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Resonances and convergence of perturbation theory for N-body atomic systems in external AC-electric field

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by

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ABSTRACT. — It is proved that the action of a weak external AC-electric field of frequency $\omega/2\pi$ shifts all non-threshold bound states and resonances of any N-body atomic system into resonances of the AC-Lo Surdo-Stark effect, defined as eigenvalues of the complex scaled Floquet Hamiltonian. In marked contrast with the $\omega = 0$, DC-field case, the Rayleigh-Schrödinger perturbation expansion converges to the resonances. The first non vanishing order for the resonance width is determined by the number of photons it takes to ionize the bound state turning into the resonance and is given by the Fermi Golden Rule. For the two-body case we also show that if the energy difference of two unperturbed bound states is $n\omega$, $n \in \mathbb{Z}$, then there is a resonant solution oscillating between them for a long time.

RÉSUMÉ. — On prouve que l'action d'un champ électrique alternatif externe faible de fréquence $\omega/2\pi$ transforme tous les états liés différents des seuils et les résonances d'un système atomique à N corps quelconque en résonances de l'effet Lo Surdo-Stark alternatif, définies comme les

valeurs propres du Hamiltonien de Floquet dilaté dans les complexes. Contrairement au cas du champ continu $\omega = 0$, la série de perturbation de Rayleigh-Schrödinger converge vers les résonances. Le premier ordre non nul de la largeur d'une résonance est déterminé par le nombre de photons nécessaire pour ioniser l'état lié transformé en cette résonance, et est donné par la règle d'or de Fermi. Dans le cas à deux corps, on montre aussi que si la différence d'énergie de deux états liés non perturbés est $n\omega$, $n \in \mathbb{Z}$, alors il existe une solution résonante qui oscille entre les deux pendant longtemps.

1. INTRODUCTION

The so-called AC-Lo Surdo-Stark effect, described by an N-body Schrödinger operator under the action of a spatially homogeneous, time-sinusoidal, external electric field is being intensely investigated since several years (see e. g. [1] [9] [15]-[18] and references therein).

Writing the external field in the Coulomb gauge, the time-dependent Schrödinger operator, acting on $L^2(\mathbb{R}^{3N})$-valued functions of time, has the form:

$$(1.1) \qquad H_1(F, t) = T + F \cos \omega t \sum_{i=1}^{N} \vec{e} \cdot \vec{r}_i$$

Here T is the N-body Schrödinger operator:

$$(1.2) \qquad T = -\frac{1}{2} \sum_{i=1}^{N} \Delta_i + \sum_{i=1}^{N} V_i(\vec{r}_i) + \sum_{i<k=1}^{N} V_{ik}(\vec{r}_i - \vec{r}_k)$$

$e = \hbar = m_i = 1$, $i = 1, \ldots, N$; $\vec{e} = (1, 0, 0)$, and $F > 0$ is the strength of the electric field of frequency $\omega/2\pi$.

Since the perturbation is time-periodic, it is well known that the non-stationary operator (1.1) can be reduced to a stationary one by introducing the Floquet Hamiltonian (the quasi-energy operator in the language of Refs. [25] [15]-[17])

$$(1.3) \qquad K_1(F) = H_1(F, t) - i \frac{\partial}{\partial t}$$

acting in the Hilbert space $\mathscr{K} = L^2(\mathbb{R}^{3N}) \otimes L^2(\mathbb{T}_\omega)$, $\mathbb{T}_\omega = \mathbb{R} \backslash (2\pi/\omega)$ the circle. If indeed λ is an eigenvalue of (1.3) with eigenvector $\phi(., t)$, then formally $\psi(., t) = e^{-i\lambda t} \phi(., t)$ solves the Schrödinger equation

$$H_1(F, t)\psi(., t) = i \frac{\partial \psi}{\partial t}(., t)$$

with ϕ periodic in time. Therefore the Floquet (or, equivalently, the quasi-energy) formalism, first implemented by Howland [10] and Yajima [24] has been at the basis also of the recent rigorous work on the subject [23] [6] (see also Tip [22] for the related but simpler case of a circularly polarized electric wave). Assuming a highly smooth potential Yajima [23] was indeed able to synthetize Floquet theory and dilation analyticity, thus obtaining in the two-body case the rigorous justification of the well known physical picture in this kind of time-periodic quantum problems (see e. g. [14, §§ [42]-[43]]): if λ is a bound state of T, then $\lambda + n\omega$, $n \in \mathbb{Z}$, is an embedded eigenvalue of $K_1(0)$ (quasi-energy state in the language of [25] [15]-[17]) turning for $F > 0$ small into a resonance (quasi-stationary quasi-energy state) of $K_1(F)$ in the standard sense of dilation analyticity. The ionization rate is the resonance width: the first non vanishing order in perturbation theory for the width is determined by the condition $\lambda + n\omega > 0$, i. e. by the number of photons it takes to ionize the bound state turning into a resonance, and is given by the Fermi Golden Rule. If two bound states of T have energy difference $n\omega$, $n \in \mathbb{Z}$, then excitation takes place in addition to ionization in the sense that there is a solution of the Schrödinger equation oscillating between the two states for a long time.

The main difficulty preventing a direct extension of these results to the atomic case lies in the fact that the Coulomb potential is not dilation analytic when considered in the moving frame (see e. g. [6] for a discussion, and for a partial extension of Yajima's results to the two-body Coulomb case by means of Simon's exterior complex scaling [21]). The point of this paper is that this difficulty does not occur by working (as in [15]-[17], [26]) in the radiation gauge for the external field. In fact, the transformation to the gauge $(\vec{A}(t), 0)$ $\vec{A}(t) = A(t)\vec{e}$, $A(t) = F\omega^{-1} \sin \omega t$, generated by $\chi = \left\langle \vec{A}(t), \sum_{i=1}^{N} \vec{r}_i \right\rangle$, is implemented in quantum mechanics by the unitary transformation $\hat{U} : \psi \to \exp(i\chi)\psi$ in $\mathcal{H}$, and the unitary image of $K_1(F)$ under $\hat{U}$ is

$$(1.4) \quad K(F) \equiv H(F, t) - i \frac{\partial}{\partial t} = \frac{1}{2} \sum_{k=1}^{N} (- i\vec{\nabla}_k - A(t)\vec{e})^2$$

$$+ \sum_{k=1}^{N} V_k + \sum_{i<k=1}^{N} V_{ik} - i \frac{\partial}{\partial t}$$

Equivalently, setting $u(., t) = \exp(i\chi)u_R(., t)$ the Schrödinger equation $H_1(F, t)u = i\dfrac{\partial u}{\partial t}$ becomes:

$$(1.5) \qquad\qquad H(F, t)u_R = i\frac{\partial}{\partial t} u_R$$

which in the limit $\omega \rightarrow 0$ reduces to the time-dependent DC-case Schrödinger equation in the moving frame

$$(1.6) \qquad i\frac{\partial u_R}{\partial t} = \left[\frac{1}{2}\sum_{k=1}^{N}(-i\vec{\nabla}_k - F\vec{e}t)^2 + \sum_{k=1}^{N}V_k + \sum_{i<k=1}^{N}V_{ik}\right]u_R$$

Relying on Yajima's basic formalism [23], in Sect. 2 we will realize the complex scaled version of K(F) as a holomorphic operator family near F = 0 (Prop. 2.3). Hence, unlike the $\omega = 0$, DC-field case [4]-[5] [7]-[8], the resonances have a convergent perturbation expansion (Thm. 2.6). This fact has been conjectured (and explicitly verified for the two-body, δ-function potential) by Manakov-Feinshtein [15] also on the basis of the different nature of the classical motions when all the V's are zero: uniformly accelerated motions for $\omega = 0$, but oscillations with increasing amplitudes for $\omega \neq 0$. Furthermore, once more unlike the $\omega = 0$ case, the resonance width is directly given by perturbation theory and is characterized as above (Thm. 3.1). Finally we will see (Thm. 3.2) that the present formalism yields, in the two-body case, the classical resonance phenomenon for a class of potentials more general than those considered in [23]. We conclude the introduction by stating the assumptions on the two-body potentials and the notation employed.

A1 : Let $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ be any two-body potential. Then $V \in C_\alpha$ for any $\alpha < \dfrac{\pi}{4}$. Here C_α is Combes' class of dilation analytic potentials (see e. g. [19, XIII.10] for the definition).

It is well known that the Coulomb potential belongs to C_α for any $\alpha > 0$. As far as the notation is concerned, $\mathscr{H}$ stands for the Hilbert space $L^2(\mathbb{R}^{3N})$, $\mathscr{K}$ for the Hilbert space $L^2(\mathbb{T}_\omega) \otimes \mathscr{H}$, $\mathbb{T}_\omega = \mathbb{R}\backslash(2\pi/\omega)$ the circle. If A is a linear operator in a Hilbert space X, we denote by D(A) its domain, by $\Theta(A)$ its numerical range, by $\rho(A)$ its resolvent set, by $\sigma(A)$ its spectrum. T_0 denotes the 3N-dimensional Laplace operator, $D(T_0) = H^2(\mathbb{R}^{3N}) \equiv H^2$;

$$T = T_0 + W, W = \sum_{i=1}^{N}V_i(\vec{r}_i) + \sum_{i<k=1}^{N}V_{ik}(\vec{r}_i - \vec{r}_k), D(T) = H^2, \text{ denotes the}$$

N-body Schrödinger operator in $\mathscr{H}$. By $\mathbb{C}_a$ we denote the complex strip

$\{z \in \mathbb{C} : |\operatorname{Im} z| < a\}$, and by $\mathbb{C}_a^{\pm}$ the open strips $\{z \in \mathbb{C}_a : 0 < \pm \operatorname{Im} z < \pm a\}$. For $\theta \in \mathbb{C}_a$ we set $T_0(\theta) = e^{-2\theta}T_0$, $T(\theta) = T_0(\theta) + W(\theta)$,

$$W(\theta) = \sum_{i=1}^{N} V_i(e^{\theta}\vec{r}_i) + \sum_{i<k=1}^{N} V_{ik}(e^{\theta}(\vec{r}_i - \vec{r}_k)), \qquad D(T(\theta)) = H^2.$$

We have $T(\theta)^* = T(\bar{\theta})$, and, if $\theta \in \mathbb{R}$, $T(\theta) = S(\theta)TS(\theta)^{-1}$, where $S(\theta)$ is the unitary dilation in $\mathscr{H}$, $(S(\theta)f)(\vec{x}) = e^{3N\theta/2}f(e^{\theta}\vec{x})$, $f \in \mathscr{H}$, $\vec{x} \in \mathbb{R}^{3N}$. For the spectral analysis of $T(\theta)$ see e.g. [19, § XIII.4, XIII.10]. We denote by K_0, $K_0(\theta)$, K, $K(\theta)$ the operators in $\mathscr{K}$ defined as the action of $T_0 - i\dfrac{\partial}{\partial t}$, $T_0(\theta) - i\dfrac{\partial}{\partial t}$, $T - i\dfrac{\partial}{\partial t}$, $T(\theta) - i\dfrac{\partial}{\partial t}$, respectively, on the common domain $L^2(\mathbb{T}_\omega) \otimes H^2 \cap H^1(\mathbb{T}_\omega) \otimes \mathscr{H}$. For $\theta \in \mathbb{R}$, $\hat{S}(\theta)K\,\hat{S}(\theta)^{-1} = K(\theta)$, $\hat{S}(\theta) = I \otimes S(\theta)$. We have $\sigma(K_0) = \bigcup_{n=-\infty}^{+\infty} (\sigma(T_0) + n\omega)$, and the same relation holds for $K_0(\theta)$. We denote by $\mathscr{F}$ the Fourier transform in $\mathscr{H}$:

$$(\mathscr{F}f)(\vec{p}) = (2\pi)^{-3N/2} \int_{\mathbb{R}^{3N}} f(\vec{x})e^{i\langle \vec{p}, \vec{x}\rangle}d\vec{x}, \qquad f \in \mathscr{H}$$

and by $\mathscr{F}_t$ the Fourier transform in $L^2(\mathbb{T}_\omega)$:

$$g(t) = \sum_{n=-\infty}^{+\infty} (\mathscr{F}_t g)(n)e^{in\omega t}$$

We refer to Yajima [23] for any further notation not explicitly specified here.

2. THE FLOQUET OPERATOR :
DILATION ANALYTICITY AND ANALYTICITY
IN THE FIELD STRENGTH

In this Section we will first establish, in analogy to Yajima [23], the connection between solutions of the time-dependent Schrödinger equation and spectral properties of the Floquet operator in the framework of dilation analyticity. The first relevant result is (compare with [23, Lemma 2.2, 2.3]):

2.1 PROPOSITION. — Let $\theta \in \mathbb{C}_a$, $a < \dfrac{\pi}{4}$, $t \in \mathbb{T}_\omega$, $F \in \mathbb{C}$, $A(t) = \omega^{-1}F \sin \omega t$, $\omega > 0$.

(1)   The operator in $\mathscr{H}$ defined as

$$(2.1)\quad H(F, \theta, t) = \frac{1}{2} \sum_{k=1}^{N} (-ie^{-\theta}\vec{\nabla}_k - \vec{e}A(t))^2 + W(\theta), \qquad D(H(.)) = H^2$$

represents for any fixed t a self-adjoint holomorphic family of type A in $(F, \theta) \in \mathbb{C} \times \mathbb{C}_a$. For $\theta \in \mathbb{R}$;

$$H(F, \theta, t) = S(\theta)H(F, t)S(\theta)^{-1}, \qquad H(F, t) \equiv H(F, 0, t).$$

(2)   There is $0 < M < \infty$ such that $\pm iH(F, \theta, t) + M$ is m-accretive and the function $(F, \theta, t) \to (iH(F, \theta, t) - z)^{-1} \in B(\mathscr{H})$ is differentiable in $(F, \theta, t) \in \mathbb{C} \times \overline{\mathbb{C}}_a^{\pm} \times \Gamma_\omega, \operatorname{Re} z < M.$

(3)   For any fixed $(F, \theta, t) \in \mathbb{R} \times \overline{\mathbb{C}}_a^{\pm} \times \mathbb{T}_\omega, \pm iH(F, \theta, t)$ generates a C_0-semi-group $\exp(\mp i\sigma H(F, \theta, t))$ in $\mathscr{H}$, with $\| \exp(\mp i\sigma H(F, \theta, t)) \| \le e^{M\sigma}$. If $\theta \in \mathbb{C}_a^{\pm}$, $\exp(\mp iH(F, \theta, t)$ is a holomorphic semigroup of type $\mathscr{H}(2\operatorname{Im}\theta - \delta, \gamma_\delta)$ for any $\delta > 0$ and some $\gamma_\delta > 0$.

(4)   The $B(\mathscr{H})$-valued function $\exp(\mp i\sigma H(F, \theta, t))$ is strongly continuous in $(\sigma, F, \theta, t) \in \mathbb{R}_+ \cup \{0\} \times \mathbb{R} \times \overline{\mathbb{C}}_a^{\pm} \times \mathbb{T}_\omega$ and is analytic for $\theta \in \mathbb{C}_a^{\pm}$. For $\phi \in \mathbb{R}$ and $\theta \in \overline{\mathbb{C}}_a^{\pm}$ we have:

$$(2.2)\qquad S(\phi) \exp(\mp i\sigma H(F, \theta, t))S(\phi)^{-1} = \exp(\mp i\sigma H(F, \theta + \phi, t))$$

Proof. — For $u \in C_0^\infty(\mathbb{R}^{3N})$ we can write:

$$(2.3)\qquad H(.)u = T(\theta)u + ie^{-\theta}A(t) \sum_{k=1}^{N} (\vec{\nabla}_k)_x u + \frac{1}{2}NA(t)^2 u$$

Now the multiplication by $A(t)$ is bounded by $|F|\omega^{-1}$, and since $D(T(\theta)) = H^2$ it is obviously seen that for any $b > 0$ there is $a > 0$ independent of (F, θ, t) in the compacts of $\mathbb{C} \times \mathbb{C}_a \times \mathbb{T}_\omega$ such that

$$(2.4)\qquad \left\| A(t). \sum_{i=1}^{N} (\vec{\nabla}_i)_x u \right\| \le b \| T(\theta)u \| + a \| u \|$$

Therefore $H(F, \theta, t)$ defined on H^2 is closed and has non empty resolvent set because $\bigcap\limits_{|\operatorname{Im}\theta| \le \pi/4} \rho(T(\theta)) \ne \phi$. Since the $\mathscr{H}$-valued function $(F, \theta, t) \to H(F, \theta, t)u$ is of course entire in $(F, \theta) \in \mathbb{C} \times \mathbb{C}$ for any fixed $t \in \mathbb{T}_\omega$, $H(F, \theta, t)$ is a type A-holomorphic family by definition [19, XII.2]. The properties

$$H(F, \theta, t)^* = H(\overline{F}, \overline{\theta}, t) \qquad \text{and} \qquad S(\theta)H(F, t)S(\theta)^{-1} = H(F, \theta, t), \quad \theta \in \mathbb{R},$$

are obvious. Assertion (2) is also obvious because the estimate (2.4)

is (F, θ, t)-independent. We show (3) and (4) for the $+$ case only. The other case follows by symmetry. For $\theta \in \overline{C}_a^+ \quad + iH(F, \theta, t) \in \mathscr{G}(1, M)$ by (2) and [12, Problem IX.1.18.]. For $\theta \in C_a^+ \quad iT_0(\theta)$ generates a holomorphic semigroup of class $\mathscr{H}(2 \operatorname{Im} \theta, 0)$ [12, Thm, IX.1.24].

By assumption A.1, $W(\theta)$ is bounded with respect to $T_0(\theta)$ with relative bound 0 uniformly with respect to θ, and the same is true for $A(t)$. $\displaystyle\sum_{i=1}^{N} (\vec{\nabla}_i)_x$ because (2.4) obviously holds with $T(\theta)$ replaced by $T_0(\theta)$. Therefore (3) follows by [12, Cor. IX.2.5]. Assertion (4) is proved exactly as Assertion (3) of [23, Lemma 2.3] (the boundedness of $(iH(F, \theta, t) - z)^{-1}$ for $z \in \Gamma_\varepsilon$ follows once more from the relative boundedness argument which implies $\operatorname{dist}(\Gamma_\varepsilon, \theta(iH(F, \theta, t)) > 0$ for $0 < \delta \leq \operatorname{Im} \theta < a, \delta > 0)$. ∎

The existence of a unique propagator generated by the Schrödinger equation $H(F, \theta, t)\psi = i\dfrac{\partial \psi}{\partial t}$ is ensured by the following Lemma whose proof, given Prop. 2.1, is identical to that of [23, Lemma 2.4] and is therefore omitted.

2.2. LEMMA. — Let $\theta \in \overline{C}_a^{\pm}$ and $F \in \mathbb{R}$. Then the time-dependent Schrödinger equation:

(2.5) $\quad H(F, \theta, t)\psi(\vec{r}, t) = i\dfrac{\partial \psi}{\partial t}(\vec{r}, t), \qquad \vec{r} = (\vec{r}_1, \ldots, \vec{r}_N) \in \mathbb{R}^{3N}, \qquad t \in \mathbb{R}$

generates a unique propagator $U(t, s, F, \theta)$ such that:

(1) $\quad U(s, s; F, \theta) = I; \quad U(t, r; F, \theta)U(r, s; F, \theta) = U(t, s; F, \theta)$

for $\pm t \geq \pm r \geq \pm s$.

(2) $\quad U(t, s; F, \theta)H^2 \subset H^2$, $U(t, s; F, \theta)$ is differentiable in (t, s) for any $f \in H^2$ and

(2.6) $\qquad i\dfrac{\partial}{\partial t} U(t, s; F, \theta)f = H(F, \theta, t)U(t, s; F, \theta)f$

(2.7) $\qquad -i\dfrac{\partial}{\partial s} U(t, s; F, \theta)f = U(t, s; F, \theta)H(F, \theta, s)f$

(2.8) $\qquad \| U(t, s; F, \theta) \| \leq e^{M|t - s|} \qquad$ for some $\quad M > 0$

(3)

(2.9) $\qquad U\left(t + \dfrac{2\pi}{\omega}, s + \dfrac{2\pi}{\omega}; F, \theta\right) = U(t, s; F, \theta)$

(4)

(2.10) $\qquad$ If $\phi \in \mathbb{R}$, $U(t, s; F, \theta + \phi) = S(\phi)U(t, s; F, \theta)S(\phi)^{-1}$ $\qquad$ (2.10)

(5) $\quad U(t, s; F, \theta)$ is strongly continuous in $(t, s; F, \theta)$ for $\pm t \geq \pm s, \theta \in \overline{C}_a^{\pm}$, $F \in \mathbb{R}$, and is analytic in $\theta \in C_a^{\pm}$ for any such fixed $(t, s; F)$.

(6)   For $(\theta, F) \in \mathbb{R} \times \mathbb{R}$ $\{ U(t, s\,; F, 0) : (t, s) \in \mathbb{R} \}$ is a unitary propagator.

Remarks. — (1) Consider the two-body case $N = 1$. Then the map

$$(R_1(t)f)(\vec{r}) = \exp \left(\frac{i}{2} \int_0^t A(\tau)^2 d\tau \right) f(\vec{r} + \vec{e} F \omega^{-2} \cos \omega t)$$

is unitary in $\mathscr{H}$ for each fixed t and the image of (2.5) under R_1 is

$$(2.11) \quad \left(-\frac{1}{2} \Delta + V(\vec{r} + \vec{e} F \omega^{-2} \cos \omega t) \right) \psi_1 = i \frac{\partial \psi_1}{\partial t}, \qquad \psi_1 = R_1 \psi$$

which is the form considered by Yajima [23]. Hence, denoting by $U_Y(.)$ the propagator generated by (2.11), we have

$$U(t, s, F, 0) = R_1(t)^{-1} U_Y(t, s\,; F, 0) R_1(t).$$

(2)   Let $N = 1$, $V = 0$, $\theta = 0$. Denote by $U_0(t, s\,; F)$ the propagator generated by $\frac{1}{2}(-i\vec{\nabla} - \vec{e} A(t))^2 \psi = i \frac{\partial \psi}{\partial t}$. Then an elementary computation yields:

$(2.12) \quad ([\mathscr{F} U_0(t, s\,; F) \mathscr{F}^{-1}] \mathscr{F} f)(\vec{p}, s) = G(\vec{p}, F\,; t, s)(\mathscr{F} f)(\vec{p}, s)$

$(2.13) \quad G(\vec{p}, F\,; t, s) = \exp [i p_x F \omega^{-2} (\cos \omega t - \cos \omega s)] \exp [i(t - s) | \vec{p} |^2 / 2]$.

$\qquad\qquad . \exp [iF^2 (\sin 2\omega s - \sin 2\omega t)/8\omega^3] . \exp [iF^2(t - s)/4\omega^2]$

As $\omega \to 0$, $s = 0$:

$(2.14) \quad G(\vec{p}, F\,; t, 0) \to \exp (-i p_x F t^2 / 2) \exp (it | \vec{p} |^2 / 2) \exp (iF^2 t^3 / 6)$

which is the propagator of $-\frac{1}{2} \Delta + F \langle \vec{r}, \vec{e} \rangle$ given by Avron-Herbst [2].

For $\theta \in \overline{\mathbb{C}}_a^{\pm}$, $F \in \mathbb{R}$, let us now define a one-parameter family of operators $\{ \mathscr{U}(\pm \sigma\,; F, \theta) : \pm \sigma \geq 0 \}$ in $\mathscr{K}$ by:

$(2.15) \quad (\mathscr{U}(\pm \sigma\,; F, \theta)f)(., t) = U(t, t - \sigma, F, \theta)f(., t - \sigma), \qquad f \in \mathscr{K}$

We will eventually see that $\{ \mathscr{U}(\pm \sigma\,; .) \pm \sigma \geq 0 \}$ is the C_0 semigroup generated by the Floquet operator. As a preparation for this, we have:

2.3 PROPOSITION. — Let $\mathscr{D} = C^1(\mathbb{T}_\omega, \mathscr{H}) \cap C(\mathbb{T}_\omega, H^2) \subset \mathscr{K}$.
For $F \in \mathbb{C}$, $\theta \in \mathbb{C}_a$, $u \in \mathscr{D}$ set:

$$(2.16) \qquad\qquad \dot{K}(F, \theta)u = H(F, \theta, t)u - i \frac{\partial u}{\partial t}$$

(1)   If $(F, \theta) \in \mathbb{R} \times \mathbb{R}$, $\dot{K}(F, \theta)$ has a self-adjoint closure $K(F, \theta)$ and

$$(2.17) \qquad K(F, \theta) = \hat{S}(\theta) K(F) \hat{S}(\theta)^{-1}, \qquad K(F) \equiv K(F, 0)$$

(2)   For $\theta \in \mathbb{C}_a^\pm$, $K(F, \theta)$ has domain $L^2(\mathbb{T}_\omega) \otimes H^2 \cap H^1(\mathbb{T}_\omega) \otimes \mathcal{H}$ and represents a (pair of) type A-holomorphic families in $(F, \theta) \in \mathbb{C} \times \mathbb{C}_a^\pm$. Furthermore, there is $M > 0$ independent of (F, θ) in the compacts of $\mathbb{C} \times \overline{\mathbb{C}}_a^\pm$ such that $\pm iK(F, \theta) + M$ is maximal accretive. For $\phi \in \mathbb{R}$:

$$(2.18) \qquad \hat{S}(\phi) K(F, \phi) \hat{S}(\phi)^{-1} = K(F, \theta + \phi)$$

(3) $K(F, \theta)$ is strongly continuous in the generalized sense as $\operatorname{Im} \theta \downarrow\uparrow 0$ uniformly on compacts in $(F, z) \in \mathbb{C} \times \{ z \in \mathbb{C} : \pm \operatorname{Im} z > M \}$.
(4)   If $\lambda \in \sigma_d(K(F, \theta))$, $\theta \in \overline{\mathbb{C}}_a^\pm$, then λ is locally independent of θ; for $F \in \mathbb{R}$, $\operatorname{Im} \lambda \le 0$ if $\theta \in \mathbb{C}_a^+$, $\operatorname{Im} \lambda \ge 0$ if $\theta \in \mathbb{C}_a^-$.

Remark. — Let $N = 1$, $\theta \in \mathbb{R}$. Then the map:

$$(\hat{R}_1 f)(\vec{r}, t) = \exp(- 4i\omega^{-2}t) \exp\left(i \int_0^t A(\tau)^2 d\tau/2 \right) f(\vec{r} + \vec{e}F\omega^{-2} \cos \omega t, \, t)$$

is unitary in $\mathcal{H}$ and leaves $\mathcal{D}$ invariant. The dilation $\hat{S}(\theta)$ enjoys the same properties. The unitary image $\hat{R}_1 \dot{K}(F, 0) \hat{R}_1^{-1}$ of $K(F, 0)$ is the action on $\mathcal{D}$ of

$$-\frac{1}{2} \Delta + V(\vec{r} + \vec{e}F\omega^{-2} \cos \omega t) - i \frac{\partial}{\partial t} + 4F\omega^{-2}. \text{ This is the form consi-}$$

dered by Yajima [23].

Proof. — (1) Let $(F, \theta) \in \mathbb{R} \times \mathbb{R}$. By Lemma 2.2 (6) the family of operators in $\mathcal{H}$ defined by (2.15) for $\sigma \in \mathbb{R}$, i. e.

$$(\mathcal{U}(\sigma; F, \theta) f)(., t) = U(t, t - \sigma; F, \theta) f(., t - \sigma), \qquad \sigma \in \mathbb{R}, \qquad f \in \mathcal{H},$$

is a unitary group. Hence by the Stone theorem $\{ \mathcal{U}(\sigma; F, \theta) : \sigma \in \mathbb{R} \}$ is generated by a self-adjoint operator $L(F, \theta)$ in $\mathcal{H}$:

$$(2.19) \qquad \mathcal{U}(\sigma, F, \theta) = e^{-i\sigma L(F, \theta)}$$

By Lemma 2.2 (2), $\mathcal{D}$ is invariant under $\{ \mathcal{U}(\sigma, F, \theta) \}$ and thus (see e. g. [19], Thm. VIII.10]) is a core for the generator $L(F, \theta)$. On the other hand by (2.7) we clearly have, if $f \in \mathcal{D}$:

$$(2.20) \qquad i \frac{d}{d\sigma} \mathcal{U}(\sigma; F, \theta) f |_{\sigma=0} = \dot{K}(F, \theta) f$$

Hence $L(F, \theta) \upharpoonright \mathcal{D} = \dot{K}(F, \theta)$. Since $K(F, \theta)$ is obviously symmetric, and $\mathcal{D}$ is a core of the self-adjoint operator $L(F, \theta)$, $\dot{K}(F, \theta)$ is essentially self-adjoint and $K(F, \theta) = L(F, \theta)$.

To see (2) (again we consider the $+$ case only), we first note that $K(0, \theta) \equiv K(\theta)$ defined as the action of

$$T(\theta) - i \frac{\partial}{\partial t} \quad \text{on} \quad L^2(\mathbb{T}_\omega) \otimes H^2 \cap H^1(\mathbb{T}_\omega) \otimes \mathcal{H}$$

is obviously type-A holomorphic for $\theta \in \mathbb{C}_a^+$. By the same argument of

Prop. 2.1, $K(F, \theta)$ will be type A-holomorphic if for any $b > 0$ there is $a > 0$ independent of θ in the compacts of $\mathbb{C}_a^+$ such that:

$$(2.21) \quad \left\| \omega^{-1} \sin \omega t \sum_{k=1}^{N} (- i\vec{\nabla}_k)_x u \right\|$$

$$\leq | \omega^{-1} | \left\| \sum_{k=1}^{N} (- i\vec{\nabla}_k)_x u \right\| \leq b \, \| K(\theta) u \| + a \, \| u \|$$

$u \in D(K(\theta))$. Since $\operatorname{Im} \theta > 0$, an elementary computation yields:

$$\lim_{\operatorname{Im} z \to +\infty} \sup_{n \in \mathbb{Z}} \left\| \sum_{k=1}^{N} (- i\vec{\nabla}_k)_x (T_0(\theta) + n\omega - z)^{-1} \right\|_{\mathscr{H}}$$

$$= \lim_{\operatorname{Im} z \to \infty} \sup_{n \in \mathbb{Z}} \| W(\theta)(T_0(\theta) + n\omega - z)^{-1} \|_{\mathscr{H}} = 0$$

uniformly on compacts in $\theta \in \mathbb{C}_a^+$. Writing:

$$\mathscr{F}_t \left(\sum_{k=1}^{N} (- i\vec{\nabla}_k)_x \right) (K(\theta) - z)^{-1} \mathscr{F}_t^{-1} = \bigoplus_{n=-\infty}^{+\infty} \sum_{k=1}^{N} (- i\vec{\nabla}_k)_x (T(\theta) + n\omega - z)^{-1}$$

by the above estimate we see that given $\varepsilon > 0$ we can find $z(\varepsilon) \in \mathbb{C}^+$ independent of (n, θ), $n \in \mathbb{Z}$, θ in the compacts of $\mathbb{C}_a^+$, such that

$$\left\| \sum_{k=1}^{N} (- i\vec{\nabla}_k)_x (T(\theta) + n\omega - z)^{-1} \right\|_{\mathscr{H}}$$

$$= \left\| \sum_{k=1}^{N} (- i\vec{\nabla}_k)_x (T_0(\theta) + n\omega - z)^{-1} [1 + W(\theta)(T_0(\theta) + n\omega - z)^{-1}]^{-1} \right\|_{\mathscr{H}} < \varepsilon$$

Hence $\left\| \sum_{k=1}^{N} (- i\vec{\nabla}_k)_x (K(\theta) - z)^{-1} \right\| < \varepsilon$ with the stated uniformity for

some $z = z(\varepsilon) \in \mathbb{C}^+$ and this implies (2.21). (2.18) is obvious. Furthermore, since $\Theta(T_0(\theta)) = \{ z \in \mathbb{C} : \arg z = - \angle \operatorname{Im} \theta \}$, by the relative boundedness argument of Prop. 2.1 $\overline{\Theta} = \bigcup_{|F| < \infty, \theta \in \overline{\mathbb{C}}_a^+} \Theta(H(F, \theta, t) \subset \{ z : \operatorname{Im} z < M \})$ for

some $M > 0$. Hence $\bigcup_{|F| < \infty, \theta \in \overline{\mathbb{C}}_a^+} \Theta(iK(F, \theta)) \subset \{ z : \operatorname{Re} z > - M \}$ and this

concludes the proof of (2). To see (3), remark that

$$\| (K(F, \theta) - z)^{-1} \| \leq \operatorname{dist} (z, \overline{\Theta})^{-1}$$

is bounded uniformly in $\theta \in \overline{\mathbb{C}}_a^{\pm}$ if $z \in \mathbb{C} \backslash \overline{\Theta}$. Since the $\mathscr{K}$-valued function $\theta \to K(F, \theta)u$ is continuous as $\operatorname{Im} \theta \downarrow 0$ for each $u \in \mathscr{D}$ which is a core of $K(F, \theta)|_{\theta \in \mathbb{R}}$, the assertion follows from a known result [12, Thm. VIII.1.5]. Finally (4) is a consequence of standard complex scaling arguments. We omit the details (see e. g. [19, XIII.10]). ∎

By Prop. 2.3, Lemma 2.2 and [23, Lemma 2.5] we immediately have:

2.4 COROLLARY. — For $\theta \in \overline{\mathbb{C}}_a^{\pm}, F \in \mathbb{R}$, the operator family in $\mathscr{K}$ defined by (2.15) is the C_0-semigroup generated by $\pm iK(F, \theta)$:

$$(2.22) \qquad \mathscr{U}(\pm \sigma, F, \theta) = \exp(\mp i\sigma K(F, \theta)), \qquad \| \mathscr{U}(\pm \sigma, F, \theta) \| \le e^{M|\sigma|}$$

Let $\theta \in \mathbb{C}_a^{\pm}$. Now $\mathscr{F}_t K(\theta) \mathscr{F}_t^{-1} = \bigoplus_{n=-\infty}^{+\infty} (T(\theta) + n\omega)$, and for any $b > 0$ there is $a > 0$ such that

$$\| W(\theta)(T_0(\theta) - z)^{-1} \| \le b \sup_{p \in \mathbb{R}} |e^{-2\theta}p^2(e^{-2\theta}p^2 - z)^{-1}|$$

$$+ a \sup_{p \in \mathbb{R}} |e^{-2\theta}p^2 - z|^{-1} < \varepsilon \quad \text{for} \quad |\operatorname{Re} z| > M(\varepsilon), \quad |\operatorname{Im} z| < \mu, \quad \mu > 0.$$

Since $(T(\theta) - z)^{-1} = (T_0(\theta) - z)^{-1}(1 + W(\theta)(T_0(\theta) - z)^{-1})^{-1}$, $z \notin \sigma(T(\theta))$, we conclude:

2.5 LEMMA. — Let $\theta \in \mathbb{C}_a^{\pm}$. Then $\sigma(K(\theta)) = \bigcup_{n=-\infty}^{+\infty} (\sigma(T(\theta)) + n\omega)$.

Assume from now on $\theta \in \mathbb{C}_a^+$ (i. e. we choose $\mathbb{C}^+$ as the physical sheet) and recall (see e. g. [19, XIII.10]) that all non-threshold bound states and resonances of T are isolated eigenvalues of $T(\theta)$, and hence of $K(\theta)$, and belong to $\overline{\mathbb{C}}^-$. Any such eigenvalue turns for $F \ne 0$ small into a resonance determined by convergent Rayleigh-Schrödinger perturbation theory. The resonance is defined as an isolated eigenvalue of dilated Floquet operator $K(F, \theta)$ and generates a solution of the dilated, time-dependent Schrödinger equation.

Specifically:

2.6 THEOREM. — Let λ be an isolated eigenvalue of $T(\theta)$, $\theta \in \mathbb{C}_a^+$, of (algebraic) multiplicity $m_0(\lambda)$. Then there is $\overline{F}(\lambda) > 0$ such that for $F \in \mathbb{C}$, $|F| < \overline{F}(\lambda)$:

(1) Let $\lambda + n_j\omega$, $n_j \in \mathbb{Z}$, $j = 1, \ldots, l$, be the eigenvalues of $T(\theta)$ of (algebraic) multiplicity $m_j(\lambda)$ which differ from λ by integer multiples of ω, and let $N(\lambda) = \sum_{j=0}^{l} m_j(\lambda)$. Then there exist exactly $N(\lambda)$ eigenvalues

$$\lambda_1(F), \ldots, \lambda_N(F) \qquad \text{of} \qquad K(F, \theta)$$

(counting multiplicity) such that $\lambda_i(F) \to \lambda$ as $|F| \to 0$, $i = 1, \ldots, N$.

For each $n_k \in \mathbb{Z}$, $k \neq j$, there exist exactly $N(\lambda)$ eigenvalues $\lambda_1^{n_k}(F), \ldots, \lambda_N^{n_k}(F)$ of $K(F, \theta)$ (counting multiplicity) such that $\lambda_i^{n_k}(F) \to \lambda + n_k \omega$, $i = 1, \ldots, N(\lambda)$.

(2) Denote by $\{ \lambda_{i,n}(F) \colon i \in \mathbb{N} \cup \{0\}, n \in \mathbb{Z} \}$ the sequence of the (repeated) eigenvalues of $K(F, \theta)$ in their natural ordering: then there are $k(i, n) \in \mathbb{N}$ such that $\lambda_{i,n}(F)$ are holomorphic function of $F^{1/k(i,n)}$ near $F = 0$. In particular if $N(\lambda) = 1$ (which occurs for almost every ω if $m_0(\lambda) = 1$) the unique eigenvalue $\lambda_{0,n}(F)$ near $\lambda + n\omega$ is holomorphic near $F = 0$ and its Rayleigh-Schrödinger perturbation expansion has therefore a positive convergence radius.

(3) Let $F \in \mathbb{R}$. Then Im $\lambda_{i,n}(F) \leq 0$ for all (i, n). Furthermore, let $\hat{D}_a = \{ \psi \in \mathcal{K} \colon$ the $\mathcal{K}$-valued function $\theta \to \hat{S}(\theta)\psi$ is holomorphic in $\mathbb{C}_a \}$. If $\{ \lambda_i \colon i \in \mathbb{N} \cup \{0\} \}$ denotes the sequence of the (repeated) non-threshold eigenvalues and resonances of T, and $\{ \lambda_i + n\omega \colon i \in \mathbb{N} \cup \{0\}, n \in \mathbb{Z} \}$ the sequence of the corresponding isolated eigenvalues of $K(\theta)$, there exist $\overline{F}_i > 0$ and neighbourhoods $\Omega_{i,n}$ of $\lambda_i + n\omega$ such that the functions

$$(2.24) \qquad f_{\phi,\psi}(z) = \langle \phi, (K(F) - z)^{-1}\psi \rangle, \qquad (\phi, \psi) \in \hat{D}_a$$

a priori holomorphic for $z \in \mathbb{C}^+$, have for $|F| < \inf\limits_{0 \leq i \leq k} \overline{F}_i$ a meromorphic continuation to $\Omega_k \equiv \overline{\mathbb{C}}^- \cap \left(\bigcup\limits_{i,|n|=0}^{k,\infty} \Omega_{i,n} \right)$, explicitly given by:

$$(2.25) \qquad f_{\phi,\psi}(z) = \langle \hat{S}(\overline{\theta})\phi, (K(F, \theta) - z)^{-1}\hat{S}(\theta)\psi \rangle$$

The set of poles of $f_{\phi,\psi}(z)$ in Ω_k as (ϕ, ψ) describe $\hat{D}_a$ coincides with

$$\sigma_d(K(F, \theta)) \cap \Omega_k \supset \bigcap\limits_{0 \leq i \leq k, -\alpha \leq n \leq \infty} \lambda_{i,n}(F).$$

(4) Let $F \in \mathbb{R}$, and let $K(F, \theta)f = \lambda(F)f, f \in D(K(F, \theta))$. Then $f = f(., t) \in C(\mathbb{T}_\omega, \mathcal{H})$ and $\psi = e^{-i\lambda(F)t}f$ solves the equation $H(F, \theta, t)\psi = i\dfrac{\partial\psi}{\partial t}$ so that

$$f(., t) = e^{i\lambda(F)(t-s)}U(t, s; F, \theta)f(., s).$$

In particular

$$U(s + 2\pi/\omega, s; F, \theta)f(., s) = e^{-(2\pi/\omega)i\lambda(F)}f(., s).$$

Conversely if $U(s + 2\pi/\omega, s; F, \theta)\phi(., s) = e^{-(2\pi/\omega)i\lambda(F)}\phi(., s)$, then

$$f(t) \equiv e^{i\lambda(F)(t-s)}U(t, s; F, \theta)\phi(., s) \in D(K(F, \theta)) \quad \text{and} \quad K(F, \theta)f = \lambda(F)f.$$

Remarks. — (1) We will see later (Thm. 3.1) that actually Im $\lambda_{i,n}(F) < 0$, $i = 0, \pm n = 0, 1, \ldots$, at least for $m_0(\lambda) = 1$, almost every ω, and a suitable class of two-body potentials.

(2) The stability result (assertion (1) above) applies to all isolated eigenvalues of $T(\theta)$, i. e. also to the non-threshold embedded eigenvalues and reso-

nances of T. The corresponding statement is not known for the $\omega = 0$, DC-field case.

Proof. — Given (2), (3), (4) of Prop. 2.3, assertions (1) and (2) are immediate consequences of analytic perturbation theory [*12*, II.1, VII.1.2]. The scalar products $\langle \hat{S}(\bar{\theta})\phi, (K(F, \theta) - z)^{-1} \hat{S}(\theta)\psi \rangle$ are θ-independent by (2.18) and equal to $f_{\phi,\psi}$ by Prop. 2.3 (4), (2.17) and the analytic continuation principle. The existence of $\Omega_{i,n}$ such that $(K(F, \theta) - z)^{-1}$ is meromorphic in $z \in \Omega_{i,n}$ is once more implied by analytic perturbation theory. This verifies (3), and (4) is proved exactly as [*23*, Lemma 2.9]. We omit the details. ∎

3. PERTURBATION SERIES AND OSCILLATORY SOLUTIONS

Our first purpose in this section is to examine more closely the perturbation theory of the problem. To avoid unnecessary complications, if λ is an isolated eigenvalue of $T(\theta)$, $\theta \in \mathbb{C}_a^+$, $\{ \lambda + n\omega : n \in \mathbb{Z} \}$ the corresponding sequence of eigenvalues of $K(\theta)$, and $\Sigma = \inf \sigma_{ess}(T)$, we consider only the following two cases in Thm. 2.5, which occur for almost every ω:

CASE A : $\lambda < \Sigma$, $m_0(\lambda) = 1$, $l = 0$, $n = 0$.

CASE B : $\lambda < \Sigma$ $m_0(\lambda) = 1$, $l = 1$, with $n_1 = \pm 1$, $m_1(\lambda) = 1$, $n_k = 0$, $k \neq \pm 1$.

We remark that the restriction to $n = 0$, $n_k = 0$ is no loss of generality. The general degenerate case can be examined along the lines of [*14*, II.2, 2.3]. We also recall that the perturbation expansion is generated by ordinary Rayleigh-Schrödinger perturbation theory in $\mathcal{K}$: the unperturbed operator is $K(\theta)$, and the perturbation is

$$e^{-\theta}A(t) \sum_{k=1}^{N} (-i\vec{V}_k)_x + \frac{1}{2}NA(t)^2 \equiv FT_1 + F^2T_2$$

$$= F(2i\omega)^{-1}(e_1(t) - e_{-1}(t)) \otimes Q_1(\theta) + F^2(2i\omega)^{-2}(e_1(t) - e_{-1}(t))^2 \otimes Q_2.$$

Here $e_k(t) = e^{ik\omega t}$, $k \in \mathbb{Z}$, $Q_1(\theta) = e^{-\theta} \sum_{k=1}^{N} (-i\vec{V}_k)_x$, $Q_2 = \frac{1}{2}I_{\mathcal{H}}$. The perturbation theory is degenerate in case B since the multiplicity of λ as an eigenvalue of $K(\theta)$ is 2. In case A, we denote by $\phi(\theta) \in H^2$, $\phi(0) = \phi$, the eigenvector of $T(\theta)$ corresponding to λ, $T(\theta)\phi(\theta) = \lambda\phi(\theta)$; in case B, the eigenvectors corresponding to λ, $\lambda \pm \omega$ will be denoted by $\phi_1(\theta)$, $\phi_2(\theta)$, $\phi_1(0) = \phi_1$, $\phi_2(0) = \phi_2$. Correspondingly, the unperturbed eigenvectors

of $K(\theta)$ will be $\Phi(t, \vec{r}, \theta) = \phi(\theta) \otimes e_0$ in case A, and $\Phi_1(.) = \Phi_1(\theta) \otimes e_0$, $\Phi_2(.) = \phi_2(\theta) \otimes e_{\pm 1}$ in case B, respectively. We also denote by $\{ E(\mu):$ $\mu \geq \inf \sigma(T) \}$ the spectral measure of T, and set $R(n\omega, \lambda, \theta) = (T(\theta) - n\omega - \lambda)^{-1}$, $n = \pm 1, \pm 2, \ldots$

Then by exactly the same argument of [23, Thm. 3.5], but with the important simplification $T_k = Q_k \equiv 0$, $k > 2$, we have:

3.1. THEOREM. — Let case A hold, and let $\displaystyle\sum_{i=0}^{\infty} C_i(\omega)F^i$, $C_0(\omega) = \lambda$, be the perturbation series of $\lambda(F)$. Then:

(1) $C_i(\omega)$ is θ-independent and $C_{2i+1} = 0$, $i = 0, 1, \ldots$
(2) Let $\lambda + n\omega < \Sigma$. Then Im $C_{2i}(\omega) = 0$ for $0 \leq i \leq n$.
(3) Let $n(\lambda)$ be the smallest integer such that $\lambda + n\omega > \Sigma$. Then

$$(3.1) \quad \text{Im } C_{2n}(\lambda, \omega) = - \pi \left\langle \frac{d E(\mu + n\omega)}{d\mu} \bigg|_{\mu = \lambda} \phi(n, \omega, \lambda, 0), \; \phi(n, \omega, \lambda, 0) \right\rangle_{\mathscr{H}}$$

and thus Im $C_{2n}(\lambda, \omega) < 0$ a. e. in ω unless it vanishes identically. Here:

$$\phi(n, \lambda, \omega, \theta) = (2\omega^{-2})^n \sum_{p=1}^{n} (-1)^p \sum_{v_1 + \ldots v_p = n} Q_{v_1}(\theta)R(n\omega - v_1\omega, \lambda, \theta)Q_{v_2}(\theta)$$

$$\ldots Q_{v_{p-1}}(\theta)R(v_p\omega, \lambda, \theta)Q_{v_p}(\theta)\phi(\theta) \qquad v_j = 1, 2; \qquad j = 1, \ldots, p.$$

Remarks. — (1) Formula (3.1) is of course the Fermi Golden Rule to first non-vanishing order in perturbation theory [20].

(2) Once more by analytic perturbation theory, we have the following representation for the resonance eigenvector $\Phi(\theta)$:

$$\Phi(\theta) = \Phi(\vec{r}, t, F, \theta) = \phi(\vec{r}, \theta)$$

$$- F(2i\omega)^{-1} \left[e_{-1}(t) \otimes (T(\theta) - \lambda + \omega)^{-1} e^{-\theta} \sum_{n=1}^{N} (- i\vec{\nabla}_k)_x \phi(\theta) \right.$$

$$\left. - e_1(t) \otimes (T(\theta) - \lambda - \omega)^{-1} e^{-\theta} \sum_{k=1}^{N} (- i\vec{\nabla}_k)_x \phi(\theta) \right] + 0(F^2)$$

where $0(F^2)$ stands for a $C(\mathbb{T}_\omega, \mathscr{H})$-function $f_\theta(\vec{r}, t, F)$ such that $\| f \|_{\mathscr{H}}$ is $0(F^2)$ uniformly in $t \in \mathbb{T}_\omega$.

(3) In case B, since $R(k\omega, \lambda, 0)$ is a real operator for $k < n$ if $n\omega + \lambda < \Sigma$, it follows from analytic, degenerate perturbation theory that the two eigenvalues $\lambda_1(F)$, $\lambda_2(F)$ admit a real Taylor expansion up to order $n - 1$.

We omit the details (see e. g. Hunziker-Pillet [11]). As in case A, analytic perturbation theory immediately yields, for $j = 1, 2$:

$$(3.3) \quad \Phi_j(\vec{r}, t, F, \theta) = \frac{1}{2} [e_0 \otimes \phi_1(\theta) + (-1)^{j-1} e_{\pm 1}(t) \otimes \phi_2(\theta)] + 0(F)$$

where $0(F)$ has the same meaning as above.

The rest of the paper is devoted to the proof of the long time behaviour of the solutions of $H(F, t)\psi = i \dfrac{\psi}{\partial t}$, having the unperturbed eigenfunctions as initial data, including the oscillation property in the resonant case B. We assume $N = 1$, and the following further requirement on the potential V:

A.2 : Let V fulfill A.1. For $\theta \in \mathbb{C}_a^+$, set $V(\theta) = V(e^\theta \vec{r})$, $A(\theta) = |V(\theta)|^{1/2}$, $B(\theta) = V(\theta) |V(\theta)|^{-1/2}$, $\hat{A}(\theta) = \mathscr{F}(A(\theta))$, $\hat{B}(\theta) = \mathscr{F}(B(\theta))$. Then there is $\varepsilon > 0$ such that

$$(A(\theta), B(\theta)) \in L^p(\mathbb{R}^3), \frac{3}{2} - \varepsilon \leq p \leq 6 + \varepsilon,$$

(3.4)

$$(\hat{A}(\theta), \hat{B}(\theta)) \in L^q(\mathbb{R}^3), \frac{3}{2} - \varepsilon \leq q \leq \frac{3}{2} + \varepsilon$$

An example of potential fulfilling A.2 but not the conditions of [23, Thm. 3.6] is $V = e^{-\beta |\vec{r}|} |\vec{r}|^{-\alpha}$, $\beta > 0$, $\alpha < 1$. The statement analogous to Thm. 3.6 of Yajima [23] is:

3.2 THEOREM. — Let V fulfill A.2, $F \in \mathbb{R}$, $U(t, s, F) = U(t, s, F, \theta = 0)$. Then

(1) In case A of Thm. 3.1

$$(3.5) \qquad \langle U(t, s, F)\phi, \phi \rangle = e^{-i\lambda(F)(t-s)} + 0(F)$$

as $F \to 0$, uniformly in $\pm t > \pm s$,

(2) In case B of Thm. 3.1

$$(3.6) \quad \langle U(t, s, F)\phi_1, \phi_1 \rangle = \frac{1}{2} [e^{-i\lambda_1(F)(t-s)} + e^{-i\lambda_2(F)(t-s)}] + 0(F)$$

$$(3.7) \quad \langle U(t, s, F)\phi_1, \phi_2 \rangle = \frac{1}{2} [e^{-i\lambda_1(F)(t-s)} - e^{-i\lambda_2(F)(t-s)}] e^{-i\omega t} + 0(F)$$

uniformly in $\pm t > \pm s$.

To prove 3.2 we first establish some preliminary results.

3.3 LEMMA. — Denote by $\{ G(t, \sigma, F, \theta): t \in \mathbb{T}, \sigma > 0, F \in \mathbb{R}, \theta \in \mathbb{C}_a^+ \}$ the four-parameter operator family in $\mathscr{H} = L^2(\mathbb{R}^3; d\vec{p})$ generated by the maximal multiplication operator by the function $G_\theta(\vec{p}, F, t, t - \sigma)$ defined by (2.13) with $\vec{p}$ replaced by $e^{-\theta}\vec{p}$ and s by $t - \sigma$. Then:

(1) There is $A_1 > 0$ independent of $(t, \sigma, F, \theta) \in \mathbb{T}_\omega \times \mathbb{R}_+ \times [-M, M] \times \overline{\mathbb{C}}_a^+$, $M > 0$, such that

(3.8) $\| G(.) \|_{\mathcal{H}} \leq A_1$

(2) There is $A_2 > 0$ independent of $t \in \mathbb{T}_\omega$ and (F, θ) in the compacts of $\mathbb{R} \times \mathbb{C}_a^+$ such that:

(3.9) $\| G_\theta(\vec{p}, .) \|_{L^s} \leq A_2 \sigma^{-3/2s}, \quad 1 \leq s \leq \infty$

(3) Let $\vec{r} \to u(\vec{r}) \in L^1(\mathbb{R}^3)$. Then:

(3.10) $\| (\mathcal{F}^{-1} G_\theta(.))(\vec{r}, .) * u(\vec{r}) \|_{L^\infty}$

$$\leq \| G_\theta(\vec{p}, .) \|_{L^1} \| u(\vec{r}) \|_{L^1} \leq A_2 \sigma^{-3/2} \| u \|_{L^1}$$

(4) Let $\vec{r} \to u(\vec{r}) \in \mathcal{H} = L^2$.

(3.11) $\| \hat{A}(\theta) * (G_\theta \mathcal{F} u)(\vec{p}, .) \|_{L^2}$

$$\leq \| \hat{A}(\theta) \|_{L^s} \| G_\theta(\vec{p}, .) \|_{L^{s/s-1}} \| u \|_{L^2}, \quad 1 \leq s \leq \infty$$

The same inequality holds with $\hat{A}(\theta)$ replaced by $\hat{B}(\theta)$.

Proof. — (1) It is enough to verify:

(3.12) $\sup\limits_{\vec{p} \in \mathbb{R}^3, t \in \mathbb{T}_\omega, \sigma \in \mathbb{R}_+} | G_\theta(\vec{p}, F, t, t - \sigma) | < A_1$

for some $A_1 > 0$ independent of $(F, \theta) \in [-M, M] \times \overline{\mathbb{C}}_a^+$. By (2.13):

$| G_\theta(.) | \leq \exp(-p_x F \omega^{-2} \sin \theta [\cos \omega t - \cos \omega(t - \sigma)]) \exp(-\sin 2\theta \sigma | \vec{p} |^2)$.

$. \exp(-F^2 8^{-1} \omega^{-3} [\sin 2\omega t - \sin 2\omega(t - \sigma)])$

$\leq \exp(M^2/4\omega^2) \exp(-F\omega^{-2} \sin \theta [\cos \omega t - \cos \omega(t - \sigma)] p_x)$

$$\exp(-\sin 2\theta | \vec{p} |^2)$$

for $| F | < M$, whence (3.5) because

$$\cos \omega t - \cos \omega(t - \sigma) = -\sin \omega t . \omega \sigma + 0(\sigma^2) \quad \text{as} \quad \sigma \to 0.$$

(2) An elementary computation yields

$\int_{\mathbb{R}^3} | G_\theta(\vec{p}, .) |^s d\vec{p}$

$= (\sigma \sin 2\theta)^{-3/2} \exp(sF^2 [\cos \omega t - \cos \omega(t - \sigma)] \sin^2 \theta/4\sigma\omega^4 \sin 2\theta)$

whence (3.9) for some $A_2 > 0$ as above.

(3) If $u \in L^1$, $(\mathcal{F} u)(\vec{p}) \in L^\infty$ and $G_\theta(\vec{p}, .)(\mathcal{F} u)(\vec{p}) \in L^\infty$ by (3.12). Then, by Young's inequality:

$$\| (\mathcal{F}^{-1} G_\theta)(\vec{r}, .) * u(\vec{r}) \|_{L^\infty} \leq \| (\mathcal{F}^{-1} G_\theta)(\vec{r}, .) \|_{L^\infty} \| u \|_{L^1}$$

and since $\| (\mathcal{F}^{-1} G_\theta)(\vec{r}, .) \|_{L^\infty} \leq \| G_\theta(\vec{p}, .) \|_{L^1}$, (3.9) yields (3.10).

(4) By the generalized Young inequality we have:

$$\| \hat{A}(\theta) * (G_\theta \mathscr{F} u)(\vec{p},.) \|_{L^2} \leq \| \hat{A}(\theta) \|_{L^s} \| (G_\theta \mathscr{F} u)(\vec{p},.) \|_{L^q},$$

$$s^{-1} + q^{-1} = \frac{3}{2}; \qquad s, q \geq 1$$

and by Hölder's inequality:

$$\| (G_\theta \mathscr{F} u)(\vec{p},.) \|_{L^q} \leq \| G_\theta(\vec{p},.) \|_{L^r} \| u \|_{L^2}, \qquad r^{-1} = q^{-1} - 2^{-1},$$

which yields (3.11). ■

3.4 LEMMA. — Let $\theta \in \mathbb{C}_a^+$, $F \in \mathbb{R}$, and V fulfill A.2. Then, if $K_0(F, \theta) = K(F, \theta)|_{V=0}$:

(1) $\| A(\theta) e^{-i\sigma K_0(F,\theta)} B(\theta) \|_{\mathscr{X}}^2$

$$\leq A_1 A_2^2 \| A(\theta) \|_{L^s}^2 \| B(\theta) \|_{L^s}^2 \sigma^{-6/s}, \qquad \sigma > 0, \frac{3}{2} - \varepsilon < s < 6 + \varepsilon$$

(2) $$\lim_{F \to 0} \| A(\theta) [e^{-i\sigma K_0(F,\theta)} - e^{-i\sigma K_0(\theta)}] B(\theta) \|_{\mathscr{X}}^2 = 0$$

uniformly on compacts in $(\theta, \sigma) \in \mathbb{C}_a^+ \times \mathbb{R}_+$.

(3) $\| A(\theta) e^{-i\sigma K_0(F,\theta)} \|_{\mathscr{X}}^2 \leq A_2^2 \| \hat{A}(\theta) \|_{L^s}^2 \sigma^{-3(s-1)/s}, \qquad \frac{3}{2} - \varepsilon < s < \frac{3}{2} + \varepsilon$

The same estimate holds with $A(\theta)$ replaced by $B(\theta)$.

Proof. — (1) By (2.12), (2.13), (2.15), for $f = f(\vec{r}, t) \in \mathscr{X}$ we have:

(3.13) $([\mathscr{F} e^{-i\sigma K_0(F,\theta)} \mathscr{F}^{-1}] \mathscr{F} f)(\vec{p}, t) = G_\theta(\vec{p}, F, t, t-\sigma)(\mathscr{F} f)(\vec{p}, t - \sigma)$

$(e^{-i\sigma K_0(F,\theta)} f)(\vec{r}, t) = (\mathscr{F}^{-1} G_\theta)(\vec{r}, F, t, t - \sigma) * f(\vec{r}, t - \sigma)$

Hence, by Fubini's theorem:

$\| A(\theta) e^{-i\sigma K_0(F,\theta)} B(\theta) f \|_{\mathscr{X}}^2$

$$= \int_0^{2\pi/\omega} \| A(\theta) [\mathscr{F}^{-1} G_\theta(., F, t, t-\sigma) * B(\theta) f(., t - \sigma)] \|_{\mathscr{X}}^2 dt$$

By (1) and (3) of Lemma 3.3 the convolution kernel $(\mathscr{F}^{-1} G_\theta)(\vec{r},.)$ is continuous both as an operator from $\mathscr{H}$ to $\mathscr{H}$, with norm bounded by A_1, and from $L^1(\mathbb{R}^3)$ to $L^\infty(\mathbb{R}^3)$, with norm bounded by $\| G_\theta(\vec{p},.) \|_{L^1} \sigma^{-3/2} \leq A_2 \sigma^{-3/2}$. Hence a direct application of Kato's interpolation argument ([*13*]; see also [*3*, Prop. 3.1]) yields:

$\| A(\theta) [(\mathscr{F}^{-1} G_\theta)(., F, t, t - \sigma) * B(\theta) f(., t - \sigma)] \|_{\mathscr{X}}^2$

$$\leq A_1 A_2^2 \| A(\theta) \|_{L^s}^2 \| B(\theta) \|_{L^s}^2 \sigma^{-6/s} \| f(., t - \sigma) \|_{\mathscr{X}}^2$$

whence:

$$\| A(\theta) [e^{-i\sigma K_0(F,\theta)} - e^{-i\sigma K_0(\theta)}]B(\theta) \|_{\mathcal{H}}^2$$
$$\leq A_1 A_2^2 \| A(\theta) \|_{L^s}^2 \| B(\theta) \|_{L^s}^2 \sigma^{-6/s} \int_0^{2\pi/\omega} \| f(.,t-\sigma) \|_{\mathcal{H}}^2 dt$$
$$= A_1 A_2^2 \| A(\theta) \|_{L^s}^2 \| B(\theta) \|_{L^s}^2 \sigma^{-6/s} \| f \|_{\mathcal{H}}^2$$

by the time-periodicity of $f \in \mathcal{H}$. This proves (1). To see (2), note that the same argument yields:

$$\| A(\theta) [e^{-i\sigma K_0(F,\theta)} - e^{-i\sigma K_0(\theta)}]B(\theta) \|_{\mathcal{H}}^2$$
$$\leq A_1 \| G_\theta(\vec{p}, F, .) - G_\theta(\vec{p}, 0, .) \|_{L^1}^2 \| A(\theta) \|_{L^s}^2 \| B(\theta) \|_{L^s}^2 \sigma^{-6/s}$$

The r. h. s. obviously vanishes as $F \to 0$ with the stated uniformities. Finally, by (4) of Lemma 3.3 and (3.9) we obtain:

$$\| A(\theta) e^{-i\sigma K_0(F,\theta)} f \|_{\mathcal{H}}^2 = \int_0^{2\pi/\omega} \| \hat{A}(\theta) * (G_\theta(\vec{p}, F, t, t-\sigma)\mathscr{F} f)(\vec{p}, t-\sigma) \|_{\mathcal{H}}^2 dt$$
$$\leq A_2^2 \| \hat{A}(\theta) \|_{L^s}^2 \sigma^{-3(s-1)/s} \| f \|_{\mathcal{H}}^2. \qquad \blacksquare$$

Proof of Theorem 3.2. — Given Remarks (1) and (2) after Theorem 3.1, we can limit ourselves to verify the assumptions of Theorem 3.6 of Yajima [23], because then the argument is the same and can be therefore omitted. For $(z, F, \theta) \in \mathbb{C} \times \mathbb{R} \times \mathbb{C}_a^+$ set:

$$(3.14) \qquad Q(z, F, \theta) = A(\theta)(K_0(F, \theta) - z)^{-1}B(\theta)$$

(1) $Q(z, F, \theta)$ is a compact-operator valued holomorphic function of $z \in \mathbb{C}^+$.
(2) $\| Q(z, F, \theta) \|_{\mathcal{H}} \to 0$ as $\mathrm{Im}\, z \to +\infty$.
(3) The $B(\mathcal{H})$-valued function $\lambda \to Q(\lambda + i\varepsilon, F, \theta)$, $\lambda \in \mathbb{R}$, has a continuous boundary value $Q(\lambda + i0, F, \theta)$ as $\varepsilon \downarrow 0$.
(4) For $n \in \mathbb{Z}$, $e^{-in\omega t}Q(z, F, \theta)e^{in\omega t} = Q(z + n\omega, F, \theta)$
(5) There is $\overline{F} > 0$ such that $(1 + Q(0 + i0, F, \theta)^{-1})^{-1} \in B(\mathcal{H})$ for $| F | < \overline{F}$ whenever $(1 + Q(0 + i0, 0, \theta)^{-1})^{-1} \in B(\mathcal{H})$.
(6) There is $C > 0$ independent of (F, θ) in the compacts of $\mathbb{R} \times \mathbb{C}_a^+$ such that

$$(3.15) \quad \sup_{\eta \geq 0} \int_{-\infty}^{+\infty} \| A(\theta)(K_0(F, \theta) - \lambda - i\eta)^{-1}f \|_{\mathcal{H}}^2 d\lambda \leq C \| f \|_{\mathcal{H}}^2$$

$$(3.16) \quad \sup_{\eta \geq 0} \int_{-\infty}^{+\infty} \| B(\theta)^*(K_0(F, \theta) - \lambda - i\eta)^{-1*}f \|_{\mathcal{H}}^2 d\lambda \leq C \| f \|_{\mathcal{H}}^2$$

To see the above assertions, first remark that by Lemma 2.4 and the semigroup theory we have, at least for $\mathrm{Im}\, z > M$, the strong Riemann integral representation:

$$(3.17) \qquad i(K_0(F, \theta) - z)^{-1} = \int_0^\infty e^{-i\sigma K_0(F,\theta)} e^{+iz\sigma} d\sigma$$

By (3.13), Lemma 3.3 (1) and the same argument of Lemma (3.4) (1) we have $\| e^{-i\sigma K_0(F,\theta)} \|_{\mathcal{X}}^2 \leq A_1^2$ and therefore $\mathbb{C}^+ \subset \rho(K_0(F, \theta))$ with

$$(3.18) \qquad \| (K_0(F, \theta) - z)^{-1} \| \leq A_1 (\mathrm{Im}\, z)^{-1}, \qquad \mathrm{Im}\, z > 0$$

Now for $z \in \mathbb{C}^+$ write:

$$(3.19) \qquad Q(z, F, \theta) = i \int_0^\infty A(\theta) e^{-i\sigma K_0(F,\theta)} B(\theta) e^{iz\sigma} d\sigma$$

By A.3 and Lemma 3.4 (1) we have:

$$(3.20) \quad \| Q(z, F, \theta) \|^2$$
$$\leq A_1 A_2^2 \left[\| A(\theta) \|_{L^{6+\varepsilon}}^2 \| B(\theta) \|_{L^{6+\varepsilon}}^2 \int_0^1 \sigma^{-6/(6+\varepsilon)} e^{-2\,\mathrm{Im}\, z\sigma} d\sigma \right.$$
$$\left. + \| A(\theta) \|_{L^{6-\varepsilon}}^2 \| B(\theta) \|_{L^{6-\varepsilon}}^2 \int_1^\infty \sigma^{-6/(6-\varepsilon)} e^{-2\,\mathrm{Im}\, z\sigma} d\sigma \right] < +\infty$$

Hence $Q(.) \in B(\mathcal{X})$, $\| Q(z, .) \| \to 0$ as $\mathrm{Im}\, z \to +\infty$, and the holomorphy for $z \in \mathbb{C}^+$ is obvious. The same argument, together with Lemma 3.4 (2) and (3.18), implies $\| Q(z, F, \theta) - Q(z, 0, \theta) \| \to 0$ as $| F | \to 0$ uniformly on compacts in $(z, \theta) \in \mathbb{C}^+ \times \mathbb{C}_a^+$, whence the compactness of $Q(z, F, \theta)$ because $Q(z, 0, \theta)$ is compact [24]. This verifies (1) and (2). Assertion (3) follows from (3.20) because the integrals in the r. h. s. converge for $\mathrm{Im}\, z = 0$. (4) is obvious, and (5) is a consequence of the norm continuity of $Q(z, F, \theta)$ at $F = 0$. Finally, by (3.17), (3.18), the Fourier inversion formula and Lemma 3.4 (3) we can write:

$$\int_{-\infty}^{+\infty} \| A(\theta)(K_0(F, \theta) - \lambda - i\eta)^{-1} f \|_{\mathcal{X}}^2 d\lambda$$
$$\leq (2\pi)^{-1} \int_0^\infty \| A(\theta) e^{-i\sigma K_0(F,\theta)} f \|_{\mathcal{X}}^2 d\sigma$$
$$\leq A_2^2 \left[\| \hat{A}(\theta) \|_{L^{3/2-\varepsilon}}^2 \int_0^1 \sigma^{-(3-6\varepsilon)/(3-2\varepsilon)} d\sigma \right.$$
$$\left. + \| \hat{A}(\theta) \|_{L^{3/2+\varepsilon}}^2 \int_1^\infty \sigma^{-(3+6\varepsilon)/(3+2\varepsilon)} d\sigma \right] \| f \|_{\mathcal{X}}^2$$

which yields (3.15). A specular argument proves (3.16). ∎

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