ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 43, nº 1 (1985), p. 1-27
<http://www.numdam.org/item?id=AIHPA_1985__43_1_1_0>

Ann. Inst. Henri Poincaré,
Vol. 43, n° 1, 1985, p. 1-27. *Physique théorique*

Star algebra and Green functions of nonlinear differential equations

by

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ABSTRACT. — An algebra generated by abstract diagrams is defined and studied. A homomorphism of this algebra into that of retarded functions allows to give an algebraic description of the solutions of nonlinear differential equations. As a result, explicit expressions of the Green functions of these equations can be derived.

RÉSUMÉ. — On définit et on étudie une algèbre engendrée par des diagrammes abstraits. Un homomorphisme de cette algèbre dans celle des fonctions retardées permet de donner une description algébrique des solutions d'équations différentielles non linéaires. On peut ainsi obtenir des expressions explicites pour les fonctions de Green de ces équations.

INTRODUCTION

The aim of this paper is to develop an algebraic formalism suited to the explicit description of the Green functions associated with nonlinear differential equations. In preceding papers, devoted to first-order equations with an arbitrary source, we showed that the Green functions could be expressed as linear combinations of products of some standard kernels [*1*]-[*3*]. These kernels were described by diagrams that we called *stars*, and their products by diagrams (different from Feynman ones) called *star diagrams.*

More particularly, the last paper emphasized the interest to adopt an algebraic point of view [*3*]. In fact the relevant algebraic operations can be formulated in an abstract framework which goes beyond the original problem, and thus admits a wider scope of applications.

The systematic study of this general framework constitutes the first part of this paper. In Paragraph I.1, we introduce the notion of abstract star diagram and of star algebra. For the applications we have in view, the main point is that the star algebra possesses properties generalizing those of the algebra of the formal power series. As exposed in Paragraph I.2, they result from the existence of two fundamental operators: a derivation ∂ and a linear operator M satisfying the commutation relation $[\partial, \mathrm{M}] = 1$. In particular, we define a right inverse of ∂ that generalizes the primitive; in the star algebra, the elements of Ker ∂ act as constants, and the Taylor formula holds. In Paragraph I.3, we define a homomorphism associating retarded functions of a real variable t with the diagrams, so that the operator ∂ corresponds to the derivative $\frac{d}{dt}$. This correspondence is basic for the treatment of the differential equations worked out in the second part. Before entering upon that treatment, we complete the abstract part by constructing operators allowing to replace, in the subsequent applications, the general diagrams by tree diagrams. These constructions are all obtained in Paragraph I.4 with the help of a projection operator on the tree diagram algebra.

The applications developed in the second part include successively first order differential equations (§ II.1), higher order equations (§ II.2), systems (§ II.3) and operator equations (§ II.4). The general principle consists in associating equations in the star algebra with given differential equations with a source. Retarded solutions for the latter then correspond to solutions of the former with convenient initial conditions. The iterative expansion of the solutions in the star algebra furnishes the Green functions for the retarded solutions in explicit form, thus generalizing the previous results mentioned at the beginning of this introduction.

Finally, the solution constructed in a previous paper by using diagrams with dressed centres [*2*] is compared with the present one in an Appendix. It appears that the star structure cannot be derived in a simple way by classical treatments of the equations.

I. STAR ALGEBRA

1. Basic definitions.

Let us call *star diagram* a schema formed by a finite set of points and crosses, together with a set of lines, each of them connecting one point to one cross, in such a way that two different lines have not the same two extremities [4]. In such a diagram any subdiagram constituted by one point and the set (eventually empty) of the crosses connected to it will be called a *star* (a *v-star* if the number of crosses in v), and the point will be called the *centre* of the star. Let $\mathscr{A}$ be the free commutative algebra with unity over the field of complex numbers, generated by the set of the connected diagrams. In $\mathscr{A}$ the product of generators $A_1 A_2 \ldots A_k$ is identified with the diagram whose connected components are $A_1, A_2, \ldots, A_k$. We denote by $\mathscr{A}_{(n,N)}$ the finite dimensional subspace of $\mathscr{A}$ spanned by the diagrams having n crosses and N points. Clearly $\mathscr{A}$ is the direct sum $\mathscr{A} = \bigoplus_{(n,N)} \mathscr{A}_{(n,N)}$, and is a bigraded algebra; in what follows, we shall need to consider also the direct product $\overline{\mathscr{A}} = \prod_{(n,N)} \mathscr{A}_{(n,N)}$ [5]. While $\mathscr{A}$ is the set of finite linear combinations of diagrams, $\overline{\mathscr{A}}$ is the set of non necessarily finite ones. This latter is also an algebra.

In $\overline{\mathscr{A}}$ we define the derivation ∂ as follows: for any diagram D the derivative ∂D is the sum of all possible diagrams obtained by removing from D one centre of star. It is obvious that ∂ maps $\mathscr{A}_{(n,N)}$ in $\mathscr{A}_{(n,N-1)}$. This mapping is in fact surjective that we prove by constructing a right inverse of ∂. For that purpose, we introduce the linear operator M such that, for any diagram D, the transformed M(D) is the diagram obtained by adding to D a point connected to all the crosses of D. We easily verify the commutation relation

$$[\partial, M] = 1 . \tag{I-1}$$

Then we define the operator

$$\int = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k\,!} M^k \partial^{k-1} \tag{I-2}$$

This expression makes sense on any element A having a given bidegree (n, N), for then $\partial^{N+1}A = 0$. Since $\int$ maps $\mathscr{A}_{(n,N)}$ in $\mathscr{A}_{(n,N+1)}$, it is therefore defined everywhere in $\overline{\mathscr{A}}$. By using the commutation relation, we verify that

$$\partial \int = 1 \tag{I-3}$$

so that ∂ is surjective and $\int$ injective.

2. Consequences of the commutation relation $[\partial, M] = 1$.

The operator $\int$ above introduced is a right inverse of ∂. Let us consider then the operator

(I-4) $$P = \int \partial = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} M^k \partial^k$$

Due to (I-3), P is a projection operator, and satisfies the relations

(I-5) $$\partial P = \partial, \qquad P \int = \int$$

Formulas (I-4) and (I-5) imply the equalities

(I-6) $$\text{Ker } \partial = \text{Ker P}, \qquad \text{Im} \int = \text{Im P}$$

From (I-1), it results that

(I-7) $$\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} M^{k-1} \partial^{k-1} \right) \partial M = 1$$

This implies at first that M is injective. Then, by multiplying (I-7) on the left by M and using (I-4), we obtain

(I-8) $$PM = M$$

and consequently, with the help of (I-4),

(I-9) $$\text{Im M} = \text{Im P}$$

Let us introduce the operator

(I-10) $$\Pi^{(m)} = \frac{1}{m!} M^m (1 - P) \partial^m = \frac{1}{m!} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} M^{k+m} \partial^{k+m}$$

From this expression and the formula

(I-11) $$\partial^p M^q = \sum_{r=0}^{\inf(p,q)} r! \, C_r^p C_r^q M^{q-r} \partial^{p-r}$$

we deduce the relations

(I-12) $$\Pi^{(m)} \Pi^{(m')} = \delta_{mm'} \Pi^{(m')}, \qquad \sum_{m=0}^{\infty} \Pi^{(m)} = 1$$

Thus the $\Pi^{(m)}$'s are projection operators and $\mathscr{A}$ is the direct sum

(I-13) $$\mathscr{A} = \bigoplus_{m=0}^{\infty} \Pi^{(m)}\mathscr{A}$$

The subspace $\Pi^{(m)}\mathscr{A}$ is the eigenspace of the operator $M\partial$ for the eigenvalue m, since we have the formula

(I-14) $$M\partial\Pi^{(m)} = m\Pi^{(m)}$$

The direct sum (I-13) leads to the following decomposition for any element A of $\mathscr{A}$

(I-15) $$A = \sum_{m=0}^{\infty} \Pi^{(m)}A = \sum_{m=0}^{\infty} \frac{1}{m!} M^m(1 - P)\partial^m A$$

where the summation contains a finite number of non vanishing terms. This formula can be called the *Taylor formula*, because it becomes identical to the usual one when $\mathscr{A}$ is replaced by the algebra $\mathbb{C}[x]$ of polynomials in one variable x, and the operator ∂ and M respectively by $\frac{d}{dx}$ and the multiplication by x; in fact in this case, the operator $(1 - P)$ acting on any polynomial f, then gives the value $f(0)$.

All the operators above introduced have a given bidegree, namely $(0, -1)$ for ∂, $(0, 1)$ for M and $\int$, and $(0, 0)$ for P and $\Pi^{(m)}$ [6]. Moreover we have $\Pi^{(m)}\mathscr{A}_{(n,N)} = \{0\}$ for $N < m$. Consequently all the preceding formulas are valid in $\overline{\mathscr{A}}$, and we have $\overline{\mathscr{A}} = \prod_{m=0}^{\infty} \Pi^{(m)}\overline{\mathscr{A}}$.

The preceding notions based on the existence of M such that $[\partial, M] = 1$, can be generalized by replacing this operator by $M - a$, $a \in \mathscr{A} \cap \operatorname{Ker} \partial$. The commutation relation remains valid and, in particular, we may define

(I-16) $$\int_a = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} (M - a)^k \partial^{k-1}$$

When $\mathscr{A}$ is replaced by the algebra $\mathbb{C}[x]$ of polynomials, with

$$a \in \operatorname{Ker}\left(\frac{d}{dx}\right) \equiv \mathbb{C},$$

the operator $\int_a$ represents the primitive vanishing for $x = a$. In $\mathscr{A}$, we also have:

(I-17) $$\Pi_a^{(m)} = \frac{1}{m!}(M - a)^m(1 - P_a)\partial^m,$$

with $P_a = \int_a \partial$, and the resulting Taylor formula « about a »:

$$A = \sum_{m=0}^{\infty} \Pi_a^{(m)} A = \sum_{m=0}^{\infty} \frac{1}{m!} (M - a)^m (1 - P_a) \partial^m A \tag{I-18}$$

The operator $1 - P_a$ plays in $\mathscr{A}$ the role of the operator $f \to f(a)$ in $\mathbb{C}[x]$. By analogy, we may define

$$\int_a^b = \int_a - \int_b \tag{I-19}$$

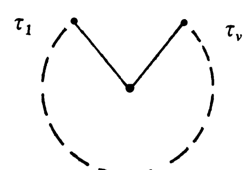
and this symbol has properties analogous to that of the usual definite integral. For example, we have

$$\int_a^b \partial A = (1 - P_b)A - (1 - P_a)A \tag{I-20}$$

3. **Correspondence between diagrams and retarded functions.**

We introduce now a homomorphism of $\overline{\mathscr{A}}$ (or $\mathscr{A}$) in a functional space. Let α and η be two functions of a real variable t, both vanishing for $t < 0$, and continuous for $t \geqslant 0$. To any diagram D we associate the function $\langle D \rangle$, called the *value of* D, defined as follows: let us label the crosses of $D \in \mathscr{A}_{(n,N)}$ by an index i, $1 \leqslant i \leqslant n$, and associate with the cross i a real variable τ_i; to the crosses and the stars constituting D, we assign the following factors:

— the function $\eta(\tau_i)$ to the cross labelled i
— the kernel $[\bar{\alpha}(t) - \bar{\alpha}(\sup(\tau_1, \ldots, \tau_\nu))]$ to the ν-star



with $\bar{\alpha}(t) = \int_0^t d\tau \alpha(\tau)$

— an overall factor $\prod_{i=1}^{n} \theta(t - \tau_i)$.

The value $\langle D \rangle(t)$ is then the integral over the τ_i of the product of all these factors [7]. The correspondence $D \to \langle D \rangle$, completed by $1 \to 1$, induces a homomorphism from $\mathscr{A}$ in the algebra of the functions of t. This homomorphism extends to $\overline{\mathscr{A}}$ provided that the value of an element of $\overline{\mathscr{A}}$ be considered as a formal functional series with respect to α and η.

It is easy to see that, except for $D = 1$, the value of a diagram D is a retarded function, i. e. is a continuous function of t, vanishing for negative t, and derivable for $t \neq 0$.

Let us now call *regular* a diagram not having any connected component reduced to a single cross. The regular diagrams span the subalgebra $\mathscr{A}^R = \bigoplus_{(n,N)} \mathscr{A}^R_{(n,N)}$ of $\mathscr{A}$ generated by all the connected diagrams except the single cross. As previously we introduce $\overline{\mathscr{A}}^R = \prod_{n,N)} \mathscr{A}^R_{(n,N)}$.

For any regular diagram D we have the formula

$$\text{(I-21)} \qquad \frac{d}{dt}(\langle D \rangle (t)) = \alpha(t) \langle \partial D \rangle (t)$$

that we establish now: since every cross of D belongs to a star, the derivatives with respect to the upper bounds of the integrals in $\langle D \rangle$ (coming from the factor $\left.\prod_i \theta(t - \tau_i)\right)$ vanish because of the form of the integrand; on the other hand, the derivative of any factor $[\bar{\alpha}(t) - \bar{\alpha}(\sup(\ldots))]$ associated with a star, giving simply $\alpha(t)$, corresponds to the term of ∂D obtained by suppressing this star in D.

In particular, $\int D$ is regular, $\forall D$, so that (I-21) gives:

$$\text{(I-22)} \qquad \frac{d}{dt}\left\langle \int D \right\rangle = \alpha \langle D \rangle, \quad \forall D$$

or equivalently

$$\text{(I-23)} \qquad \left\langle \int D \right\rangle (t) = \int_0^t d\tau \alpha(\tau) \langle D \rangle (\tau), \quad \forall D .$$

These relations are the key of the treatment of the differential equations presented in Section II.

We shall say that two elements A and $B \in \overline{\mathscr{A}}$ are equivalent, and write $A \simeq B$, iff $\langle A \rangle = \langle B \rangle$ (that is iff $A\text{-}B \in \operatorname{Ker} \langle \ \rangle$). If $A \in \overline{\mathscr{A}}^R$, it results from (I-21) that the two propositions $\partial A \simeq 0$ and $A \simeq c \in \mathscr{A}_{(0,0)}$ are equivalent. In particular the relation $\partial A = 0$ implies $A \simeq C$, thus

$$\overline{\mathscr{A}}^R \cap \operatorname{Ker} \partial \subset \mathscr{A}_{(0,0)} \oplus \operatorname{Ker} \langle \ \rangle .$$

In the next paragraph, we show that any diagram is equivalent to a combination of tree diagrams.

4. Tree operators.

In a previous paper we constructed, for any diagram D and any loop b contained in D, a combination of diagrams equivalent to D, such that each of these diagrams is obtained from D by suppressing some lines of b (opening of b) [3]. We start from this result, that we first restate: after choosing a circulation sense on b, let us call (+)-lines (resp. (−)-lines) of b those contained in b and beginning by a point (resp. a cross) and ending by a cross (resp.-a point) according to the chosen circulation sense, and let $\mathcal{L}_b^\pm$ be the set of the (±)-lines of b; for any subset c of $\mathcal{L}_b^\pm$ with $|c|$ elements, we denote by D^c the diagram obtained from D by suppressing the lines belonging to c; lastly we define

(I-24) $$t_b^\pm D = \sum_{\substack{c \subset \mathcal{L}_b^\pm \\ c \neq \phi}} (-1)^{|c|-1} D^c .$$

It is clear that if D is regular, the same is true for D^c and thus for $t_b^\pm D$.
We first established the equivalences [3]

(I-25) $$t_b^\pm D \simeq D$$

Next, when denoting the points of D by an index k, $1 \leqslant k \leqslant N$, and representing by $\partial_k D$ the part of ∂D obtained by the derivation on the only point k, we proved

(I-26) $$\begin{cases} \partial_k t_b^\pm D = \partial_k D & \text{if} \quad k \in b \\ \partial_k t_b^\pm D = t_b^\pm \partial_k D & \text{if} \quad k \notin b \end{cases}$$

Let us now define

(I-27) $$t_b D = \frac{1}{2}\left(t_b^+ D + t_b^- D\right)$$

Clearly, $t_b D$ does not depend on the circulation sense above-chosen on b, and satisfies the same relations as $t_b^\pm D$ in (I-26). This allows us to introduce the linear operator $T: \mathcal{A} \to \mathcal{A}$ such as

(I-28) $$TD = \sum_{b \in \mathcal{B}(D)} t_b D$$

where $\mathcal{B}(D)$ denotes the set of *all* the loops of D. When D is a tree, that is when $\mathcal{B}(D) = \emptyset$. (I-28) has to be understood as $TD = 0$. Let ν be the « loop-number operator » defined by

(I-29) $$\nu D = \nu_D D$$

where $\nu_D = |\mathcal{B}(D)|$ is the number of loops of D. For a tree we also have $\nu D = 0$. Since T and ν both have bidegree (0, 0), they extend to $\overline{\mathcal{A}}$, and we readily verify that they are derivations on $\overline{\mathcal{A}}$.

Owing to (I-26), we have

$$\partial_k \mathrm{TD} = \sum_{b \in \mathscr{B}(\mathrm{D})} \partial_k t_b \mathrm{D} = \sum_{b \ni k} \partial_k \mathrm{D} + \sum_{b \not\ni k} t_b \partial_k \mathrm{D} \tag{I-30}$$

Since the relation $k \notin b \in \mathscr{B}(\mathrm{D})$ is equivalent to $b \in \mathscr{B}(\partial_k \mathrm{D})$, (I-30) gives

$$\partial_k \mathrm{TD} = (\nu_{\mathrm{D}} - \nu_{\partial_k \mathrm{D}})\partial_k \mathrm{D} + \mathrm{T}\partial_k \mathrm{D}\,. \tag{I-31}$$

It follows that

$$\partial_k(\mathrm{T} - \nu)\mathrm{D} = (\mathrm{T} - \nu)\partial_k \mathrm{D} \tag{I-32}$$

and, by summing on k,

$$[\partial, \mathrm{T} - \nu] = 0 \tag{I-33}$$

Let $\mathscr{A}_\tau$ (resp. $\overline{\mathscr{A}}_\tau$) be the subalgebra of $\overline{\mathscr{A}}$ constituted by the finite (resp. arbitrary) linear combinations of tree diagrams.

PROPOSITION 1. — There exists a unique linear operator $\tau : \overline{\mathscr{A}} \to \overline{\mathscr{A}}$, satisfying the conditions

$$\tau = 1 \quad \text{on} \quad \mathscr{A}_\tau \tag{I-34}$$

$$\tau\nu = \tau \mathrm{T} \tag{I-35}$$

Proof. — We at first construct τ on $\mathscr{A}$.

Let $\mathscr{A}^{(\nu)}$ be the eigenspace of eigenvalue ν of the loop-number operator. We have $\mathscr{A}^{(0)} = \mathscr{A}_\tau$, and $\mathscr{A} = \bigoplus_{\nu=0}^{\infty} \mathscr{A}^{(\nu)}$. The operator T maps $\mathscr{A}^{(\nu_0)}$ in $\bigoplus_{\nu=0}^{\nu_0 - 1} \mathscr{A}^{(\nu)}$ so that τ can be obtained by an inductive process: indeed, τ is uniquely determined on $\mathscr{A}^{(0)}$ due to (I-34); let us assume that the same is true on $\bigoplus_{\nu=0}^{\nu_0 - 1} \mathscr{A}^{(\nu)}$; by applying (I-35) to $\mathrm{D} \in \mathscr{A}^{(\nu_0)}$, $\nu_0 \geqslant 1$, we get

$$\tau \mathrm{D} = \frac{1}{\nu_0} \tau(\mathrm{TD}) \tag{I-36}$$

Therefore $\tau\mathrm{D}$ is determined since $\mathrm{TD} \in \bigoplus_{\nu=0}^{\nu_0 - 1} \mathscr{A}^{(\nu)}$. We easily check that τ has bidegree (0, 0), so that it extends to $\overline{\mathscr{A}}$. Finally we verify that the so constructed operator satisfies (I-35). Q.E.D.

PROPOSITION 2. — The operator τ has the following properties:

a) τ is a projection operator,

b) $\operatorname{Im} \tau = \overline{\mathscr{A}}_\tau$,

c) $\tau \mathscr{A}^{\mathrm{R}} \subset \overline{\mathscr{A}}^{\mathrm{R}}$,

d) $\tau A \simeq A, \forall A \in \overline{\mathscr{A}}$,
e) τ is an algebra homomorphism,
f) τ satisfies the commutation relation

$$[\partial, \tau] = 0\,. \tag{I-37}$$

Proof. — *a*) τ^2 satisfies (I-34) and (I-35), and is then equal to τ from Prop. 1.

b, c, d) The definition of T implies that $T\overline{\mathscr{A}}^R \subset \overline{\mathscr{A}}^R$ and that $TD \simeq \nu_D D$ (or, more generally, $TA \simeq \nu A, \forall A \in \overline{\mathscr{A}}$). Properties *b, c, d* then immediately result from the construction of τ by the inductive relation (I-36).

e) The relation $\tau(A_1A_2) = \tau(A_1)\tau(A_2)$ is true when A_1 and A_2 belong to $\mathscr{A}_\tau = \mathscr{A}^{(0)}$. Let us assume that it is true for $A_1A_2 \in \bigoplus_{\nu=0}^{\nu_0-1} \mathscr{A}^{(\nu)}$, and let D_1 and D_2 be two diagrams such as $D_1D_2 \in \mathscr{A}^{(\nu_0)}$. We have, from (I-35)

$$\begin{aligned} \tau(D_1D_2) &= \frac{1}{\nu_0}\tau(T(D_1D_2)) \\ &= \frac{1}{\nu_0}[\tau(TD_1 . D_2) + \tau(D_1 . TD_2)] \end{aligned} \tag{I-38}$$

Since $\nu_{D_i} \leqslant \nu_0$, $i = 1, 2$, we have $TD_i \in \bigoplus_{\nu=0}^{\nu_0-1} \mathscr{A}^{(\nu)}$.

If, for example, $\nu_{D_2} = 0$, the second term in the last member of (I-38) vanishes, and we get

$$\begin{aligned} \tau(D_1D_2) &= \frac{1}{\nu_{D_1}}\tau(TD_1 . D_2) = \frac{1}{\nu_{D_1}}\tau(TD_1)\tau(D_2) \\ &= \tau(D_1)\tau(D_2) \end{aligned} \tag{I-39}$$

If ν_{D_1} and ν_{D_2} are different from zero (and then $< \nu_0$), we have

$$\begin{aligned} \tau(D_1D_2) &= \frac{1}{\nu_0}[\tau(TD_1)\tau(D_2) + \tau(D_1)\tau(TD_2)] = \\ &= \frac{1}{\nu_0}[\nu_{D_1}\tau(D_1)\tau(D_2) + \nu_{D_2}\tau(D_1)\tau(D_2)] = \\ &= \tau(D_1)\tau(D_2)\,. \end{aligned} \tag{I-40}$$

Since τ is linear the result for D_1D_2 is valid for $A_1A_2 \in \bigoplus_{\nu=0}^{\nu_0} \mathscr{A}^{(\nu)}$, and thus by induction for $A_1, A_2 \in \mathscr{A}$.

Finally, it extends for any two elements A_1 and A_2 of $\overline{\mathscr{A}}$, because τ has bidegree (0, 0). Q. E. D.

f) The relation $\partial\tau A = \tau\partial A$ is valid for $A \in \mathscr{A}_\tau = \mathscr{A}^{(0)}$. Let us assume that it is true in $\bigoplus_{\nu=0}^{\nu_0-1} \mathscr{A}^{(\nu)}$; for $A \in \mathscr{A}^{(\nu_0)}$, we have, using (I-33) and (I-35),

$$\text{(I-41)} \qquad \partial\tau A = \frac{1}{\nu_0}\partial\tau TA = \frac{1}{\nu_0}\tau\partial TA = $$
$$= \frac{1}{\nu_0}\tau[(T-\nu)\partial A + \nu_0\partial A] = \tau\partial A .$$

As previously this establishes the result in $\bigoplus_{\nu=0}^{\nu_0} \mathscr{A}^{(\nu)}$, thus in $\mathscr{A}$, afterwards in $\overline{\mathscr{A}}$. Q. E. D.

Let us note that a more detailed form of (I-37) holds, namely

$$\text{(I-42)} \qquad \partial_k\tau D = \tau\partial_k D, \quad \forall k,$$

where D is any diagram and k the index of a point of D.

The operator τ being a projection operator onto $\overline{\mathscr{A}}_\tau$ allows to define operators leaving $\overline{\mathscr{A}}_\tau$ invariant, and having properties similar to those of paragraphs 1 and 2. In particular, to ∂, M and $\int$ correspond ∂, $M_\tau = \tau M\tau$ and $\tau\int\tau$. This last operator is a right inverse of ∂ in $\overline{\mathscr{A}}_\tau$, that is satisfies the relation

$$\text{(I-43)} \qquad \partial\left(\tau\int\tau\right) = \tau$$

Moreover, a commutation relation identical to (I-1) holds in $\overline{\mathscr{A}}_\tau$, because

$$\text{(I-44)} \qquad [\partial, M_\tau] = \tau$$

From that relation we can repeat the constructions of paragraphs 1 and 2, and more particularly define the operator

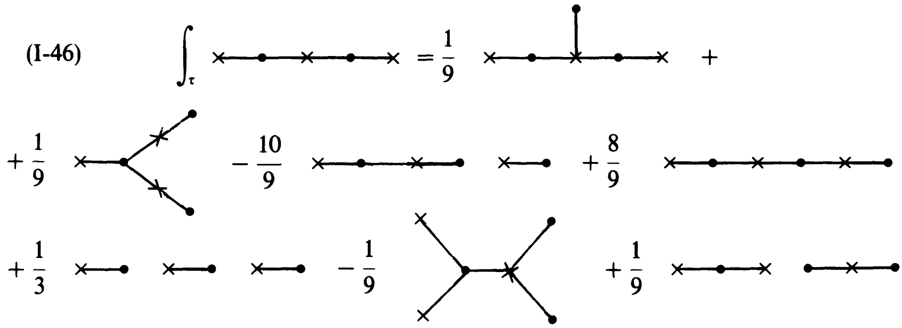
$$\text{(I-45)} \qquad \int_\tau = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!}(M_\tau)^k\partial^{k-1}$$

It is too a right inverse of ∂ in $\overline{\mathscr{A}}_\tau$. Furthermore, just as $\int$, the two operators $\tau\int\tau$ and $\int_\tau$ satisfy (I-23), so that we have the equivalences

$$\tau\int\tau A \simeq \int_\tau A \simeq \int A, \qquad \forall A \in \overline{\mathscr{A}} .$$

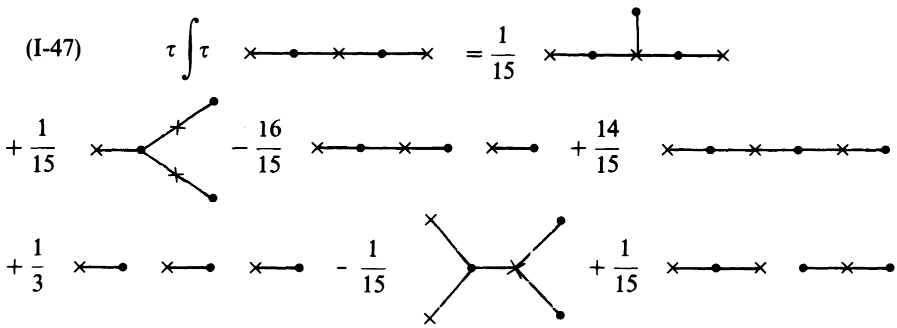
Let us stress, however, that the operators $\int_\tau$ and $\tau\int\tau$ are different; for example we have

(I-46)



and

(I-47)



The difference between these two expressions is a combination of regular tree diagrams, belonging to Ker ∂, having a vanishing component in $\mathscr{A}_{(0,0)}$, and consequently equivalent to zero.

II. APPLICATION TO GREEN FUNCTIONS

In this section, we are going to show that the solutions of one variable differential equations with an arbitrary source can be constructed with the help of solutions of differential equations in $\overline{\mathscr{A}}$, or in some generalizations of it. In fact, the star algebra structure is suited to the explicit description of the Green functions, namely to their algebraic construction in terms of standard kernels, those associated to the stars in Paragraph I.3 [3].

1. First-order differential equations.

Let us consider in $\overline{\mathscr{A}}$ the equation

(II-1) $$\partial X = \mathscr{P}(X)$$

where $\mathscr{P}$ is a polynomial, and the « initial condition »

(II-2) $$(1 - P)X = c \in \operatorname{Ker} \partial$$

By applying the operator $\int$ to (II-1), we obtain the integral form

$$\text{(II-3)} \qquad X = c + \int \mathscr{P}(X)$$

Due to (I-3) and (I-5), this equation is equivalent to (II-1) and (II-2). Taking its value, as defined in I-3, using (I-23) and the fact that $\langle\ \rangle$ is an algebra homomorphism, we now get

$$\text{(II-4)} \qquad \langle X \rangle (t) = \langle c \rangle (t) + \int_0^t d\tau \alpha(\tau) < \mathscr{P}(X) \rangle (\tau) = \\ = \langle c \rangle (t) + \int_0^t d\tau \alpha(\tau) \mathscr{P}(\langle X \rangle (\tau)) .$$

The function $\langle X \rangle (t)$ will be identical to the retarded solution of the equation

$$\text{(II-5)} \qquad \frac{d}{dt} x(t) = \eta(t) + \alpha(t) \mathscr{P}(x(t))$$

provided that the following condition be satisfied

$$\text{(II-6)} \qquad \langle c \rangle (t) = \overline{\eta}(t) \equiv \int_0^t d\tau \eta(\tau) ,$$

or, in other words,

$$\text{(II-7)} \qquad c \simeq \times$$

Since Ker $\langle\ \rangle \neq \emptyset$, the solution of this last equation is not unique. However, due to the classical existence theorems, the retarded solution $x(t)$ is uniquely defined in a neighbourhood of the origin. Thus it is sufficient to choose [8]

$$\text{(II-8)} \qquad c = \times$$

To the resolution of (II-5) is therefore substituted that of the equation

$$\text{(II-9)} \qquad X = \times + \int \mathscr{P}(X)$$

The solution of this latter can be obtained by an iterative process. Let us introduce the expansion

$$\text{(II-10)} \qquad X = \sum_{N=0}^{\infty} X_N$$

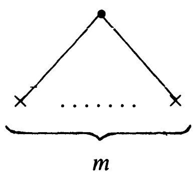
where X_N is a combination of diagrams, each having N points. Since $\int$ has bidegree (0, 1), equation (II-9) gives

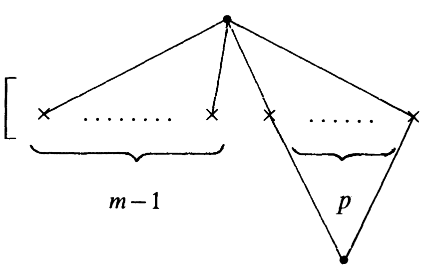
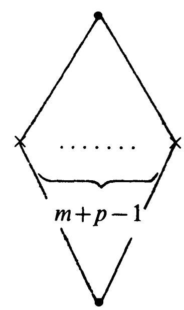
$$\text{(II-11)} \qquad \begin{cases} X_0 = \times \\ X_N = \int (\mathscr{P}(X))_{N-1}, \qquad N \geqslant 1 \end{cases}$$

so that the X_N's are recursively defined. It is clear that X_N is homogeneous of degree N with respect to the coefficients of the polynomial $\mathscr{P}$. Moreover, the induction defined by (II-11) implies that X_N has a finite number of terms, $\forall$N. So we have: $X_N \in \bigoplus_n \mathscr{A}_{(n,N)}$. By writing

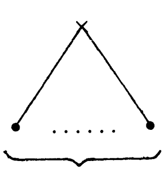
$$\text{(II-12)} \qquad \mathscr{P}(X) = \sum_m \frac{a_m}{m!} X^m$$

the first few terms of X are

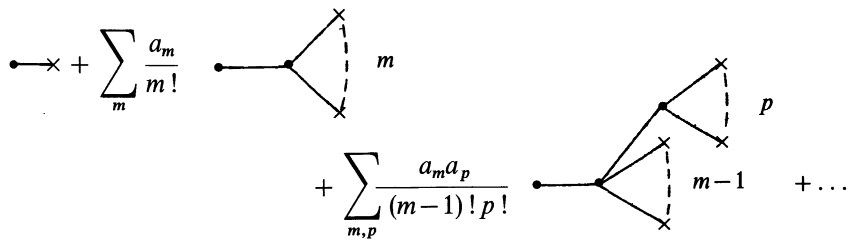
(II-13) $X = \times + \sum_m \frac{a_m}{m!}$  $+ \sum_{m,p} \frac{a_m a_p}{(m-1)!\,p!}$

$\Bigg[$  $-\frac{1}{2}$  $\Bigg] + \dots$

In particular, for $\mathscr{P}(X) = aX$, the complete solution is

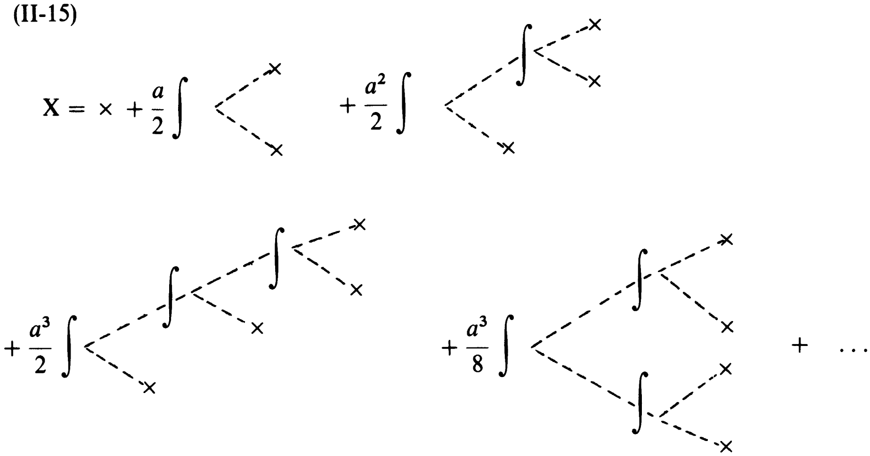
(II-14) $X = \sum_{m=0}^{\infty} \frac{a^m}{m!}$  $= e^{aM}(\times)$

In the general case, however, although the X_N's are well defined by (II-11), it seems difficult to express them in closed form. Let us simply give two alternative forms of (II-11). Firstly, the formal similarity between (II-9) and the integrated form of (II-5) implies that their iteration schemes are identical. As it is known, the second one can be described with the help of a series of Feynman diagrams of the form



in which the number of lines starting from each vertex is equal to one of the exponents appearing in $\mathcal{P}$. For the equation (II-9), the analogous expansion, written for example in the case $\mathcal{P}(X) = \frac{a}{2}X^2$, is represented by

(II-15)



that means

(II-16) $$X = \times + \frac{a}{2}\int \times\times + \frac{a^2}{2}\int\times\int\times\times + \frac{a^2}{2}\int\times\int\times\times + \frac{a^2}{8}\int\left(\int\times\times\right)^2 + \dots$$

A second way to iterate (II-9) uses the explicit expression of the operator $\int$: from (II-1) one easily derives the formula

(II-17) $$\partial^n X = \mathcal{Q}_n(X)$$

where the polynomial $\mathcal{Q}_n(x)$ is defined by

(II-18) $$\mathcal{Q}_n(x) = \left(\mathcal{P}(x)\frac{d}{dx}\right)^n(x)\,;$$

the equation (II-9) then reads

(II-19) $$X = \times + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} M^k \mathcal{Q}_k(X)$$

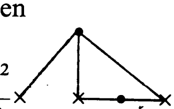
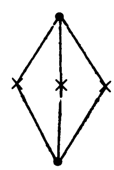
For $\mathcal{P}(X) = aX$, this expression reduces to (II-14). For $\mathcal{P}(X) = \frac{a}{2}X^2$ it gives

(II-20) $$X = \times - \sum_{k=1}^{\infty} \left(-\frac{a}{2}M\right)^k (X^{k+1})$$

The iteration of this equation furnishes the series

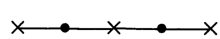
(II-21) $$X = \times + \frac{a}{2}M(\times \times) + \frac{a^2}{4}(2M(\times M(\times \times)) - M^2(\times \times \times)) + \dots$$

that can be diagrammatically written

(II-22) $X = \times + \frac{a}{2}$  $+ \frac{a^2}{2}$  $- \frac{a^2}{4}$  $+ \dots$

As previously explained, the expansion of X in terms of star diagrams furnishes for the retarded Green functions of Eq. (II-5) explicit algebraic expressions constructed with the help of the kernels associated with the stars [*3*].

Following the results of Paragraph I.4, the solution $x(t)$ of (II-5) is as well represented by τX, which only contains tree diagrams. In particular (II-22) is equivalent to

(II-23) $\tau X = \times + \frac{a}{2}$  $+ \frac{a^2}{4}$  $+ \dots$

Due to the properties of the tree operator, τX still satisfies equations of the type (II-1) and (II-2), and is thus well determined by them; however the corresponding element $C_0 = (1 - P)\tau X \in \text{Ker}\,\partial$ differs from $\times$ (although being equivalent to it), and does not seem likely to be easily predetermined; nevertheless, it remains true that (II-1) and (II-2) are able to furnish a tree solution with a convenient choide of C_0. Two others ways of obtaining tree solutions, keeping the simplicity of the initial condition (II-8), consists in solving either of the equations

(II-1′-II-2′) $\partial Y = \mathcal{P}(Y), \quad (1 - \tau P \tau)Y = \times$

(II-1″-II-2″) $\partial Z = \mathcal{P}(Z), \quad (1 - P_\tau)Z = \times$

The corresponding integrated forms of these equations, from which the iterative solutions result, are respectively

(II-9′) $$Y = \times + \tau \int \tau \mathscr{P}(Y)$$

and

(II-9″) $$Z = \times + \int_{\tau} \mathscr{P}(Z) .$$

It is easy to prove that the tree solutions τX, Y and Z are equivalent.

To end this paragraph, let us remark that the preceding method allows to treat the case where the initial condition $x(0)$ is different from zero. If $x(0) = x_0$, it suffices to replace $\eta(t)$ in (II-5) by $\eta(t) + x_0\delta(t)$, while maintaining the condition $x(t) = 0$ for $t < 0$. In this substitution the values of the diagrams are only modified by the change of the factors associated with the crosses. Finally, the classical Cauchy problem for $t \geqslant 0$, is recovered by putting afterwards $\eta = 0$.

2. **Higher order differential equations.**

The preceding methods can be generalized to higher order equations in the case where $\alpha(t) = \theta(t)$ (or simply $\alpha(t) = 1$ if $P(0) = 0$). Let us first examine in $\bar{\mathscr{A}}$ the second-order equation

(II-24) $$\partial^2 X = \mathscr{P}(X)$$

with the initial conditions

(II-25) $$(1 - P)X = 0, \qquad (1 - P)\partial X = \times$$

which, corresponding to the conditions (II-2) and (II-8), represent the simplest choice for our purpose. By applying twice the operator $\int$ to (II-24) we get

(II-26) $$X = \int \times + \int\int \mathscr{P}(X) = \bullet\!\!-\!\!\!-\!\!\times + \int\int \mathscr{P}(X)$$

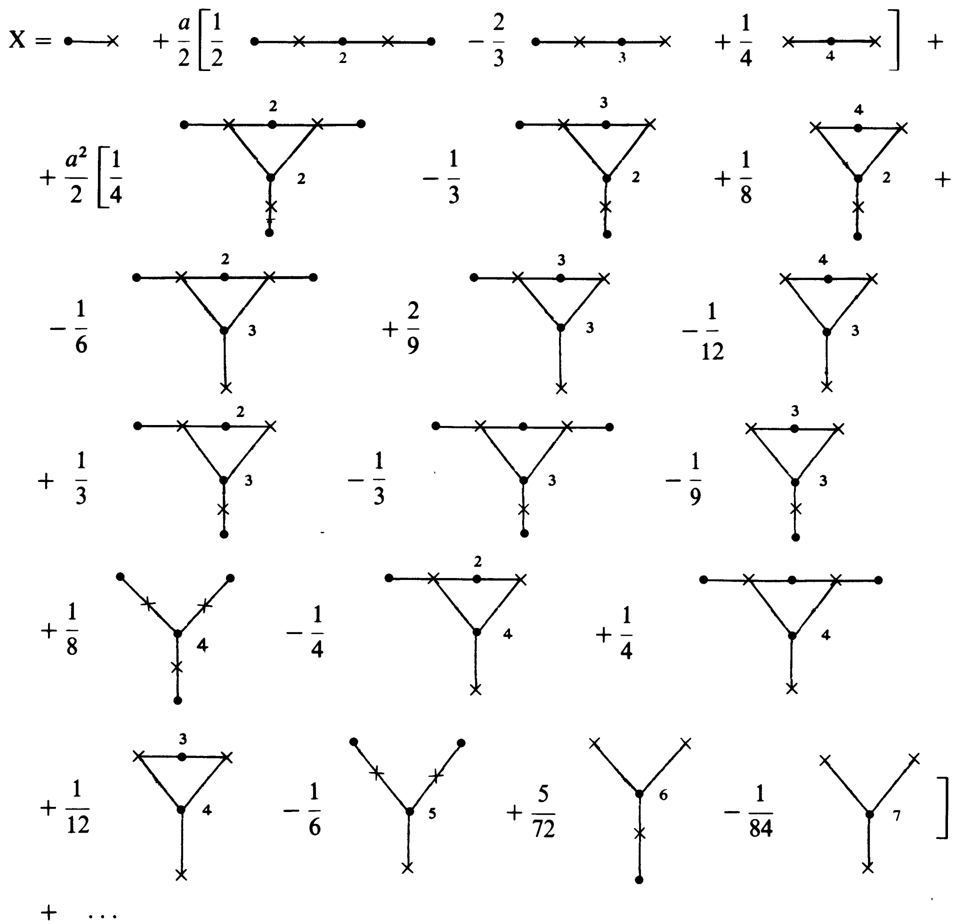
Taking the second derivative with respect to t of the value of (II-26) gives the equation

(II-27) $$\frac{d^2}{dt^2} x(t) = \eta(t) + \theta(t)\mathscr{P}(x(t)), \qquad x(t) = \langle X \rangle (t)$$

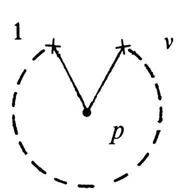
Like that of (II-9) in the first order case, the iterative solution of (II-26) furnishes an expansion of the retarded solution $x(t)$ of (II-27), involving explicit expressions of the Green functions.

If for example, $\mathscr{P}(X) = \frac{a}{2} X^2$, the iterative solution of (II-26) is represented by an expansion analogous to (II-15) or (II-16), the operator $\iint$ replacing $\int$ and •—× replacing ×. That gives for the first few terms

(II-28)



Here, to simplify, the schema



stands for a set of p stars of order ν having the same extremities. That expression is rather complicated as compared to the result (II-22) for the first-order equation. Let us remark however that these two expressions are constructed with the help of the *same* pieces, namely the stars, the same star corresponding to the same value in both cases. On the contrary, the Feynman diagrams associated with (II-5) or (II-27), though identical (since coming from identical iteration schemes) have to be evaluated with propagators of different values.

The most general equation that we treat now is

$$\partial(\partial - \lambda_1)(\partial - \lambda_2) \dots (\partial - \lambda_n)X = \mathscr{P}(X) \tag{II-29}$$

with

$$\begin{cases} (1 - P)X = (1 - P)\partial X = \dots = (1 - P)\partial^{n-1}X = 0 \\ (1 - P)\partial^n X = \times \end{cases} \tag{II-30}$$

We need to introduce the operator

$$G_\lambda = e^{\lambda M} \int e^{-\lambda M} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} M^k(\partial - \lambda)^{k-1} \tag{II-31}$$

It satisfies the relations

$$(\partial - \lambda)G_\lambda = 1, \qquad G_\lambda(\partial - \lambda) = e^{\lambda M}Pe^{-\lambda M} \tag{II-32}$$

Let us apply on (II-29) the product of operators $G_{\lambda_n} \dots G_{\lambda_1}G_0$; on account of (II-30) and (II-32), this gives

$$X = G_{\lambda_n} \dots G_{\lambda_1}(\times) + G_{\lambda_n} \dots G_{\lambda_1}G_0\mathscr{P}(X) \tag{II-33}$$

The first term in the right hand side can be explicitly calculated. For any element C of Ker ∂, we have the formula

$$G_\lambda(C) = \frac{e^{\lambda M} - 1}{\lambda}(C) = \sum_{k=0}^{\infty} \frac{\lambda^k}{(k+1)!} M^{k+1}(C) \tag{II-34}$$

Furthermore the following composition law is valid

$$G_{\lambda_2}G_{\lambda_1}(C) = \left(\frac{G_{\lambda_1}}{\lambda_1 - \lambda_2} + \frac{G_{\lambda_2}}{\lambda_2 - \lambda_1}\right)(C) \tag{II-35}$$

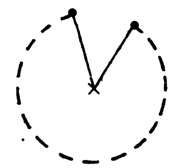
By induction this implies

$$G_{\lambda_n} \dots G_{\lambda_2}G_{\lambda_1}(C) = \sum_{k=1}^{n} \frac{G_{\lambda_k}}{\prod_{j \neq k} (\lambda_k - \lambda_j)}(C) \tag{II-36}$$

By using (II-36) and (II-34), the first term in the right hand side of (II-33) can be written under the form

(II-37) $$G_{\lambda_n}\ldots G_{\lambda_1}(\times)=\sum_{i=0}^{\infty}\frac{1}{(i+1)!}\sum_{k=1}^{n}\frac{(\lambda_k)^i}{\prod_{j\neq k}(\lambda_k-\lambda_j)}\,M^{i+1}(\times)$$

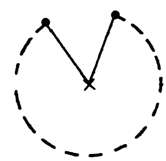
$$=\sum_{i=n-1}^{\infty}\frac{1}{(i+1)!}\left[\sum_{\substack{(m_1,\ldots,m_n)\\ \sum_{k=1}^{n} m_k=i-(n-1)}}\ \prod_{k=1}^{n}(\lambda_k)^{m_k}\right]$$



$i+1$

If, for example, $\lambda_k=0$, $1\leqslant k\leqslant n$, it simply remains

(II-38) $$\left(\int\right)^n(\times)=\frac{1}{n!}$$



n

Finally, as in the previous cases, the equation for $x=\langle X\rangle$ corresponding to (II-29) is obtained by differentiation of the value of (II-33), and reads

(II-39) $$\frac{d}{dt}\left(\frac{d}{dt}-\lambda_1\right)\cdots\left(\frac{d}{dt}-\lambda_n\right)x(t)=\eta(t)+\theta(t)\mathscr{P}(x(t))$$

Of course, the same treatment can be applied to the equations of the form

(II-40) $$\mathscr{D}x=\eta+\theta f(x)$$

in which $\mathscr{D}$ denotes a linear differential operator with constant coefficients, and f an entire function.

In the last paragraphs, we sketch the extension of the preceding methods to two other cases which require some generalizations of the star algebra $\overline{\mathscr{A}}$.

3. Systems of differential equations.

Let us consider the system

(II-41) $$\frac{dx_i}{dt}=\eta_i+\alpha(t)\mathscr{P}_i(x),\qquad 1\leqslant i\leqslant K$$

in which the K functions η_i are arbitrary sources, and the $\mathscr{P}_i$ some given polynomials in K variables. To describe the Green functions of (II-41) we need to generalize the preceding diagrams by admitting the occurrence of K types of crosses, distinguished by an index i, corresponding to the K sources η_i, $1 \leqslant i \leqslant K$. The notions introduced in Section I are immediately generalized; in particular, the rule defining the value of a diagram now involves the correspondence

$$\underset{i}{\times} \rightarrow \eta_i, \qquad 1 \leqslant i \leqslant K \tag{II-42}$$

In the algebra, the system of equations associated with (II-41) is

$$\partial X_i = \mathscr{P}_i(X), \qquad X = (X_i), \qquad 1 \leqslant i \leqslant K \tag{II-43}$$

with the initial conditions

$$(1 - P)X_i = \underset{i}{\times} \tag{II-44}$$

The corresponding solution is determined by

$$X_i = \underset{i}{\times} + \int \mathscr{P}_i(X), \qquad 1 \leqslant i \leqslant K \tag{II-45}$$

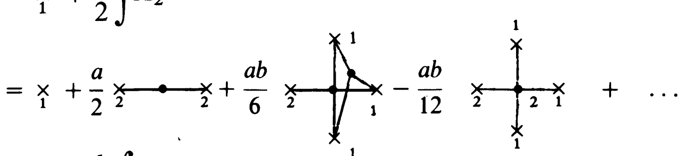
and one easily verifies that the retarded solution of (II-41) is given, in terms of this one, by $x_i(t) = \langle X_i \rangle (t)$, $1 \leqslant i \leqslant K$. The diagrammatic expansions are still obtained by iteration of (II-45). For example the expansions associated with the system

$$\left\{ \begin{aligned} \frac{dx_1}{dt} &= \eta_1 + \frac{a}{2}\alpha(t)x_2^2 \\ \frac{dx_2}{dt} &= \eta_2 + \frac{b}{6}\alpha(t)x_1^3 \end{aligned} \right. \tag{II-46}$$

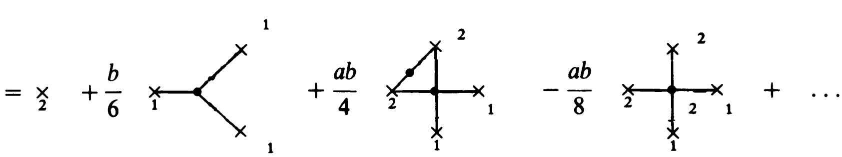
are

(II-47)

$$X_1 = \underset{1}{\times} + \frac{a}{2}\int X_2^2 =$$



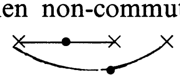
$$X_2 = \underset{2}{\times} + \frac{b}{6}\int X_1^3 =$$



Let us note that our treatment of (II-41) works because, for any i, the coefficient of $\mathscr{P}_i(x)$ is the same function $\alpha(t)$. For $\alpha(t) \neq \theta(t)$ however, these systems correspond to differential equations slightly more general than (II-39), obtained from (II-39) by the replacing

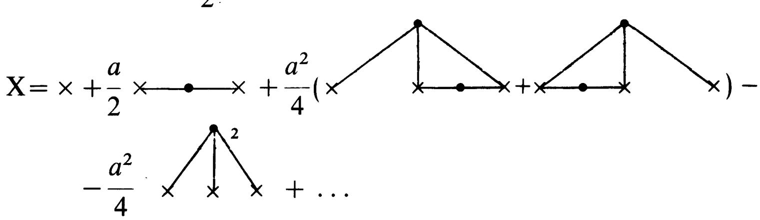
$$\begin{cases} \dfrac{d}{dt} \to \dfrac{1}{\alpha}\dfrac{d}{dt} \\ \eta \to \eta/\alpha \end{cases}$$

4. **Operator equations.**

Let us now examine the case of an equation, for instance (II-5), in which the source η and the solution $x(t)$ belong to a non-commutative algebra, for example an operator algebra. The diagrams we have to consider here are defined similarly to those of $\mathscr{A}$, with the additional condition that the set of the crosses of each diagram be totally ordered. Accordingly, the product D_1D_2 of two diagrams is obtained by putting the crosses of D_1 before those of D_2, i. e. to the left of the drawing. The corresponding star algebra is then non-commutative. For example, the diagrams ×—•—×—•—× and  have to be distinguished, just as the products × ×—• and ×—• ×. Lastly, the value of a diagram is given by the same rule as in I.3, taking care to write the factors $\eta(\tau)$ in the same order as the corresponding crosses, and the formulas (I-21)-(I-23) remain valid.

Under these conditions one immediately verifies that, as previously, the solution of (II-5) is given by $x = \langle X \rangle$, where X is the solution of (II-9).

For example, if $\mathscr{P}(X) = \frac{a}{2} X^2$, the first few terms of the expansion of X are

(II-48) 

The expansion reduces to (II-22) when the algebra is commutative.

These methods can be easily extended to the cases of higher order equations or systems. For example, the equation

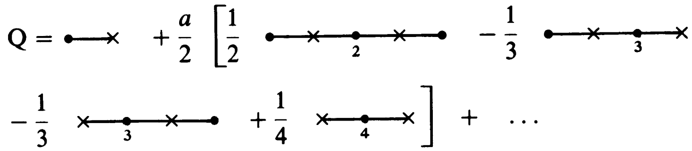
(II-49) $$\frac{d^2q}{dt^2} = \eta + \frac{a}{2} q^2$$

is solved by the formulas

(II-50) $$q = \langle Q \rangle, \qquad Q = \times\!\!-\!\!\bullet + \frac{a}{2}\iint Q^2$$

the corresponding expansion, that must be compared to (II-28), being

(II-51)



The solution $q(t)$ of (II-49) is as well obtained as the first component of the system

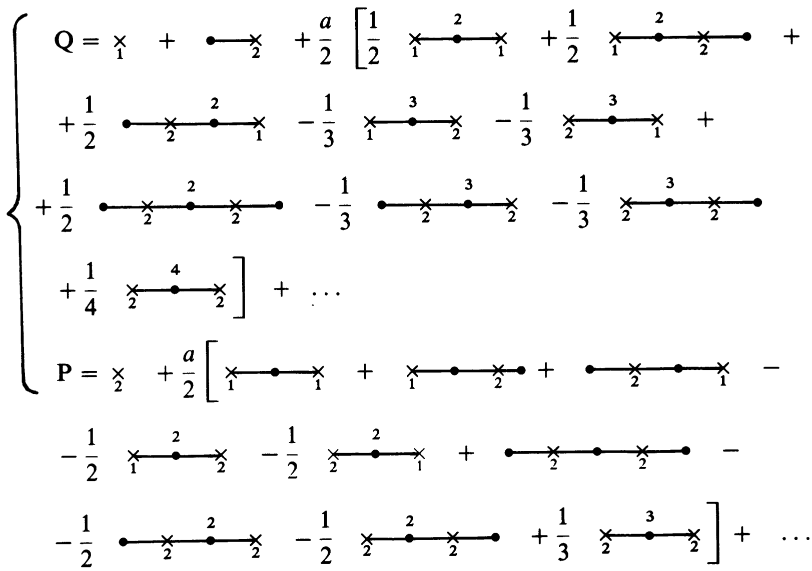
(II-52)
$$\begin{cases} \dfrac{dq}{dt} = \eta_1 + p \\[2mm] \dfrac{dp}{dt} = \eta_2 + \dfrac{a}{2} q^2 \end{cases}$$

provided that $\eta = \eta_1' + \eta_2$, that corresponds to $q = \langle \mathrm{Q} \rangle$ and $p = \langle \mathrm{P} \rangle$ with

(II-53)
$$\begin{cases} \mathrm{Q} = \underset{1}{\times} + \displaystyle\int \mathrm{P} \\[2mm] \mathrm{P} = \underset{2}{\times} + \dfrac{a}{2} \displaystyle\int \mathrm{Q}^2 \end{cases}$$

The expansions read

(II-54)



When the sources have the form

(II-55) $$\eta_1(t) = \xi_1(t) + q_0\delta(t), \qquad \eta_2(t) = \xi_2(t) + p_0\delta(t)$$

where $q_0 = x$ and $p_0 = -i\hbar\frac{d}{dx}$ are the quantum operators, and ξ_1 and ξ_2 classical sources, the expansions (II-54) furnish the Green functions of the Heisenberg equations corresponding to the Hamiltonian with sources

(II-56) $$\mathrm{H} = \frac{p^2}{2} - \frac{a}{6}q^3 + \xi_1 p - \xi_2 q$$

In particular, these expansions contain the solution of the usual (i. e. without sources) Cauchy problem, namely

(II-57) $$\begin{cases} q(t) = q_0 + p_0 t + \dfrac{a}{2}\left[\dfrac{1}{2}q_0^2t^2 + \dfrac{1}{6}(q_0p_0 + p_0q_0)t^3 + \dfrac{1}{12}p_0^2t^4\right] + \dots \\ p(t) = p_0 + \dfrac{a}{2}\left[q_0^2t + \dfrac{1}{2}(q_0p_0 + p_0q_0)t^2 + \dfrac{1}{3}p_0^2t^3\right] + \dots \end{cases}$$

These equations are simply obtained from (II-54) by the replacing

(II-58) $$\begin{cases} \underset{1}{\times} \to q_0 \\ \underset{2}{\times} \to p_0 \\ \bullet \to t \end{cases}$$

APPENDIX

COMPARISON WITH PREVIOUS RESULTS [2].

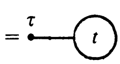
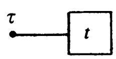
The retarded solution $x(t)$ of the equation (II-5) with the choice $\mathscr{P}(x) = \frac{a}{2}x^2 + \frac{b}{6}x^3$, was previously obtained under the form [2]

(A-1) $$x(t) = \int_0^t \eta(\tau)X(t, \tau)Y(t, \tau)d\tau$$

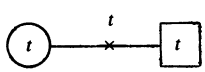
where X and Y are solutions of the equations

(A-2) $$\begin{cases} \theta(t-\tau)X(t, \tau) = \theta(t-\tau)\left[1 + \frac{a}{2}\int_\tau^t d\tau'\alpha(\tau')\int_0^{\tau'} d\tau''\eta(\tau'')X(t, \tau'')Y(\tau', \tau'')\right] \\ \theta(t-\tau)Y^2(t, \tau) \\ = \theta(t-\tau)\left[1 + \frac{b}{3}\int_\tau^t d\tau'\alpha(\tau')\int_0^{\tau'} d\tau''d\tau'''\eta(\tau'')\eta(\tau''')X(\tau', \tau'')X(\tau', \tau''') \times Y(t, \tau'')Y(t, \tau''')\right] \end{cases}$$

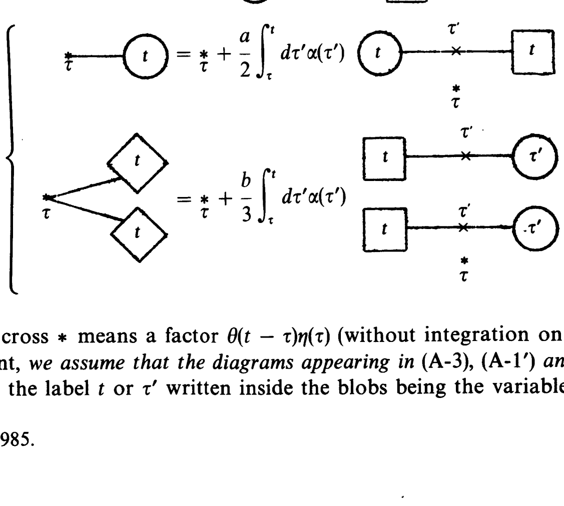
The iteration of these equations naturally led to diagrams analogous to present star diagrams, but in the value of which the function $\bar{\alpha}$ associated with the centres had to be replaced by a functional of the source function η (*star diagrams with dressed centres*) [2]. Then, the expressions so obtained for the Green functions were not completely explicit. Hence the question is posed to obtain an explicit result analogous to those of Paragraph II.1 from the equations (A-1) and (A-2). To this end let us introduce the following graphical representation

(A-3) $\theta(t - \tau)X(t, \tau) =$ ; $\theta(t-\tau)Y(t, \tau) =$ 

The formulas (A-1) and (A-2) (after multiplications by $\eta(\tau)$ for these last) then may be written

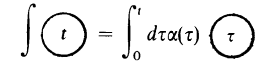
(A-1′) $x(t) =$ 

(A-2′)

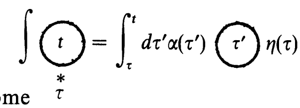


in which the cross $*$ means a factor $\theta(t - \tau)\eta(\tau)$ (without integration on τ).

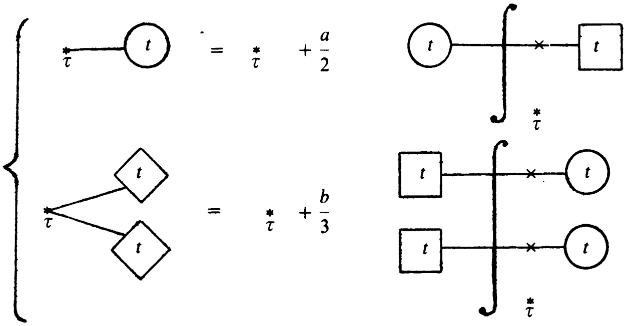
At this point, *we assume that the diagrams appearing in* (A-3), (A-1′) *and* (A-2′) *have the star structure*, the label t or τ' written inside the blobs being the variable attached to the centres according to the rules given in Paragraph I.3. With these conventions, Formula (I-23) reads

(A-4) 

the symbol $\int$ being that defined in I.1. Moreover, for $t \geqslant \tau$, we find

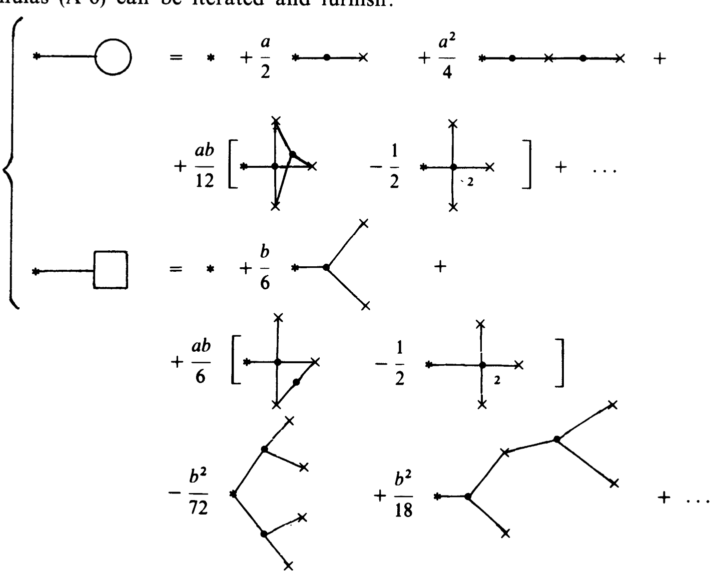
(A-5) 

Formulas (A-2′) then become

(A-6) 

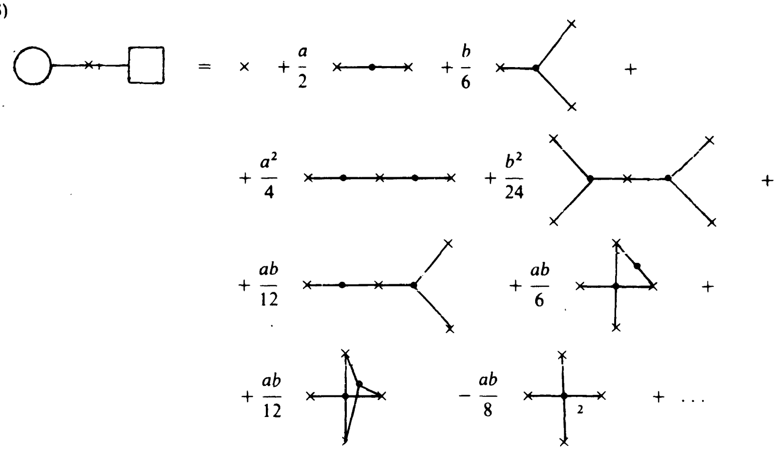
In these formulas, only the part of the diagrams lying on the right of the symbol $\int$ have to be integrated. In all the preceding formulas, the diagrams represent in fact the values. We now make the additional hypothesis that they remain true for the abstract diagrams. Compared with the diagrams of previous paragraphs, these last have a distinguished cross $*$. It is then not difficult to prove that, by substituting the diagrammatic solutions of (A-6) in (A-1′), we obtain a solution of (II-1) with $\mathcal{P}(x) = \frac{a}{2}x^2 + \frac{b}{6}x^3$.

Formulas (A-6) can be iterated and furnish:

(A-7) 

This finally gives for the diagrammatic solution corresponding to $x(t)$:

(A-8)



Thus the star structure can be extracted from the equations (A-1) and (A-2) provided that it be explicitly postulated. Let us remark however that, although equivalent to it, the solution (A-8) differs from that resulting from the method of Paragraph II.1.

ACKNOWLEDGMENTS

One of us (M. I. A.) thanks M. Ginocchio for discussions.

NOTES AND REFERENCES

[1] J. C. HOUARD, *Lett. Nuovo Cimento*, t. **33**, 1982, p. 519.

[2] J. C. HOUARD and M. IRAC-ASTAUD, *J. Math. Phys.*, t. **24**, 1983, p. 1997.

[3] J. C. HOUARD and M. IRAC-ASTAUD, Two theorems on star diagrams (Preprint Feb. 1984), to appear in *J. Math. Phys.*

[4] Each star diagram thus defines an equivalence class of isomorphic *systems*, according to the definition of J. E. GRAVER and M. E. WATKINS, *Combinatorics with emphasis on the theory of Graphs*, Springer-Verlag, New York, Heidelberg, Berlin, 1977. The diagrams having no isolated point or cross correspond to the classes of *hypergraphs*, see C. BERGE, *Graphes et hypergraphes*, Dunod, Paris, 1970.

[5] N. BOURBAKI, *Éléments de Mathématique*, Livre II, Chap. II, Hermann, Paris, 1955.

[6] An operator of bidegree (p, q) is an operator mapping $\mathscr{A}_{(n,N)}$ in $\mathscr{A}_{(n+p,N+q)}$, $\forall n$, N.

[7] The product of the kernels appearing in the integrand of $\langle$ D $\rangle$ is an algebraic expression constructed in terms of $\bar{\alpha}$. The same is true for the Green functions corresponding to the solutions of the equations considered later since, by the treatment of Section II, they will appear as linear combinations of such products.

[8] Let us note however that a different choice may have an interest, for example to obtain tree solutions, as explained below.

(Manuscrit reçu le 17 octobre 1984)