i,j_1,\ldots,j_n}([.\mathrm{S}^{\mathrm{M}-1}_{\cdot}]^{\circledast(n+1)}[\ldots\ldots - \mathrm{U}])$$

with

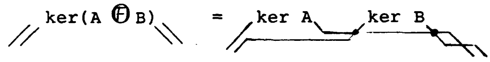
$${}_{i'_1,j'_1,\ldots,j'_n} - \mathrm{U} \equiv \mathrm{P}_{i'} \sum_{\sum_{l=1}^{n} i'_l = i'} \binom{i'}{i'_1, \ldots, i'_n}\delta_{i',i'_1+\ldots+i'_n}\Big(\prod_{l=1}^{n} {}_{i'_l j'_l}\mathrm{S}\Big)\sigma^{[\Sigma i'_l(\sum_{m<l} j'_m)]}$$

Since $\mathrm{R}(j_n + \ldots + j_1) = (\mathrm{R}_{j_1} + \ldots + \mathrm{R}_{j_n})$, the statement of the Lemma now follows immediately. ■

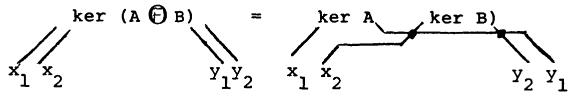
At this point we find it useful to introduce some graphical notation for our various functions and integrals. In particular, we follow the usual procedure of representing the variables associated to a function by lines (or « legs ») emanating from a location (or « vertex ») that is labelled by a symbol representing the function itself. For $m \in \mathcal{M}$, lines corresponding to variables $\vec{x}_m$ will terminate at a label m in the name of the function, and if there is no such label then it is considered to be part of the variable. In that case m will run over $\mathcal{M}$ for a smooth line and over $\overline{\mathcal{M}}$ for a wiggly line. (The legs associated to a function's variables will generally project downwards from the function name on either the left or right but will do so consistently for a given function. The choice of sides has not been formalized and is made according to the taste of the authors. In particular, for example, for a (left or right) functional derivative of a fps, the corresponding variables will (usually) be on the same side of the fps name, and kernels of integral operators will often be split with « incoming » variables on the right and « outgoing » on the left.) As usual, sums and integrals of products will be indicated by joining legs that correspond to the same integration variable and removing the label. (So, for example

$$[\mathrm{AF}](x) = \int \mathrm{A}(x, y) f(y) dy \text{ may be written as}$$ 

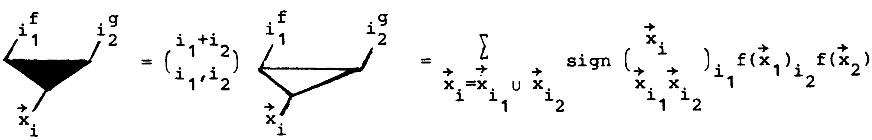
Because of the anticommuting fermi calculus we do not feel free to arbitrarily rearrange the positions of vertices in a graph, and in fact will always arrange the vertices in a line in the order in which the product is to be taken and will terminate the unintegrated (« external ») lines in a row below in the order in which we wish them to be considered as arguments of the product. This process may involve crossing of lines. The presence of a dot on such a crossing of lines representing variables $\vec{x}_i$ and $\vec{y}_j$ will serve to indicate a factor of σ^{ij}. So, for example, for the ⓕ tensor product we have



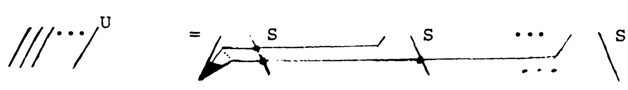
(i. e.



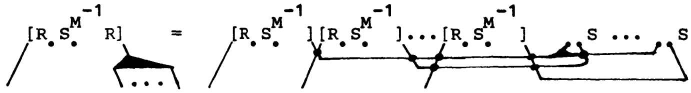
Sometimes (as in the above Lemma) a product is to be thought of as a function of which a single argument $\vec{x}_i$ corresponds to several factor arguments $\vec{x}_{i_l}$. This will be indicated by joining the corresponding lines in the obvious way (with the order of components $\vec{x}_{1_l}$ in $\vec{x}_i$ being that in which the corresponding lines come together). Appropriate symmetrisation of such a product will be indicated by joining the lines via a hollow blob, and the occurrence of a cominatoric factor $\binom{i_1 + \ldots + i_n}{i_1, \ \ldots, \ i_n}$ in such a symmetric product will be indicated by filling in the blob. So, for example



where sign $\begin{pmatrix} & \vec{x}_i & \\ \vec{x}_{i_1} & & \vec{x}_{i_2} \end{pmatrix}$ is the sign of the permutation which puts the odd arguments of $\vec{x}_{i_1}$ ahead of those of $\vec{x}_{i_2}$ but otherwise respects the order of $\vec{x}_i$. As another example, the definition (A.4 b) of U becomes

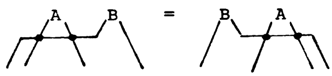


and the statement of Lemma A.2 then reads

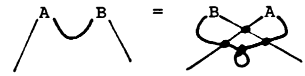


(A.4′)

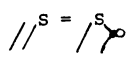
If .A. and .B. are function valued fps with « appropriate parity » (i. e. same fps parity as total numerical parity of all fermi arguments) then


(A.5 *a*)

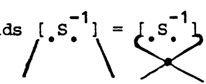
and


(A.5 *b*)

Also the particular symmetry properties of S and S^{-1} are conveniently represented in this notation. For example,

$${}_{nm}S = \sigma^{m}\ {}_{n}S_{m} \quad \text{reads}$$

(A.6 *a*)

and

$${}_{n}[.S.^{-1}]_{m} = \sigma^{mn}\ {}_{m}[.S.^{-1}]_{n} \quad \text{reads}$$

(A.6 *b*)

These ingredients (together with $S^{M^{-1}}SP^{M} = P^{M}$) provide a simple graphical version of the proof of Lemma A.2.

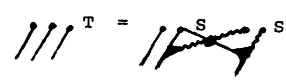
LEMMA A.3. — If M is gapless, and $i \in M$, then

a)
$$A^{R}_{i}({}_{jk}S\{J^{M}\}) = {}_{i}[R.S.^{M^{-1}}\{J^{M}\}]_{i'}\ {}_{i'jk}T\{J^{M}\} \tag{A.7 a}$$

and

b)
$$A^{R}[R.S.^{M^{-1}}\{J^{M}\}R] = -\ [R.S.^{M^{-1}}\{J^{M}\}]^{\circledR 3}\dots T\{J^{M}\} \tag{A.7 b}$$

where, in the graphical notation just introduced,


(A.7 *c*)

Remark. — These results are also true with the R's omitted (since the R's occur in exactly the same locations on the two sides). In fact they play no role and we shall prove the results without them. (We keep them in the statement of the lemma because this is the form that is most compatible with our factorization results for S^{-1}).

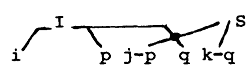
Proof of Lemma. — Starting with the case $M = \mathcal{M}$ we compute

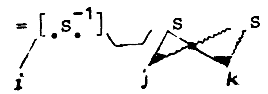
$$\begin{aligned}
A_i({}_{jk}S \circ J\{A\}) &= \frac{\delta}{\delta A_i}\left(\frac{\delta}{\delta f}\right)^{j}\left(\frac{\delta}{\delta g}\right)^{k} \mathcal{S}\,|_{f=g=0} \\
&= \sigma^{i(j+k)}\left[\left(\frac{\delta}{\delta f}\right)^{j}\left(\frac{\delta}{\delta f}\right)^{k}\left(\left[\frac{\delta}{\delta A_i}\sum_{p,q\in\mathcal{M}} (A^{0}+A)_{p+q}R\,\frac{f^{p}}{p\,!}\frac{g^{q}}{q\,!}\right]\right)\mathcal{S}\right)\Bigg|_{f=g=0} \\
&= \sigma^{i(j+k)}\left[\left(\frac{\delta}{\delta f}\right)^{j}\left(\frac{\delta}{\delta g}\right)^{k}\left(\sum_{\substack{p,q\in\mathcal{M}\\ q+p=i}} P_iR\,\frac{f^{p}}{p\,!}\frac{g^{q}}{q\,!}\,\mathcal{S}\right)\right|_{f=g=0}
\end{aligned}$$

$$= \sigma^{i(j+k)} \sum_{\substack{p,q \in \mathcal{M} \\ q+p=i \\ p \leq j\ q \leq k}} \mathrm{P}_i \binom{j}{p} \mathrm{P}_j \binom{k}{p} \mathrm{P}_k \sigma^{kp} \delta_{j,j_1 p} \delta_{k,k_1 q} \left[\left(\frac{\delta}{\delta f} \right)^{j_1} \left(\frac{\delta}{\delta g} \right)^{j_2} \mathscr{S} \right|_{f=g=0}$$

$$= \sum_{\substack{p,q \in \mathcal{M} \\ q+p=i \\ p \leq j,\ q \leq k}} \mathrm{P}_i \binom{j}{p} \mathrm{P}_j \binom{k}{q} \mathrm{P}_{ki} \mathrm{I}_{q+p\ (j-p)(k-q)} \mathrm{S} \sigma^{i(j+k)+kp}$$

$$= \sum_{\substack{p,q \in \mathcal{M} \\ q+p=i \\ p \leq j,\ q \leq k}} \mathrm{P}_i \binom{j}{p} \mathrm{P}_j \binom{k}{q} \mathrm{P}_k \sigma^i$$





(since by (A.6 a), $\sigma_i^i \mathrm{I}_{p+q} = {}_i[.\mathrm{S}.^{-1}]_l\, {}_l\mathrm{S}_{p+q} \sigma^{p+q} = {}_i[.\mathrm{S}.^{-1}]_{l\ l\ p+q}\mathrm{S}$).

Reversing the $\vec{x}_i$ argument on both sides then gives (a) for $\mathrm{M} = \mathcal{M}$.

Now we turn to the cases with $\mathrm{M} \neq \mathcal{M}$ where instead of $\mathrm{S} \circ \mathrm{J}$ we consider $\mathrm{S} \circ \mathrm{J}^{\mathrm{M}}$ (and to avoid confusion denote the argument by A^{M}). By the chain rule (Theorem II.8) we have

$$\underset{\vec{x}}{\mathrm{A}^{\mathrm{M}}} \Big(\underset{\vec{y}\ \vec{z}}{..} \mathrm{S} \circ \mathrm{J}^{\mathrm{M}} \{ \mathrm{A}^{\mathrm{M}} \} \Big) = \underset{\vec{x}}{\mathrm{A}^{\mathrm{M}}} \Big(\underset{\vec{y}\ \vec{z}}{..} (\mathrm{S} \circ \mathrm{J}) \{ (\mathrm{A} \circ \mathrm{J}^{\mathrm{M}}) \{ \mathrm{A}^{\mathrm{M}} \} \} \Big)$$

$$= \underset{\vec{x}\quad \vec{x}'}{\mathrm{A}^{\mathrm{M}}} (\mathrm{A} \circ \mathrm{J}^{\mathrm{M}} \{ \mathrm{A}^{\mathrm{M}} \}) \Big[\Big(\underset{\vec{x}'}{\mathrm{A}} \underset{\vec{y}\ \vec{z}}{..} \mathrm{S} \circ \mathrm{J}) \{ (\mathrm{A} \circ \mathrm{J}^{\mathrm{M}}) \{ \mathrm{A}^{\mathrm{M}} \} \} \Big]$$

$$= \underset{\vec{x}}{\mathrm{A}^{\mathrm{M}}} (\underset{\vec{x}'}{\mathrm{A}} \circ \mathrm{J}^{\mathrm{M}} \{ \mathrm{A}^{\mathrm{M}} \}) (\underset{\vec{x}'\ \vec{x}''}{\mathrm{S}^{-1}} \{ \mathrm{J} \} \underset{(\vec{x}'', \vec{y}, \vec{z})}{\mathrm{T}^{(3)}} \{ \mathrm{J} \}) \circ (\mathrm{A} \circ \mathrm{J}^{\mathrm{M}}) \{ \mathrm{A}^{\mathrm{M}} \}$$

$$= [(\underset{\vec{x}}{\mathrm{A}^{\mathrm{M}}} \underset{\vec{z}}{\mathrm{J}^{\mathrm{M}}}) [(\underset{\vec{z}}{{}_{\mathrm{J}}\mathrm{A}})_{\vec{x}} \circ \mathrm{J}^{\mathrm{M}}] (\underset{x'\ x''}{\mathrm{S}^{-1}} \circ \mathrm{J}^{\mathrm{M}}) (\underset{\vec{x}''\vec{y}\vec{z}}{\mathrm{T}^{(3)}} \circ \mathrm{J}^{\mathrm{M}})] \{ \mathrm{A}^{\mathrm{M}} \}$$

But now, by Corollary III.10, ${}_{\mathrm{J}}\mathrm{A} = \rangle\!\langle\mathrm{S}$, and if M is gapless, then

$$\mathrm{P}^{\mathrm{M}}_{\mathrm{A}^{\mathrm{M}}} \mathrm{J}^{\mathrm{M}} = (\mathrm{S}^{\mathrm{M}} \{ \mathrm{J}^{\mathrm{M}} \})^{-1} (\rangle\!\langle^{\mathrm{M}} \{ \mathrm{J}^{\mathrm{M}} \})^{-1}.$$

Also $\mathrm{P}^{\mathrm{M}} \rangle\!\langle = \mathrm{P}^{\mathrm{M}} \rangle\!\langle \mathrm{P}^{\mathrm{M}}$ so $(\rangle\!\langle^{\mathrm{M}})^{-1}(\rangle\!\langle) = \mathrm{P}^{\mathrm{M}}$. Thus if $i \in \mathrm{M}$, we have

$${}_{\mathrm{A}^{\mathrm{M}}}\mathrm{J}^{\mathrm{M}} [({}_{\mathrm{J}}\mathrm{A}) \circ \mathrm{J}^{\mathrm{M}}] \mathrm{S}^{-1} \circ \mathrm{J}^{\mathrm{M}} = [(\mathrm{S}^{\mathrm{M}})^{-1} (\rangle\!\langle^{\mathrm{M}})^{-1} \rangle\!\langle \mathrm{S}\mathrm{S}^{-1}] \circ \mathrm{J}^{\mathrm{M}} = (\mathrm{S}^{\mathrm{M}})^{-1} \circ \mathrm{J}^{\mathrm{M}}.$$

Since $.\mathrm{S}.^{-1} = \mathrm{S}^{-1}\underline{\sigma}$, this gives

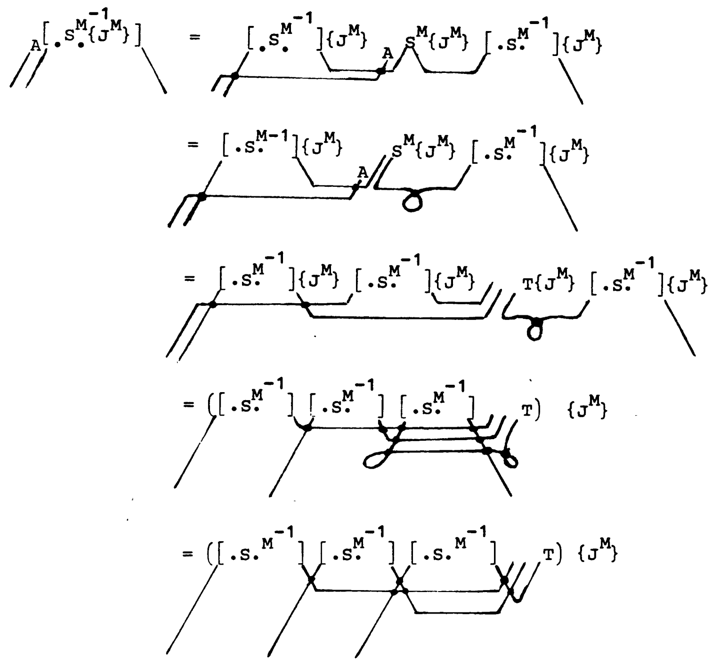
$${}_{\mathrm{A}^{\mathrm{M}}}(..\mathrm{S} \circ \mathrm{J}^{\mathrm{M}} \{ \mathrm{A}^{\mathrm{M}} \}) = [(\mathrm{S}^{\mathrm{M}^{-1}} \circ \mathrm{J}^{\mathrm{M}}) \underline{\sigma}] (\mathrm{T} \circ \mathrm{J}^{\mathrm{M}}) \{ \mathrm{A}^{\mathrm{M}} \} = (.\mathrm{S}.^{\mathrm{M}^{-1}} \circ \mathrm{J}^{\mathrm{M}}) (\mathrm{T} \circ \mathrm{J}^{\mathrm{M}}) \{ \mathrm{A}^{\mathrm{M}} \},$$

as required.

For part (b) we start by using (2.28) to get

$${}_{\mathrm{A}_i}[.\mathrm{S}.^{\mathrm{M}-1}] \{ \mathrm{J}^{\mathrm{M}} \}]_k = - {}_j[.\mathrm{S}.^{\mathrm{M}^{-1}}]_{j'}\ {}_{\mathrm{A}_i}({}_j\mathrm{S}_{k'} \{ \mathrm{J}^{\mathrm{M}} \})_{k'} [.\mathrm{S}.^{\mathrm{M}^{-1}}]_k \sigma^{i(j+j')}.$$

Using the graphical notation we have introduced, this gives



(by (A.6 *a*))

(by (A.7 *a*))

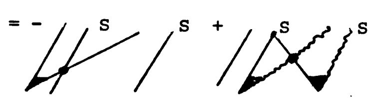
(by (A.5 *b*))

(by (A.6 *b*)) ■

By applying these two lemmas to the last two terms in (A.3), we find that

$$A^R_i A^R_j A^R_k \Gamma^M = \begin{pmatrix} i+j+k \\ i,\ j,\ k \end{pmatrix}_{A^R_{i+j+k}} \Gamma^M + {}_{i,j,k}\{\, [.S^M_. \{ J^M \}^{-1}]^{\circledast 3} \ldots V^{(3)} \} \qquad (A.8\,a)$$

where $\ldots V^{(3)} = \ldots U + \ldots T$



(A.8 *b*)

and so, by the comments preceeding (3.34), is a sum of products of G's with each factor G_n connecting more than one cluster of variables. (The symmetry properties of $V^{(3)}$ that are obvious from (A.8 *a*) are less apparent in the explicit formula (A.8 *b*) but can be restored to view either by expanding each S as a product of G's, or by defining generating functionals $\mathcal{U}$ and $\mathcal{T}$ for U's and T's with $\mathcal{U} + \mathcal{T}$ appropriately symmetric in its arguments as we did in III. and IV for the boson case.)

To illustrate (A.8 *b*) we consider the case of ${}_{ijk}V^{(3)}$ with $|i| = |j| = |k| = 2$. Writing $l = (l_1, l_2)$ for $l = i, j, k$ we have

$$\begin{aligned} {}_{ijk}V^{(3)} = & - [({}_{i_1,j}S)({}_{i_2,k}S)\sigma^{i_2 j} + ({}_{i_2,j}S)({}_{i_1,k}S)\sigma^{i_1 j + i_1 i_2}] \\ & + ({}_{i,j_1k_1}S)({}_{j_2,k_2}S)\sigma^{k_1 j_2} + ({}_{i,j_2k_1}S)({}_{j_1,k_2}S)\sigma^{k_1 j_1 + j_1 j_2} \\ & + ({}_{i,j_1k_2}S)({}_{j_2,k_1}S)\sigma^{j_2k_2 + k_1k_2} + ({}_{i,j_2k_2}S)({}_{j_1,k_1}S)\sigma^{j_1 k_2} \\ & + {}_{i,jk}S \end{aligned}$$

At 0, since $P_0SP_\pm = 0$ and $P_+SP_- = 0$ we have

$$_{i_+i_-,j_+j_-,k_+k_-}V^{(3)} = {}_{i_+i_-j_+j_-,k_+k_-}S\,.$$

If we now apply (3.29) we see that, at 0,

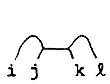
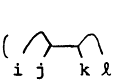
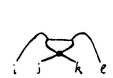
$$_{i_+i_-,j_+j_-,k_+k_-}V^{(3)} = {}_{i_+j_+k_+}G\,{}_{i_-j_-k_-}G\,\sigma^{l-(j_- - k_-)}\,. \tag{A.9}$$

In fact (A.2) and Lemmas A.2 and A.3 provide us with a machine with which to express arbitrarily high derivatives of Γ in terms of first derivatives of Γ and integrals (of tree graph type) involving factors of $[.S^M_{\cdot}]^{-1}$ (for the legs) and sums of products of G's (for the vertices).

For example

$$\begin{aligned} A^R_i A^R_j A^R_k A^R_l \Gamma^M &= \binom{i+j+k+l}{i,\ j,\ k,\ l}_{A^R_{i+j+k+l}} \Gamma^M \\ &+ {}_{ijkl}[R.S^{M^{-1}}_{\cdot}]^{\circledast 4} \ldots.\ V^{(4)} \\ &- ({}_{ij}[R.S^{M^{-1}}_{\cdot}]^{\circledast 2} \ldots\ V^{(3)}_{\cdot}[.S^{M^{-1}}_{\cdot}].V^{(3)}_{\cdot\cdot}[.S^{M^{-1}}_{\cdot}R]^{\circledast 2}_{kl} + \text{perms.}) \end{aligned} \tag{A.10}$$

where

(i) ( + perms.) means ( +  + )

(i. e. the sum over « topologically distinct » assignments of variables to external legs with appropriate σ factors included).

ii) $..V^{(3)}_{\cdot}$ and $.V^{(3)}_{\cdot\cdot}$ are defined in the same way as $\ldots V^{(3)}$ but with appropriately modified S factors (i. e. variables on left or right of $V^{(3)}$ corresponding to left or right derivatives on $\mathscr{S}$), and

iii) $\ldots.\ V^{(4)}$ is a polynomial in the G's with each G factor in a term connecting at least two clusters.

This result follows quite easily by differentiating the right hand side of (A.8 *a*) (using (A.2) and Lemma A.2 on the first term, Lemma A.3 *b* on the S^{-1} factors in the second term, and Lemma A.3 *a* with (A.8 *b*) to compute ${}_AV^{(3)}$), and then collecting terms in the manner indicated.

With (A.2), (A.8) and (A.10) in hand, the proof of Theorem A.1 is, as remarked earlier, so similar to the pure boson case of IV that we omit the details.

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