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Multiple tunnelings in *d*-dimensions: a quantum particle in a hierarchical potential

by

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ABSTRACT. — We study the properties of the Schrödinger equation in d dimensions for a class of potentials, introduced in an earlier paper, exhibiting a geometrical hierarchical structure. The main feature of such models is that for low energy the particle can move to infinity only by tunneling through a sequence of barriers of increasing length. The qualitative properties of these models may be similar to those arising in periodic potentials perturbed over different scales. The main result which holds for the whole class of potentials is that quantum evolution is very slow and can be characterized by: $r^2(t) \leqslant C(\ln t)^\beta$ where $r(t)$ is the distance traveled by a wave packet of sufficiently low energy initially localized near the origin. By imposing symmetries compatible with the hierarchical structure we obtain the remarkable result that $r^2(t) \geqslant C'(\ln t)^{\beta'}$ at least for a sequence of increasing times, i. e. the motion is actually characterized by a logarithmic growth. For these symmetric cases the spectral properties of the Hamiltonian are studied in detail in the low energy region and we show that the spectrum is not discrete but of zero Lebesgue measure. Finally we add an arbitrarily weak random perturbation and we show that in all cases $r^2(t) \leqslant$ const with probability one.

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RÉSUMÉ. — On étudie les propriétés de l'équation de Schrödinger en d-dimensions pour une classe de potentiels hiérarchiques décrits dans un article précédent. La caractéristique principale de ces modèles est le fait que la particule peut se déplacer vers l'infini seulement par effet tunnel à travers des barrières de longueur croissante. D'un point de vue qualitatif, ces modèles peuvent simuler les propriétés de potentiels périodiques perturbés à des échelles différentes. Le résultat principal, valable pour tous les potentiels considérés, est la lenteur de l'évolution quantique qui peut être caractérisée par $r^2(t) \leqslant C(\ln t)^{\beta}$ où $r(t)$ est la distance traversée par un paquet d'ondes d'énergie suffisamment basse et initialement localisée près de l'origine. Si on impose des symétries compatibles avec la structure hiérarchique, on obtient le résultat intéressant $r^2(t) \geqslant C'(\ln t)^{\beta'}$ pour une séquence de temps croissant vers l'infini. Cela signifie que le mouvement est en effet caractérisé par une croissance logarithmique. Dans les cas symétriques, on analyse en détail les propriétés spectrales de l'Hamiltonien dans la région de basse énergie et on trouve que le spectre n'est pas discret mais a une mesure de Lebesgue égale à zéro. Si on ajoute une perturbation stochastique arbitrairement faible, on trouve que pour tous les potentiels considérés $r^2(t) \leqslant$ const avec probabilité un.

1. INTRODUCTION

In this work we study the properties of the Schrödinger equation in d-dimensions for a class of potentials exhibiting a geometrical hierarchical structure. A description of these potentials is contained in a previous paper [1] and will be given in greater detail in section II of the present article.

The reason for studying such models is that they represent a first step towards a detailed understanding of the behaviour of a quantum particle in complicated potentials such as those that may result from perturbing the structure of an ideal cristal. We believe that our approach may shed new light on the physical mechanisms leading to localization of the wave functions and/or absence of diffusion in disordered systems. To illustrate our point of view we begin by discussing a simple example of a particle moving initially in a periodic potential on a segment of finite length (see fig. 1) with Dirichlet boundary conditions. In this situation there is a lowest band of N eigenvalues differing one from the other by an amount of the order $\exp(-A/\hbar)$ where $A = 2\int_0^{\pi} \sqrt{2V(x)}dx$.

Suppose now that we lower the height of some barriers, for example

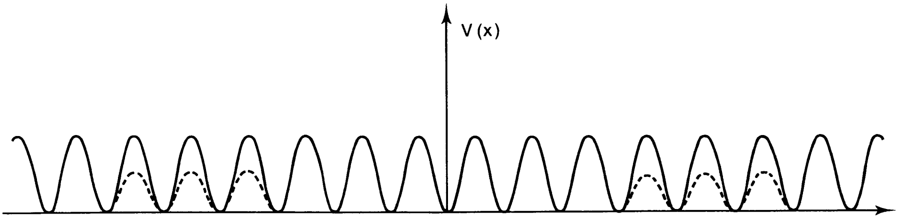

FIG. 1.

we lower all the barriers between $[a_1, a_2] \subset [0, L]$ and $[-a_2, -a_1] \subset [-L, 0]$ by multiplying by a factor $\alpha < 1$ the potential in these intervals. The lowest part of the spectrum and the shape of the lowest eigenfunctions in the new situation can be easily analyzed using the methods of [2]. The effects of the perturbation are the following:

1) The ground state is lowered. There is an isolated state closest to the ground state at a distance of the order $\exp\left(-\int_0^{a_1} 2\sqrt{2V(x)}dx/\hbar\right)$ corresponding to a direct tunneling between the regions with lower potential. There are bands with energy splittings of the order of

$$\exp(-\sqrt{\alpha}A/\hbar) \gg \exp(-A/\hbar)$$

corresponding to tunnelings across the small barriers. They are in number $N_1 < N$ where N_1 is of the order of the number of low barriers. A number of levels $N - N_1$ of the original band is shifted at a distance $\gg e^{-\sqrt{\alpha}A/\hbar}$ above the ground state.

2) The eigenfunctions corresponding to the ground state and to the $N_1 + 1$ levels within a distance $0(e^{-\sqrt{\alpha}A/\hbar})$ are localized in the regions of low potential and decrease exponentially in the regions of high barriers.

The lesson we learn from the above example is that the perturbed potential behaves roughly like a double well with a barrier which in terms of equivalent action has a height of $2\int_0^{a_1}\sqrt{2V(x)}dx$.

This type of argument can be easily extended to the case where the segments of low potential have different lengths and one finds, as expected, that the situation is equivalent to an unsymmetric double well as those studied in [2].

This example suggests naturally that as far as the low energy states are concerned, a periodic potential over the whole line perturbed by lowering the potential over a sequence of arbitrary segments can be studied in first approximation as a system of wells separated by barriers of different length corresponding to the non perturbed regions.

In this way one is led to the introduction of a simpler effective potential for the qualitative study of low energy properties [3]. The models we construct in the following section are motivated by this idea. The principle of the construction is very simple. We start from a constant potential $V=\lambda$ over the whole space and we dig holes in it (i. e. we set $V=0$ in certain regions) in such a way that the resulting potential is approximatively self similar over a sequence of rapidly increasing length scales d_k.

This imposes a geometrical structure that we call hierarchical and that can be realized in arbitrary dimension d. However for $d > 1$ the models share with the $d = 1$ case the property that communication among the wells can take place only via tunneling through a barrier, i. e. we do not admit corridors of low potential.

The main general result we prove for such potentials is that

$$r^2(t) = \int x^2 \,|(e^{-itH}\phi)(x)|^2 dx \leqslant C(\ln t)^\beta$$

for an initially localized wave packet ϕ superposition of states of sufficiently low energy.

If we restrict further the models by imposing symmetries compatible with the hierarchical structure we can prove that for a suitable sequence of times t_k, $t_k \xrightarrow[k\to\infty]{} \infty$

$$r^2(t_k) \geqslant C'(\ln t_k)^{2-\varepsilon}$$

with ε arbitrarily small. The reason why for the lower bound one has to consider a sequence of t_k is that between t_k and t_{k+1} the wave packet may contract.

For models with symmetries the analysis can be carried out much further.

In particular in the proof of the lower bound one shows that there exist delocalized states including the ground state. As far as the spectrum is concerned we show that for energies below the height of the barriers the Lebesgue measure of any spectral interval is zero and there is no isolated point of finite multiplicity.

In one dimension this implies that this part of the spectrum is a Cantor set. At this point the natural question which however we do not discuss in this paper is whether this spectrum is singular continuous. Our models for $d = 1$ and $E < \lambda$, where λ is the height of the potential, are in some sense complementary to the one dimensional models constructed by Pearson [4] for which he proves the existence of singular continuous spectrum. Pearson's methods should allow to conclude that in our case for $d = 1$ but $E > \lambda$ the spectrum is singular continuous.

However, they are not applicable for $E < \lambda$. For $d > 1$ and $E > \lambda$ our symmetric models have a component of absolutely continuous spectrum.

These models can be analyzed in great detail also if we add a stochastic perturbation. We can show in fact that for an arbitrarily small random perturbation all the low lying states become exponentially localized and $r^2(t) \leqslant$ const with probability one.

All our models are constructed on the lattice $\mathbb{Z}^d$ but as we will discuss later everything extends naturally to the continuum .The paper is divided in two parts.

The first part, sections II to VI, contains a description of the models and of the results. The second part, which consists of section VII, is much more technical in character, and contains all the proofs.

II. DESCRIPTION OF THE MODELS

We give here the precise definition of the class of models we are going to analyze.

Let $d_0 > 1$, $a > 1$ and set $d_k = d_0^{a^k}$. In what follows the numbers $\{ d_k \}_{k=0}^{\infty}$ will play the role of length scales characteristic of the models in consideration. For concreteness and simplicity we take $a = 5/4$ and $d_0 \geqslant 20$; with this choice some numerical inequalities that will appear in the proofs will be satisfied with no extra efforts. This particular choice of length scales, which is the same as the one of Fröhlich and Spencer paper [5] on Anderson localization, is not essential for the results discussed in this paper. For our purposes an increase like $d_k \sim \exp(k^{1+\varepsilon})$, $\varepsilon > 0$ would be sufficient.

Let now $\Lambda_k \subset \mathbb{Z}^d$ be the cube centered at the origin of size $4\,[3d_k]$ with its faces parallel to the coordinate axes. Here $[\,.\,]$ stands for the integer part.

DEFINITION. — A function $V : \mathbb{Z}^d \to \{ 0, \lambda \}$, $\lambda > 4d$ is said to be a « hierarchical potential » if for any $k \geqslant 0$ the set

$$A_k = \{ j \in \Lambda_{k+1} \backslash \Lambda_k \,;\, V(j) = 0 \} \neq \phi$$

and it can be written as union of components C_k^α, $A_k = \bigcup_\alpha C_k^\alpha$, with the following two properties:

$$a) \qquad \operatorname{diam} C_k^\alpha \leqslant 4\,[3d_k]$$

$$b) \qquad \operatorname{dist}\,(C_k^\alpha, (A_k \backslash C_k^\alpha) \cup \Lambda_k) \geqslant 2d_{k+1}$$

We observe here that the components $\{ C_k^\alpha \}_\alpha$ need not to be connected.

An interesting example can be constructed inductively as follows: let $\{ \Lambda_k^\alpha \}$ $\alpha = 1, \ldots, 2d$ be the cubes obtained by translating Λ_k along the coordinate axes by a distance $\pm 2([3d_{k+1}] - [3d_k])$ and let V: $\mathbb{Z}^d \to \{0, \lambda\}$ be defined by:

$$i) \qquad V(j) = \lambda \qquad j \in \Lambda_{k+1} \setminus \left(\bigcup_{\alpha=1}^{2d} \Lambda_k^\alpha \cup \Lambda_k \right)$$

$$V(j) = V(j^*) \qquad j \in \Lambda_k^\alpha \quad \alpha = 1, \ldots, 2d$$

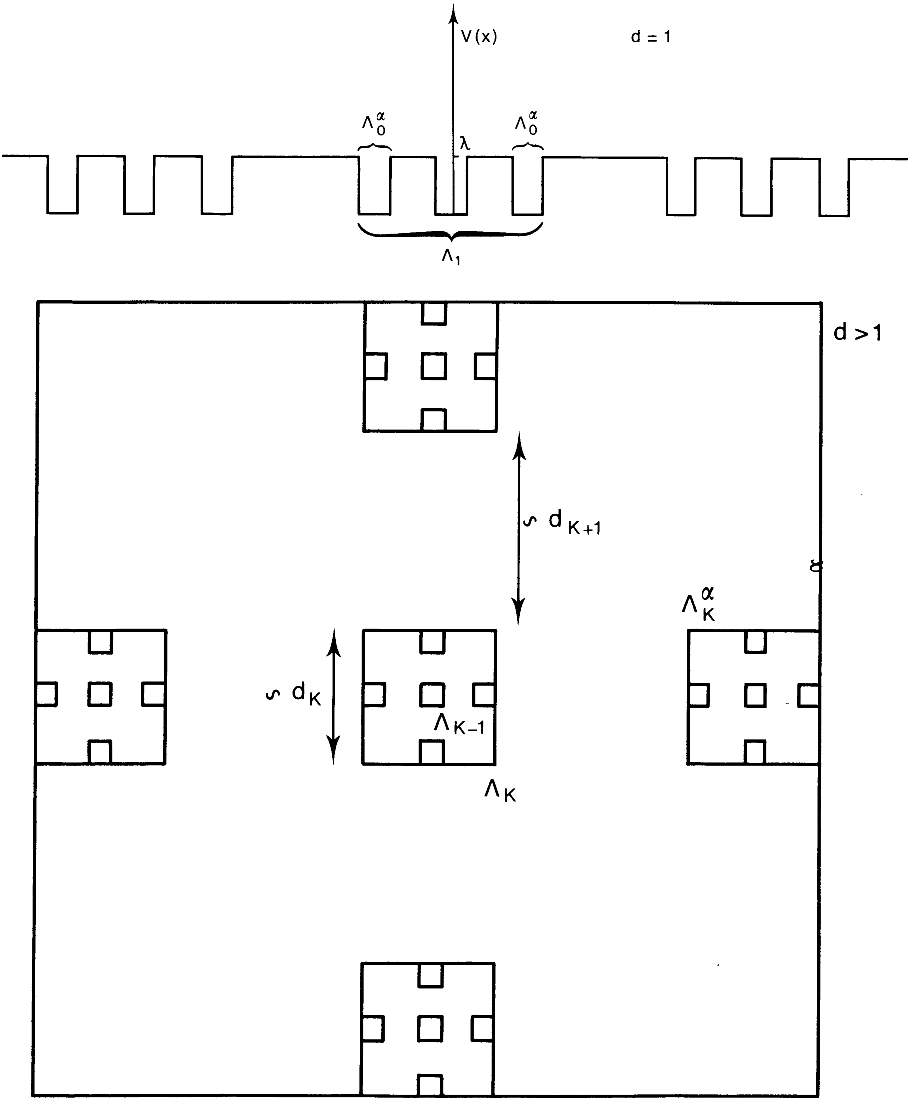

FIG. 2.

where $j^* \in \Lambda_k$ is obtained from $j \in \Lambda_k^\alpha$ by translating back the cube Λ_k^α till its center coincides with the origin.

ii) $$V(j) = 0 \qquad \forall j \in \Lambda_0.$$

It is clear that such a potential V satisfies the conditions of the general definition if we take

$$C_k^\alpha = \{ j \in \Lambda_k^\alpha ; V(j) = 0 \}$$

For reader's convenience the structure of the function V is sketched in fig. 2:

Because of its special symmetries we will henceforth refer to this example as the « symmetric hierarchical potential ».

Let now, for a hierarchical potential V, H be the tight-binding Hamiltonian:

$$\begin{aligned}(H\psi)(j) &= (-\Delta\psi)(j) + V(j)\psi(j) \\ &= \sum_{|j'-j|=1} \{ \psi(j) - \psi(j') \} + V(j)\psi(j) \qquad \psi \in l^2(\mathbb{Z}^d) \qquad (2.1)\end{aligned}$$

The Hamiltonian H defines a bounded selfadjoint operator on $l^2(\mathbb{Z}^d)$ and our aim is to study its spectral properties and in particular its time evolution.

In conclusion we would like to emphasize that most of the results of this paper hold for a class of potentials which is considerably richer than that fulfilling the general definition of this section.

As an example the barriers of length d_{k+1} appearing in a box Λ_{k+1} can be modified to contain chains of wells of diameter d_{k-1} at a distance one from the other not less than d_k. However in this paper we have not aimed at the maximum generality in order not to obscure the basic ideas.

III. UPPER BOUND ON r^2 (t)

In this section we analyse the long time behaviour of the quantity

$$r^2(t) = \langle e^{-itH}\phi, x^2 e^{-itH}\phi \rangle \qquad (3.1)$$

where $H = -\Delta + V$, V being a hierarchical potential and ϕ is an initial wave packet well localized in space *and* in energy. Before we start this analysis we need to locate the spectrum of the Hamiltonian H.

PROPOSITION 3.1. — Let V be a hierarchical potential and $\lambda = |V|_\infty$. Then :

a) $$\sigma(H) \cap [\lambda, +\infty) = [\lambda, \lambda + 4d]$$

b) $$\sigma(H) \cap [0, \lambda] \subseteq [0, 4d]$$

Proof. — *a*) Because of the hierarchical structure of the potential V the set $B = \{ j \in \mathbb{Z}^d ; V(j) = \lambda \}$ contains spheres of arbitrary radius. Thus the statement follows from Weyl's criterium.

b) Let $\Lambda \subset \mathbb{Z}^d$ be an arbitrary finite box centered at the origin. It clearly suffices to prove the result for H_Λ where for an arbitrary region $A \subset \mathbb{Z}^d$ we denote H_A the restriction of H to $l_2(A)$ with Dirichlet boundary conditions, since subsequently we can pass to the limit $\Lambda \uparrow \mathbb{Z}^d$ and use the strong resolvent convergence of H_Λ to H. Let then $\lambda_n(H_\Lambda)$ and $\lambda_n(V_\Lambda)$ be the eigenvalues of the matrices H_Λ, V_Λ respectively, V_Λ being the restriction of V to Λ. From the min-max principle we get:

$$\lambda_n(V_\Lambda) \leqslant \lambda_n(H_\Lambda) \leqslant \lambda_n(V_\Lambda) + 4d$$

since $\| - \Delta \| = 4d$. Since V is either zero or λ the proposition follows.

Remark. — It follows from the proof that

$$\# \{ n ; \lambda_n(H_\Lambda) \in [0, 4d] \} = \# \{ n \in \Lambda ; V(n) = 0 \}$$

We are now in a position to state precisely the main result of this section:

THEOREM 3.1. — Let $\Delta \equiv [0, 4d]$ and $P_\Delta(H)$ be the spectral projection of H associated to Δ. Define

$$r^2(t) = \sum_{x \in \mathbb{Z}^d} x^2 \, | (e^{-itH} P_\Delta(H) \delta_0)(x) |^2 \tag{3.2}$$

Then for large t, $r^2(t)$ satisfies the bound:

$$r^2(t) \leqslant \text{const} \, (\log t)^{2(5/4)^2}$$

Proof. — Let $\bar{\lambda} = (\lambda + 4d)/2$ and let Γ_t be the contour in the complex plane clockwise oriented drawn in fig. 3:

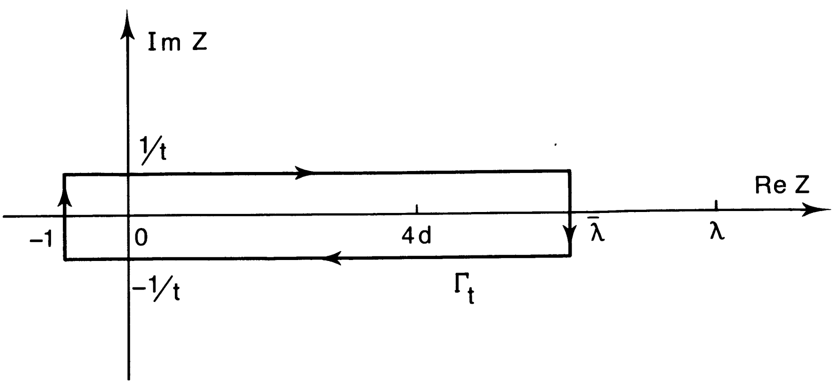

FIG. 3.

From the spectral theorem we get:

$$e^{-itH}P_\Delta(H)(x, y) = 1/2\pi i \oint_{\Gamma_t} dZ e^{-itZ}(G(Z)P_\Delta(H))(x, y) \qquad (3.3)$$

where $G(z) = (H - z)^{-1}$.

Thus we have to estimate the kernel of $G(z)$ $P(H)$ for $z \in \Gamma_t$. This is the content of the next lemma:

LEMMA 3.1. — There exist positive constants $m(\lambda)$, $K(\lambda)$, t_0, such that: for any $t > t_0$:

$$\sup_{Z \in \Gamma_t} | G(Z)P_\Delta(H)(0, y) | \leqslant e^{-(m(\lambda)/4)|y|}$$

provided $| y | \geqslant K(\lambda) (\ln t)^{(5/4)^2}$.

Furthermore $m(\lambda)$ is estimated by:

$$m(\lambda)^2 \geqslant (\lambda - 4d)/2$$

The proof is given in the second part of the paper and relies heavily on a theorem of Fröhlich and Spencer [5].

Assuming the lemma we can complete the proof of the theorem. We divide the sum appearing in the definition of $r^2(t)$ into two parts:

$$\begin{aligned} r^2(t) = & \sum_{\substack{x \in \mathbb{Z}^d \\ |x| \leqslant K(\lambda)(\ln t)^{(5/4)^2}}} | x |^2 | (e^{-itH}P_\Delta(H)\delta_0)(x) |^2 \\ + & \sum_{\substack{x \in \mathbb{Z}^d \\ |x| \geqslant K(\lambda)(\ln t)^{(5/4)^2}}} | x |^2 | (e^{-itH}P_\Delta(H)\delta_0)(x) |^2 \end{aligned} \qquad (3.4)$$

The first term in (3.4) is bounded by:

$$\text{const } (\ln t)^{2(5/4)^2}$$

while the second one, using the lemma and (3.3) is bounded by:

$$\text{const } e^{-(m(\lambda)/2)K(\lambda)(\ln t)^{(5/4)^2}}$$

Thus the theorem is proved.

This result requires some comments. First of all we want to give some intuitive reasons why the $\ln t$ appears in the estimate of theorem (3.1). In the energy range we are considering the spreading of the wave packet can take place only through tunneling from one minimum of the potential to the others. The mean time to overcome a barrier of length d_l is of order of $\exp(cd_l)$ for some $c > 0$. The time necessary therefore to reach the boundary of the box Λ_k is of order $\sum_{l=0}^{K} \exp(cd_l)$. This argument is legitimate because, due to the rapid increase of the d_k, tunneling on scale d_k is very weakly coupled with tunneling on scale d_{k+1}.

Replacing now the sum with its last term leads to a simple logarithmic relationship between space and time.

Lemma 3.1 expresses rigorously this fact as it implies the exponential decay of the wave packet outside the sphere of radius $K(\lambda)(\ln t)^{(5/4)^2}$. The exponent $(5/4)^2$ instead of 1 is purely of technical origin as it will be discussed at the end of the proof of the lemma.

IV. THE SYMMETRIC HIERARCHICAL POTENTIAL

1. Spectral properties.

In this section we analyse more closely the Hamiltonian $H = -\Delta + V$ where V is a symmetric hierarchical potential introduced in section II. The particular symmetries of this model will allow us to investigate in full detail the structure of the spectrum of H below $4d$ and its time evolution properties.

The basic result on the structure of the spectrum of H below $4d$ is:

THEOREM 4.1. — Let $I \equiv \sigma(H) \cap [0, 4d]$, where $H = -\Delta + V$, V a symmetric hierarchical potential.

Then:

a) The Lebesgue measure of I is zero.

b) I contains no isolated point of finite multiplicity i. e. $I \cap \sigma_{\text{dis}}(H) = \varnothing$ where $\sigma_{\text{dis}}(H)$ denotes the discrete part of the spectrum of H.

The proof of this theorem is given in the second part. The result however can be understood in simple terms. Suppose we consider first the Hamiltonian restricted to the box Λ_k with Dirichlet boundary conditions H_{Λ_k}. Its spectrum consists of discrete eigenvalues. When we go to the next scale d_{k+1} and we consider $H_{\Lambda_{k+1}}$ its part of the spectrum below $4d$ arises from the splittings of the eigenvalues of H_{Λ_k} due to the tunnelings among the equal boxes Λ_k^α contained in Λ_{k+1} over the scales d_{k+1}. Each level of H_{Λ_k} splits into $2d + 1$ levels whose spacing is at most of order $\exp(-Cd_{k+1})$ for some $C > 0$ (see Fig. 4). Therefore the spectrum below $4d$ of $H_{\Lambda_{k+1}}$ is contained in a neighborhood of order $\exp(-Cd_{k+1})$ of the spectrum of H_{Λ_k}.

From this it follows that I is contained in a neighborhood of order $\sum_{l \geq k}^{\infty} \exp(-Cd_l)$ of the spectrum of H_{Λ_k} for arbitrary k.

Going to the limit *a*) and *b*) follow.

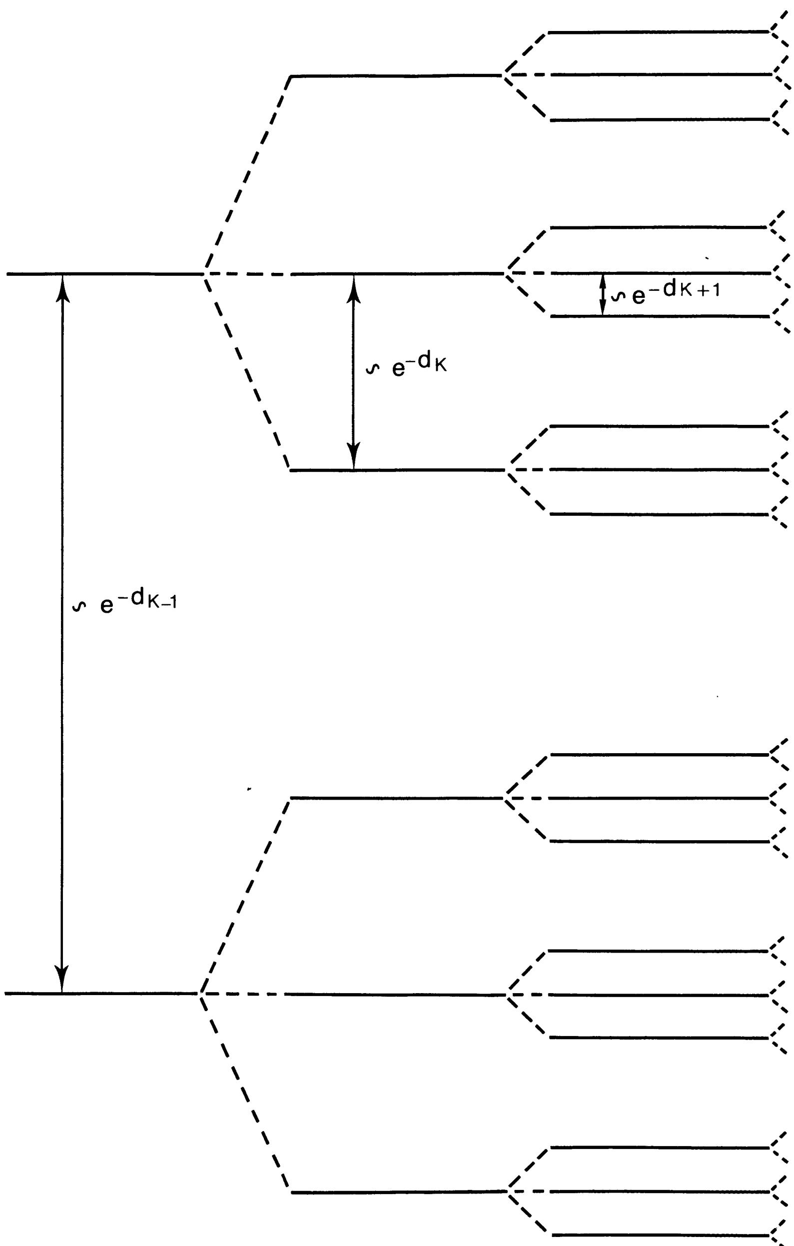

FIG. 4.

COROLLARY 4.1. — In the one dimensional case the set I is a Cantor set (closed, no-where dense, with no isolated point).

Proof. — In one dimension the spectral multiplicity of H is at most 2. Hence, using theorem 4.1 I has no isolated points. I is by definition closed and no-where dense since $| \mathrm{I} | = 0$.

For the part of the spectrum above λ which is not discussed in detail in this paper, as we mentioned in the introduction, for $d = 1$ the methods of Pearson [*4*] should imply that we have singular continuous spectrum. However the situation is different in $d > 1$ where it is possible to construct wave packets which are asymptotically free for large times. They evolve in the complement of the cones

$$\mathrm{C}_{\gamma,j} \equiv \{ \underline{x} \in \mathbb{Z}^d ; x . \underline{a}_j \geqslant | x | \cos \gamma \}, \qquad j = 1 \ldots 2d$$

where $\underline{a}_j$ are unit vectors associated to each direction of the coordinate axes and $\operatorname{tg} \gamma = \frac{1}{4} d_1/d_0$.

On the complement of the above cones the potential is constant by construction. This implies that above λ there is a component of absolutely continuous spectrum.

2. Lower bound on $r^2(t)$.

We turn here to the analysis of the time evolution of $- \Delta + \mathrm{V}$, and in particular to the long time behaviour of $r^2(t)$. Our main result is summarized in the following theorem:

THEOREM 4.2. — Let:

$$r^2(t) = \sum_{x \in \mathbb{Z}^d} | x |^2 \, | (e^{-it\mathrm{H}} \mathrm{P}_{[0,4d]}(\mathrm{H}) \delta_0)(x) |^2$$

where δ_0 is the Kronecker delta at $x = 0$ and $\mathrm{H} = - \Delta + \mathrm{V}$, V a symmetric hierarchical potential. Then there exists a sequence of times $\{ t_k \}$ such that:

$$r^2(t_k) \geqslant \frac{\text{const}}{(2d+1)^k} (\ln t_k)^2$$

Furthermore the times t_k's satisfy the bounds:

$$\exp(\sqrt{d_k}) \leqslant t_k \leqslant \exp(m'_0 d_k)$$

for any k sufficiently large, where m'_0 is a positive constant independent of k.

The proof of this theorem is based on a simple idea. If our system has delocalized eigenfunctions, and an important part of the proof consists in showing that this is the case, one can extract from the wave packet a linear combination of two eigenfunctions with the following properties: at $t=0$ it is completely localized near the origin while at a time $t \sim \pi/\Delta E$, where ΔE is the difference between the corresponding eigenvalues, is localized at a distance $0(\ln t)$. This is completely analogous to what happens in a symmetric double well potential if one considers for instance the sum of the ground state and of the first excited state.

In order to prove the theorem we need more informations about the low lying states of the Hamiltonian H_{Λ_k} for k large enough. These informations are provided by the next proposition;

PROPOSITION 4.2. — Let $\{ E_n^k \}$ and $\{ \psi_n^k \}$ be the eigenvalues and eigenfunctions of $H_{D_{k+1}}$ where D_{k+1} is a cube of side $4\left[\frac{d_{k+1}}{5}\right]$ centered at the origin. Then for k large enough the following holds:

a) Among the first $(2d+1)$ eigenfunctions there exists one, denoted by $\psi^{(k)}$ with eigenvalue $E^{(k)}$, which is different from the ground state $\psi_0^{(k)}$ and which is left invariant by reflections $X_i \to -X_i$ $i = 1 \dots d$, and by rotation of $\pi/2$ of the coordinate axes. Furthermore there are constants $a_0^{(k)}$, $a^{(k)}$ and $m_0 \geqslant \sqrt{2d}$ with

$$1 \geqslant (a_0^{(k)})^2 \geqslant 1/2d+1 - e^{-m_0 d_k/10}$$

$$1 \geqslant (a^{(k)})^2 \geqslant \frac{1}{2(2d+1)} - e^{-m_0 d_k/10}$$

such that for all $x \in D_k$:

$$| \psi_0^{(k)}(x) - a_0^{(k)}\psi_0^{(k-1)}(x) | \leqslant e^{-m_0 d_k/10}$$

$$| \psi^{(k)}(x) - a^{(k)}\psi_0^{(k-1)}(x) | \leqslant e^{-m_0 d_k/10}$$

b) There exists $\sqrt{\lambda} \geqslant m_0' \geqslant \sqrt{2d}$ such that

$$E_1^{(k)} - E_0^{(k)} \geqslant e^{-m_0' d_k} \qquad E_{2d}^{(k)} - E_0^{(k)} \leqslant e^{-\sqrt{d_k}}$$

c) $\psi_0^{(k)} \geqslant 0$ and $\psi_0^{(k)}(0) \geqslant \text{const}/(2d+1)^k$.

We assume the proposition and prove theorem 4.2. Define

$$t_k \equiv \pi/(E^{(k)} - E_0^{(k)}).$$

Clearly, using the proposition, t_k satisfies the stated bounds for k large enough. The next lemma shows that up to time t_k the dynamics generated by $e^{-itH}P_{(0,4d)}(H)$ and $e^{-itH_{D_{k+1}}}P_{(0,4d)}(H_{D_{k-1}})$ are very close to each other.

LEMMA 4.1. — For any k large enough, any $t \leqslant t_k$ and $x \in \Lambda_k$:

$$| e^{-itH}P_{(0,4d)}(H)(x,0) - e^{-itH_{D_{k+1}}}P_{(0,4d)}(H_{D_{k+1}})(x,0) | \leqslant e^{-\overline{m}d_{k+1}}$$

for some $\overline{m} > 0$.

The proofs of the proposition and of the lemma will be given in the second part.

Using the lemma we can bound $r^2(t_k)$ from below by:

$$r^2(t_k) \geqslant \sum_{x \in \Lambda_k} |x|^2 \, |e^{-it_k H_{D_{k+1}}} P_{(0,4d)}(H_{D_{k+1}})(0, x)|^2 - e^{-\bar{m} d_{k+1}}. \quad (4.2.1)$$

Define now the function $\overline{\psi}^{(k)} = \psi_0^{(k)} - (a_0^{(k)}/a^{(k)})\psi^{(k)}$.
Using $a)$ of proposition 4.2) we have:

$$|\overline{\psi}^{(k)}(x)| \leqslant 2e^{-m_0 d_k/10} \qquad \text{for} \quad x \in D_k \quad (4.2.2)$$

Furthermore, being a linear combination of eigenfunctions with energy less than $4d$, $\overline{\psi}^{(k)}$ decays exponentially outside the boxes D_k^α, $\alpha = 0, \ldots, 2d$.

$$\begin{aligned} |\langle \overline{\psi}^{(k)}, e^{-it_k H_{D_{k+1}}} P_{(0,4d)}(H_{D_{k+1}})\delta_0 \rangle| &= |\langle e^{it_k E_0^{(k)}} \psi_0^{(k)} - (a_0^{(k)}/a^{(k)}) e^{it_k E^{(k)}} \psi^{(k)}, \delta_0| \\ &= |\psi_0^{(k)}(0) + (a_0^{(k)}/a^{(k)})\psi^{(k)}(0)| \quad (4.2.3) \end{aligned}$$

by the definition of t_k.

The estimates $a)$ and $c)$ of proposition 4.2) imply that the r.h.s. of (4.2.3) is bounded from below by:

$$\text{const}/(2d+1)^{k/2} \quad (4.2.4)$$

On the other hand the l.h.s. of (4.2.3) is bounded above by:

$$\begin{aligned} &\Big\{ \sum_{x \in \bigcup_{\alpha=1}^{2d} D_k^\alpha} |(e^{-it_k H_{D_{k+1}}} P_{(0,4d)}(H_{D_{k+1}})\delta_0)(x)|^2 \Big\}^{1/2} \|\overline{\psi}^{(k)}\| \\ &\qquad + \sum_{x \notin \bigcup_{\alpha=1}^{2d} D_k^\alpha} |(e^{-it_k H_{D_{k+1}}} P_{(0,4d)}(H_{D_{k+1}})\delta_0)(x)| \, |\overline{\psi}^{(k)}(x)| \quad (4.2.5) \end{aligned}$$

The second term in (4.2.5) by (4.2.2) is exponentially small in d_k i. e. of order $\exp(-m_0 d_k/10)$ while the first one, using again proposition 4.2), is less than or equal:

$$\Big\{ \sum_{x \in \bigcup_{1=\alpha}^{2d} D_k^\alpha} |e^{-it_k H_{D_{k+1}}} P_{(0,4d)}(H_{D_{k+1}})\delta_0(x)|^2 \Big\}^{1/2} (1 + (a_0^{(k)}/a^{(k)})^2)^{1/2} \quad (4.2.6)$$

Combining (4.2.4) ... (4.2.6) we see that the restriction of the function $e^{-it_k H_{D_{k+1}}} P_{(0,4d)}(H_{D_{k+1}})\delta_0$ to the boxes D_k^α, $\alpha = 1 \ldots 2d$ has norm greater or equal than const $(2d+1)^{-k/2}$. This, together with (4.2.1) proves the theorem.

V. RANDOM HIERARCHICAL POTENTIALS

So far we have considered situations in which the potential wells had the same depth producing in this way resonances over all the length scales. In this section we want to study the case in which the bottoms of the wells may be at different heights.

A natural choice is to let them fluctuate under the effect of a random perturbation. The perturbation is taken in such a way that the hierarchical structure of V is preserved. More precisely we let $d\lambda(v)$ to be a probability measure on the reals with a bounded density with respect to the Lebesgue measure and support on $[0, 1]$ and consider the probability space $\Omega = \prod_{j \in \mathbb{Z}^d} ([0, 1], d\lambda(v_j))$. We then call a random field $V_v : Z^d \to R$, $v \in \Omega$ a « *hierarchical random potential* » if there exists a hierarchical potential V such that:

$$V_v(j) = V(j) + \beta v(j) \qquad \forall j \in \mathbb{Z}^d \tag{5.1}$$

for some $\beta \in (0, 1)$. Here β measure the strength of the perturbation. We then consider the stochastic tight-binding Hamiltonian

$$H(v) = -\Delta + V_v, \tag{5.2}$$

The main consequence of the introduction of a stochastic perturbation is that all the states with energy less than λ, λ being the maximum value of the deterministic part of V_v, become exponentially localized and in turn the quantity $r^2(t)$ will stay bounded uniformly in t, for *any* $0 < \beta \leqslant 1$. This is the content of the two main theorems of this section. In what follows the coupling constant β will be supposed to be fixed within zero and one.

THEOREM 5.1. — There exists a set of realizations of the potential $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that if $v \in \Omega_0$ and $H(v)$ is the corresponding Hamiltonian, the following holds:
let $E < \lambda$ be an energy for which

$$(H(v) - E)\psi = 0$$

has a polynomially bounded solution ψ_E; then there exists a $\overline{K}(E, v)$ with the property that any such ψ_E is exponentially localized in the box $\Lambda_{\overline{K}(E,v)}$ in the sense that:

$$|\psi_E(x)| \leqslant \exp(-m \operatorname{dist}(x\partial^{ext}\Lambda_{\overline{K}(E,v)}))d^{d-1}_{\overline{K}(E,v)} \sup_{y \in \Lambda_{\overline{K}(E,V)}} |\psi_E(y)|$$

for all x with $\operatorname{dist}(x, \partial^{ext}\Lambda_{\overline{k}(E,v)}) \geqslant \frac{1}{5} d_{\overline{k}(E,v)+1}$. Here $m \geqslant \frac{1}{2}\sqrt{\lambda - E}$.

THEOREM 5.2. — Let for $\lambda > \varepsilon > 0$:

$$r^2(t, v) \equiv \langle \delta_0, e^{itH(v)}P_{(0,\lambda-\varepsilon)}(H(v))x^2e^{-itH(v)}P_{(0,\lambda-\varepsilon)}(H(v))\delta_0 \rangle$$

Then with probability one there exists a finite constant $C(v)$ such that

$$r^2(t, v) \leqslant C(v) \qquad \forall t \geqslant 0$$

The proof of the above theorem is rather technical and is given in the second part. Here we want to emphasize that the general random hierarchical models represent the first multidimensional case in which $r^2 < C$ has been proved. The specific features of our models allow to prove localizations for any $d \geqslant 1$ independently of the strength of the stochastic perturbation as it happens in the one dimensional Anderson model [6].

VI. EXTENSION TO THE CONTINUUM CASE

To conclude we wish to briefly discuss the extension of our analysis to the Schrödinger operator in the continuous case. The simplest version of a hierarchical potential V on $\mathbb{R}^d$ can be obtained by imposing that V is a constant V_i on each unit cube C_i around the site $i \in \mathbb{Z}^d$ and that $\{ V_i \}_{i\in\mathbb{Z}^d}$ is itself a hierarchical potential. This corresponds to a situation where one has square wells separated by constant barriers. One could also smoothen out the shape of the wells to get it C^∞, while keeping the bottoms of the wells all at the same level. Since the main ingredient of our analysis, namely the Fröhlich-Spencer bound on the Green's function, has been extended to the continuous case in [7] the above analysis can be carried out without too much trouble also for $-\Delta + V$ on $L^2(\mathbb{R}^d)$, V a hierarchical potential. A little complication arise when one tries to choose the height λ of the barriers among the wells in such away that for some $\delta > 0$ $(\lambda-\delta, \lambda)$ is a gap for $\sigma(H)$. However this can be satisfactory solved, at least for the simplest case $V \upharpoonright_{C_i} = V_i$, by means of the Dirichlet-Neumann bracketing. Also the random case is within the reach of our analysis provided we perturb a hierarchical potential V by means of random fields such that: $v \upharpoonright_{C_i} = v(i)$ and the $v(i)$'s, $i \in \mathbb{Z}^d$, are i. i. d. random variables.

VII. PROOFS

In this part of the paper we give detailed proofs of the results explained in the first part.

An important tool will be a recent result obtained by Fröhlich and Spencer [5] in their analysis of the Anderson model concerning the exponential decay of the Green's function of tight-binding Hamiltonians restricted to bounded regions with Dirichlet boundary conditions. For completeness we recall here their notations and main result in a form more suitable for our purposes.

1) The Theorem of Fröhlich and Spencer.

Let $v: \mathbb{Z}^d \to \mathrm{R}$, $v \geqslant 0$ be bounded and let $\mathrm{H} = -\Delta + v$ on $l^2(\mathrm{Z}^d)$. For any set $\mathrm{A} \subset \mathrm{Z}^d$ we define:

$$\begin{aligned} \partial \mathrm{A} &= \{ (n, n'), n \in \mathrm{A}, n' \notin \mathrm{A}, | n - n' | = 1 \} \\ \partial^{\mathrm{int}} \mathrm{A} &= \{ n \in \mathrm{A} ; \exists n' \notin \mathrm{A} ; | n - n' | = 1 \} . \\ \partial^{\mathrm{ext}} \mathrm{A} &= \partial^{\mathrm{int}}(\mathbb{Z}^d \backslash \mathrm{A}) \end{aligned}$$

and set H_A to be the restriction of H to $l^2(\mathrm{A})$ with Dirichlet boundary conditions at $\partial \mathrm{A}$. We will also define:

$$\mathrm{G}_\mathrm{A}(\mathrm{E}, x, y) \equiv (\mathrm{H}_\mathrm{A} - \mathrm{E})^{-1}(x, y)$$

whenever it exists.

For any positive α and m and any E let $\mathrm{S}_j(\mathrm{E}, v, m, \alpha) \subset \mathbb{Z}^d$ be a sequence of sets defined inductively as follows:

$$\begin{aligned} \mathrm{S}_0 (\mathrm{E}, v, m, \alpha) &= \{ j \in \mathrm{Z}^d ; v(j) - \mathrm{Re}\, \mathrm{E} < m^2 \} \\ \mathrm{S}_{j+1}(\mathrm{E}, v, m, \alpha) &= \mathrm{S}_j(\mathrm{E}, v, m, \alpha) \backslash \mathrm{S}_j^g(\mathrm{E}, v, m, \alpha) \end{aligned}$$

where $\mathrm{S}_j^g(\mathrm{E}, v, m, \alpha) = \bigcup_\beta \mathrm{C}_j^\beta$ is the *maximal* union of components of $\mathrm{S}_j(\mathrm{E}, v, m, \alpha)$ such that:

$$\begin{array}{ll} i) & \operatorname{diam} \mathrm{C}_j^\beta \leqslant 12\, l_j \\ ii) & \operatorname{dist} (\mathrm{C}_j^\beta, \mathrm{S}_j(\mathrm{E}, v, m, \alpha) \backslash \mathrm{C}_j^\beta) \geqslant 2 l_{j+1} \\ iii) & \operatorname{dist} (\sigma(\mathrm{H}_{\overline{\mathrm{C}_j^\beta}}), \mathrm{E}) \geqslant e^{-\alpha l_j} \end{array}$$

where $\overline{\mathrm{C}_j^\beta} = \{ i \in \mathrm{Z}^d ; \operatorname{dist} (i, \mathrm{C}_j^\beta) \leqslant 4 l_j \}$.

Here the length scales l_j satisfy:

$$l_{j+1} = l_j^{5/4} ; \qquad l_0 = d_{k_0}, \qquad k_0 \geqslant 0$$

A set $\mathrm{A} \subset \mathrm{Z}^d$ will be said to be k-admissible iff $\partial \mathrm{A} \cap \overline{\mathrm{C}_j^\beta} = \varnothing \ \forall j = 0, 1 \ldots k$ and any β.

We are now in a position to state the modified Fröhlich-Spencer result:

THEOREM 7.1. — For any real $\varepsilon \neq 0$ and all $m > 0$ there exist constants $k_0(m) > 1$, $\alpha_0(m) < 1$ *independent* of ε such that if $l_0 \equiv d_{k_0(m)}$, and $\alpha \leqslant \alpha_0(m)$ then for all k-admissible sets A with $\mathrm{A} \cap \mathrm{S}_{k+1}(\mathrm{E}, v, m, \alpha) = \varnothing$:

$$\begin{array}{lll} a) & | \mathrm{G}_\mathrm{A}(\mathrm{E} + i\varepsilon, x, y) | \leqslant e^{-\frac{m}{2} |x-y|} & \text{if} \quad | x - y | \geqslant \frac{1}{5} l_{k+1} \\ b) & \| \mathrm{G}_\mathrm{A}(\mathrm{E}) \| < +\infty . & \end{array}$$

The proof of the theorem is omitted since it follows word by word the proof of theorem 2.1 in [5].

Remark. — 1) In their original paper Fröhlich and Spencer expressed the non-resonance condition as: dist (E, $\sigma(\mathrm{H}_{\overline{\mathrm{C}_k^\beta}})) \geqslant \exp(-\sqrt{l_k})$. It is easy to check that as far as the exponential decay of the Green's function the two conditions are equivalent provided α is small enough. However with our choice we will get a better estimate on the long time behaviour of the mean square displacement.

2) In their proof Fröhlich and Spencer used a sequence of length scales increasing like the l_k above. This choice was dictated by the needs of their probabilistic estimates. The theorem on the decay of the Green's function however remains true for $l_k \sim \exp(k^{1+\varepsilon})$, $\varepsilon > 0$.

2) **Proof of lemma 3.1.**

We begin the proof by proving the following estimate on the Green's function G(Z):

$$\sup_{\mathrm{Z}\in\Gamma_t} |\,\mathrm{G}(\mathrm{Z}, 0, y)\,| \leqslant 2\exp\left(-\frac{m}{2}|\,y\,|\right) \tag{7.2.1}$$

for $|\,y\,| \geqslant k(\lambda)(\ln t)^{(5/4)^2}$ for a suitable constant $k(\lambda)$.

In what follows the Fröhlich-Spencer result in the form given in theorem 7.1 will play a crucial role. Let $m^2 = \lambda - \bar{\lambda} = \dfrac{\lambda - 4d}{2}$ and let $\alpha \equiv \alpha_0(m)$, $k_0 \equiv k_0(m)$ be the corresponding constants appearing in theorem 7.1. Define now $k(t)$ as the smallest integer such that:

$$\exp(-\alpha_0 d_{k(t)}) \leqslant 1/t \qquad \text{for} \qquad t \geqslant \tilde{t}_0 \equiv \exp(\alpha_0 d_{k_0+1})$$

With this position we see that using the hierarchical structure of the potential V, for any $k > k(t) > k_0$ the box D_{k+1} centered at the origin of side $4\left[\dfrac{d_{k+1}}{5}\right]$ is $k - k_0$ admissible and satisfies:

$$\mathrm{D}_{k+1} \cap \mathrm{S}_{k-k_0}(\mathrm{Z}, \mathrm{V}, \alpha_0, m) = \varnothing \qquad \forall \mathrm{Z} \in \Gamma_t$$

Thus from theorem 7.1 we get:

$$|\,\mathrm{G}_{\mathrm{D}_{k+1}}(\mathrm{Z}, 0, y)\,| \leqslant \exp\left(-\frac{m}{2}|\,y\,|\right) \tag{7.2.2}$$

provided $|\,y\,| \geqslant (1/5)l_{k-k_0} \equiv (1/5)d_k$.

Let $k \equiv k(y)$ be the smallest integer such that $y \in \tilde{\mathrm{D}}_{k+1}$ where $\tilde{\mathrm{D}}_{k+1}$ is a box centered at the origin of side $2\left[\dfrac{d_{k+1}}{5}\right]$.

We then see that there exists a constant $k(\lambda) \sim \alpha_0^{-(5/4)^2}$ such that condition $|y| \geqslant k(\lambda)(\ln t)^{(5/4)^2}$ implies $k(y) \geqslant k(t) + 1$. For $Z \in \Gamma_t$ we can write, using the first resolvent identity (see Sect. 2 of [5]):

$$G(Z, 0, y) = G_{D_{k+1}}(Z, 0, y) + G_{D_{k+1}}(Z)\Gamma_{\partial D_{k+1}}G(Z)(0, y) \quad (7.2.3)$$

where for any set $A \subset Z^d$ the boundary operator $\Gamma_{\partial A}$ is defined by:

$$\Gamma_{\partial A}(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \partial A \\ 0 & \text{otherwise} \end{cases} \quad (7.2.4)$$

Using (7.2.2) it follows that the r.h.s. of (7.2.3) is bounded by:

$$e^{-\frac{m}{2}|y|} + e^{-\frac{m}{2}\frac{2}{5}d_{k+1}} d_{k+1}^{d-1} e^{\alpha_0 d_{k(t)}} \leqslant 2e^{-\frac{m}{2}|y|} \quad (7.2.5)$$

where we have used the estimates:

i) $\quad |y| \geqslant (1/5)d_k$ from the definition of $k = k(y)$

ii) $\quad \operatorname*{Sup}_{Z \in \Gamma_t} \sup_{x,y \in \mathbb{Z}^d} |G(Z, x, y)| \leqslant \sup_{Z \in \Gamma_t} \|G(Z)\| \leqslant t \leqslant e^{\alpha_0 d_{k(t)}}$

iii) $\quad \exp\{-md_{k+1}/10 + \alpha_0 d_{k(t)}\} d_{k+1}^{d-1} \leqslant 1$

which holds for t sufficiently large.

Thus inequality (7.2.1) is proved.

In order to conclude the proof of the lemma it remains to estimate the kernel of $P_\Delta(H)$. From the spectral theorem we have:

$$P_\Delta(H)(x, y) = \frac{1}{2\pi i} \oint_\gamma dZ G(Z, x, y) \quad (7.2.6)$$

where γ is a contour enclosing $[0, 4d]$ in the complex plane such that

$$\operatorname{dist}(\gamma, [0, 4d] \cup \Gamma_t \cup [\lambda, \lambda + 4d]) \geqslant \frac{m^2}{2}$$

Clearly from the definition of Γ_t and proposition 3.1 such a contour always exists.

Using now the Combes-Thomas argument (see e. g. [8]) we get

$$\sup_{Z \in \gamma} |G(Z, x, y)| \leqslant \frac{2}{m^2} \exp\left\{ -\frac{m}{2\sqrt{2}}|x - y| \right\} \quad (7.2.7)$$

We are now in position to estimate the kernel of $G(Z)P_\Delta(H)$ for $Z \in \Gamma_t$. From (7.2.6) and the resolvent identity we have:

$$G(Z)P_\Delta(H)(0, y) = \frac{1}{2\pi i} \oint_\gamma \frac{dZ'}{(Z - Z')} [G(Z, 0, y) - G(Z', 0, y)] \quad (7.2.8)$$

Using now (7.2.7) together with (7.2.1) and the bound

$$\sup_{\substack{Z' \in \gamma \\ Z \in \Gamma_t}} | Z' - Z | \geqslant \frac{m^2}{2}$$

we have:

$$\sup_{Z \in \Gamma_t} | G(Z)P_\Delta(H)(0, y) | \leqslant C(\lambda)e^{-\frac{m}{2\sqrt{2}}|y|} \leqslant e^{-\frac{m}{4}|y|} \tag{7.2.9}$$

provided $| y | \geqslant k(\lambda)(\ln t)^{(5/4)^2}$ and t is large enough.

The lemma is thus proved.

We would like at this point to make some comments both on the result and on the assumptions of the theorem.

We first emphasize that the appearance of $(5/4)^2$ in the exponent is essentially due to the fact that we have not attempted to give the best possible estimate. With some extra effort one can show that the exponential decay of the Green's function actually starts on the scale $d_{k(t)}$, and this allows to change $(5/4)^2$ into 5/4.

This residual dependence on the way the distances d_k increase is due to the fact that to any time $s \in [e^{\alpha_0 d_{k-1}}, e^{\alpha_0 d_k})$ is associated the same $k(s) = k$. In fact if one estimates $r^2(t)$ at the discrete times $t_k = e^{\alpha_0 d_k}$ one can infer the bound:

$$r^2(t_k) \leqslant C(\lambda)(\ln t_k)^2$$

As a second remark we point out that the assumption $\lambda > 4d$ which insures the existence of a gap in the spectrum of H, is important for the exponential decay of the kernel of $P_\Delta(H)$.

3) **Proof of Theorem 4.1.**

Let D_{k+1} be the cube centered at the origin of side $4\left[\frac{d_{k+1}}{5}\right]$ and let $I_k \equiv \sigma(H_{D_{k+1}}) \cap [0, 4d]$. We also denote by $\tilde{I}_k$ the closure of the $\exp(-\sqrt{d_{k+1}})$-neighborhood of I_k. We will prove the following inclusions (see fig. 4):

$$\tilde{I}_{k+1} \subset \tilde{I}_k \tag{7.3.1}$$

and

$$I \equiv \sigma(H) \cap [0, 4d] \subset \tilde{I}_k \tag{7.3.2}$$

for all k sufficiently large.

Since $| \tilde{I}_k | \leqslant \exp(-\sqrt{d_{k+1}})d_k^d$, where $| \, . \, |$ denotes the Lebesgue measure, (7.3.2) implies (*a*).

To prove (7.3.2) it clearly suffices, using (7.3.1) that we prove after, to show that:

$$I \subset \bigcup_{k \geqslant k_0} \tilde{I}_k = \tilde{I}_{k_0} \tag{7.3.3}$$

The above inclusion (7.3.3) follows immediately from the strong resolvent convergence of $H_{D_{k+1}}$ to H as $k \to +\infty$.

It remains to prove (7.3.1) and part *b*) of theorem 4.1. Let us fix $k \gg 1$ and consider a covering of I_k by disjoint, closed intervals $\{ \Delta_i^{(k)} \}_{i=1}^{N_k}$ with the following properties:

i) $$\operatorname{diam} \Delta_i^{(k)} \leqslant \frac{1}{2} \exp(-\sqrt{d_{k+1}})$$

ii) $$e^{-2\sqrt{d_{k+1}}} \leqslant \operatorname{dist}(\partial\Delta_i^{(k)}, I_k \cup I_{k+1})$$

The existence of such a covering is immediate if the points of the set $I_k \cup I_{k+1}$ are spaced one from the other by more than $2\exp(-2\sqrt{d_{k+1}})$. In this case each $\Delta_i^{(k)}$ contains only one point of I_k. If there exist clusters of points of $I_k \cup I_{k+1}$ whose spacing is less than $2\exp(-2\sqrt{d_{k+1}})$ then these clusters have a length at most

$$2\exp(-2\sqrt{d_{k+1}})((2d+1)^{k+1}+(2d+1)^k)\ \#\ \{ E \in I_0 \} \leqslant \frac{1}{2}\exp(-\sqrt{d_{k+1}})$$

for k large enough, where we have used that $\#\{E \in I_k) = (2d+1)^k \# \{ E \in I_0 \}$ by the remark after proposition (3.1) and the definition of the symmetric hierarchical potential. Thus the existence of a covering satisfying *i*) and *ii*) follows.

For the proof of (7.3.1) and *b*) we need the following basic lemma:

LEMMA 7.1.

$$\forall i \qquad \# \{ E \in I_{k+1} \cap \Delta_i^{(k)} \} = (2d+1)\ \# \{ E \in I_k \cap \Delta_i^{(k)} \} \qquad (7.3.4)$$

The meaning of this lemma is that each eigenvalue of the Hamiltonian $H_{D_{k+1}}$ splits into $(2d+1)$ eigenvalues with a spacing of at most $1/2\exp(-\sqrt{d_{k+1}})$. The symbol $\#$ counts also the multiplicity.

Suppose now that k was chosen so large that

$$\exp(-\sqrt{d_{k+2}}) \ll 1/2 \exp(-\sqrt{d_{k+1}})\,;$$

then from the lemma and the relationship:

$$\# \{ E \in I_{k+1} \} = (2d+1)\ \# \{ E \in I_k \} \qquad (7.3.5)$$

we get (7.3.1).

Proof of lemma 7.1. — This is obtained if we prove that:

$$\operatorname{Tr} P_{\Delta_i^{(k)}}(H_{D_{k+2}}) = (2d+1) \operatorname{Tr} P_{\Delta_i^{(k)}}(H_{D_{k+1}}) \qquad (7.3.6)$$

where Tr stands for trace and $P_\Delta(H_{D_k})$ is the spectral projection of H_{D_k} associated to Δ. To prove (7.3.6) let us fix $\Delta_i^{(k)}$, henceforth denoted by Δ, and let us consider a circle γ in the complex plane with diameter equal to the length of Δ and center in the middle point of Δ. By the spectral theorem we have:

$$P_\Delta(H_{D_{k+2}}) = \frac{1}{2\pi i}\oint_\gamma dZ G_{D_{k+2}}(Z) \tag{7.3.7}$$

Let now $\{ D^\alpha_{k+1} \}^{2d}_{\alpha=0}$, $D^0_{k+1} \equiv D_{k+1}$ be the cubes obtained by translating D_{k+1} along the coordinate axes by an amount of $\pm 2 \{ [3d_{k+1}] - [3d_k] \}$, let $\Gamma = \Gamma_{\bigcup_{\alpha=0}^{2d} \partial D^\alpha_{k+1}}$ be the boundary operator associated to their boundaries and $\hat{G}(Z)$ the Green's function of $H_{D_{k+2}}$ with additional Dirichlet boundaries condition at $\bigcup_{\alpha=0}^{2d} \partial D^\alpha_{k+1}$.

Then from (7.3.7) and the first resolvent identity we get:

$$P_\Delta(H_{D_{k+2}}) = \bigoplus_{\alpha=0}^{2d} P_\Delta(H_{D^\alpha_{k+1}}) + \frac{1}{2\pi i}\oint_\gamma dZ \hat{G}(Z)\Gamma G_{D_{k+2}}(Z) \tag{7.3.8}$$

Since the operators $H_{D^\alpha_{k+1}}$ differ only by a translation (7.3.8) implies:

$$\operatorname{Tr} P_\Delta(H_{D_{k+2}}) = (2d+1)\operatorname{Tr} P_\Delta(H_{D_{k+1}}) + \operatorname{Tr}\frac{1}{2\pi i}\oint_\gamma dZ \hat{G}(Z)\Gamma G_{D_{k+2}}(Z) \tag{7.3.9}$$

We estimate now the r. h. s. of (7.3.9). Let $\{ \varphi_n \}$ and $\{ \psi_n \}$ be the eigenfunctions of $H_{D_{k+2}}$ and $\bigoplus_{\alpha=0}^{2d} H_{D^\alpha_{k+1}}$ respectively with eigenvalues in Δ.

Then, from (7.3.8), (7.3.9) we have:

$$\begin{aligned}\operatorname{Tr} P_\Delta(H_{D_{k+2}}) &\leqslant (2d+1)\operatorname{Tr} P_\Delta(H_{D_{k+1}}) \\ &+ \sum_n \left\langle \varphi_n, \frac{1}{2\pi i}\oint_\gamma dZ \hat{G}(Z)\Gamma G_{D_{k+2}}(Z)\varphi_n \right\rangle \end{aligned} \tag{7.3.10}$$

and

$$\begin{aligned}\operatorname{Tr} P_\Delta(H_{D_{k+2}}) &\geqslant (2d+1)\operatorname{Tr} P_\Delta(H_{D_{k+1}}) \\ &+ \sum_n \left\langle \psi_n, \frac{1}{2\pi i}\oint_\gamma dZ \hat{G}(Z)\Gamma G_{D_{k+2}}(Z)\psi_n \right\rangle \end{aligned} \tag{7.3.11}$$

where $\langle , \rangle$ is the scalar product in $l^2(D_{k+2})$ and we have extended $\{ \psi_n \}$ outside $\bigcup_{\alpha=0}^{2d} D^\alpha_{k+1}$ by setting them equal to zero outside $\bigcup_{\alpha=0}^{2d} D^\alpha_{k+1}$. We discuss only the second term in the r.h.s. of (7.3.10) since the case (7.3.11) is completely analogous. Using the fact that $\{ \varphi_n \}$ are eigenfunctions of $H_{D_{k+1}}$ with eigenvalues $E_n \in \Delta$, the n-th term in the sum in (7.3.10) is equal to:

$$\left\langle \varphi_n, \frac{1}{2\pi i} \oint_\gamma dZ\hat{G}(Z)\Gamma\varphi_n/(E_n - Z) \right\rangle \tag{7.3.12}$$

Since $E_n \in (0, 4d)$ and the potential V is equal to $\lambda > 4d$ in

$$D_{k+2} \backslash \bigcup_{\alpha=0}^{2d} D_{k+1}^\alpha$$

it is easy to show that:

$$\sup_{y \in \bigcup_\alpha \partial D_{k+1}^\alpha} |\varphi_n(y)| \leqslant \exp(-md_{k+1}/10) \qquad \text{with} \qquad m^2 \geqslant \lambda - 4d. \tag{7.3.13}$$

Furthermore from the definition of $\{\Delta_i^{(k)}\}$, $\inf_{Z\in\gamma} |E_n - Z| \geqslant \exp(-2\sqrt{d_{k+1}})$. Hence (7.3.12) is estimated by:

$$d_{k+2}^d e^{-md_{k+1}/10} d_{k+1}^{d-1} e^{4\sqrt{d_{k+1}}} < e^{-\sqrt{d_{k+1}}} \tag{7.3.14}$$

for k large enough.

(7.3.4) implies in turn that the sum in the r.h.s. of (7.3.10) is bounded by:

$$\#\{E \in I_{k+1}\} e^{-\sqrt{d_{k+1}}} < 1 \tag{7.3.15}$$

Hence $\operatorname{Tr} P_\Delta(H_{D_{k+2}}) \leqslant (2d + 1) \operatorname{Tr} P_\Delta(H_{D_{k+1}})$.

Analogously one proves:

$$\operatorname{Tr} P_\Delta(H_{D_{k+2}}) \geqslant (2d + 1) \operatorname{Tr}_\Delta P(H_{D_{k+1}})$$

and the lemma is proved.

To prove part b) let E be an isolated point of I. There exists then a δ-neighborhood of E, Δ_δ such that: dist $(I\backslash\{E\}, \partial\Delta_\delta) > 2\delta$ and by the previous discussion a k_0 such that dist $(E, I_{k_0}) \leqslant e^{-\sqrt{d_{k_0+1}}} < \delta/4$. Using (7.3.4) we now get:

$$\#\{E \in I_{k_0+1} \cap \Delta_{\delta/4} + e^{-\sqrt{d_{k_0+1}}}\} \geqslant 2d + 1 \tag{7.3.16}$$

A k-times iteration of (7.3.16) gives:

$$\#\left\{E \in I_{k_0+k} \cap \Delta_{\delta/4} + \sum_{j=k_0+1}^{k+1} e^{-\sqrt{d_j}}\right\} \geqslant (2d + 1)^k \tag{7.3.17}$$

Since $\sum_{j\geqslant k_0+1}^{\infty} \exp(-d_j) < \delta/4$ for k_0 large enough (7.3.17) implies:

$$\#\{E \in I_{k_0+k} \cap \Delta_{\delta/2}\} \geqslant (2d + 1)^k \tag{7.3.18}$$

It is easy to show that (7.3.18) implies:

$$\mathrm{Tr}\, P_{\Delta_\delta}(H) = +\infty$$

i. e. E has infinite multiplicity.

4) Proof of Lemma 4.1 and of Proposition 4.1.

From the spectral theorem, (7.2.3) and (7.2.8):

$$e^{-itH}P_{(0,4d)}(H)(x,0) = \frac{1}{2\pi i}\oint_{\Gamma_t} dZ e^{-itZ}(G(Z)P_{(0,4d)}(H))(x,0)$$
$$= e^{-itH_{D_{k+1}}}P_{(0,4d)}(H_{D_{k+1}})(x,0) + \frac{1}{(2\pi)^2}\oint_{\Gamma_t} dZ e^{-itZ}\oint_\gamma \frac{dZ'}{(Z'-Z)}$$
$$[G(Z)\Gamma_{D_{k+1}}G_{D_{k+1}}(Z)(x,0) - G(Z')\Gamma_{D_{k+1}}G_{D_{k+1}}(Z')(x,0)] \quad (7.4.1)$$

where Γ_t is the contour drawn in Fig. 3 and γ is as in (7.2.6).

For $t \leqslant t_k \leqslant \mathrm{const}\, e^{m'_0 d_k}$ the second term in the r. h. s. of (7.4.1) is bounded by:

$$\mathrm{const}\; d_{k+1}^{d-1} e^{m'_0 d_k} \sup_{\substack{y\in\partial D_{k+1}\\ Z\in\Gamma_t\cup\gamma}} |G_{D_{k+1}}(Z,0,y)| \quad (7.4.2)$$

Since the potential V is equal to $\lambda > 4d$ in $D_{k+1}\backslash\Lambda_k$ it is easy to see that:

$$|G_{D_{k+1}}(Z,0,y)| \leqslant e^{m'_0 d_k} e^{-m d_{k+1}/10} \quad (7.4.3)$$

for all $y \in \partial D_{k+1}$, $Z \in \Gamma_t \cup \gamma$ and k large enough. As usual $m \geqslant \sqrt{\lambda - 4d}$.

Hence (7.4.2) is smaller than $e^{-\bar m d_{k+1}}$ for some $\bar m > 0$; this, together with (7.4.1), implies:

$$|e^{-itH}P_{(0,4d)}(H)(x,0) - e^{-itH_{D_{k+1}}}P_{(0,4d)}(H_{D_{k+1}})(x,0)| \leqslant e^{-\bar m d_{k+1}} \quad (7.4.4)$$

for k large enough and $t \leqslant t_k$.

We are left with the proof of proposition (4.1).

Part c). — It is quite easy to check that $\psi_0^{(0)}(0) \geqslant |\Lambda_0|^{-1/2} \equiv \delta_0$. Using part *a)* we can proceed by induction. In fact assume that:

$$\psi_0^{(k-1)}(0) \geqslant \delta_{k-1}/(2d+1)^{(k-1)/2}$$

with

$$\delta_{k-1} \equiv \delta_0 - \sum_{j=1}^{k-1} 2e^{-m_0 dj/10}(2d+1)^{j/2}$$

Then from *a)* we obtain:

$$\psi_0^{(k)}(0) \geqslant \delta_{k-1}/(2d+1)^{k/2} - 2e^{-m_0 dk/10} = \delta_k/(2d+1)^{k/2}$$

Hence if d_0 is so large that:

$$\sum_{j=1}^{\infty} (2d+1)^{j/2} e^{-m_0 d j/10} \leqslant (1/2)\delta_0 \tag{7.4.5}$$

Then $\delta_k \geqslant \frac{1}{2}\delta_0$ for any k.

For $d = 1, 2, 3$ our choice $d_0 \geqslant 20$ is sufficient to satisfy (7.4.5). As far as the positivity of $\psi_0^{(k)}$ is concerned we refer to [9].

Part b). — The bound $E_{2d}^{(k)} - E_0^{(k)} \leqslant e^{-\sqrt{d_k}}$ actually follows from the proof of theorem 4.1. To estimate $E_1^{(k)} - E_0^{(k)}$ we observe that because of the symmetries of the potential the eigenfunction $\psi_1^{(k)}$ can be chosen to be antisymmetric with respect to the plane $\pi = \{ x_1, \dots, x_d ; x_1 = 0 \}$. Hence $\psi_1^{(k)} \restriction_\pi = 0$ so that the restriction of $\psi_1^{(k)}$ to the set $D_{k+1} \cap \{ x \in Z^d ; x_1 < 0 \} \equiv D_{k+1}^-$ is an eigenfunction of the Hamiltonian $H_{D_{k+1}}$ with an additional Dirichlet boundary condition on π. Therefore:

$$E_1^{(k)} = E_0(H_{D_{k+1}^-}) \tag{7.4.6}$$

If now $\overline{V}$ denote the function:

$$\overline{V} = \begin{cases} \lambda & \text{if} \quad x_1 > 0 \quad \text{and} \quad V(x) = 0 \\ 0 & \text{otherwise} \end{cases} \tag{7.4.7}$$

we get from the monotonicity of the eigenvalues with respect to the potential :

$$E_1^{(k)} = E_0(H_{D_{k+1}^-}) \geqslant E_0(H_{D_{k+1}} + \overline{V}) \tag{7.4.8}$$

If $\varphi^{(k)}$ denotes the ground state wave function of $H_{D_{k+1}} + \overline{V}$ we finally get from (7.4.8):

$$E_1^{(k)} - E_0^{(k)} \geqslant \sum_{x \in D_{k+1}} \overline{V}(x) \mid \varphi^{(k)}(x) \mid^2 \tag{7.4.9}$$

It is now easy to check [*10*] that the exponential decay of $\varphi^{(k)}$ cannot be faster than: $\exp\left(-m_0' \operatorname{dist}\left(x, \bigcup_{\alpha=0}^{2d} D_k^\alpha \cap \{ x ; x_1 < 0 \} \right)\right)$; with:

$$m_0' = (\lambda - E_0(H_{D_{k+1}} + \overline{V}))^{1/2}$$

i. e. $\sqrt{\lambda} \geqslant m_0' \geqslant \sqrt{2d}$.

since $E_0(H_{D_{k+1}} + \overline{V}) \leqslant 2d$.

Hence $b)$ follows from (7.4.9).

Part a). — For simplicity we discuss only the two-dimensional case $d=2$. Let γ be a circle in the complex plane centered at $E_0^{(k-1)}$ of radius

$\delta = 1/2 \exp(-m'_0 d_{k-1})$ where m'_0 is the constant appearing in b). From the proof of theorem 4.1 and part b) we know that there are exactly $5 = 2d+1$ eigenvalues of $H_{D_{k+1}}$ enclosed by γ at distance from $E_0^{(k-1)}$ less or equal than $\exp(-\sqrt{d_k})$.

We can therefore write for $n=0, \dots, 4$:

$$\psi_n^{(k)}(x) = \frac{1}{2\pi i} \oint_\gamma dZ(G_{D_{k+1}}(Z)\psi_n^{(k)}(x) \qquad (7.4.10)$$

For $x \in D_k$ we now expand $G_{D_{k+1}}(Z)$ as:

$$G_{D_{k+1}}(Z) = G_{D_k}(Z) + G_{D_k}(Z)\Gamma_{\partial D_k} G_{D_{k+1}} \qquad (7.4.11)$$

and insert (7.4.11) into (7.4.10). Since $E_1^{(k-1)}$ is not enclosed in γ this gives:

$$\psi_n^{(k)}(x) = a_n^{(k)} \psi_0^{(k-1)}(x) + \frac{1}{2\pi i} \oint_\gamma dZ(G_{D_k}(Z)\Gamma_{\partial D_k} G_{D_{k+1}}(Z)\psi_n^{(k)})(x) \quad (7.4.12)$$

where $a_n^{(k)} = \langle \psi_n^{(k)}, \psi_0^{(k-1)} \rangle$.

As in the proof of theorem (4.1) it is easy to see that the second term in the r. h. s. of (7.4.12) can be estimated for k large enough by:

$$\exp(-m_0 d_k/10), \qquad m_0 \geqslant \sqrt{2d}$$

Clearly because of the symmetries of the problem the same analysis can be repeated for each of the cubes D_k^α, $\alpha = 1, \dots, 4$ provided we change the constant $a_n^{(k)}$ into $a_n^{(k,a)}$, and we translate the function $\psi_0^{(k-1)}$ by an amount $\pm 2([3d_{k+1}] - [3d_k])$ along the coordinate axes. We now observe that with symmetry arguments one can construct the five eigenfunctions $\psi_n^{(k)}$ with eigenvalues enclosed in γ as follows:

$\psi_0^{(k)}$ is symmetric with respect to the coordinate axes and to the two diagonals $x = y$, $x = -y$.

$\psi_1^{(k)}$ is antisymmetric with respect to the plane $x = 0$ and symmetric with respect to $y = 0$.

$\psi_2^{(k)}$ is antisymmetric with respect to $y = 0$ and symmetric with respect to $x = 0$.

That this is actually the correct ordering of the first three eigenfunctions can be proved using the monotonicity of the eigenvalues of H_Λ with respect to Λ.

A fourth eigenfunction must be antisymmetric with respect to both the diagonals $x=y$, $x=-y$. A fifth eigenfunction which we call $\psi^{(k)}$ (*) must be symmetric with respect to the planes $x=0$, $y=0$, $x=-y$. To see this

(*) Note that with respect to the ordering of the eigenvalues $E_n^{(k)}$ in increasing order, $\psi^{(k)}$ can be either $\psi_3^{(k)}$ or $\psi_4^{(k)}$; however this ambiguity is irrelevant for our discussion.

it is enough to observe that using the normalization condition on the $\psi_n^{(k)}$'s and (7.4.12) one gets:

$$1 \geqslant \sum_{\alpha=0}^{4} | a_n^{(k,\alpha)} |^2 \geqslant 1 - 5 \exp\left\{ - m_0 \frac{d_k}{10} \right\} \tag{7.4.13}$$

$n = 0, \ldots, 4$ where $a_n^{(k,0)} \equiv a_n^{(k)}$, and subsequently to impose the orthogonality of $\psi^{(k)}$ with the other eigenfunctions. Let now $a^{(k,\alpha)}$ be the constants $a_n^{(k,\alpha)}$ computed for $\psi^{(k)}$. Because of the symmetries of both $\psi_0^{(k)}$ and $\psi^{(k)}$ we have:

$$\begin{aligned} a_0^{(k,\alpha)} &\equiv \bar{a}_0^{(k)} \qquad \forall \alpha = 1, \ldots, 4 \\ a^{(k,\alpha)} &\equiv \bar{a}^{(k)} \qquad \forall \alpha = 1, \ldots, 4 \end{aligned} \tag{7.4.14}$$

Furthermore we must have:

$$\begin{aligned} 1 &\geqslant (a_0^{(k)})^2 + 4(\bar{a}_0^{(k)})^2 \geqslant 1 - \mathcal{O}\left(\exp\left\{ - m_0 \frac{d_k}{10} \right\} \right) \\ 1 &\geqslant (a^{(k)})^2 + 4(\bar{a}^{(k)})^2 \geqslant 1 - \mathcal{O}\left(\exp\left\{ - m_0 \frac{d_k}{10} \right\} \right) \end{aligned} \tag{7.4.15}$$

and by the orthogonality condition:

$$| a_0^{(k)} a^{(k)} + 4 \bar{a}_0^{(k)} \bar{a}^{(k)} | \leqslant \mathcal{O}\left(\exp\left\{ - m_0 \frac{d_k}{10} \right\} \right)$$

This gives:

$$a^{(k)} \geqslant \left(1 + \frac{a_0^{(k)}}{\bar{a}_0^{(k)}} \right)^{-1} - \mathcal{O}\left(\exp\left\{ - m_0 \frac{d_k}{10} \right\} \right) \tag{7.4.16}$$

To estimate $a_0^{(k)}$ and $\bar{a}_0^{(k)}$ from below we proceed as follows: (see fig. 5).

Let $X^* \equiv (2([3d_k] - [3d_{k-1}]), 0)$, and let Λ, Λ^* be the cubes of side $2([3d_k] - [3d_{k-1}]) - 2$ centered at $x = 0$ and $x = x^*$ respectively.

Then $\psi_0^{(k)}$ is the solution of the two Dirichlet problems

$$\begin{aligned} (\mathrm{H} - \mathrm{E}_0^{(k)})u &= 0 \qquad \text{in } \Lambda \\ u \restriction_{\partial} \mathrm{ext}_\Lambda &= \psi_0^{(k)} \\ (\mathrm{H} - \mathrm{E}_0^{(k)})u &= 0 \qquad \text{in } \Lambda^* \\ u \restriction_{\partial} \mathrm{ext}_{\Lambda^*} &= \psi_0^{(k)} \end{aligned} \tag{7.4.17}$$

Since $\mathrm{E}_0^{(k)} < \mathrm{E}_0(\mathrm{H}_\Lambda) = \mathrm{E}_0(\mathrm{H}_{\Lambda^*})$ we can write, using the symmetry of $\psi_0^{(k)}$:

$$\begin{aligned} \psi_0^{(k)}(0) &= 4 \sum_{(y,y') \in \bar{\partial}} \mathrm{G}_\Lambda(\mathrm{E}_0^{(k)}, 0, y) \psi_0^{(k)}(y') \\ \psi_0^{(k)}(x^*) &= \sum_{(y,y') \in \partial\Lambda^*} \mathrm{G}_{\Lambda^*}(\mathrm{E}_0^{(k)}, x^*, y) \psi_0^{(k)}(y') \end{aligned} \tag{7.4.18}$$

where $\bar{\partial} \equiv \partial^{\mathrm{ext}} \Lambda \cap \partial^{\mathrm{ext}} \Lambda^*$.

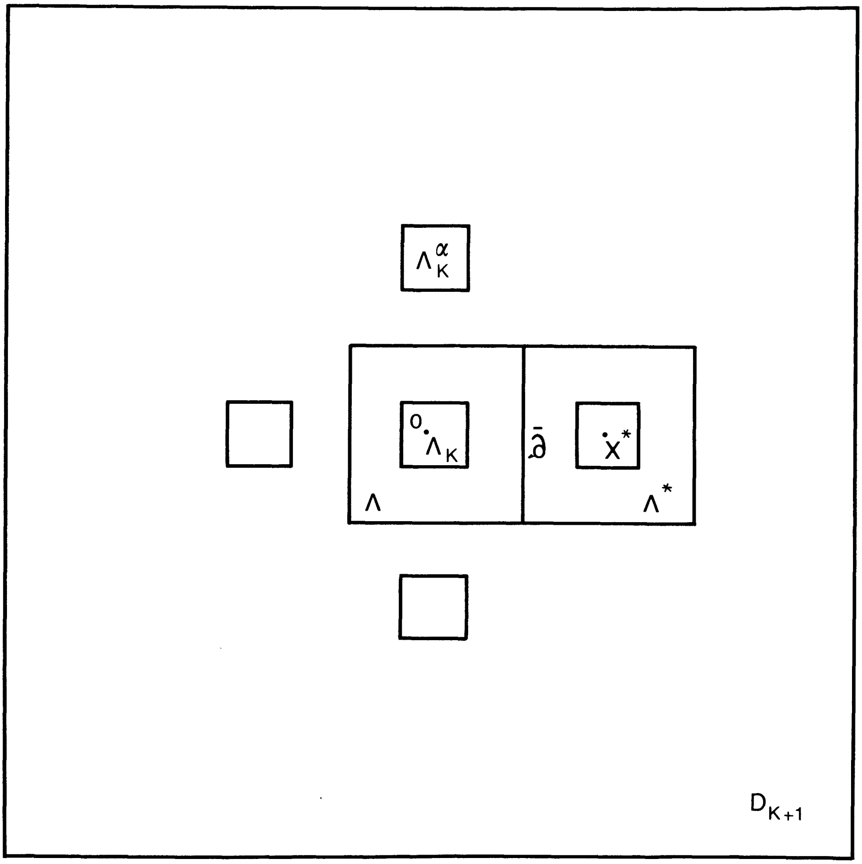

FIG. 5.

The Green function $G_{\Lambda^*}(E_0^{(k)}, x, y)$ and $\psi_0^{(k)}$ are both positive; furthermore $G_{\Lambda^*}(E_0^{(k)})$ differs from $G_\Lambda(E_0^{(k)})$ only by a translation. Hence (7.4.18) gives:

$$\psi_0^{(k)}(x^*) \geqslant \frac{1}{4}\,\psi_0^{(k)}(0)$$

that is:

$$\bar{a}_0^{(k)} \geqslant \frac{1}{4}\,a_0^{(k)} - \mathcal{O}\left(\exp\left\{-m_0\frac{d_k}{10}\right\}\right) \tag{7.4.19}$$

A similar trick shows that

$$a_0^{(k)} \geqslant \bar{a}_0^{(k)} - \mathcal{O}\left(\exp\left\{-m_0\frac{d_k}{10}\right\}\right)$$

These last two inequalities together with the normalization condition (7.4.15) give finally:

$$(a_0^{(k)})^2 \geqslant \frac{1}{5} - \mathcal{O}\left(\exp\left\{-m_0\frac{d_k}{10}\right\}\right)$$

$$(a^{(k)})^2 \geqslant \frac{1}{9} - \mathcal{O}\left(\exp\left\{-m_0\frac{d_k}{10}\right\}\right) \tag{7.4.20}$$

and the proposition is proved for $d = 2$. The proof for $d > 2$ goes along the same lines.

5) **Proof of theorems 5.1 and 5.2.**

We divide the proof of the theorem (5.1) into several lemmas:

LEMMA 5.1. — Let V be the deterministic part of the random potential V_v, i. e. $V_v(x) = V(x) + \beta v(x)$, and let C_k^α, $k = 1, 2, \ldots$ be the corresponding sets of points $x \in \Lambda_{k+1} \backslash \Lambda_k$ where $V(x) = 0$.

Let also for any α and k

$$\overline{C_k^\alpha} \equiv \{ x ; \mathrm{dist}\,(x, C_k^\alpha) \leqslant 4d_k \}, \qquad \overline{\Lambda_k^\alpha} \equiv \{ x ; \mathrm{dist}\,(x, \Lambda_k) \leqslant 4d_k \}$$

Then there exists a set $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that $\forall v \in \Omega_0$ we can find a $k_0(v) < +\infty$ with the property that:

$$\mathrm{dist}\,(\sigma(H_{\overline{\Lambda_k}}(v)), \sigma(H_{\overline{C_k^\alpha}}(v)) \geqslant 2e^{-\sqrt{d_k}}$$

for any α and any $k > k_0(v)$.

Proof. — By the Borel-Cantelli lemma it is enough to show that:

$$P(\exists\alpha ; \mathrm{dist}\,(\sigma(H_{\overline{\Lambda_k}}(v)), \sigma(H_{\overline{C_k^\alpha}}(v))) < 2e^{-\sqrt{d_k}}) \tag{7.5.1}$$

is summable in k.

Using Wegner argument (see [*11*] and also lemma 2.4 in [*5*]) for a fixed α:

$$P\,(\mathrm{dist}\,(\sigma(H_{\overline{\Lambda_k}}(v)), \sigma(H_{\overline{C_k^\alpha}}(v))) < 2e^{-\sqrt{d_k}}) \leqslant C(\beta)d_{k+1}^{2d}e^{-\sqrt{d_k}} \tag{7.5.2}$$

Since $\# \{ C_k^\alpha ; C_k^\alpha \subset \Lambda_{k+1} \backslash \Lambda_k \} \leqslant \mathrm{const}\; d_{k+1}^d$ (7.5.1) is clearly summable and the lemma is proved.

The following is the key step of the proof:

LEMMA 5.2. — Let (E, ψ_E) be as in theorem (5.1). Then there exists a $k_1(v, E)$ such that for any $k > k_1(v, E)$:

$$\mathrm{dist}\,(\sigma(H_{\overline{\Lambda_k}}(v)), E) < e^{-\sqrt{d_k}}$$

Proof. — We suppose the contrary, namely that there exists a sequence $\{ k_n \}$, $k_n \to +\infty$, as $n \to +\infty$ such that:

$$\mathrm{dist}\,(\sigma(H_{\overline{\Lambda_{k_n}}}(v)), E) > e^{-\sqrt{d_{k_n}}} \qquad \forall n \tag{7.5.3}$$

Take now n so large that $k_n > k_0(m)$ and $e^{-\sqrt{d_{k_n}}} \geqslant e^{-\alpha_0(m)d_{k_n}}$ where $m^2 = \lambda - \mathrm{E}$ and $\alpha_0(m)$, $k_0(m)$ are the constants appearing in theorem 7.1.

For such n's ψ_{E} is also the unique solution of the Dirichlet problem:

$$\begin{aligned} &(\mathrm{H}(v) - \mathrm{E})u = 0 \qquad \text{in} \quad \mathrm{D}_{k_n+1} \\ &u \restriction_{\partial\,\mathrm{ext}_{\mathrm{D}_{k_n+1}}} = \psi_{\mathrm{E}} \end{aligned} \tag{7.5.4}$$

where D_{k+1} is the cube of side $4\left[\dfrac{d_{k+1}}{5}\right]$ centered at the origin. Using now (7.5.3) and theorem 7.1 we get:

$$|\, \mathrm{G}_{\mathrm{D}_{k_n+1}}(\mathrm{E}, v, x, y) \,| \leqslant e^{-\frac{m}{2}|x-y|} \tag{7.5.5}$$

for all $|\, x - y \,| \geqslant d_{k_n+1}/5$, where $m^2 \geqslant \lambda - \mathrm{E}$.

Thus, expressing the solution ψ_{E} of the problem (7.5.4) in terms of $\mathrm{G}_{\mathrm{D}_{k_n+1}}$ as:

$$\psi_{\mathrm{E}}(x) = (\mathrm{G}_{\mathrm{D}_{k_n+1}}(\mathrm{E})\Gamma_{\partial \mathrm{D}_{k_n+1}}\psi_{\mathrm{E}})(x) \tag{7.5.6}$$

and using (7.5.5) together with the polynomial boundedness of ψ_{E}, by taking the limit $n \to +\infty$ in (7.5.6), we get $\psi_{\mathrm{E}} = 0$. Therefore (7.5.3) is false and the lemma is proved.

We define now $\tilde{k}(\mathrm{E}, v) \equiv \max(k_0(v), k_1(\mathrm{E}, v), k_0(m))$, $k_0(v)$, $k_1(\mathrm{E}, v)$ being defined by lemma 5.1, lemma 5.2 respectively.

Finally we let $\bar{k}(\mathrm{E}, v) \geqslant \tilde{k}(\mathrm{E}, v)$ to be the smallest $k \geqslant \tilde{k}(\mathrm{E}, v)$ such that $\exp\left(-\dfrac{\sqrt{\lambda - \mathrm{E}}}{2}\, d_k\right) d_k^d \leqslant 1$. With this definition we have:

Lemma 5.3. — *i*) For any $k > \bar{k}(\mathrm{E}, v)$, which by lemma 5.1 is finite with probability one,

$$\|\, \mathrm{G}_{\mathrm{D}_{k+1}/\Lambda_{\bar{k}(\mathrm{E},v)}}(v, \mathrm{E}) \,\| < +\infty$$

ii) For any $k > \bar{k}(\mathrm{E}, v)$, any x, y with $y \in \partial^{\mathrm{ext}}\Lambda_{\bar{k}(\mathrm{E},v)}$ and

$$\mathrm{dist}\,(x, \Lambda_{\bar{k}(\mathrm{E},v)}) \geqslant 1/5 d_{\bar{k}(\mathrm{E},v)+1}, \qquad x \in \mathrm{D}_{k+1}$$

$$|\, \mathrm{G}_{\mathrm{D}_{k+1}\setminus\Lambda_{\bar{k}(\mathrm{E},v)}}(\mathrm{E}, v, x, y) \,| \leqslant 2e^{-\frac{m}{2}|x-y|}$$

Proof. — *i*) From the definition of $\bar{k}(\mathrm{E}, v)$ we have that for any $k > \bar{k}(\mathrm{E}, v)$

$$\mathrm{dist}\,(\sigma(\mathrm{H}_{\mathrm{C}_\alpha^k}), \mathrm{E}) \geqslant e^{-\sqrt{d_k}} \geqslant e^{-\alpha_0(m)d_k} \qquad \forall \alpha \tag{7.5.6}$$

which implies: $(\mathrm{D}_{k+1}\setminus\Lambda_{\bar{k}(\mathrm{E},v)}) \cap \mathrm{S}_{k-k_0(m)}(\mathrm{E}, \mathrm{V}_v) = \varnothing$ where $\mathrm{S}_j(\mathrm{E}, \mathrm{V})$ are the set of singular sites defined in Section VII § 1. Thus *i*) follows from theorem 7.1.

ii) Using (7.5.6) and theorem 7.1 we also get that for $|\, x - y \,| \geqslant \dfrac{d_k}{5}$:

$$|\, \mathrm{G}_{\mathrm{D}_{k+1}\setminus\Lambda_{\bar{k}(\mathrm{E},v)}}(\mathrm{E}, v, x, y) \,| \leqslant e^{-\frac{m}{2}|x-y|} \qquad m^2 = \lambda - \mathrm{E} \tag{7.5.7}$$

In order to extend (7.5.7) to the range of x and y described in the lemma we expand $G_{D_{k+1}\setminus\bar{\Lambda}_{k(E,v)}}(E, v)$ as:

$$G_{D_{k+1}\setminus\bar{\Lambda}_{\bar{k}(E,v)}}(E, v, x, y) = G_{D_{j+1}\setminus\bar{\Lambda}_{\bar{k}(E,v)}}(E, v, x, y) + (G_{D_{j+1}\setminus\bar{\Lambda}_{\bar{k}(E,v)}}(E, v)\Gamma_{\partial D_{j+1}}G_{D_{k+1}\setminus\bar{\Lambda}_{\bar{k}(E,v)}})(x, y) \quad (7.5.8)$$

where x and y are as in the statement of the lemma and $j \equiv j(x) > \bar{k}(E, v)$ is such that:

$$1/5d_j \leqslant \text{dist}\,(x, \bar{\Lambda}_{\bar{k}(E,v)}) \leqslant 1/5d_{j+1} \quad (7.5.9)$$

By the choice of j and (7.5.7) the first term in (7.5.8) has the correct behaviour: $\exp\left(-\frac{m}{2}|x-y|\right)$. We observe now that by construction $\text{dist}\,(x, \partial D_{j+1}) \geqslant \frac{1}{5}d_{j+1}$ and $\text{dist}\,(\partial D_{j+1}, \partial D_{j+2}) \geqslant \frac{1}{5}d_{j+2}$. Hence if we expand the second term in (7.5.8) as:

$$\begin{aligned}&(G_{D_{j+1}\setminus\bar{\Lambda}_{\bar{k}(E,v)}}(E, v)\Gamma_{\partial D_{j+1}}G_{D_{j+1}\setminus\bar{\Lambda}_{\bar{k}(E,v)}}(E, v))(x, y)\\ &+(G_{D_{j+1}\setminus\bar{\Lambda}_{k(E,v)}}(E, v)\Gamma_{\partial D_{j+1}}G_{D_{j+2}\setminus\bar{\Lambda}_{k(E,v)}}(E, v)\Gamma_{\partial D_{j+2}}G_{D_{j+2}\setminus\bar{\Lambda}_{k(E,v)}}(E, v))(x, y)\\ &+ \dots \end{aligned} \quad (7.5.10)$$

we obtain that each term $G_{D_{j+1}\setminus\bar{\Lambda}_{\bar{k}(E,v)}}(E, v, x', y')$ is computed at points x', y' with $|x'-y'| \geqslant \frac{1}{5}d_{j+i-1}$ and therefore they are estimated as (7.5.7). Hence we can bound (7.5.8) by:

$$e^{-\frac{m}{2}\text{dist}\,(x,\bar{\Lambda}_{\bar{k}(E,v)})} + e^{-\frac{m}{10}d_{j+1}}\left\{\sum_{i=1}^{k-j}\prod_{l=1}^{i} e^{-\frac{m}{10}d_{j+l+1}}(Cd_{j+l})^{d-1}\right\}$$
$$\leqslant 2e^{-\frac{m}{2}\text{dist}\,(x,\bar{\Lambda}_{\bar{k}(E,v)})} \quad (7.5.11)$$

by the choice of j and of $\bar{k}(E, v)$.

The lemma is proved.

We are now in a position to give the proof of theorem 5.1. Fix $k > \bar{k}(E, v)$ and consider in $D_{k+1}\setminus\bar{\Lambda}_{\bar{k}(E,v)}$ the Dirichlet problem:

$$\begin{aligned}(H(v) - E)u &= 0\\ u \restriction_{\partial}\, \text{int}_{\bar{\Lambda}_{\bar{k}(E,v)}\cup\partial}\, \text{ext}_{D_{k+1}} &= \psi_E\end{aligned} \quad (7.5.12)$$

where ψ_E is one of the polynomially bounded solutions of $(H(v)-E)\psi = 0$.

By lemma 5.3, problem (7.5.12) has a unique solution which coincides with ψ_E. Thus:

$$\psi_E(x) = (G_{D_{k+1}\setminus\bar{\Lambda}_{\bar{k}(E,v)}}(E, v)\Gamma_{\partial\bar{\Lambda}_{\bar{k}(E,v)}\cup\partial D_{k+1}}\psi_E)(x) \quad (7.5.13)$$

Clearly $\Gamma_{\partial\overline{\Lambda}_{\overline{k}(\mathrm{E},v)}\cup\partial \mathrm{D}_{k+1}} = \Gamma_{\partial\overline{\Lambda}_{\overline{k}(\mathrm{E},v)}} + \Gamma_{\partial \mathrm{D}_{k+1}}$ and in turn:

$$\psi_{\mathrm{E}}(x) = (\mathrm{G}_{\mathrm{D}_{k+1}\backslash\overline{\Lambda}_{\overline{k}(\mathrm{E},v)}}(\mathrm{E}, v)\Gamma_{\partial\overline{\Lambda}_{\overline{k}(\mathrm{E},v)}}\psi_{\mathrm{E}})(x) + (\mathrm{G}_{\mathrm{D}_{k+1}\backslash\overline{\Lambda}_{\overline{k}(\mathrm{E},v)}}(\mathrm{E}, v)\Gamma_{\partial \mathrm{D}_{k+1}}\psi_{\mathrm{E}})(x) \tag{7.5.14}$$

The first term in the r. h. s. of (7.5.14), for $\mathrm{dist}(x, \partial\Lambda_{\overline{k}(\mathrm{E},v)}) \geqslant \frac{1}{5} d_{\overline{k}(\mathrm{E},v)+1}$ is estimated using lemma 5.3 by:

$$\mathrm{const}\; e^{-\frac{m}{2}\,\mathrm{dist}\,(x, \overline{\Lambda}_{\overline{k}(\mathrm{E},v)})} d^{d-1}_{\overline{k}(\mathrm{E},v)} \sup_{x' \in \Lambda_{\overline{k}(\mathrm{E},v)}} |\,\psi_{\mathrm{E}}(x)\,| \tag{7.5.15}$$

To estimate the second term we choose k so large that

$$\mathrm{dist}\,(x, \partial^{\mathrm{int}}\mathrm{D}_{k+1}) \geqslant \frac{1}{5} d_k.$$

We can then apply (7.5.7) and the polynomial boundedness of ψ_{E} to get that the second term vanishes in the limit $k \to +\infty$. The theorem is now proved.

We now turn to the proof of theorem 5.2. We have to estimate a temporal evolution and the natural approach is to perform a spectral decomposition of $\exp(-\,it\,\mathrm{H}(v))\mathrm{P}_{(0,\lambda-\varepsilon)}(\mathrm{H}(v))$ in terms of generalized eigenfunctions [*8*] [*12*]:

$$(e^{-it\,\mathrm{H}(v)}\mathrm{P}_{(0,\lambda-\varepsilon)}(\mathrm{H}(v))\delta_0)(x) = \int_0^{\lambda-\varepsilon} d\rho_v(\mathrm{E})e^{-it\mathrm{E}}\mathrm{F}(x, 0, \mathrm{E}, v) \tag{7.5.16}$$

where $d\rho_v(\mathrm{E})$ is the spectral measure of $\mathrm{H}(v)$ and the kernel $\mathrm{F}(x, 0, \mathrm{E}, v)$ is defined for $\mathrm{E} \in \Delta_v$ with $\rho_v([0, \lambda - \varepsilon]\backslash\Delta_v) = 0$ by:

$$\mathrm{F}(x, 0, \mathrm{E}, v) = (1 + |\,x\,|^2) \sum_{j=1}^{\delta/2\mathrm{N(E)}} f_j(x, \mathrm{E})\bar{f}_j(0, \mathrm{E}) \tag{7.5.17}$$

Here $\delta > d/2$ and $\{ f_j \}_{j=1}^{\mathrm{N(E)}}$ are orthogonal functions in $l^2(\mathbb{Z}^d)$ such that $\varphi_j(x) = (1 + x^2)^{\delta/2} f_j(x)$ are solutions of the Schrödinger equation:

$$(\mathrm{H}(v) - \mathrm{E})\varphi = 0$$

The normalization is chosen in such a way that

$$\sum_{j=1}^{\mathrm{N(E)}} \|\, f_j \,\|^2 = 1 \tag{7.5.18}$$

and N(E) counts the multiplicity.

The usefulness of the decomposition of (7.5.16) in this case comes from the fact that the φ_j's are polynomially bounded and by theorem 5.1 exponentially localized in the box $\Lambda_{\overline{k}(\mathrm{E},v)}$. We shall be able in fact to show that $|x| F(x, 0, E, v) \in l^2(\mathbb{Z}^d)$ for v in a set of measure one, uniformly in E for $E \in \Delta_v$. From theorem 5.1 we see that each $|\varphi_j|$ attains its maximum value in $D_{\bar{k}(E,v)+1}$; hence using the fact that $|f_j|_\infty \leqslant 1$ we obtain:

$$|\varphi_j|_\infty \leqslant ((1 + d_{\bar{k}(E,v)+1})^2)^{\delta/2} \tag{7.5.19}$$

We now estimate $|F(x, 0, E, v)|$ for $E \in \Delta_v$. We distinguish two different cases:

a) $\bar{k}(E, v) > k_1(E, v) \equiv \inf\{ k ;\ \forall j \geqslant k \ \text{dist}\ (\sigma(H_{\bar{\Lambda}_j}(v)), E) \geqslant e^{-\sqrt{d_j}} \}$

b) $\bar{k}(E, v) = k_1(E, v)$.

We observe that for all $E \in \Delta_v$ such that *a*) is verified $\bar{k}(E, v) = \bar{k}(v)$. The geometrical meaning of *a*) is therefore that the eigenfunctions φ_j are exponentially localized in the box $\bar{\Lambda}_{\bar{k}(v)}$ which is independent of E. In case *b*) the eigenfunctions φ_j are localized in the box $\bar{\Lambda}_{k_1(E,v)}$ which may increase with E, but as we will show, they are exponentially small at the origin.

Therefore their contribution to the initial wave packet is small.

a) In this case using theorem 5.1 and (7.5.19) we get:

$$\begin{aligned} |F(x, 0, E, v)| &\leqslant N(E) |\varphi_j|_\infty^2 d_{\bar{k}(v)}^{d-1} e^{-\frac{m}{2}\text{dist}\,(x,\bar{\Lambda}_{\bar{k}(v)})} \\ &\leqslant CN(E) d_{\bar{k}(v)+1}^{2\delta} e^{-\frac{m}{2}\text{dist}\,(x,\bar{\Lambda}_{\bar{k}(v)})} \end{aligned} \tag{7.5.20}$$

with $m^2 = \lambda - E \geqslant \varepsilon$, $x \notin D_{\bar{k}(v)+1}$ and $C > 1$ a numerical constant. For $x \in D_{\bar{k}(v)+1}$ we bound $|F(x, 0, E, v)|$ by:

$$|F(x, 0, E, v)| \leqslant CN(E) d_{\bar{k}(v)+1}^{2\delta} \tag{7.5.21}$$

It remains to estimate the multiplicity N(E).

LEMMA 5.4. — $N(E) \leqslant \text{const}\, d_{\bar{k}(E,v)+1}^{d}$ for $E \in \Delta_v$. Assuming the lemma we complete the discussion of case *a*). We write:

$$\begin{aligned} \| xF(x, 0, E, v) \|^2 &= \sum_{x \in D_{k(v)+1}} x^2 |F(x, 0, E, v)|^2 \\ &+ \sum_{x \notin D_{k(v)+1}} |x|^2 |F(x, 0, E, v)|^2 \end{aligned} \tag{7.5.22}$$

Using (7.5.20), (7.5.21) and lemma 5.4, the r. h. s. of (7.5.22) is bounded uniformly in E for all E such that *a*) is verified.

b) In this case, from the definition of $k_1(E, v)$ we have:

$$\text{dist}\,(\sigma(H_{\bar{\Lambda}_{k_1(E,v)-1}}(v)), E) \geqslant \exp\{ -\sqrt{d_{k_1(E,v)-1}} \} \tag{7.5.23}$$

We can therefore apply to the cube $D_{k_1(E,v)}$ the argument used in the proof of lemma 5.2 to get:

$$| \varphi_j(0) | \leqslant d^{d-1}_{k_1(E,v)} | \varphi_j |_\infty \exp \left\{ -\frac{m}{10} d_{k_1(E,v)} \right\} \tag{7.5.24}$$

which in turn implies:

$$| F(0, x, E, v) | \leqslant CN(E) d^{2\delta}_{k_1(E,v)+1} \exp \left\{ -\frac{m}{10} d_{k_1(E,v)} \right\} \tag{7.5.25}$$

for $x \in D_{k_1(E,v)+1}$;

$$| F(0, x, E, v) | \leqslant CN(E) d^{2\delta}_{k_1(E,v)+1} \exp \left\{ -\frac{m}{10} d_{k_1(E,v)} - \frac{m}{2} \operatorname{dist}(x, \overline{\Lambda}_{k_1(E,v)}) \right\} \tag{7.5.26}$$

for $x \notin D_{k_1(E,v)+1}$.

Again, using lemma 5.4 we infer from (7.5.25) and (7.5.26) that:

$$\| xF(0, x, E, v) \|^2$$

is bounded uniformly in E for $E \in \Delta_v$.

Writing now:

$$r^2(t) = \sum_{x \in \mathbb{Z}^d} x^2 \, | e^{-itH} P_{[0,\lambda-\varepsilon]}(H(v)) \delta_0(x) |^2$$

and using Schwartz inequality in (7.5.16) we obtain:

$$r^2(t) \leqslant \rho_v([0, \lambda - \varepsilon]) \int_0^{\lambda-\varepsilon} d\rho_v \, \| xF(0, x, E, v) \|^2 \leqslant \text{const.}$$

We are left with the proof of lemma 5.4.

Proof of lemma 5.4. — Let us consider the functions $\tilde{f}_j = f_j / \| f_j \|$. For $x \notin D_{\overline{k}(E,v)+1}$ we get from theorem 5.1, (7.5.19) and the relationship $f_j(x) = \varphi_j(x)(1 + x^2)^{-\delta/2}$ the estimate:

$$| f_j(x) | \leqslant d^{d-1}_{k(E,v)} \exp \left\{ -\frac{m}{2} \operatorname{dist}(x, \overline{\Lambda}_{\overline{k}(E,v)}) \right\} \tag{7.5.27}$$

Choose now L so large that on the cube Λ of side L centered at the origin one has:

i) $\| \tilde{f}_j \|_{l^2(\Lambda)} \geqslant 1 - \varepsilon \qquad j = 1, \ldots, N(E)$

ii) $| \langle \tilde{f}_j, \tilde{f}_i \rangle_{l^2(\Lambda)} | \leqslant \varepsilon \qquad \Lambda \neq j$

with $\varepsilon = \exp(-\sqrt{L})$.

Clearly, since the $\tilde{f}_j$'s are orthonormal in $l^2(\mathbb{Z}^d)$, using (7.5.27) such an L always exists and actually it can be taken equal to $2d_{\overline{k}(E,v)+1}$. Using the Gram-Schmidt orthogonalization procedure we define:

$$\psi_i = \tilde{f}_i - \sum_{k=1}^{i-1} \langle \tilde{f}_i, \psi_k \rangle_{l^2(\Lambda)} \psi_k / \| \psi_k \|^2_{l^2(\Lambda)} \tag{7.5.28}$$

and
$$\delta_i \equiv \| \psi_i - \tilde{f}_i \|_{l^2(\Lambda)}$$

By construction the ψ_k's are orthogonal and from (7.5.28) we get:

$$\delta_i \leqslant \sum_{k=1}^{i-1} \frac{1}{\| \psi_k \|_{l^2(\Lambda)}} \{ | \langle \tilde{f}_i, \tilde{f}_k \rangle_{l^2(\Lambda)} | + | \langle \tilde{f}_i - \psi_i, \tilde{f}_k - \psi_k \rangle_{l^2(\Lambda)} | \} \tag{7.5.29}$$

which implies, together with i) and ii):

$$\delta_i \leqslant \sum_{k=1}^{i-1} \frac{1}{\| \psi_k \|_{l^2(\Lambda)}} \{ \varepsilon + \delta_i \delta_k \} \tag{7.5.30}$$

Since $\dim l^2(\Lambda) = L^d$ it follows that $\psi_i = 0$ for any $i > L^d$ i. e. $\delta_i = \| \tilde{f}_i \|$ for $i > L^d$.

In what follows we will show that $\delta_i \leqslant 4\varepsilon i$ for all $i \leqslant N$ with $N^2 = (16\varepsilon)^{-1}$. Since $\varepsilon = e^{-\sqrt{L}}$ this implies that: $\delta_i < \sqrt{\varepsilon}$ for all $i \leqslant \frac{1}{4} e^{\sqrt{L}/2}$ which gives, using the normalization of the $\tilde{f}_i$, $\tilde{f}_i \equiv 0$ $\forall i \in \left(L^d, \frac{1}{4} e^{\sqrt{L}/2} \right]$, i. e. $N(E) \leqslant L^d$.

The estimate on δ_i is proved by induction. First we observe that $\delta_1 = 0$. So let us assume the bound to be true up to $i < N$ and let us estimate δ_{i+1}. From (7.5.30) we get:

$$\delta_{i+1} \leqslant \varepsilon \sum_{k=1}^{i} \frac{1}{\| \psi_k \|_{l^2(\Lambda)}} \{ 1 + \delta_{i+1} 4k \} \tag{7.5.31}$$

Furthermore, by the choice of N,

$$\| \psi_k \|_{l^2(\Lambda)} \geqslant \| \tilde{f}_k \| - \delta_k \geqslant 1 - \varepsilon - 4k\varepsilon > 1/2 \tag{7.5.32}$$

Thus, inserting (7.5.32) into (7.5.31) we get:

$$\delta_{i+1} \leqslant \frac{2\varepsilon(i-1)}{1 - 8\varepsilon N^2} \leqslant 4\varepsilon(i+1)$$

The lemma is proved.

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Note added in proof:

After this work was submitted for publication the localization of the eigenstates of the Anderson model for large disorder was proved by showing that the typical configurations have the structure of the random hierarchical models studied in this work. This result will appear in a joint paper by J. Fröhlich, F. Martinelli, E. Scoppola, T. Spencer.