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Complex Scaling Technique in Non-relativistic Massive QED

by

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ABSTRACT. — We study the resonance problem for Hamiltonians of non-relativistic massive quantum electrodynamics. Applying the complex scaling technique of Balslev-Combes, we shall prove that for generic potentials all the embeeded eigenvalues off thresholds of non-interacting systems will dissolve into continuum, forming resonance poles in the unphysical Riemann sheets and that the imaginary parts of the poles may be computed by Fermi's Golden rule.

RÉSUMÉ. — On étudie le problème des résonances pour des hamiltoniens d'électrodynamique quantique non relativiste. En appliquant la méthode de dilatation complexe de Balslev-Combes, on montre que pour des potentiels génériques, toutes les valeurs propres plongées dans le continu et différentes des seuils des systèmes sans interaction se dissolvent dans le continu, formant des pôles de résonances dans les feuillets de Riemann non physiques, et que les parties imaginaires des pôles peuvent être calculées par la règle d'or de Fermi.

1. INTRODUCTION

We consider a non-relativistic spinless electron in a potential $V(x)$ interacting with the quantized *massive* electromagnetic field. We adopt the Coulomb gauge for the field with the ultraviolet cut-off and describe the states of the system by elements of the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^3) \otimes \mathscr{F}_{eM}$, $L^2(\mathbb{R}^3)$ for the electron and $\mathscr{F}_{eM} = \mathscr{F}_b \otimes \mathscr{F}_b$ for

« photons », $\mathscr{F}_b$ the boson Fock space. When we regard an element of $\mathscr{H}$ as an $\mathscr{F}_{eM}$-valued square integrable function of $x \in \mathbb{R}^3$, we denote it as $f(x, .)$. The dynamics of the system is governed by the Hamiltonian $H(\lambda)$ which is formally given (after mass renormalization) as

$$(1.1)\quad H(\lambda)f(x, .) = (2(m - \delta m(\lambda)))^{-1} \sum_{j=1}^{3} (-i\partial/\partial x_j - \lambda A_j(x))^2 f(x, .) + H_0^{eM} f(x, .) + V(x)f(x, .), \qquad f \in \mathscr{H}$$

in terms of the field operator $\vec{A}(x) = (A_1(x), \ldots, A_3(x))$ and the free energy operator H_0^{eM} of the « photons »; $m > 0$ and $M > 0$ are masses of the electron and the « photon », respectively, and $\lambda \in \mathbb{R}$ is the coupling parameter. When $\hat{\rho}(k)$ denotes the momentum cut-off function of the interaction, $\vec{A}(x)$ and H_0^{eM} are given in conventional forms

$$(1.2)\quad \vec{A}(x) = 2^{-1/2} \sum_{j=1}^{2} \int \vec{e}(k,j)\{ \hat{\rho}(k)e^{-ikx}a^*(k,j) + \hat{\rho}(-k)e^{ikx}a(k,j) \} \times dk/\omega(k)^{1/2}$$

and

$$(1.3)\quad H_0^{eM} = \sum_{j=1}^{2} \int \omega(k)a^*(k,j)a(k,j)dk, \qquad \omega(k) = (k^2 + M^2)^{1/2},$$

using the creation (and annihilation) operator $a^*(k,j)$ (and $a(k,j)$) of the « photon » of momentum k and polarization $\vec{e}(k,j) \in \mathbb{R}^3$ ($j = 1, 2$) (see [*7*]).

$$(1.4)\qquad \delta m(\lambda) = \frac{2}{3}\left(\frac{\lambda}{m}\right)^2 \int |\hat{\rho}(k)|^2 \omega(k)^{-2} dk$$

is the mass renormalization constant. Here the normalization is made so that $\rho(x) = \delta(x)$ corresponds to the field without cut-off. The spectral property of the Hamiltonian $H(\lambda)$, in particular, the resonance problem associated with it is the objet of the study in this paper. Technically we wish to present another application of the complex scaling technique which has been very successful in the quantum mechanics of finitely many degrees of freedom ([*1*] [*11*] [*12*]. See also [*5*]).

We fix our assumption on the potential $V(x)$ and the cut-off function $\hat{\rho}(k)$ first and record some of the known results. $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$, with the domain $D(\Delta) = H^2(\mathbb{R}^3)$, the Sobolev space of second order. For $a > 0$, $\mathbb{C}_a = \{ z \in \mathbb{C} : |\operatorname{Im} z| < a \}$ is an open strip in the complex plane. For Banach spaces X and Y, $\mathscr{B}(X, Y)$ is the Banach space of all bounded (linear) operators from X to Y.

ASSUMPTION (A). — 1) $V(x)$ is a real-valued measurable function of $x \in \mathbb{R}^3$.

2) For each $\theta \in \mathbb{R}$, the multiplication operator V_θ by the function $V(e^\theta x)$ is a compact operator from $H^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. Moreover there exists $a > 0$ such that the function $\theta \to V_\theta$ can be extended to $\mathbb{C}_a$ as a $\mathscr{B}(H^2(\mathbb{R}^3), L^2(\mathbb{R}^3))$-valued analytic function.

ASSUMPTION (B). — 1) $\hat{\rho}(k)$ is the Fourier transform of a real-valued spherically symmetric function $\rho(x) \in L^2(\mathbb{R}^3)$.

2) The L^2-valued function $\mathbb{R} \ni \theta \to e^{-3\theta/2}\hat{\rho}(e^{-\theta}k) \in L^2(\mathbb{R}^3)$ can be extended to $\mathbb{C}_a$ as an analytic function $\hat{\rho}_\theta(k)$ of $\theta \in \mathbb{C}_a$.

We shall regard $H(\lambda)$ as a perturbed operator of $H(0) \equiv H_0$ and we write

(1.5) $$H(\lambda) = H_0 + H_I(\lambda)$$

(1.6) $$H_0 = \{ -(2m)^{-1}\Delta + V \} \otimes \mathbb{1} + \mathbb{1} \otimes H_0^{eM} = H^{el} \otimes \mathbb{1} + \mathbb{1} \otimes H_0^{eM}.$$

Under the Assumption (A), the operator H^{el} with $D(H^{el}) = H^2(\mathbb{R}^3)$ is selfadjoint and is bounded from below; the essential spectrum $\sigma_{ess}(H^{el}) = [0, \infty)$ and the discrete spectrum $\sigma_d(H^{el}) = \{ E_0 < E_1 < \ldots < 0 \}$ with 0 as the only possible accumulation point. H_0^{eM} is selfadjoint with its natural domain (see Sect. 2) and $\sigma(H_0^{eM}) = \{ 0 \} \cup [M, \infty)$. Thus H_0 is selfadjoint and bounded from below in $\mathscr{H}$ with the domain

$$D(H_0) = D(-\Delta \otimes \mathbb{1}) \cap D(\mathbb{1} \otimes H_0^{eM})$$

and its spectrum is given as

(1.7) $$\sigma(H_0) = \{ E_0 < E_1 < \ldots < 0 \} \cup [\min \{ 0, M + E_0 \}, \infty).$$

We note that if $M > 0$ is small, all the eigenvalues of H_0 but E_0 appear as embedded eigenvalues and they are expected to be very sensitive to perturbations. On the other hand, the perturbation $H_I(\lambda)$ is H_0-bounded ([3]) so that for sufficiently small λ, $H(\lambda)$ with $D(H(\lambda)) = D(H_0)$ is selfadjoint on $\mathscr{H}$ and $H(\lambda)$ has an isolated eigenvalue $E_0(\lambda)$ at the bottom of the spectrum. Furthermore, by virtue of the existence of the asymptotic field ([2] [6]), but under a little stronger assumption, we know

$$[E_0(\lambda) + M, \infty) \subset \sigma_{ac}(H(\lambda)).$$

However these seem to be almost all we know about $\sigma(H(\lambda))$ and many interesting questions are left open.

Among those we wish to study here the problem of perturbation of the embedded eigenvalues. We shall show, applying the complex scaling technique and the perturbation theory, that (under a suitable implicit assumption on the potential $V(x)$) all the eigenvalues E_j ($j \geqq 1$) embedded in the continuum of $\sigma(H_0)$ will after perturbation turn into resonances, the poles of the resolvent of the scaled Hamiltonian $H(\lambda, \theta)$ in the unphysical Riemann sheet and there will be no embedded eigenvalues near E_j for $H(\lambda)$.

Moreover the location of the poles can be computed by means of the perturbation series. This, of course, is accounted for, in physics language, the spontaneous emission of light and the Lamb shift of the spectral line of the atom ([4] [7]). We should remark here that the implicit assumption on $V(x)$ mentioned above seems to be satisfied by most potentials though no proofs exist. We also remark that we always assume $M > 0$ and for $M = 0$ our method does not apply. Nonetheless the resonances are uniformly away from the real line as $M \to 0$ and this suggests the existence of such resonance poles also for the massless field.

Thus this model (1.1) $\sim$ (1.4), which is obtained from the nonrelativistic QED by placing the ultraviolet cut off ρ and by replacing the energy $|k|$ by the massive $(k^2 + M^2)^{1/2}$, is mathematically tractable and gives some insight into the resonance phenomena in the nonrelativistic QED. However we should warn the reader that this model as it stands is not physical because besides the ultraviolet cut off it has the gauge condition (the Coulomb gauge) which is not compatible with the massive field equation. We also remark here that, as we shall not try to remove the cut off in this paper, the mass renormalization term is irrelevant to our theory and replacing $\delta m(\lambda)$ by some other $C\lambda^2$ term (C may be equal to zero) or including higher order terms will not change our mathematics at all. This term is chosen as in (1.4) only for the later convenience where one may try to remove the cut off.

The content of this paper is as follows. In Sect. 2, we introduce the dilation group in the Hilbert space $\mathcal{H}$ and examine the dilation analyticity of the free Hamiltonian H_0. The dilation analyticity of the total Hamiltonian $H(\lambda)$ will be shown in Sect. 3 where also the embedded eigenvalues of $H(\lambda)$ near E_j are identified with real eigenvalues of the scaled Hamiltonian $H(\lambda, \theta)$. In § 4, we study $\sigma(H(\lambda, \theta))$ by means of perturbation theory.

2. DILATION ANALYTICITY

In this section we examine the dilation analyticity of the operators to be used in the following sections. We define the dilation group $\{ U_e(\theta) : \theta \in \mathbb{R}^1 \}$ on $L^2(\mathbb{R}^3_x)$ by

$$(2.1) \qquad (U_e(\theta)f)(x) = e^{3\theta/2} f(e^\theta x), \qquad x \in \mathbb{R}^3, \quad \theta \in \mathbb{R}^1 .$$

THEOREM 2.1 (*Aguilar-Combes*). — Let $H^{el}(\theta) = -\dfrac{1}{2m} e^{-2\theta}\Delta + V_\theta$ with $D(H^{el}(\theta)) = H^2(\mathbb{R}^3)$ for $\theta \in \mathbb{C}_a$. Then $\{ H^{el}(\theta) : \theta \in \mathbb{C}_a \}$ is a selfadjoint holomorphic family of type (A) in the sense of Kato and satisfies the following properties:

1) For $\theta \in \mathbb{R}$, $U_e(\theta) H^{e1} U_e(\theta)^{-1} = H^{e1}(\theta)$.
2) $\sigma_{ess}(H^{e1}(\theta)) = e^{-2\theta}\mathbb{R}^+$.
3) $\sigma_d(H^{e1}(\theta))$ is invariant in θ: $\sigma_d(H^{e1}(\theta)) = \bigcup_{0 < \pm \operatorname{Im}\theta' < \pm \operatorname{Im}\theta} \sigma_d(H^{e1}(\theta'))$.

4)
$$\sigma_d(H^{e1}(\theta)) \cap \mathbb{R} = \sigma_p(H^{e1})$$

and
$$\sigma_d(H^{e1}(\theta)) \backslash \mathbb{R} \subset \{ z \in \mathbb{C} : \mp \operatorname{Im}\theta < \pm \arg z < 0 \}.$$
$$\overline{\sigma_d(H^{e1}(\theta))} \backslash \sigma_d(H^{e1}(\theta)) \subset \{ 0 \}.$$

5) The eigenfunction $\phi(x)$ of H^{e1} with eigenvalue $E < 0$ is dilation analytic, i. e. $\phi_\theta(x) = e^{3\theta/2}\phi(e^\theta x)$, $\theta \in \mathbb{R}$ can be extended to $\mathbb{C}_a$ as an $L^2(\mathbb{R}^3)$-valued analytic function of θ. $\phi_\theta(x)$ is the eigenfunction of $H^{e1}(\theta)$ with the same eigenvalue $E : H^{e1}(\theta)\phi_\theta = E\phi_\theta$, $\theta \in \mathbb{C}_a$.

6) Each eigenvalue of $H^{e1}(\theta)$ in $\sigma_d(H^{e1}(\theta)) \cap \mathbb{R}$ is semi-simple.

We also define unitary group of dilation $U_b(\theta)$ (and $U_{eM}(\theta)$) on the Boson Fock space $\mathscr{F}_b = \bigoplus_{n=0}^{\infty} L_s^2(\mathbb{R}^{3n})$ (and $\mathscr{F}_{eM} = \mathscr{F}_b \otimes \mathscr{F}_b$) as follows:

$L_s^2(\mathbb{R}^{3n}) = \{ \phi = \phi(k_1, \ldots, k_n) \in L^2(\mathbb{R}^{3n}) : \phi(k_1, \ldots, k_n) = \phi(k_{\sigma(1)}, \ldots, k_{\sigma(n)})$

for all $\sigma \in \mathscr{S}(n)$, n-th symmetric group $\}$

and $L_s^2(\mathbb{R}^0) \equiv \mathbb{C}$. For $\Psi = \bigoplus_{n=0}^{\infty} \Psi_n$, $\Psi_n \in L_s^2(\mathbb{R}^{3n})$,

$$(2.2) \quad \begin{cases} (U_b(\theta)\Psi)_n(k_1, \ldots, k_n) = e^{-3\theta n/2}\Psi_n(e^{-\theta k_1}, \ldots, e^{-\theta k_n}), \; n \neq 0 \backslash \\ (U_b(\theta)\Psi)_0 = \Psi_0 \end{cases}$$

and

$$(2.3) \qquad U_{eM}(\theta) = U_b(\theta) \otimes U_b(\theta).$$

For a measurable function $g(k)$ on the conjugate space $\mathbb{R}_k^3$ of $\mathbb{R}_x^3$, we write for $\theta \in \mathbb{R}$

$$(2.4) \qquad g_\theta(k) = e^{-3\theta/2} g(e^{-\theta}k) \qquad k \in \mathbb{R}^3.$$

The operator $d\Gamma(g)$ generated by g on $\mathscr{F}_b$ is defined as

$$(2.5) \qquad d\Gamma(g) = \bigoplus_{n=0}^{\infty} g^{(n)},$$

$$(2.6) \quad g^{(n)}\Psi_n(k_1, \ldots, k_n) = \begin{cases} \sum_{j=1}^{n} g(k_j)\Psi_n(k_1, \ldots, k_n), & n = 1, 2, \ldots, \\ 0 & n = 0. \end{cases}$$

with its natural domain. We obviously see that

$$(2.7) \qquad U_b(\theta) d\Gamma(g) U_b(\theta)^{-1} = d\Gamma(g_\theta), \qquad \theta \in \mathbb{R}.$$

The free photon energy operator H_0^{eM} may be written as

$$H_0^{eM} = d\Gamma(\omega) \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma(\omega), \tag{2.8}$$

with $\omega(k) = \sqrt{k^2 + M^2}$. By (2.7) and (2.8) we have for $\theta \in \mathbb{R}$,

$$U_{eM}(\theta)H_0^{eM}U_{eM}(\theta) = d\Gamma(\omega_\theta) \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma(\omega_\theta), \tag{2.9}$$

which will be written as $H_0^{eM}(\theta)$. Now the function of θ $\omega_\theta(k)=\sqrt{e^{-2\theta}k^2+M^2}$, for fixed k, has an analytic extension to $\mathbb{C}_{\pi/2}$ which has $\operatorname{Im} \omega_\theta(k) < 0$ for $\operatorname{Im} \theta > 0$. Using this fact, we have the following

LEMMA 2.2. — Define for $\theta \in \mathbb{C}_{\pi/2}$ $H_0^{eM}(\theta) = d\Gamma(\omega_\theta) \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma(\omega_\theta)$ with the domain $D(H_0^{eM}(\theta)) = D(H_0^{eM})$. Then

1) $\{ H_0^{eM}(\theta) : \theta \in \mathbb{C}_{\pi/2} \}$ is a selfadjoint holomorphic family of type (A).
2) $H_0^{eM}(\theta)$ is strictly m-sectorial with the semi-angle $|\operatorname{Im} \theta|$.
3) $\sigma_{ess}(H_0^{eM}(\theta)) = \bigcup_{n=1}^{\infty}\left\{ \sum_{j=1}^{n} \sqrt{e^{-2\theta}t_j + M^2} \mid t_j \geqq 0 \right\}$.
4) $\sigma_d(H_0^{eM}(\theta)) = \{ 0 \}$ is the unique eigenvalue of $H_0^{eM}(\theta)$ with the corresponding simple eigenfunction $\Omega_0 \otimes \Omega_0$, Ω_0 being the Fock vacuum.

Proof. — Since the other case may be proved similarly, we prove the case $0 \leqq \operatorname{Im} \theta < \pi/4$ only. For these θ, we have

$$0 < c_1(\theta) \leqq | \omega_\theta(k)/\omega(k) | \leqq c_2(\theta) < \infty \tag{2.8}$$

$$- \operatorname{Im} \theta < \arg \omega_\theta(k) \leqq 0 \tag{2.9}$$

with constants $c_1(\theta)$ and $c_2(\theta)$ which are independent of k and are taken uniformly on every compact set of $\theta \in \mathbb{C}_{\pi/4}$. All the statements but the analyticity obviously follow from (2.8), (2.9) and the definition of $H_0^{eM}(\theta)$. For proving the analyticity, we first note that $\mathscr{F}_{eM} = \mathscr{F}_b \otimes \mathscr{F}_b = \bigoplus_{n,m=0}^{\infty} (L_s^2(\mathbb{R}^{3n}) \otimes L_s^2(\mathbb{R}^{3m}))$ and for any $\Psi \in D(H_0^{eM}(\theta))$, $\Psi = \bigoplus_{n,m} \Psi_{n,m}$ the sequence $\Psi^{(l)} \in D(H_0^{eM}(\theta))$ defined as

$$\Psi_{m,n}^{(l)} = \begin{cases} \Psi_{m,n} & m + n \leqq l. \\ 0, & \text{otherwise}, \end{cases}$$

satisfies as $l \to \infty$

$$\| H_0^{eM}(\theta) [\Psi - \Psi^{(l)}] \|_{\mathscr{F}_{eM}} \to 0$$

uniformly in θ on every compact subset of $\mathbb{C}_{\pi/4}$. Thus by Weierstrass theorem, it sufficies to show that $H_0^{eM}(\theta)\Psi$ is analytic for $\Psi \in D(H_0^{eM}(\theta))$ with $\Psi_{n,m} = 0$ except for finite (n, m). But the analyticity for such Ψ follows from the following estimates: For any fixed $\theta \in \mathbb{C}_{\pi/4}$ and sufficiently small $\varepsilon > 0$. We have for all $|h| < \varepsilon$,

$$(2.10)\quad |\omega_{\theta+h}(k) - \omega_\theta(k) - h(d/d\theta)\omega_\theta(k)| = |h|^2 \left| \int_0^1 (1-t)\frac{d^2}{d\theta^2}\omega_{\theta+th}(k)dt \right| \leqq 3|h|^2 \int_0^1 (1-t)|\omega_{\theta+th}(k)|dt \leqq c(\theta,\varepsilon)|h|^2\omega(k).$$

$$(2.11)\quad |(d/d\theta)\omega_\theta(k)| \leqq \left| e^{-2\theta}k^2/\sqrt{e^{-2\theta}k^2 + \mathrm{M}^2} \right| \leqq |\omega_\theta(k)|.$$

This completes the proof of Lemma 2.3.

Using $\mathrm{U}_e(\theta)$ on $\mathrm{L}^2(\mathbb{R}_x^3)$ and $\mathrm{U}_{e\mathrm{M}}(\theta)$ on $\mathscr{F}_{e\mathrm{M}}$, we define the dilation group $\mathscr{U}(\theta)$ on $\mathscr{H} = \mathrm{L}^2(\mathbb{R}^3) \otimes \mathscr{F}_{e\mathrm{M}}$:

$$(2.12)\quad \mathscr{U}(\theta) = \mathrm{U}_e(\theta) \otimes \mathrm{U}_{e\mathrm{M}}(\theta),\ \mathrm{H}_0(\theta) = \mathscr{U}(\theta)\mathrm{H}_0\mathscr{U}(\theta)^{-1}.$$

LEMMA 2.3. — For $\theta \in \mathbb{C}_a$, let $\mathrm{H}_0(\theta) = \mathrm{H}^{e1}(\theta) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{H}_0^{e\mathrm{M}}(\theta)$ with $\mathrm{D}(\mathrm{H}_0(\theta)) = \mathrm{D}((-\Delta) \otimes \mathbb{1}) \cap \mathrm{D}(\mathbb{1} \otimes \mathrm{H}_0^{e\mathrm{M}})$. Suppose H^{e1} has the eigenvalues $\mathrm{E}_0 < \mathrm{E}_1 < \dots$ with the eigenfunctions $\phi_j(x)$ as in Lemma 2.1. Then

0) For $\theta' \in \mathbb{R}$, $\mathscr{U}(\theta')\mathrm{H}_0(\theta)\mathscr{U}(\theta') = \mathrm{H}_0(\theta + \theta')$.

1) $\{ \mathrm{H}_0(\theta) : \theta \in \mathbb{C}_a \}$ is a selfadjoint holomorphic family of type (A) on $\mathscr{H}$.

2) $\mathrm{H}_0(\theta)$ is maximal sectorial and

$$\sigma_{\mathrm{ess}}(\mathrm{H}_0(\theta)) = \{ \sigma(\mathrm{H}^{e1}(\theta)) + \sigma_{\mathrm{ess}}(\mathrm{H}_0^{e\mathrm{M}}(\theta)) \} \cup e^{-2\theta}\mathbb{R}_+ .$$

3) For $\theta \in \mathbb{C}_a^\pm$, $\sigma_d(\mathrm{H}_0(\theta)) \cap \mathbb{R} = \sigma_p(\mathrm{H}_0)\backslash\Sigma$, $\Sigma = \{0, n\mathrm{M}, \mathrm{E}_j + n\mathrm{M} : n = 1, 2, \dots\ j = 0, 1, 2, \dots \}$ is the threshold. Each $\mu \in \sigma_d(\mathrm{H}_0(\theta)) \cap \mathbb{R}$ is a semisimple eigenvalue of $\mathrm{H}_0(\theta)$.

4) $\mathrm{H}_0(\theta)$ has eigenvalues $\mathrm{E}_0 < \mathrm{E}_1 < \dots < 0$ with the eigenfunction $\Phi_j(\theta) = \phi_{j\theta} \otimes \Omega_0 \otimes \Omega_0$ and they are the only eigenvalues which are possibly isolated.

Proof. — We set for $\theta \in \mathbb{C}_a$

$$(2.13)\quad \mathrm{H}_{00}(\theta) = \left(-\frac{e^{-2\theta}}{2m}\Delta \right) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{H}_0^{e\mathrm{M}}(\theta)$$

with $\mathrm{D}(\mathrm{H}_{00}(\theta)) = \mathrm{D}((-\Delta) \otimes \mathbb{1}) \cap \mathrm{D}(\mathbb{1} \otimes \mathrm{H}_0^{e\mathrm{M}})$. Since $(-e^{-2\theta}/2m)\Delta$ is unitarily equivalent to the multiplication operator by $e^{-2\theta}p^2/2m$ on $\mathrm{L}^2(\mathbb{R}^3)$, via Fourier transform, an argument similar to that of the proof of Lemma 2.2 shows that $\{ \mathrm{H}_{00}(\theta) \}$ is a selfadjoint holomorphic family of type (A) and that $\mathrm{H}_{00}(\theta)$ is m-sectorial. Since V_θ is $(-\Delta + \mathbb{1})$-compact in $\mathrm{L}^2(\mathbb{R}^3)$, $\mathrm{H}_0(\theta) = \mathrm{H}_{00}(\theta) + \mathrm{V}_\theta \otimes \mathbb{1}$ is also m-sectorial for each $\theta \in \mathbb{C}_a$ and $\{ \mathrm{H}_0(\theta) : \theta \in \mathbb{C}_a \}$ is a selfadjoint holomorphic family of type (A) (see, Kato [*8*], p. 338). Now we may regard $\mathrm{H}_0(\theta) = \mathrm{H}^{e1}(\theta) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{H}_0^{e\mathrm{M}}(\theta)$. Then it follows from Ichinose's lemma (see, Reed-Simon [*10*]) that

$$(2.15)\quad \sigma(\mathrm{H}_0(\theta)) = \sigma(\mathrm{H}^{e1}(\theta)) + \sigma(\mathrm{H}_0^{e\mathrm{M}}(\theta)).$$

This proves the statements (1), (2) and the first half of (3). The semi-simplicity of the real eigenvalues $\mu \in \sigma_d(\mathrm{H}(\theta)) \cap \mathbb{R}$ may be proved along the line of the well-known argument of Aguilar-Combes [1] and we omit its proof here.

Now we look at how $\mathrm{U}_b(\theta)$ acts on the creation and annihilation operators, $a^*(f) = \int a^*(k)f(k)dk$ and $a(f) = \int a(k)f(k)dk$, $f \in \mathrm{L}^2(\mathbb{R}^3)$.

PROPOSITION 2.4. — Let $f \in \mathrm{L}^2(\mathbb{R}^3)$ and let $\mathrm{U}_b(\theta)$ be the dilation group on the Fock space $\mathscr{F}_b$ defined by (2.2). Then

$$(2.16)\quad \mathrm{U}_b(\theta)a(f)\mathrm{U}_b(\theta)^{-1} = a(f_\theta), \qquad \mathrm{U}_b(\theta)a^*(f)\mathrm{U}_b(\theta)^{-1} = a^*(f_\theta), \quad \theta \in \mathbb{R}.$$

(*Proof*). — Let $\Psi = \bigoplus_{n=0}^{\infty} \Psi_n$ with $\Psi_n \in \mathscr{S}(\mathbb{R}^{3n})$. Then

$$\begin{aligned}(2.17)\quad & (\mathrm{U}_b(\theta)a(f)\mathrm{U}_b(\theta)^{-1}\Psi)_n(p_1, \ldots, p_n) \\ &= e^{-3n\theta/2}[a(f)\mathrm{U}_b(\theta)^{-1}\Psi]_n(e^{-\theta}p_1, \ldots, e^{-\theta}p_n) \\ &= \sqrt{n+1}\,e^{-3n\theta/2}\int (\mathrm{U}_b(\theta)^{-1}\Psi)_{n+1}(e^{-\theta}p_1, \ldots, e^{-\theta}p_n, p_{n+1})f(p_{n+1})dp_{n+1} \\ &= \sqrt{n+1}\,e^{-3\theta/2}\int \Psi_{n+1}(p_1, \ldots, p_n, p_{n+1})f(e^{-\theta}p_{n+1})dp_{n+1} \\ &= (a(f_\theta)\Psi)_n(p_1, \ldots, p_n).\end{aligned}$$

Similarly

$$(2.18)\quad (\mathrm{U}_b(\theta)a^*(f)\mathrm{U}_b(\theta)^{-1}\Psi)_n(p_1, \ldots, p_n) = (a^*(f_\theta)\Psi)_n(p_1, \ldots, p_n).$$

Since $\mathrm{U}_b(\theta)$ is unitary, (2.17), (2.18) and the standard limiting argument imply (2.16).

PROPOSITION 2.5. — Suppose that f_θ and g_θ $(\theta \in \mathbb{R})$ can be extended to $\mathbb{C}_a$ as $\mathrm{L}^2(\mathbb{R}^3)$-valued analytic functions of θ such that $\sqrt{\omega}f_\theta, \sqrt{\omega}g_\theta \in \mathrm{L}^2(\mathbb{R}^3)$. Then for any $\Phi \in \mathrm{D}(d\Gamma(\omega))$, $a(f_\theta)\Phi$, $a(f_\theta)a(g_\theta)\Phi$, $a(f_\theta)a^*(g_\theta)\Phi$, $a^*(f_\theta)\Phi$, $a^*(f_\theta)a(g_\theta)\Phi$ and $a^*(f_\theta)a^*(g_\theta)\Phi$ are $\mathscr{F}_b$-valued analytic functions of $\theta \in \mathbb{C}_a$ and are bounded in norm by $(\|f_\theta\| + \|\sqrt{\omega}^{-1}f_\theta\| + \|\sqrt{\omega}f_\theta\|)(1 + \|g_\theta\| + \|\sqrt{\omega}^{-1}g_\theta\| + \|\sqrt{\omega}g_\theta\|)(\|\Phi\| + \|d\Gamma(\omega)\Phi\|)$.

Proof. — By standard estimates on creation-annihilation operators,

$$(2.18)\qquad \|a^*(f_\theta)\Phi\|^2 \leqq \|f_\theta\|^2\|\Phi\|^2 + \|f_\theta/\sqrt{\omega}\|^2\|\sqrt{d\Gamma(\omega)}\Phi\|^2$$

$$(2.19)\qquad \|a(f_\theta)\Phi\|^2 \leqq \|f_\theta/\sqrt{\omega}\|^2\|\sqrt{d\Gamma(\omega)}\Phi\|^2.$$

On the other hand the estimates

$$\int dk_1 \ldots dk_n \left| \sqrt{n+1} \int \Phi_{n+1}(k, k_1, \ldots, k_n) f_\theta(k) dk \right|^2 \sum_{j=1}^{n} \omega(k_j)$$

$$\leqq (n+1) \int dk_1 \ldots dk_n \left(\sum_{j=1}^{n} \omega(k_j) \right) \int | \Phi_{n+1}(k, \ldots, k_n) |^2 \omega(k) dk \, \| f_\theta/\sqrt{\omega} \|^2$$

$$\leqq \| (d\Gamma(\omega)\Phi)_{n+1} \|^2 \, \| f_\theta/\sqrt{\omega} \|^2$$

and

$$\int dk_1 \ldots dk_n \left| \frac{1}{\sqrt{n}} \sum_{l} \Phi_{n-1}(k_1, \ldots, \check{k}_l, \ldots, k_n) f_\theta(k_l) \right|^2 \sum_{j=1}^{n} \omega(k_j)$$

$$\leqq \frac{1}{n} \int dk_1 \ldots dk_n \left\{ \sum_{l} | \Phi_{n-1}(k_1, \ldots, \check{k}_l, \ldots, k_n) |^2 \, | f_\theta(k_l) |^2 \right\} \sum_{j=1}^{n} \omega(k_j)$$

$$+ \frac{1}{n} \int dk_1 \ldots dk_n \sum_{i \neq j} | \Phi_{n-1}(k_1, \ldots, \check{k}_i, \ldots, k_n) \Phi_{n-1}(k_1, \ldots, \check{k}_j, \ldots, k_n) \, | f_\theta(k_i) f_\theta(k_j) | \sum_{l} \omega(k_l)$$

$$\leqq \| \Phi_{n-1} \|^2 \, \| \sqrt{\omega} f_\theta \|^2 + \| \sqrt{d\Gamma(\omega)} \Phi_{n-1} \|^2 \, \| f_\theta \|^2$$

$$+ (n-1) \int \sum_{j=1}^{n} \omega(k_j) \, | \Phi_{n-1}(k_1, k_3, \ldots, k_n) | \, | \Phi_{n-1}(k_2, k_3, \ldots, k_n) | \, | f_\theta(k_1) f_\theta(k_2) | \, dk_1 \ldots dk_n$$

$$\leqq \| \Phi_{n-1} \|^2 \| \sqrt{\omega} f_\theta \|^2 + \| \sqrt{d\Gamma(\omega)} \Phi_{n-1} \|^2 \| f_\theta \|^2$$
$$+ 2 \| f_\theta/\sqrt{\omega} \| \, \| \sqrt{\omega} f_\theta \| \, \| \sqrt{d\Gamma(\omega)} \Phi_{n-1} \|^2 + \| d\Gamma(\omega) \Phi_{n-1} \|^2 \| f_\theta/\sqrt{\omega} \|^2$$

imply

$$\| \sqrt{d\Gamma(\omega)} a(f_\theta) \Phi \|^2 \leqq \| d\Gamma(\omega) \Phi \|^2 \, \| f_\theta/\sqrt{\omega} \|^2 \tag{2.20}$$

$$\| \sqrt{d\Gamma(\omega)} a^*(f_\theta) \Phi \|^2 \leqq \| \Phi \|^2 \, \| \sqrt{\omega} f_\theta \|^2 + \| \sqrt{d\Gamma(\omega)} \Phi \|^2 (\| f_\theta \|^2 + 2 \| \sqrt{\omega} f_\theta \| \, \| f_\theta/\sqrt{\omega} \|) + \| d\Gamma(\omega) \Phi \|^2 \, \| f_\theta/\sqrt{\omega} \|^2 . \tag{2.21}$$

(2.18) $\sim$ (2.21) obviously imply the statement about the estimate of the norms. Once one gets these estimates, it suffices to show the analyticity for $\Phi = \oplus \Phi_n$ such that $\Phi_n = 0$ for $n \geqq n_0$ for some n_0 and $\Phi_n \in C_0^\infty(\mathbb{R}^{3n})$. For such Φ, however, the analyticity of the vectors $a(f_\theta)\Phi$, etc. is obvious and we omit the proof.

3. DILATION ANALYTICITY OF THE TOTAL HAMILTONIAN

Being prepared with the lemmas of the previous section, we study in this section analyticity of the Hamiltonian $H(\lambda)$. We assume Assumption (B) in this section and consider for $\theta \in \mathbb{C}_a$,

$$(3.1)\quad \vec{A}(\theta, x) = \frac{1}{\sqrt{2}} \sum_{j=1}^{2} \{ a^*(\hat{\rho}_\theta(k) e^{-ikx} \vec{e}(k, j)/\sqrt{\omega_\theta}, j) + a(\hat{\rho}_\theta(-k) e^{ikx} \vec{e}(k, j)/\sqrt{\omega_\theta}, j) \}.$$

LEMMA 3.1. — Let $\Phi \in D(H_0)$. Then for $\theta \in \mathbb{C}_a$

$$(3.2)\quad \| \vec{A}(\theta, x) \cdot \vec{\nabla} e^{-\theta} \Phi \|_{\mathcal{H}} \leqq \{ 1/\cos(\operatorname{Im}\theta) \} c_1(\rho) \{ \| H_{00}(\theta)\Phi \| + \| \Phi \| \},$$

$$(3.3)\quad \| \vec{A}(\theta, x)^2 \Phi \|_{\mathcal{H}} \leqq \{ 1/\cos(\operatorname{Im}\theta) \} c_2(\rho) \{ \| H_{00}(\theta)\Phi \| + \| \Phi \| \}.$$

Here the constants $c_1(\rho)$, $c_2(\rho)$ are dependent only on ρ. Moreover the functions $\theta \to \vec{A}(\theta, x) e^{-\theta} \vec{\nabla}\Phi$ and $\theta \to \vec{A}(\theta, x)^2 \Phi$ are $\mathcal{H}$-valued analytic functions of $\theta \in \mathbb{C}_a$ and satisfy for $\theta \in \mathbb{R}$

$$(3.4)\quad \vec{A}(\theta, x) e^{-\theta} \vec{\nabla}\Phi = \mathcal{U}(\theta) \vec{A}(x) \cdot \vec{\nabla} \mathcal{U}(\theta)^{-1} \Phi,$$

$$(3.5)\quad \vec{A}(\theta, x)^2 \Phi = \mathcal{U}(\theta) \vec{A}(x)^2 \mathcal{U}(\theta)^{-1} \Phi.$$

Proof. — Since $\| \nabla e^{-\theta} \Phi \|_{\mathcal{H}}^2 = | e^{-2\theta} | (-\Delta\Phi, \Phi)$, it is clear that $\nabla e^{-\theta} \Phi \in D(| d\Gamma(\omega_\theta) |^{1/2})$ with

$$\| | d\Gamma(\omega_\theta) |^{1/2} \nabla e^{-\theta} \Phi \|^2 \leqq | e^{-2\theta} | \| -\Delta\Phi \| \| | d\Gamma(\omega_\theta) | \Phi \| \leqq (m / | \cos(\operatorname{Im}\theta) |)^2 \| H_{00}(\theta)\Phi \|^2.$$

Thus by Proposition 2.5, we have (3.2). (3.3) follows similarly and its proof is omitted. Note that $\| \hat{\rho}_\theta / \omega_\theta \|$, $\| \hat{\rho} \|$, $\| \hat{\rho}_\theta / \sqrt{\omega_\theta} \|$ are independent of $\operatorname{Re}\theta$ and hence are bounded for $| \operatorname{Im}\theta | \leqq a$.

The equations (3.4) and (3.5) follows from the definition of $\mathcal{U}(\theta)$ and Proposition 2.4. The analyticity of the operators are also clear from Proposition 2.4 and the estimates (3.2) and (3.3).

By Lemma 3.1, the operator

$$(3.6)\quad H_I(\theta, \lambda) = \frac{1}{2(m - \delta m(\lambda))} \left\{ \frac{\delta m(\lambda)}{m} (-e^{-2\theta}\Delta) - 2\lambda e^{-\theta} \vec{A}(\theta, x) \cdot (-i\nabla) + \lambda^2 \vec{A}(\theta, x)^2 \right\}, \qquad \theta \in \mathbb{C}_a,$$

is well-defined on $D(H_{00}(\theta)) = D(H_0(\theta))$ and is $H_0(\theta)$-bounded with bound dependent only on ρ and $\cos(\mathrm{Im}\,\theta)$. Therefore there exists a constant λ_0 depending only on ρ such that the operator $H(\theta, \lambda) = H_0(\theta) + H_I(\lambda, \theta)$ with domain $D(H(\theta, \lambda)) = D(H_0(\theta)) = D(H_{00}(\theta)) = D(H_{00})$ is a well-defined closed operator for $\theta \in \mathbb{C}_a$, $|\lambda| < \lambda_0$.

LEMMA 3.2. — For any fixed $\lambda \in \mathbb{R}$ with $|\lambda| < \lambda_0$ the operator $H(\theta, \lambda)$ $(\theta \in \mathbb{C}_a)$ is a selfadjoint holomorphic family of type (A). Moreover for $\tilde{\theta}$ real,

$$\mathcal{U}(\tilde{\theta})H(\theta, \lambda)\mathcal{U}(\tilde{\theta})^* = \mathcal{H}(\theta + \tilde{\theta}, \lambda). \tag{3.7}$$

Proof. — The first part of the lemma is obvious by Lemma 3.1. The equation (3.7) is clear for θ real by (3.4) and (3.5). Since both sides of (3.7) are analytic in θ on $D(H_{00})$, (3.7) holds for all $\theta \in \mathbb{C}_a$.

As a small perturbation of a maximal sectorial operator $H(\theta, \lambda)$ is also a quasi-maximal sectorial operator in $\mathcal{H}$ and the spectrum of $H(\theta, \lambda)$ may be analyzed by the bounded perturbation theory, although our complex scaling may not isolate all the singular spectrum and there may be eigenvalues which remain embedded after scaling, in contrast to the usual case where the perturbations are relatively compact (cf. [*1*]).

4. APPEARANCE OF THE RESONANCES

We know the structure of the spectrum of the non-interacting system H_0 very well: $\sigma(H_0) = \sigma_p(H_0) \cup [E_0 + M, \infty)$ and $\sigma_p(H_0) = \{ E_0 < E_1 < \dots \}$ is embedded in the continuum $[E_0 + M, \infty)$ except for the lowest E_0. As was mentioned at the introduction much is not known about the structure of the spectrum of $H(\lambda)$ and what we intend to show here is that for most potentials these embedded eigenvalues will disappear after the interaction is switched on and they will form resonances in the usual sense of the complex dilation theory. Let us first recall what is known: Since the interaction $H_I(\lambda)$ is H_0-bounded the isolated eigenvalue E_0 of H_0 is stable, i. e. there exists $\varepsilon > 0$ such that for $|\lambda| < \varepsilon$, there exists an eigenvalue $E_0(\lambda)$ of $H(\lambda)$ such that $E_0(\lambda) \to E_0$ as $\lambda \to 0$. (Note that E_0 is simple under our assumption on V, see Reed-Simon [*10*]). On the other hand the existence of the asymptotic field yields

$$[E_0(\lambda) + M, \infty) \subset \sigma_{ac}(H(\lambda)). \tag{4.1}$$

In the following discussions, we assume that H^{e1} has negative eigenvalues $E_0 < E_1 < E_2 < \dots$ with multiplicity m_j ($j = 0, 1, 2, \dots$) and the mass M of the « photon » satisfies the conditions

$$(4.2) \qquad 0 < M < E_1 - E_0$$

$$(4.3) \qquad E_j \notin \Sigma = \{ 0, nM, E_j + nM : j = 0, 1, 2, \dots, n = 1, 2, \dots \}.$$

Note that $\Sigma \cap (-\infty, 0]$ is a discrete set of $(-\infty, 0]$ and there are plenty of M which satisfy (4.2) and (4.3). We also assume

$$(4.4) \qquad \operatorname{Im} \theta \geqq 0,$$

since the other case can be treated by a similar method. We first prove the following theorem which shows that for proving the disappearance of the eigenvalues, it suffices to work with $H(\lambda, \theta)$ with $\operatorname{Im} \theta \neq 0$, in place of $H(\lambda)$.

THEOREM 4.1. — For each j, there exists $\varepsilon_j > 0$ such that for $|\lambda| < \varepsilon_j$ there exist eigenvalues $E_j^{(k)}(\lambda)$, $k = 1, \dots, m_j$, of $H(\lambda, \theta)$ such that $E_j^{(k)}(\lambda) \to E_j$ as $\lambda \to 0$. Moreover $E_j^{(k)}(\lambda)$ is independent of θ as long as it does not touch other parts of the spectrum of $H(\lambda, \theta)$. If $\operatorname{Im} E_j^{(k)}(\lambda) \neq 0$ for all $k = 1, \dots, m_j$, there are no eigenvalues for $H(\lambda)$ near E_j.

Proof. — Since $\sigma(H_0(\theta))$ is as is given by Lemma 2.3 and E_j is an isolated eigenvalue of $H_0(\theta)$, hence by the standard perturbation theory there exist exactly m_j eigenvalues $E_j^{(k)}(\lambda)$, $k = 1, \dots, m_j$, with $E_j^{(k)}(\lambda) \to E_j$ as $\lambda \to 0$. Since they are the eigenvalues of type (A) holomorphic family $\{ H(\lambda, \theta) \}$ and are independent of $\operatorname{Re} \theta$, $E_j^{(k)}(\lambda)$ is θ-independent (remember the standard dilation analyticity argument). Moreover the well-known Aguilar-Combes' proof for the identification $\sigma_p(H) = \sigma_d(H(\theta)) \cap \mathbb{R}$ also implies $\sigma_p(H(\lambda)) = \sigma_d(H(\lambda, \theta)) \cap \mathbb{R}$ under our assumption (4.3). This proves the theorem.

Since E_j is isolated, the usual perturbation theory provides the way to compute $E_j^{(k)}(\lambda)$. It is particularly simple when E_j is a simple eigenvalue.

COROLLARY 4.2. — Suppose that E_j is a simple eigenvalue of H^{e1} with the eigenfunction $\phi_j(x)$ (normalized). Then

$$(4.5) \qquad E_j(\lambda) = E_j + \lambda^2 E_{j,2} + 0(\lambda^3)$$

and $E_{j,2}$ is given as

$$(4.6) \qquad E_{j,2} = (T_2 \Phi_j, \Phi_j)_{\mathscr{H}} - \lim_{z \to E_j + i0} (T_1 (H_0 - z)^{-1} (1 - P_j) T_1 \Phi_j, \Phi_j)$$

in terms of the function $\Phi_j = \phi_j \otimes \Omega_0 \otimes \Omega_0$, the perturbations

$$(4.7) \qquad T_1 = \frac{i}{m} \sum_{j=1}^{3} A_j(x)\partial/\partial x_j$$

$$(4.8) \qquad T_2 = \frac{1}{2m} \sum_{j=1}^{3} A_j(x)^2 + \frac{\delta m(1)}{2m}(-\Delta) \otimes \mathbb{1},$$

and P_j the projection in $\mathscr{H}$ onto the space spanned by Φ_j.

Proof. — The standard computation shows (cf. Kato [8], p. 78) that

$$(4.9) \qquad E_j(\lambda) = E_j + \lambda E_{j,1} + \lambda^2 E_{j,2} + 0(\lambda^3)$$

with

$$(4.10) \qquad E_{j,1} = \operatorname{tr} P_j(\theta)T_1(\theta)P_j(\theta)$$

$$(4.11) \qquad E_{j,2} = \operatorname{tr} [P_j(\theta)T_2(\theta)P_j(\theta) - P_j(\theta)T_1(\theta)S(\theta)T_1(\theta)P_j(\theta)]$$

where $P_j(\theta)$, $T_1(\theta)$ and $T_2(\theta)$ are obvious correspondings to P_j, T_1 and T_2 given by (4.7) ~ (4.8) for $H(\lambda, \theta)$, $S(\theta)$ is the reduced resolvent of $H_0(\theta)$ at E_j and $P_j(\theta) = | \Phi_{j,\theta} \rangle\langle \Phi_{j,\bar{\theta}} |$. Then by the θ-independence of the following inner products, we have

$$(4.12) \quad E_{j,1} = (T_1(\theta)\Phi_{j,\theta}, \Phi_{j,\bar{\theta}}) = (T_1\Phi_j, \Phi_j) = 0$$

$$(4.13) \quad E_{j,2} = (T_2(\theta)\Phi_{j,\theta}, \Phi_{j,\bar{\theta}}) - (T_1(\theta)S(\theta)T_1(\theta)\Phi_{j,\theta}, \Phi_{j,\bar{\theta}}) = (T_2\Phi_j, \Phi_j) - \lim_{z\to E_j + i0} (T_1(H_0 - z)^{-1}(1 - P_j)T_1\Phi_j, \Phi_j).$$

(4.9) ~ (4.13) obviously prove (4.5).

The expression (4.5) and (4.6) can be used to compute the Fermi-Golden rule:

COROLLARY 4.3. — Suppose that E_j is a simple eigenvalue of H^{e1}. Then

$$(4.14) \quad \operatorname{Im} E_{j,2} \equiv \Gamma_j = -\pi \sum_{E_m \leq E_j - M} \sum_{l=1}^{2} \hat{\rho}^2(\sqrt{\mu^2 - M^2})\sqrt{\mu^2 - M^2} \int_{S^2} |(\phi_m, \tilde{\phi}_l(\cdot, \sqrt{\mu^2 - M^2}\,\hat{k}))_{L^2}|^2 d\hat{k}$$

with $\mu = E_j - E_m$. Here $\tilde{\phi}_l(x, v\hat{k}) = e^{-ivk\cdot x} \sum_{\alpha=1}^{3} e_\alpha(\hat{k}, l)\partial\phi_j/\partial x_\alpha$ and $(\ ,\ )_{L^2}$ is is the inner product of ϕ_m and $\tilde{\phi}_l(x, v\hat{k})$ w. r. t. x.

Proof. — Taking the imaginary parts in (4.5) and (4.6), we see that

$$(4.15)\quad \operatorname{Im} E_j(\lambda) = -\operatorname{Im} \lim_{z \to E_j + i0} \lambda^2 (T_1(H_0 - z)^{-1}(1 - P_j)T_1\Phi_j, \Phi_j)_{\mathscr{H}} + 0(\lambda^3)$$
$$= -\lim_{\varepsilon \downarrow 0} \varepsilon\lambda^2(((H_0 - E_j)^2 + \varepsilon^2)^{-1}(1 - P_j)T_1\Phi_j, T_1\Phi_j)_{\mathscr{H}} + 0(\lambda^3).$$

To compute the inner product in the RHS of (4.15) we first note that

$$(4.16)\quad ia^*(e^{-ixk}\hat{\rho}(k)/\sqrt{\omega(k)}e_\alpha(k, l), l)((\partial/\partial x_\alpha) \otimes \mathbb{1})\Phi_j, \qquad l = 1, 2$$

contains only one photon state of l-polarization. Hence $(1 - P_j)$ in the inner product may be omitted and we obtain, writing the spectral measure for H^{e1} as $dE_{H^{e1}}(\mu)$,

$$(((H_0 - E_j)^2 + \varepsilon^2)^{-1} ia^*(e^{-ikx}\hat{\rho}(k)/\sqrt{\omega(k)}e_\alpha(k, l), l)((\partial/\partial x_\alpha)\otimes\mathbb{1})\Phi_j,$$
$$ia^*(e^{-ikx}\hat{\rho}(k)/\sqrt{\omega(k)}e_{\tilde{\alpha}}(k, l), l)((\partial/\partial x_\alpha)\otimes\mathbb{1})\Phi_j)_{\mathscr{H}}$$
$$= \int_{\mathbb{R}^3} ((H^{e1} + \omega(k) - E_j)^2 + \varepsilon^2)^{-1}(e^{-ikx}\hat{\rho}(k)/\sqrt{\omega(k)}e_\alpha(k, l)(\partial\phi_j/\partial x_\alpha)),$$
$$e^{-ikx}\hat{\rho}(k)/\sqrt{\omega(k)}e_{\tilde{\alpha}}(k, l)(\partial\phi_j/\partial x_{\tilde{\alpha}}))_{L^2(\mathbb{R}^3_x)}d^3k$$

$$= \int_{\mathbb{R}^3} \frac{|\hat{\rho}(k)|^2}{\omega(k)} d^3k \left\{ \int_{-\infty}^{\infty} \frac{1}{(\mu + \omega(k) - E_j)^2 + \varepsilon^2} d\left(E_{H^{e1}}(\mu)\left(e^{-ikx}e_\alpha(k, l)\frac{\partial\phi_j}{\partial x_\alpha}\right),\right.\right.$$
$$\left.\left. e^{-ikx}e_{\tilde{\alpha}}(k, l)\frac{\partial\phi_j}{\partial x_\alpha}\right)\right\}$$

and we see that the inner product of (4.15) may be written as

$$(4.17)\quad \int_{\mathbb{R}^3} \frac{|\hat{\rho}(k)|^2}{\omega(k)} d^3k \int_{-\infty}^{\infty} \frac{\varepsilon}{(\mu + \omega(k) - E_j)^2 + \varepsilon^2}$$
$$\sum_{l=1}^{2} d\left(E_{H^{e1}}(\mu)\left(e^{-ikx}\sum_\alpha e_\alpha(k, l)\frac{\partial\phi_j}{\partial x_\alpha}\right), \left(e^{-ikx}\sum_\alpha e_\alpha(k, l)\frac{\partial\phi_j}{\partial x_\alpha}\right)\right).$$

We write

$$(4.18)\quad e^{-ikx}\sum_\alpha e_\alpha(k, l)\partial\phi_j/\partial x_\alpha = e^{-ikx}\partial\phi_j/\partial x_{e(k,l)}(x) = \tilde{\phi}_l(x, k),$$

where $\partial\phi_j/\partial x_{e(k,l)}$ is the $e(k, l)$-directional derivative of $\phi_j(x)$, and rewrite (4.17) as

$$(4.20)\quad \lim_{\varepsilon \downarrow 0} \int_0^{\infty} \frac{|\hat{\rho}(v)|^2 v^2}{(v^2 + M^2)^{1/2}} dv$$
$$\left\{ \int_{E_0}^{\infty} \frac{\varepsilon}{(\mu + \sqrt{v^2 + M^2} - E_j)^2 + \varepsilon^2} \int_{S^2} \sum_{l=1}^{2} d(E_{H^{e1}}(\mu)\tilde{\phi}_l(., v\hat{k}), \tilde{\phi}_l(., v\hat{k}))d\hat{k} \right\}.$$

When $\mu > E_j - (1-\delta)M$, $|\varepsilon/[(\mu + \sqrt{v^2+M^2} - E_j)^2 + \varepsilon^2]| \leqq \varepsilon/\delta M$, and we see

$$\lim_{\varepsilon\downarrow 0}\int_0^\infty \frac{|\hat\rho(v)|^2 v^2}{(v^2+M^2)^{1/2}}dv \int_{E_j-(1-\delta)M}^\infty \frac{\varepsilon}{(\mu+\sqrt{v^2+M^2}-E_j)^2+\varepsilon^2}\int_{S^2} d\hat k$$
$$\sum_{l=1}^{2} d(E_{H^{el}}(\mu)\tilde\phi_l(.,v\hat k), \tilde\phi_l(.,v\hat k))$$
$$\leqq \overline{\lim_{\varepsilon\downarrow 0}}\,(\varepsilon/\delta M)\int_0^\infty \frac{|\hat\rho(v)|^2 v^2}{(v^2+M^2)^{1/2}}dv\int_{S^2} d\hat k (E_{H^{el}}([E_j-(1-\delta)M,\infty)) \tilde\phi_l(.,v\hat k), \tilde\phi_l(.,v\hat k))_{L^2}$$
$$\leqq \overline{\lim_{\varepsilon\downarrow 0}}\,(\varepsilon/\delta M)\int_0^\infty |\hat\rho(v)|^2 v^2 (v^2+M^2)^{-1/2} dv \int_{S^2} d\hat k\, \|\nabla\phi_j\|^2 = 0.$$

On the other hand for $\mu < E_j - (1-\delta)M$, H^{el} has only point spectrum and (4.20) is written as

$$(4.21)\quad \lim_{\varepsilon\downarrow 0}\int_0^\infty \frac{|\hat\rho(v)|^2 v^2}{(v^2+M^2)^{1/2}}dv\int_{S^2} d\hat k \sum_{E_m\leqq E_j-(1-\delta)M} \varepsilon \sum_{l=1}^{2} |(\phi_m(x), \tilde\phi_l(x, v\hat k))|^2 ((E_m - E_j + \sqrt{v^2+M^2})^2 + \varepsilon^2)^{-1}$$
$$= \lim_{\varepsilon\downarrow 0}\sum_{E_m\leqq E_j-M}\sum_{l=1}^{2}\int_M^\infty \frac{\varepsilon|\hat\rho(\sqrt{\mu^2-M^2})|^2\sqrt{\mu^2-M^2}}{((\mu+E_m-E_j)^2+\varepsilon^2)}\left\{\int_{S^2} d\hat k |(\phi_m(x), \tilde\phi_l(x,\sqrt{\mu^2-M^2}\hat k))_{L^2(\mathbb{R}^3_x)}|^2\right\} d\mu$$
$$= \pi \sum_{E_m\leqq E_j-M}\sum_{l=1}^{2} |\hat\rho(\sqrt{(E_j-E_m)^2-M^2})|^2\sqrt{(E_j-E_m)^2-M^2}\int_{S^2} d\hat k |(\phi_m(x), \tilde\phi_l(x,\sqrt{\mu^2-M^2}\hat k))|^2.$$

(4.21) obviously implies the desired expression (4.14).

It follows from the expression (4.14) that if $\operatorname{Im} E_j(\lambda) = 0$,

$$(4.22)\quad (\phi_m(x), e^{-i\sigma k.x}\partial\phi_j/\partial v_k) = 0, \qquad \sigma = \sqrt{(E_j - E_m)^2 - M^2},$$

for all eigenfunction $\phi_m(x)$ of H^{el} with eigenvalue $E_m < E_j - M$, all $\hat k \in S^2$ and all the directional derivative $\partial\phi_j/\partial v_k$ of ϕ_j in the direction orthogonal to $\hat k$. Note that the LHS of (4.22) is a real analytic function of $\sigma > 0$ for any fixed $\hat k$ and v_k. In fact,

$$(\phi_m(x), e^{-i\sigma e^\theta k.x}\partial\phi_j/\partial v_k(x))_{L^2(\mathbb{R}^3)} = e^{-3\theta}(\phi_m(e^{-\theta}x), e^{-i\sigma\hat k.x}(\partial\phi_j/\partial v_k)(e^{-\theta}x))$$

and the analyticity of $e^{-3\theta/2}\phi_m(e^{-\theta}x)$ and $e^{-3\theta}(\partial\phi_j/\partial v_k)(e^{-\theta}x)$ imply the analyticity of the LHS of (4.22). Thus it can not be zero except for countable σ's, except for the case when it is identically zero, in which case, however, by Plancherel's inversion formula, we have

$$(4.23)\quad 0 = \int_{-\infty}^{\infty} \phi_m(\hat{k}\rho + x)\frac{\partial\phi_j}{\partial\rho}(\hat{k}\rho + x)d\rho, \qquad x \in \mathbb{R}^3, \quad \hat{k} \in S^2 .$$

Summing up, we have the following

COROLLARY 4.4. — Suppose that for some $E_m < E_j$, $k \in S^2$ and some $x \in \mathbb{R}^3$ $\int_{-\infty}^{\infty} \phi_m(k\rho + x)\frac{\partial\phi_j}{\partial\rho}(k\rho + x)d\rho \neq 0$. Then for almost all small $M > 0$, there exists $\lambda_0 > 0$ such that for all $|\lambda| < \lambda_0$, Im $E_j(\lambda) < 0$ and $H(\lambda)$ has no eigenvalues near E_j.

Finally the case when E_j is degenerated.

COROLLARY 4.5. — Suppose that E_j has multiplicity m_j and $\phi_j^{(1)}(x), \ldots, \phi_j^{(m_j)}(x)$ are the orthonormalized eigenfunctions of H^{e1} with the eigenvalue E_j. We set $\Phi_j^{(k)} = \phi_j^{(k)} \otimes \Omega_0 \otimes \Omega_0$. Then $E_j^{(k)}(\lambda), k = 1, 2 \ldots, m_j$ are asymptotically

$$E_j^{(k)}(\lambda) = E_j + \lambda^2 E_{j,2}^{(k)} + 0(\lambda^3), \qquad \text{as} \quad \lambda \to 0$$

where $E_{j,2}^{(k)}$ are the eigenvalues of the $m_j \times m_j$-matrix

$$\{ (T_2\Phi_j^{(\tilde{l})}, \Phi_j^{(l)}) - \lim_{z \to E_j + i0} (T_1(H_0 - z)^{-1}(1 - P_0)T_1\Phi_j^{(\tilde{l})}, \Phi_j^{(l)}) \}_{(\tilde{l},l)}.$$

Proof. — The proof may be carried out in a way similar to that of Corollary 4.2 by using the standard perturbation theory and the invariance of the inner product in θ. We omit the detail here.

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REFERENCES

[1] J. AGUILAR, J. M. COMBES, A class of analytic perturbations for one body Schrödinger Hamiltonians, *Commun. Math. Phys.*, t. **22**, 1971, p. 269-279.

[2] S. ALBEVERIO, An introduction to some mathematical aspects of scattering in models of quantum fields, In *Scattering theory in Mathematical Physics*, Lavita, J. A. and Marchand, J. P. (eds.): Dordrecht; Reidel Publishing Company, 1974.

[3] A. ARAI, Selfadjointness and spectrum of Hamiltonians in non-relativistic quantum electrodynamics, *J. Math. Phys.*, t. **22**, 1981, p. 534-537.

[4] H. A. BETHE, The electromagnetic shift of energy levels, *Phys. Rev.*, t. **72**, 1947, p. 339-341.

[5] A. GROSSMANN, A. TIP, Hydrogen atoms interacting with a quantized radiation mode, *J. Phys. A. Math. Gen.*, t. **13**, 1980, p. 3381-3397.

[6] R. HOEGH-KROHN, On the spectrum of the space cut-off: $P(\phi)$: Hamiltonian in two-space-time dimensions, *Commun. Math. Phys.*, t. **21**, 1971, p. 256-260.

[7] J. M. JAUCH, F. ROHRLICH, *The theory of photons and electrons* (2nd ed.), New York, Springer, 1976.

[8] T. KATO, *Perturbation theory for linear operators* (2nd ed.), New York, Springer, 1976.

[9] Y. KATO, N. MUGIBAYASHI, Regular perturbation and asymptotic limits of operators in quantum field theory, *Prog. Theor. Phys.*, t. **30**, 1963, p. 103-133.

[10] M. REED, B. SIMON, *Method of modern mathematical physics, IV, Analysis of Operators*, New York, Academic Press, 1978.

[11] B. SIMON, Resonances in *n*-body quantum systems with dilation analytic potentials and the foundations of time-dependent perturbation theory, *Ann. Math.*, t. **97**, 1973, p. 247-274.

[12] K. YAJIMA, Resonances for the AC-Stark Effect, *Commun. Math. Phys.*, t. **87**, 1982, p. 331-352.

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