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Propagation and local-decay properties for long-range scattering of quantum three-body systems

by

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ABSTRACT. — We consider quantum three-body systems interacting via long-range two-body potentials that have arbitrary $|x|^{-\varepsilon}$ decrease at infinity and a few regularity assumptions (Hörmander's potentials). We prove that, in the continuous spectral subspace of the hamiltonian, the states that are orthogonal to all two-cluster channels represent particles which, asymptotically in time

— get arbitrarily far separated from each other
— are outgoing relative to each other (or incoming if time is reversed).

We borrow Hörmander's stationary phase estimates, and ideas from Kitada-Yajima for the systematic use of suitable Fourier integral operators in long-range problems. The above results are major steps on the time dependent route proposed by Enss for the proof of three-body asymptotic completeness. The final steps, still for Hörmander's potentials, will be given in a subsequent publication.

RÉSUMÉ. — On considère des systèmes quantiques à 3 corps interagissant par des potentiels de paire à longue portée de décroissance arbitraire $|x|^{-\varepsilon}$ à l'infini et satisfaisant quelques conditions de régularité locale (potentiels de Hörmander). On montre, que dans le sous-espace spectral continu de l'hamiltonien, les états qui sont orthogonaux à tous

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les sous-canaux à deux corps représentent des particules qui, asymptotiquement dans le temps

— s'éloignent asymptotiquement deux à deux

— sont deux à deux dans un état « sortant » (ou « entrant » si l'on change le signe du temps).

On emprunte les estimations de phase stationnaire de Hörmander, et des idées dues à Kitada-Yajima pour l'utilisation systématique dans les problèmes à longue portée de certains opérateurs Fourier intégraux. Les résultats ci-dessus sont des étapes majeures de la méthode « dépendant du temps » proposée par Enss pour la preuve de la complétude asymptotique à 3 corps. Les étapes finales, toujours pour les potentiels de Hörmander seront données dans un article ultérieur.

1. INTRODUCTION

In this paper we begin a study of quantum three-body scattering for long-range two-body potentials of arbitrary power decrease at infinity. Traditionally, quantum spectral and scattering theory is organised through the following set of questions:

1) Under what conditions on the potentials can the hamiltonian H of the system be defined as a self-adjoint operator?

2) What is the essential spectrum of H?

3) Does the spectral continuous subspace of H coincide with the absolutely continuous subspace?

4) Accumulation of the point spectrum of H only at zero and at the two-body thresholds.

5) Can we define (in this case modified) wave operators associated to each scattering channel?

6) Asymptotic completeness: does the direct sum over all possible scattering channels of the ranges of the corresponding wave operators coincide with the absolutely continuous subspace of H?

Questions (1) and (2) can easily be dealt with for a large class of interactions:

— self adjointness of the hamiltonian mainly requires a control over the local singularities of the potentials as for the two-body case [*28*, vol. II],

— the answer to question (2) is the content of H.V.Z. theorem (see for example [*28*, vol. IV]): provided each two-body potential goes to zero (arbitrarily slowly) when its two-body variable goes to infinity, the essential spectrum of H is $[E_0, \infty)$ where E_0 is the minimum energy of all subsystems.

Suitable « modified wave operators » for N-body Coulomb systems are defined by Dollard [*6*], see also [*21*], thus providing an answer to problem (5) for Coulomb interactions. For the two-body problem in the time-dependent framework, suitably defined « modified wave operators » have been introduced and proven to be complete for more general long range interactions [*7*] [*26*] [*19*] [*20*] [*24*] [*25*] [*26*], and we expect that these results easily extend to the N-body case (For similar results on the two-body long range operators by the stationary methods, see [*1*] [*18*] and references therein contained, together with the bibliography of [*26*].)

As in the two-body problem, question (3), (4) and (6) are more difficult to handle. A general proof of (3) and (4) in the N-body case for a large class of long range two-body interactions has been given in [*27*]. Since the pioneering work of Faddeev on three-body problem, various proofs of question (6) have been provided in the short range case (see [*15*] for a review up to 1976). Several improvements have been made in recent years to those proofs, and among them works by Mourre [*22*] [*23*] and Enss [*8*] [*12*] (see also [*32*]). Both approaches are related by the fact that they extend to the three-body problem the study of the asymptotic direction of flight for the particles; that leads to a simultaneous proof of (6) and (3). Apart from their underlying intuition, which is physically appealing, the main merit of these works is that they cover the case of arbitrarily short range interactions, and they do not require any assumptions on the two-body subsystems at thresholds. This was not the case in the previous approaches, neither in the more recent works by Hagedorn and Perry [*16*] or Sigal [*29*] [*30*]. Question (6) for long range interactions has been given very few answers in the three-body case, except for the work by Merkuriev [*21*] for Coulomb plus short range interactions.

Our aim is to present a proof of steps (5) and (6) in the three-body problem, for the class of long range interactions (of arbitrary power decrease at infinity) introduced by Hörmander [*17*], yielding (3) as a subproduct. The method mainly follows that of Enss [*8*] for the three-body short range problem, but deviates from his by the systematic use of some « Fourier-integral-operators » just tailored specifically for applying the stationary phase method of [*17*]. Thus it bears some similarity with Kitada-Yajima's version of two-body Enss'method for long range time-dependent potentials [*20*]. As in [*20*] a fundamental use will be made of

A) the stationary phase method [*17*]

B) L^2 estimates of some oscillatory integrals [*3*] [*13*]

C) classical orbits associated to our three-body problem, and associated Hamilton-Jacobi equations.

In this paper we present A, B and C, and show how they can be used in the study of the asymptotic evolution of some observables, in particular to prove the following fact:

D) if ψ is a state of the continuous subspace of H, orthogonal to all two-cluster channels, then asymptotically in time in a suitable sense, the particles represented by $e^{-itH}\psi$ get *far separated from each other.*

We shall then be in a position to prove that such a ψ is actually in the range of the three-cluster « modified wave operator », thus proving asymptotic completeness (6) and absence of singular continuous spectrum (3). However for reasons of length we shall give the proof of these two statements in a separate paper. But we claim that a separate proof of D, and a description of how points A-C naturally occur in our scattering problem, have their own interest.

In *section 2* we give the assumptions on the long-range two-body potentials, which are the same as Hörmander's [*17*], and study the classical orbits of the system of 3 incoming and outgoing particles scattered by the long-range regular part of the potential. Then using the generating functions of these orbits, we give in *section 3* the « modified wave operators » of the quantum problem for each scattering channel; we then prove that the complement in the continuous spectral subspace of the hamiltonian of all two-cluster channels represents, as suggested by intuition, 3 particles which get arbitrarily far separated, asymptotically in time. We have collected in the *last section* results extracted from [*3*] [*13*] on L^2-estimates of some oscillatory integrals, and a slight generalisation of them.

2. CLASSICAL ORBITS FOR THE THREE-BODY PROBLEM

In this section we shall study three classical particles, interacting via two-body long-range potentials V_α, that go farther and farther from each other as time goes to $+$ or $-\infty$ (let us call them « outgoing » or « incoming » particles). We introduce the generating functions of the corresponding orbits. For this purpose, it is convenient to built time-dependent potentials $V_{\alpha,t}$ such that the orbits for these time dependent potentials are identical with those for the original potentials as far as the outgoing or incoming particles are concerned. Thus we mimic Kitada-Yajima's approach for two-body long-range scattering [*20*], but we incorporate in it the more general class of long-range interactions of Hörmander [*17*]. More precisely the assumptions on the potentials are as follows:

$$\forall x \qquad V'_\alpha = V_\alpha + V^s_\alpha \tag{2.1}$$

where the decrease requirement on the short range par V^s_α is given in the next section (formula (3.2)) and where the long range part is such that

$$\forall x \in \mathbb{R}^\nu \qquad | D^i V_\alpha(x) | \leqslant (1 + | x |)^{-m(|i|)} \qquad | i | \leqslant 3 \tag{2.2}$$

(i is any multiindex $(i_1, \ldots, i_\nu)$ and $|i| = i_1 + \ldots + i_\nu$), $m(0), \ldots, m(3)$ are positive, $D^i = \frac{\partial^{i_1}}{\partial x_1} \cdots \frac{\partial^{i_\nu}}{\partial x_\nu}$ and

$$m(1) + m(3) > 4 \tag{2.3}$$

REMARK 2.1. — Making use of lemmas 3.2 and 3.3 of Hörmander [17] V_α can in turn be split into a short range part plus a highly regular long range par $\tilde{V}_\alpha$ satisfying (2.2) for every multiindex i for a new $m(j)$:

$$(\mathrm{H}) \left\{ \begin{array}{l} |D^i V_\alpha(x)| \leqslant C(1 + |x|)^{-m(|i|)} \qquad \forall i \\ \text{where the new function } m(j) \text{ is such that} \\ m(1) + m(3) > 4 \\ m(j) \text{ is concave} \\ m(j) - j \text{ is decreasing} \\ m(j) - \delta j \text{ is increasing for } j < 3, \text{ and constant for } j \geq 3 \text{ for some } \delta > 1/2 \end{array} \right.$$

Thus in what follows we shall replace V_α by its regularized part $\tilde{V}_\alpha$, denoted V_α for simplicity, and we assume (H) instead of (2.2) and (2.3).

REMARK 2.2. — (2.3) together with concavity of m imply:

$$m(2) > \frac{m(1) + m(3)}{2} > 2.$$

Let now a be a real positive constant which will be used throughout the paper, and let $\chi \in \mathscr{C}^\infty(\mathbb{R}^\nu)$ be such that $0 \leq \chi \leq 1$, $\chi(x) = 1$ for $|x| \geq 1$ and $\chi(x) = 0$ for $|x| \leq 1/2$. We set

$$V_{\alpha,t}(x) = V_\alpha(x)\chi(x/at) \tag{2.4}$$

Consider three particles whose positions and momenta are denoted x_i and p_i respectively and belong to $\mathbb{R}^\nu$. In order to avoid notational complexity we assume they have equal masses 1. The two-body interactions V_α and V_α^s only depend on $x_\alpha = x_i - x_j$ for $\alpha = (ij)$; thus we can get rid of the center of mass motion. Then if y_α denotes the relative position of the third particle k w. r. to the center of mass $\frac{x_i + x_j}{2}$ of the pair α, and if p_α and q_α are the conjugate momenta of x_α and y_α respectively, the (classical) hamiltonian of the relative motion of the *two* particles i, j in the potential V_α is:

$$h_\alpha(x_\alpha, p_\alpha) = p_\alpha^2 + V_\alpha(x_\alpha)$$

and that of the *three* particles in the time-dependent potentials $V_{\alpha,t}$:

$$H_t(X, P) = p_\alpha^2 + \frac{3}{4} q_\alpha^2 + \sum_\beta V_{\beta,t}(x_\beta) \qquad (\text{any } \alpha) \tag{2.5}$$

where
$$X = (x_\alpha, y_\alpha) \quad \text{position}$$
$$P = (p_\alpha, q_\alpha) \quad \text{momentum} \qquad (\text{any } \alpha) \qquad (2.6)$$
$$V = \left(2p_\alpha, \frac{3}{2} q_\alpha\right) \quad \text{velocity}$$

Then the classical orbit (X, V) (s, t; Z, K) is the solution of the Hamilton equations:

$$\frac{dX}{dt}(s, t) = V(s, t)$$
$$\frac{dp_\alpha}{dt}(s, t) = -\nabla_{x_\alpha} V_{\alpha,t}(x_\alpha(s, t)) \qquad \forall\alpha \qquad (2.7)$$

with initial conditions:

$$X(s, s) = Z$$
$$V(s, s) = K \qquad (2.8)$$

Let $i = (i_1, \ldots, i_{2\nu})$ be a multi-index whose components i_k are non-negative integers. Then for $Z, K \in \mathbb{R}^{2\nu}$, we use the following notations:

$$D_Z^i = \frac{\partial^{i_1}}{\partial Z_1} \cdots \frac{\partial^{i_{2\nu}}}{\partial Z_{2\nu}}, \qquad D_K^i = \frac{\partial^{i_1}}{\partial K_1} \cdots \frac{\partial^{i_{2\nu}}}{\partial K_{2\nu}} \qquad \nabla_Z = \left(\frac{\partial}{\partial Z_1}, \ldots, \frac{\partial}{\partial Z_{2\nu}}\right)_{\text{etc}}.$$

PROPOSITION 2.1. — Let V_α satisfy assumption (H) and let $V_{\alpha,t}$ be defined by (2.4). Then there exists $T > 0$ such that equations (2.7-8) admit a unique solution satisfying for any $t \geq s \geq T$:

i)
$$|X(s, t) - Z| \leqslant C(t - s)(|K| + s^{1-m(1)})$$
$$|V(s, t) - K| \leqslant Cs^{1-m(1)}$$

ii)
$$|\nabla_Z X(s, t) - 1| \leqslant Cs^{1-m(2)}(t - s)$$
$$|\nabla_Z V(s, t) \leqslant Cs^{1-m(2)}$$

iii)
$$|\nabla_K X(s, t) - (t - s)| \leqslant C(t - s)s^{2-m(2)}$$
$$|\nabla_K V(s, t) - 1| \leqslant Cs^{2-m(2)}$$

iv) $\forall i$ and j multiindices s. t. $|i| + |j| \geq 2$ $\exists C_{ij}$ s. t.

$$|D_Z^i D_K^j X(s, t)| \leqslant C_{ij} \operatorname{Max}[1, (t - s)^{1+\mu(|j|)}]$$
$$|D_Z^i D_K^j V(s, t)| \leqslant C_{ij} \operatorname{Max}[1, (t - s)^{\mu(|j|)}]$$

where
$$\mu(k) = \operatorname{Max}(0, k + 1 - m(k + 1)) \qquad (2.9)$$

v) $$|D_K^i(X(s, t) - (t - s)V(s, t) - Z)| \leqslant C \operatorname{Min}((t - s)^{2+|i|-m(|i|+1)}, (t - s)s^{1+|i|-m(|i|+1)}) \qquad \forall |i| \leqslant 1.$$

Proof. — By successive approximations, and by noting that $m(j) - j$ is a decreasing sequence.

PROPOSITION 2.2. — For T sufficiently large, and $t \geq s \geq T$ or $t \leq s \leq -T$, there exist the inverse $\mathscr{C}^\infty$ diffeomorphisms

$$Z \mapsto \Xi^{\pm}(t, s; Z, K)$$
$$K \mapsto \Pi^{\pm}(s, t; Z, K)$$

of the mappings

$$\Xi \mapsto X(t, s; \Xi, K) = Z$$
$$\Pi \mapsto V(s, t; Z, \Pi) = K$$

respectively. $\Xi^{\pm}$ and $\Pi^{\pm}$ belong to $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}) \times \mathscr{C}^\infty(\mathbb{R}^{2\nu} \times \mathbb{R}^{2\nu})$ and satisfy the following properties, for $t \geq s \geq T$ or $t \leq s \leq -T$:

i)
$$X(s, t; Z, \Pi^{\pm}(s, t; Z, K)) = \Xi^{\pm}(t, s; Z, K)$$
$$X(t, s; \Xi^{\pm}(t, s; Z, K), K) = Z$$
$$V(s, t; Z, \Pi^{\pm}(s, t; Z, K)) = K$$
$$V(t, s; \Xi^{\pm}(t, s; Z, K), K) = \Pi^{\pm}(s, t; Z, K)$$

ii)
$$|\Xi^{\pm}(t, s; Z, K) - Z - (t-s)K| \leqslant C\,\mathrm{Min}(|t-s|^{2-m(1)}, |t-s||s|^{1-m(1)})$$
$$|\nabla_Z \Xi^{\pm}(t, s; Z, K) - 1| \leqslant C|s|^{2-m(2)}$$
$$|\nabla_K \Xi^{\pm}(t, s; Z, K) - (t-s)| \leqslant C|t-s||s|^{2-m(2)}$$

iii)
$$|\Pi^{\pm}(s, t; Z, K) - K| \leqslant C|s|^{1-m(1)}$$
$$|\nabla_Z \Pi^{\pm}(s, t; Z, K)| \leqslant C|s|^{1-m(2)}$$
$$|\nabla_K \Pi^{\pm}(s, t; Z, K) - 1| \leqslant C|s|^{2-m(2)}$$

iv) $\forall i$ and j multiindices such that $|i| + |j| \geq 2$, there exist constants C_{ij} such that

$$|D_Z^i D_K^j \Xi^{\pm}(t, s; Z, K)| \leqslant C_{ij}\,\mathrm{Max}\,[1, |t-s|^{1+\mu(|i|+|j|)}]$$
$$|D_Z^i D_K^j \Pi^{\pm}(s, t; Z, K)| \leqslant C_{ij}\,\mathrm{Max}\,[1, |t-s|^{\mu(|j|)}]$$

The proof closely follows that of proposition 2.2 in [*19*] using the contraction mapping principle, and we do not reproduce it here.

The classical action along the classical orbit which starts from the phase-space point (Ξ, p_K) at time s $\left(\text{where } P_K = \left(\frac{k\alpha}{2}, \frac{2}{3} k'_\alpha\right)\right)$ is the momentum associated to the velocity $K = (k_\alpha, k'_\alpha)$ is

$$u(s, t; \Xi, P_K) - \Xi \cdot P_K = \int_t^s d\tau (P \cdot \nabla_P H_\tau - H_\tau)(X(t, s; \Xi, K), \quad P(t, s; \Xi, K))$$

We define:

$$W^{\pm}(s, t; Z, P_K) \equiv u(s, t; \Xi^{\pm}(t, s; Z, K), P_K) \tag{2.10}$$

$W^{\pm}(s, t; Z, P_K) - \Xi^{\pm}(t, s; Z, K) \cdot P_K$ is therefore the classical action along the orbit which is characterized by the momentum P_K at time s, and the position Z at time t, for the evolution associated to the time-dependent hamiltonian H_t. It satisfies:

PROPOSITION 2.3. — For $t \geq s \geq T$ or $t \leq s \leq -T$ and T large enough, $W^{\pm}(s, t; Z, P_K)$ is the unique solution of the Hamilton-Jacobi equations

$$\frac{\partial}{\partial s} W^{\pm}(s, t; Z, P_K) = - H_s(Z, \nabla_Z W^{\pm}(s, t; Z, P_K))$$

or

$$\frac{\partial}{\partial t} W^{\pm}(s, t; Z, P_K) = H_t(\nabla_K W^{\pm}(s, t; Z, P_K), P_K) \tag{2.11}$$

satisfying

$$W^{\pm}(s, s; Z, P_K) = Z \cdot P_K \tag{2.12}$$

Moreover

$$\begin{aligned} \nabla_P W^{\pm}(s, t; Z, P_K) &= \Xi^{\pm}(t, s; Z, K) \\ \nabla_Z W^{\pm}(s, t; Z, P_K) &= \Pi^{\pm}(s, t; Z, K) \end{aligned} \tag{2.13}$$

The proof follows from standard calculations in classical mechanics, that we do not reproduce here.

COROLLARY 2.4. — *i*) For any $|s|$ sufficiently large, any $|t| \geqslant |s|$ with $|t - s|$ sufficiently large, and any $P = (p_\alpha, q_\alpha)$ such that $|p_\alpha| > a, \forall\alpha$, we have:

$$\frac{\partial}{\partial t} W^{\pm}(s, t; 0, P) = H(\nabla_P W^{\pm}(s, t; 0, P), P)$$

ii) For $|s|$ sufficiently large, any $|t| \geqslant |s|$ and any $P = (p_\alpha, q_\alpha)$, $Z = (z_\alpha, z'_\alpha)$ such that $|p_\alpha| > a \ \forall\alpha$

$$\left| \frac{Z}{s} - V \right| < a \qquad \left(V = \left(2p_\alpha, \frac{3q_\alpha}{2} \right) \right)$$

we have

$$\frac{\partial}{\partial t} W^{\pm}(s, t; Z, P) = H(\nabla_P W^{\pm}(s, t; Z, P), P) \tag{2.14}$$

Furthermore for any multiindices i, j:

$$| D_Z^j D_K^i V_\alpha(\nabla_{P_\alpha} W^{\pm}(s, t; Z, P)) | \leqslant C_{ij} \operatorname{Max} [1, |t-s|^{\nu(|i|)}] \tag{2.15}$$

where
$$\nu(k) = \operatorname{Max} (1 - m(1) + \mu(k), k - m(k)) \tag{2.16}$$

Proof. — We only prove (*ii*) for the + sign case. The proof of (*i*), and of (*ii*) for the − sign case are similar. Let

$$\Xi^{\pm}(s, t; Z, V) = (\xi_\alpha, \eta_\alpha) \qquad (\text{some } \alpha)$$

Then

$$\begin{aligned} |\xi_\alpha| &> |z_\alpha + (t-s)2p_\alpha| - |\xi_\alpha - (t-s)2p_\alpha - z_\alpha| \\ &> 2t|p_\alpha| - |z_\alpha - 2sp_\alpha| - C|t-s|s^{1-m(1)} \\ &> at + (a - Cs^{1-m(1)})(t-s) \\ &> at \end{aligned}$$

for s sufficiently large and $t > s$.

This proves (2.14) because $V_\alpha(\xi_\alpha) = V_{\alpha,s}(\xi_\alpha)$ for $|\xi_\alpha| > as$ by (2.4). Now (2.15) follows from assumption (H), from proposition 2.2 (*iii*), (*iv*), and from lemma 3.6 in [*17*] noting that:

$$|D_Z^j D_P^i W^\pm(s,t;Z,P)| \leqslant \begin{cases} |t-s|^{1+\mu(|i|-1)} & \text{if } j=0 \\ |t-s|^{\mu(|i|)} & \text{if } |j| \geqslant 1 \end{cases}$$

We shall also be interested in the following problem of classical mechanics that will be useful for a proof of 3-body long range asymptotic completeness [*33*]: let x_α be a given real number, possibly depending upon t. Let:

$$H_t^\alpha(X;q_\alpha) = \frac{3q_\alpha^2}{2} + \sum_{\beta \neq \alpha} V_{\beta,t}\left(y_\alpha + \varepsilon_\beta \frac{x_\alpha}{2}\right)$$

where ε_β takes the values $+1$ or -1 and let $y_\alpha(t,s')$ and $q_\alpha(t,s')$ be solutions of the Hamilton equation associated with the time dependent hamiltonian $H_{s'}^\alpha(x_\alpha, y_\alpha(t,s'); q_\alpha(t,s'))$:

$$\begin{aligned} \frac{dy_\alpha}{ds'}(t,s') &= \nabla_{q_\alpha} H_{s'}^\alpha(x_\alpha, y_\alpha(t,s'); q_\alpha(t,s')) \\ \frac{dq_\alpha}{ds'}(t,s') &= -\nabla_{y_\alpha} H_{s'}^\alpha(x_\alpha, y_\alpha(t,s'); q_\alpha(t,s')) \end{aligned} \tag{2.17}$$

with initial conditions

$$\begin{aligned} y_\alpha(t,t) &= \eta_\alpha \\ q_\alpha(t,t) &= q_\alpha \end{aligned} \tag{2.18}$$

Then the analogs of propositions 2.1, 2.2 and 2.3 hold true, yielding existence and uniqueness of the solution $y_\alpha(t,s;\eta_\alpha,q_\alpha)$, $q_\alpha(t,s;\eta_\alpha,q_\alpha)$ of (2.17-2.18) (for s' replaced by s), of the inverse $\mathscr{C}^\infty$-diffeomorphism

$$z'_\alpha \mapsto \eta_\alpha^\pm(t,s;z'_\alpha,q_\alpha) \tag{2.19}$$

and of the solution $\tilde{W}_{\alpha 0}^\pm(s,t;z'_\alpha,q_\alpha)$ of the Hamilton-Jacobi equation:

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{W}_{\alpha 0}^\pm(s,t;z'_\alpha,q_\alpha) &= H_t^\alpha(x_\alpha, \nabla_{q_\alpha}\tilde{W}_{\alpha 0}^\pm; q_\alpha) \\ \tilde{W}_{\alpha 0}^\pm(s,s;z'_\alpha,q_\alpha) &= z'_\alpha \cdot q_\alpha \end{aligned} \tag{2.20}$$

which satisfies:

$$\nabla_{q_\alpha} \tilde{W}^{\pm}_{\alpha 0}(s, t ; z'_\alpha, q_\alpha) = \eta^{\pm}_\alpha(t, s ; z'_\alpha, q_\alpha) \tag{2.21}$$

Then the following property holds:

PROPOSITION 2.5. — For T sufficiently large, and $t \geq s \geq T$ or $t \leq s \leq -T$ there exists the inverse $\mathscr{C}^\infty$-diffeomorphism

$$z'_\alpha \mapsto \eta^{\pm}_\alpha(t, s ; z'_\alpha, q_\alpha)$$

of the mapping

$$\eta_\alpha \mapsto y_\alpha(t, s ; \eta_\alpha, q_\alpha)$$

$\eta^{\pm}_\alpha$ belongs to $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}) \times \mathscr{C}^\infty(\mathbb{R}^\nu \times \mathbb{R}^\nu)$ and satisfies:

i) $\left| \eta^{\pm}_\alpha(t, s ; z'_\alpha, q_\alpha) - z'_\alpha - \frac{3}{2} q_\alpha(t - s) \right| \leqslant C \, | t - s | \, | s |^{1 - m(1)}$

ii) $| \nabla_{z'_\alpha} \eta^{\pm}_\alpha(t, s ; z'_\alpha, q_\alpha) - 1 | \leqslant C \, | s |^{2 - m(2)}$

iii) $\left| \nabla_{q_\alpha} \eta^{\pm}_\alpha(t, s ; z'_\alpha, q_\alpha) - \frac{3}{2}(t - s) \right| \leqslant C \, | t - s | \, | s |^{2 - m(2)}$

iv) $\forall i$ and j multiindices such that $| i | + | j | \geq 2$ there exist constants C_{ij} such that:

$$| D^i_{z'_\alpha} D^j_{q_\alpha} \eta^{\pm}_\alpha(t, s ; z'_\alpha, q_\alpha) | \leqslant C_{ij} \operatorname{Max} \, [1, | t - s |^{1 + \mu(|j|)}]$$

v) $\forall i$ multiindex such that $1 \leqslant | i | \leqslant 2$

$$| D^i_{x_\alpha} \eta^{\pm}_\alpha(t, s ; z'_\alpha, q_\alpha) | \leqslant C \, | s |^{2 - m(|i| + 1)}$$

Proof. — We do not give the proof of (*i*)-(*iv*) which is similar to that of proposition 2.2. For (*v*) we note that, from (2.17):

$$\left| \frac{d^2}{ds'^2} D^i_{x_\alpha} y_\alpha(t, s') \right| \leqslant C \, | s' |^{-m(|i| + 1)}$$

and therefore:

$$\begin{aligned} | D^i_{x_\alpha} y_\alpha(t, s) | &\leqslant C \int_s^t ds' \int_{s'}^t ds'' \, | s'' |^{-m(|i| + 1)} \\ &\leqslant C \, | s |^{2 - m(|i| + 1)} \end{aligned} \tag{2.22}$$

Similarly

$$| D^i_{\eta_\alpha} y_\alpha(t, s) | \leqslant C' \, | s |^{2 - m(|i| + 1)} \tag{2.23}$$

Now using the fact that, given x_α and q_α, $\eta^{\pm}_\alpha$ is the inverse mapping of $\eta_\alpha \mapsto y_\alpha(t, s ; \eta_\alpha, q_\alpha)$ for $t \geq s \geq T$ (or $t \leq s \leq -T$ resp.), (2.22-2.23) and remark 2.2 yield the result.

The orbits defined by (2.17-2.18) are mappings

$$(t, \eta_\alpha, q_\alpha) \mapsto (s, y_\alpha, q_\alpha)$$

from $\mathbb{R} \times \mathbb{R}^{\nu} \times \mathbb{R}^{\nu}$ to itself which depend on some parameter x_α. If in turn x_α is given by a suitably chosen mapping from some phase-space variables at time t to other phase-space variables at time s, then the orbits defined by (2.17-2.18) can be compared with those of the threebody problem (2.7-2.8). This is precisely the purpose of the two following propositions, in which case the answer is quite simple and physically natural.

PROPOSITION 2.6. — Let $Z = (z_\alpha, z'_\alpha)$ and $K = (p_\alpha, q_\alpha)$ be such that $| p_\alpha | > a > 0, | z_\alpha | > \sqrt{2}\, a \, | s |, z_\alpha \cdot k_\alpha > 0$ for some k_α with $| p_\alpha - k_\alpha | < a/20$. Then if $t \geq s \geq T$ ($t \leq s \leq -T$ resp) and T is large enough, and if

$$x_\alpha = \nabla_{p_\alpha} W^{\pm}(s, t; Z, K)$$

we have

$$\eta_\alpha^{\pm}(t, s; z'_\alpha, q_\alpha) = \nabla_{q_\alpha} W^{\pm}(s, t; Z, K)$$

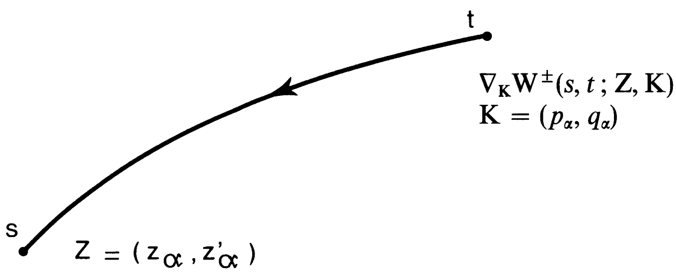

FIG. 1. — A portion of $\mathbb{R}^{4\nu}$ phase space orbit.

Before giving the proof of proposition 2.6 we state the next result which is closely related:

PROPOSITION 2.7. — Let Γ be the uniquely determined orbit in the $\mathbb{R}^{4\nu}$ phase space defined through equations (2.7-2.8), whose x_α-component of the starting position Ξ at time t is ξ_α, which has momentum $p=(p_\alpha, q_\alpha)$ at time $s < t$, and where y_α-component of the position at time $\tau < s$ is z'_α. Thus the y_α-component of Ξ is a uniquely determined fonction of ξ_α, z'_α, P, τ, s and t and if

$$| \xi_\alpha | > 3a_1 t \qquad (\text{some } a_1 > a)$$
$$| P_\alpha | < 3a/2$$

τ is large enough and

$$x_\alpha = \nabla_{p_\alpha} W^{\pm}(t, s; \Xi, P)$$

then we have

$$\eta_\alpha^\pm(s, \tau ; z'_\alpha, q_\alpha) = \nabla_{q_\alpha} W^\pm(t, s ; \Xi, P)$$

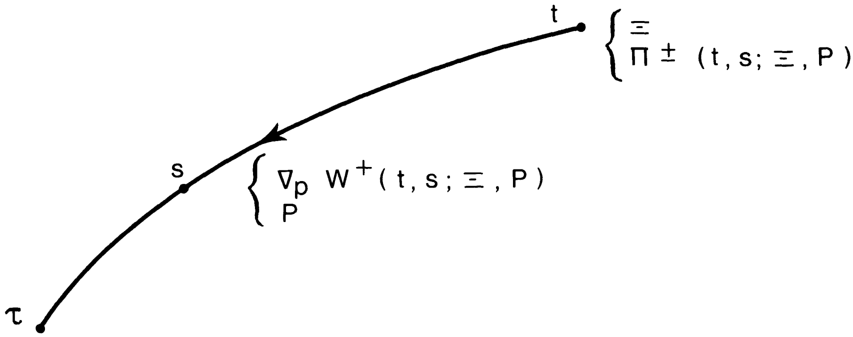

FIG. 2. — Γ.

Proof of proposition 2.6. — We only prove the + sign case, the − sign case being similar. We consider the orbit Γ in the phase-space $\mathbb{R}^{4\nu}$ starting at time $t > s$ with momentum K and ending at time s with position Z (see fig. 1). Then if $X(s')$ and $P(s')$ are the position and momentum on Γ at some time s' between s and t, it makes no difference to arrive at the position Z at time s:

— by starting from the phase-space point

$$(X(s'), P(s')) \quad \text{at time } s'$$

— or by starting from $(\nabla_K W^+(s, t ; Z, K), K))$ at time t.
Therefore

$$Z = X(s', s ; X(s'), P(s'))$$

and the inverse mapping

$$Z \mapsto \nabla_P W^+(s, s' ; Z, P(s'))$$

of

$$X(s') \mapsto X(s', s ; X(s'), P(s')) = Z$$

is nothing but

$$X(s') \equiv X(t, s' ; \nabla_K W^+(s, t ; Z, K), K) ;$$

$$X(t, s' ; \nabla_K W^+(s, t ; Z, K), K) = \nabla_P W^+(s, s' ; Z, P(s'))$$

Thus if one lets

$$x_\alpha(s') = \nabla_{p_\alpha} W^+(s, s' ; Z, P(s'))$$

in equ. (2.17), then:

$$h_\alpha(x_\alpha(s'), p_\alpha(s')) + H_{s'}^\alpha(x_\alpha(s'), y_\alpha(s') ; q_\alpha(s')) = H_{s'} \quad (X(t, s'), P(t, s'))$$

where the initial conditions are:

$$X(t, t) = (\nabla_{p_\alpha} W^+(s, t; Z, K), \eta_\alpha)$$
$$P(t, t) = K$$

Namely if is easy to check that the conditions on p_α and z_α imply that:

$$|\nabla_{p_\alpha} W^+(s, s'; Z, P(s'))| > as' \tag{2.24}$$

for any $s' \geq s$, and s large enough, so that:

$$(V_\alpha - V_{\alpha,s'})(\nabla_{p_\alpha} W^+(s, s'; Z, P(s')) = 0 \tag{2.25}$$

Thus (2.17) reduces to the y_α-component of equation (2.7) which implies for the inverse mappings:

$$\eta_\alpha^+(t, s'; \nabla_{q_\alpha} W^+(s, s'; Z, P(s')), q_\alpha) = \nabla_{q_\alpha} W^+(s', t; X(s'), K)$$

for any s' between s and t. Letting $s' = s$ implies the result

Proof of proposition 2.7. — Let Γ be the orbit in the $\mathbb{R}^{4\nu}$ phase-space specified by proposition 2.7 (see fig. 2). Then using proposition 2.2 (*i*) we have

$$\nabla_{q_\alpha} W^+(t, s; \Xi, P) = y_\alpha(t, s; \Xi, \Pi^+(t, s; \Xi, P))$$

But one easily checks that the conditions on p_α, z_α imply the analogs of (2.24-2.25) so that, as above, $\eta_\alpha^+(s, \tau; z'_\alpha, q_\alpha)$ is the y_α-component of the position at time s on Γ. But is is equivalent to arrive at position (, z'_α) at time τ:

— by starting from the phase-space point $(\Xi, \Pi^+(t, s; \Xi, P))$ at time t
— or by starting from the position (, $\eta_\alpha^+(s, \tau; z'_\alpha, q_\alpha)$) at time s;

so that:

$$\eta_\alpha^+(s, \tau; z'_\alpha, q_\alpha) = \nabla_{q_\alpha} W^+(t, s; \Xi, P) = y_\alpha(t, s; \Xi, \Pi^+(t, s; \Xi, P))$$

which completes the proof.

By analogy with (2.10) and proposition 2.3 we have defined a classical action $W^\pm_{\alpha 0}$ along the orbits in the $\mathbb{R}^{2\nu}$ phase-space, which satisfies equ. (2.20) with a time dependent hamiltonian where x_α, z'_α and q_α are some parameters. However when these parameters lie in suitable phase-space regions, i. e. intuitively as far as « outgoing » or « incoming » particles are concerned, this hamiltonian with time dependent potentials can be replaced with its analog with the time-independent ones. This is the content of the following lemma which, together with previous ones, will intensively be used in the proof of 3-body asymptotic completeness [*33*].

LEMMA 2.8. — Let x_α, z'_α and q_α be such that

$$|x_\alpha| < 3a|s|, \qquad \left|\frac{z'_\alpha}{s} - \frac{3q_\alpha}{2}\right| < a, \qquad |q_\alpha| > 10a/3$$

Then for any $t \gtrless s \gtrless 0$, $|s|$ large enough, we have:

$$|\eta_\alpha^\pm(s, t; z'_\alpha, q_\alpha)| > 4at \tag{2.26}$$

so that the first equation in (2.20) becomes

$$\frac{\partial}{\partial t} W_{\alpha 0}^\pm(s, t; z'_\alpha, q_\alpha) = \frac{3q_\alpha^2}{4} + \sum_{\beta \neq \alpha} V_\beta\left(\nabla_{q_\alpha} W_{\alpha 0}^\pm + \varepsilon_\beta \frac{x_\alpha}{2}\right)$$

Thus under the above conditions, equ. (2.27) has a unique solution $W_{\alpha 0}^\pm$ satisfying $W_{\alpha 0}^\pm(s, s; z'_\alpha, q_\alpha) = z'_\alpha \cdot q_\alpha$, which coincides with $\tilde{W}_{\alpha 0}^\pm$.

Proof. — The argument follows Hörmander's [*17*]; we only consider the case of $\eta_\alpha^+, t > s$ (the other case can be dealt with similarly). One can choose s large enough, so that

$$\left| \sum_{\beta \neq \alpha} \int_s^t \nabla V_{\beta,s'}(x_\beta) ds' \right| < 2a/3$$

Therefore by (2.17) the component along the q_α direction of $q_\alpha(t, s)$ and of η_α^+ are respectively greater than $8a/3$ and

$$5as - as + 4a(t - s) = 4at$$

This implies that

$$|\eta_\alpha^+ + \varepsilon_\beta x_\alpha/2| > 4at - 3as > at \qquad (t \geqslant s)$$

and therefore that for any $t \geqslant s$

$$V_{\beta,t}(\nabla_{q_\alpha}\tilde{W}_{\alpha 0}^+ + \varepsilon_\beta x_\alpha/2) = V_\beta(\nabla_{q_\alpha}\tilde{W}_{\alpha 0}^+ + \varepsilon_\beta x_\alpha/2)$$

This completes the proof.

In order to complete this section about classical orbits, we state here (without proofs because they bear a strong similarlity with previous ones) some results about an auxiliary two-body problem that will be needed in the study of three-body quantum scattering in « two-cluster subchannels ».

Assume $v_t(y): \mathbb{R}^\nu \mapsto \mathbb{R}$ is a $\mathscr{C}^\infty$ function depending on the time parameter t, in such a way that if $t \geqslant 1$:

$$|D_y^i v_t| \leqslant C_i t^{-m(|i|)} \tag{2.28}$$

with $m(j)$ specified by conditions (H). Then if

$$h_t(y, q) = \frac{3}{4} q^2 + v_t(y) \tag{2.29}$$

there exists T sufficiently large such that for $t \gtrless s \gtrless \pm T$ the equation

$$\frac{\partial}{\partial t} \omega^\pm(s, t; z', q) = h_t(\nabla_q \omega^\pm(s, t; z', q), q) \tag{2.30}$$

admits a unique solution with initial condition

$$\omega^{\pm}(s, s; z', q) = z' \cdot q \qquad (2.31)$$

Furthermore:

PROPOSITION 2.9. — For T sufficiently large and any $t \gtrless s \gtrless \pm T$, we have

i) $\left| \nabla_q \omega^{\pm}(s,t;z',q) - z' - \frac{3}{2} q(t-s) \right| \leqslant C \operatorname{Min}(|t-s|\,|s|^{1-m(1)}, |t-s|^{2-m(1)})$

ii) $\left| \frac{\partial^2}{\partial z' \partial q} \omega^{\pm}(s, t; z', q) - 1 \right| \leqslant C |s|^{2-m(2)}$

$\left| \frac{\partial^2}{\partial q^2} \omega^{\pm}(s, t; z', q) - \frac{3}{2}(t-s) \right| \leqslant C |t-s|\ |s|^{2-m(2)}$

iii) $\forall i$ and j multiindices such that $|i| + |j| \geqslant 2$ there exist constants C_{ij} such that

$$| D^i_{z'} D^j_q \nabla_q \omega^{\pm}(s, t; z', q) | \leqslant C_{ij} \begin{matrix} \operatorname{Max}(1, |t-s|^{1+\mu(|j|)}) & \text{if } i = 0 \\ \operatorname{Max}(1, |t-s|^{\mu(|j|+1)}) & \text{otherwise} \end{matrix}$$

iv) For any multiindices i and j

$$| D^i_{z'} D^j_q v_t(\nabla_q \omega^{\pm}(s, t; z', q)) | \leqslant C_{ij}(|t - s|^{\nu(|j|)} + 1)$$

3. LOCAL DECAY AND PROPAGATION PROPERTIES OF SOME « SCATTERING STATES »

The generating functions for the two-body and three-body classical trajectories that have been introduced in the preceding section will be a powerful tool in the study of three-body quantum scattering in the long range case.

We start this section by introducing the notations and assumptions that are needed in order to unambiguously specify our quantum problem. We consider a system of three particles in their center of mass frame in ν-dimensional space. The physical Hilbert space of quantum states is $\mathcal{H} = L^2(\mathbb{R}^{3\nu})$, and for notational convenience we assume all particles have mass equal to 1. The free hamiltonian H_0 is therefore the usually quantified kinetic energy ($\hbar = 1$)

$$H_0 = p_\alpha^2 + \frac{3}{4} q_\alpha^2 \qquad (\text{any } \alpha) \qquad (3.1)$$

(where α labels the pairs (ij) of particles), which is obviously defined as a self-adjoint operator in $\mathcal{H}$. We assume the three particles interact via translational invariant pair potentials V'_α. According to Remark 2.1, we assume further that V'_α can be split into $V_\alpha + V_\alpha^s$ where the long range part V_α satisfies assumption (H) and the short range part V_α^s obeys:

$$\| V_\alpha^s(1 + H_0)^{-1}F(|x_\alpha| > R) \| \in L^1(\mathbb{R}_+, dR) \tag{3.2}$$

where $\| \cdot \|$ denotes the vector or operator norm in $\mathcal{H}$ (here the operator norm), and where $F(|x_\alpha| > R)$ is the multiplication operator in configuration space representation by the characteristic function of the set $\{ X = (x_\alpha, y_\alpha) : |x_\alpha| > R \}$. In what follows we shall use freely this notation, and similar ones for momentum space instead of configuration space. It is then a standard result [*28*] that the full hamiltonian

$$H = H_0 + \sum_\alpha V'_\alpha \tag{3.3}$$

is uniquely determined as a self-adjoint operator in $\mathcal{H}$ with the same domain as H_0. Thus it follows from the functional calculus that $f(H)$ can be defined for a large class of well behaved functions f, whose Fourier transform will always be denoted $\hat{f}$. For any pair α, the two-body hamiltonians

$$h_\alpha = p_\alpha^2 + V'_\alpha \tag{3.4}$$

is also uniquely determined as a self-adjoint operator in $\mathcal{H}$. Furthermore under our assumptions (3.2) and (H), the following properties hold true:

(P_1) h_α has no singularly continuous spectrum

(P_2) h_α has no point spectrum in $(0, \infty)$

(P_3) h_α is bounded below and the projector P_α over its point spectrum is

$$P_\alpha = \sum_i P_{\alpha,i} \tag{3.5}$$

where the sum over i is in general infinite, where

$$P_{\alpha,i} = \sum_{k=n_1}^{n_i} \varphi_{\alpha,k} \rangle \langle \varphi_{\alpha,k} = \sum_{k=n_1}^{n_i} P_{\alpha,k} \tag{3.6}$$

is the (finite dimensional) projector over the eigenspace of h_α of eigenvalue $E_{\alpha,i}$, where $E_{\alpha,i}$ are non-positive and can only accumulate at zero. Note that $E_{\alpha,0}$ may be zero, and thus the corresponding eigenspace may be infinite dimensional. Moreover for any non zero eigenvalue $\varphi_{\alpha,k}$ has exponential decrease at infinity.

(P_4) The modified Möller wave operators exist and are asymptotically complete.

(P_1, P_2, P_3) are by now well established (see references in [*28*]) but (P_2)

would require a few additional regularity assumption on V^s_α. In order to keep the full generality of assumption (3.2) we do not add this assumption to (H)U (3.2), but we'd rather keep (P2) as an additional assumption on subsystems. The first statement in (P_4) is the results of Hörmander's paper [*17*]. Since Dollard's approach of Coulomb quantum systems, there is a large amount of literature dealing with two-body long range asymptotic completeness, especially by the « stationary methods », and we have not checked precisely whether their results cover our class of potentials. However the method developed in this paper, and in the following, for the proof of three-body long range asymptotic completeness provides a proof of two body asymptotic completeness as well, under our assumptions (3.2) and (H) (see [*33*]). We shall further need the following additional property on subsystems:

(P_5) let $\varphi_{\alpha,k}$ be an eigenfunction of h_α with zero eigenvalue (if any!). Then we require

$$\varphi_{\alpha,k} \in L^2_\delta(\mathbb{R}^\nu) = \{ \varphi : (1 + x^2)^{\delta/2}\varphi \in L^2(\mathbb{R}^\nu) \} \quad \text{some} \quad \delta > 1$$

There is, up to now, no complete result about the exact decrease of threshold eigenstates for the general class of potentials considered in this paper. Thus we keep (P_5) as an additional assumption on subsystems, although we know that for some particular V_α's (P_5) actually requires the absence of a zero eigenvalue, or the dimension ν not being too small.

Each eigenstate $\varphi_{\alpha,k}$ of each two-body subsystem gives rise to a scattering channel for the corresponding three-body system. Given

$$v_t(y_\alpha) = \sum_{\beta \neq \alpha} V_{\beta,t}(y_\alpha) \tag{3.7}$$

which obviously satisfies (2.28), we define the corresponding time-dependent two-body hamiltonian h_t, and solutions $\omega_\pm$ of Hamilton-Jacobi equations as in (2.29-2.30), denoting the latter $\omega^\pm_\alpha(s, t; z'_\alpha, q_\alpha)$. This function is particularly convenient for the study of the $(\alpha, k)^{th}$ scattering channels in the three-body problem.

Using corollary 2.4 (*i*) we can proceed as in [*17*, theorem 3.8] to prove the existence of a solution $W^\pm(s, t; 0, P)$ constructed independently of the cut off a on the potentials, and similarly for $\omega^\pm_\alpha(s, t; 0, q_\alpha)$. In the momentum space representation of $\mathscr{H}$, they are multiplicative operators that we denote $W^\pm(s, t)$ and $\omega^\pm_{\alpha,k}(s, t)$, for simplicity. Then we have the following:

PROPOSITION 3.1. — *i*) There exist the strong limits

$$\Omega^\pm_0(s) = \underset{t \to \pm\infty}{s\text{-lim}}\ e^{i(t-s)H} e^{-iW^\pm(s,t)} \tag{3.8}$$

$$\Omega^\pm_{\alpha,k}(s) = \underset{t \to \pm\infty}{s\text{-lim}}\ e^{i(t-s)H} e^{-i\omega^\pm_\alpha(s,t) - i(t-s)E_{\alpha,k}} P_{\alpha,k} \tag{3.9}$$

where $P_{\alpha,k}$ and $E_{\alpha,k}$ are defined by (3.6) and property (P_3)

ii) $\Omega_0^{\pm}$ intertwines H and H_0, and

$\Omega_{\alpha,k}^{\pm}$ intertwines H and $\frac{3}{4} q_\alpha^2 + E_{\alpha,k}$ (any α, k)

iii) Moreover their ranges do not depend on s.

Proof. — We do not give the proof of (*i*) which would be a word by word repetition of Hörmander's [*17*], modulo the replacement of $V_\beta(x_\beta)$ by $V_\beta(y_\alpha)$ for $\Omega_{\alpha,k}^{\pm}$, which can be dealt with as in the proof of lemma 3.5 (*i*) (see below). Again as in [*17*], (*ii*) easily follows from the dominated convergence theorem and the fact that for almost every P (resp. q_α) and every s':

$$\lim_{t\to\pm\infty} (W^{\pm}(s, t + s'; 0, P) - W^{\pm}(s, t; 0, P)) - s'H_0(P) = 0$$

$$\left(\text{resp. } \lim_{t\to\pm\infty} (\omega_\alpha^{\pm}(s, t + s'; 0, q_\alpha) - \omega_\alpha^{\pm}(s, t; 0, q_\alpha)) - \frac{3}{4} q_\alpha^2 s' = 0\right)$$

Now using (*ii*), we see that (*iii*) reduces to proving the following lemma:

LEMMA 3.2. — For any s, s' ($|s|$ and $|s'| \geqslant T$ as in prop. 2.2 or 2.5) and any $P \neq 0$ (resp. $q_\alpha \neq 0$) the following limits exist:

$$\lim_{t\to\pm\infty} [W^{\pm}(s, t; 0, P) - W^{\pm}(s', t; 0, P)]$$

$$\left(\text{resp. } \lim_{t\to\pm\infty} [\omega_\alpha^{\pm}(s, t; 0, q_\alpha) - \omega_\alpha^{\pm}(s', t; 0, q_\alpha)]\right).$$

For a proof of similar statements, see [*20*, prop. 2.8].

In the rest of this section, we shall prove that the complement in the continuous spectral subspace $\mathcal{H}^{\text{cont}}$ of H of all two-cluster channels represents, intuitively, three asymptotically free particles. More precisely:

— the particles get, asymptotically in time, arbitrarily far separated,

— for any pair, the particles in the pair are outgoing relative to each other (or incoming if time is reversed).

PROPOSITION 3.3. — Let $\psi \in \mathcal{H}^{\text{cont}} \setminus \bigcup_{\alpha,k} (\Omega_{\alpha,k}^{\pm})$ be such that there exists a $\mathcal{C}_0^\infty(A, \infty)$ function G, $A > \frac{4(3+\sqrt{2})}{3} a^2$, with $\psi = G(H)\psi$. Then for any sequence $\rho_n \to \infty$ ($n \to \infty$) there exists a sequence τ_n^+ (resp. τ_n^-) converging to $+\infty$ (resp. $-\infty$) as $n \to \infty$ such that:

$$\lim_{n\to\infty} \| F(|x_\alpha| < \rho_n) e^{-i\tau_n^{\pm} H} \psi \| = 0 \tag{3.10}$$

(one statement for the + sign case, and one statement for the − sign).

Proof. — It is enough to prove convergence in the time-mean, i. e.

$$\lim_{T\to\pm\infty} \frac{1}{T}\int_0^T dt \,\| F(|x_\alpha| < \rho)e^{-itH}\psi \| = 0 \tag{3.11}$$

for any $\rho < \infty$. The proof splits into several lemmas, along the same lines as in the short range case [*8*]:

There exists a $\mathscr{C}_0^\infty(\mathbb{R})$ function G', with $G' = 1$ on supp G and supp $G' \subset [4(1 + \sqrt{2/3})a^2, \infty)$. Thus $G'G = 1$ and it is easy to check that $F(|x_\alpha| < \rho)[G'(H) - G'(H_\alpha)]G(H)$ is a compact operator, so that $\| F(|x_\alpha| < \rho)[G'(H) - G'(H_\alpha)]e^{-itH}\psi \|$ goes to zero in the time mean. Thus we only have to prove for any $\rho < \infty$:

$$\lim_{T\to\pm\infty} \frac{1}{T}\int_0^T dt \,\| F(|x_\alpha| < \rho)G'(H_\alpha)e^{-itH}\psi \| = 0 \tag{3.12}$$

We now split this term via projection operators in y_α-space. We define them using « generalized coherent states » as proposed by [*5*]. Let $\hat{\eta}$ be a $\mathscr{C}_0^\infty(\mathbb{R}^\nu)$ function with support contained in $|q| < 2a/3$ and let

$$\langle \varphi, \eta_{z'_\alpha k'_k} \rangle (p_\alpha) \equiv \int dq_\alpha \bar{\hat{\varphi}}(p_\alpha, q_\alpha)e^{-iq_\alpha . z'_\alpha}\hat{\eta}(q_\alpha - k'_\alpha) \tag{3.13}$$

for any $\varphi \in \mathscr{H}$. Then the following operators

$$P_R^\pm(y_\alpha) = \int_{\substack{z'_\alpha . k'_\alpha \gtrless 0 \\ |z'_\alpha| > R}} dz'_\alpha dk'_\alpha \eta_{z'_\alpha k'_\alpha} \rangle \langle \eta_{z'_\alpha k'_\alpha} \tag{3.14}$$

are known to be bounded operators in $L^2(\mathbb{R}^{2\nu})$ such that

$$P_R(y_\alpha) \equiv P_R^+(y_\alpha) + P_R^-(y_\alpha)$$

is multiplication operator in y_α-space by $\int_{|z'_\alpha| > R} dz'_\alpha |\eta(y_\alpha - z'_\alpha)|^2$, and $\underset{R\to\infty}{\text{s-lim}}\, P_R(y_\alpha) = 0$.

Now we write:

$$\begin{aligned}
&\| F(|x_\alpha| < \rho)G'(H_\alpha)e^{-itH}\psi \|^2 \\
&\quad = \langle G'(H_\alpha)e^{-itH}\psi, F(|x_\alpha| < \rho)(1 - P_R(y_\alpha))G'(H_\alpha)e^{-itH}\psi \rangle \\
&\quad + \langle G'(H_\alpha)e^{-itH}\psi , F(|x_\alpha| < \rho)P_\alpha P_R(y_\alpha)G'(H_\alpha)e^{-itH}\psi \rangle \\
&\quad + \langle G'(H_\alpha)e^{-itH}\psi, F(|x_\alpha| < \rho)(1 - P_\alpha)P_R(y_\alpha)G'(H_\alpha)e^{-itH}\psi \rangle
\end{aligned} \tag{3.15}$$

where $P_\alpha = \sum_k P_{\alpha,k}$ is the projector on the point spectrum of h_α. But as $F(|x_\alpha| < \rho)\,(1 - P_R(y_\alpha))G'(H_\alpha)$ is a compact operator, the contribution to (3.12) of the first term of (3.15) is O. K. because ψ lies in $\mathscr{H}^{\text{cont}}$. For the second and third terms, we use lemma 3.5 below. Let

$$\Omega^{\pm}_{\alpha,\mathrm{R}} = \underset{t\to\pm\infty}{s\text{-lim}} \sum_k e^{i(t-s)\mathrm{H}} \mathrm{E}_{\alpha k,\mathrm{R}}(s, t) \tag{3.16}$$

where the sum lies over all bound states $\mathrm{E}_{\alpha,k}$ of h_α, and where for any $\varphi \in \mathscr{H}$:

$$(\mathrm{E}^{\pm}_{\alpha,k,\mathrm{R}}(s, t)\varphi)(x_\alpha, q_\alpha) \equiv \int_{\substack{z'_\alpha\cdot k'_\alpha \gtrless 0 \\ |z'_\alpha|>\mathrm{R}}} dz'_\alpha dk'_\alpha e^{-i\omega^{\pm}_\alpha(s,t;z'_\alpha,q_\alpha)-i(t-s)\mathrm{E}_{\alpha,k}} \hat{\eta}(q_\alpha - k'_\alpha)\varphi_{\alpha,k}(x_\alpha)\mathrm{G}'\left(\frac{3q_\alpha^2}{4} + \mathrm{E}_{\alpha,k}\right)\langle \varphi_{\alpha,k} \otimes \eta_{z'_\alpha k'_\alpha}, \varphi \rangle \tag{3.17}$$

LEMMA 3.4. — $\Omega^{\pm}_{\alpha,\mathrm{R}}$ is a well-defined, bounded operator in $\mathscr{H}$, whose range is independent of s and contained in $\bigoplus_k \mathscr{R}(\Omega^{\pm}_{\alpha,k})$.

Remark. — $\Omega^{\pm}_{\alpha,\mathrm{R}}$ is a long range generalization of

$$\Omega^{\pm}_\alpha \mathrm{G}'(\mathrm{H}_\alpha)\mathrm{P}^{\pm}(y_\alpha)\mathrm{F}(\,|\,y_\alpha\,| > \mathrm{R})$$

where $\Omega^{\pm}_\alpha$ is the sum of all wave operators for the channel α and $\mathrm{P}^{\pm}(y_\alpha)$ are the projectors over the positive and negative parts of the spectrum of the operator $y_\alpha \cdot q_\alpha + q_\alpha \cdot y_\alpha$.

LEMMA 3.5. — Let ψ be as in proposition 3.2. Then

i) $\| (\Omega^{\pm}_{\alpha,\mathrm{R}} - \mathrm{G}'(\mathrm{H}_\alpha)\mathrm{P}_\alpha\mathrm{P}^{\pm}_{\mathrm{R}}(y_\alpha))\mathrm{F}(\,|\,x_\alpha\,| < \rho) \| \to 0$ as $\mathrm{R} \to \infty$

ii) $\forall\varepsilon$, $\exists\mathrm{T}(\varepsilon)$ and $\mathrm{R}(\varepsilon)$ such that $\mathrm{T} > \mathrm{T}(\varepsilon)$ and $\mathrm{R} > \mathrm{R}(\varepsilon)$ imply

$$\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} dt \,\| \mathrm{F}(\,|\,x_\alpha\,| < \rho)(1-\mathrm{P}_\alpha)\mathrm{P}_{\mathrm{R}}(y_\alpha)\mathrm{G}'(\mathrm{H}_\alpha)e^{-it\mathrm{H}}\psi \| < \varepsilon \tag{3.18}$$

Admitting these two lemmas for the moment, we complete the proof of proposition 3.3:

$$\begin{aligned}
&\langle \mathrm{G}'(\mathrm{H}_\alpha)e^{-it\mathrm{H}}\psi, \mathrm{F}(\,|\,x_\alpha\,| < \rho)\mathrm{P}_{\mathrm{R}}(y_\alpha)\mathrm{G}'(\mathrm{H}_\alpha)\mathrm{P}_\alpha e^{-it\mathrm{H}}\psi \rangle \\
&= \langle \mathrm{G}'(\mathrm{H}_\alpha)e^{-it\mathrm{H}}\psi, \mathrm{F}(\,|\,x_\alpha\,| < \rho)(\mathrm{P}^{+}_{\mathrm{R}}(y_\alpha)\mathrm{P}_\alpha\mathrm{G}'(\mathrm{H}_\alpha) - \Omega^{+*}_{\alpha,\mathrm{R}})e^{-it\mathrm{H}}\psi \rangle \\
&+ \langle \mathrm{G}'(\mathrm{H}_\alpha)e^{-it\mathrm{H}}\psi, \mathrm{F}(\,|\,x_\alpha\,| < \rho)\Omega^{+*}_{\alpha,\mathrm{R}}e^{-it\mathrm{H}}\psi \rangle \\
&+ \sum_k \langle \mathrm{G}'(\mathrm{H}_\alpha)e^{-it\mathrm{H}}\psi, \mathrm{F}(\,|\,x_\alpha\,| < \rho)(\mathrm{P}_{\alpha,k}\mathrm{P}^{-}_{\mathrm{R}}(y_\alpha)\mathrm{G}'(\mathrm{H}_\alpha)e^{-it\mathrm{H}} \\
&\qquad\qquad - \mathrm{E}^{-}_{\alpha,k,\mathrm{R}}(s, s-t)^*)\psi \rangle \\
&+ \sum_k \langle \mathrm{G}'(\mathrm{H}_\alpha)e^{-it\mathrm{H}}\psi, \mathrm{F}(|x_\alpha|<\rho) \quad \mathrm{E}^{-}_{\alpha,k,\mathrm{R}}(s, s-t)^*\mathrm{F}(\,|\,y_\alpha\,| > \mathrm{R}/2)\psi \rangle \\
&+ \sum_k \langle \mathrm{G}'(\mathrm{H}_\alpha)e^{-it\mathrm{H}}\psi, \mathrm{F}(|x_\alpha|<\rho) \quad \mathrm{E}^{-}_{\alpha,k,\mathrm{R}}(s, s-t)^*\mathrm{F}(|y_\alpha|<\mathrm{R}/2)\psi\rangle
\end{aligned} \tag{3.19}$$

It follows immediately from lemma 3.5 (*i*) that the first term satisfies

$$\lim_{R\to\infty} \operatorname{Sup}_{t>0} | \cdot | = 0 \tag{3.20}$$

and the proof of lemma 3.5 (*i*) easily implies that so does the third term of (3.19). But the second is zero because ψ (and thus $e^{-itH}\psi$) is orthogonal to $\mathscr{R}(\Omega^{\pm}_{\alpha,k})$ for any k. The fourth term is dominated by $\| F | y_\alpha | > R/2)\psi \|$ which obviously satisfies (3.20), and the fifth term is dominated by $\| F(| y_\alpha | < R/2)E^{-}_{\alpha,k,R}(s, s-t) \|$ which satisfies (3.20) by lemma 3.6 below. Thus given $\varepsilon > 0$ there exists $R'(\varepsilon)$ such that the second term of (3.15) is dominated by ε for $R > R'(\varepsilon)$. Let us choose $R > \operatorname{Max}(R(\varepsilon), R'(\varepsilon))$, where $R(\varepsilon)$ is as in lemma 3.5 (*ii*). Then there exists $T'(\varepsilon)$ (depending on R) such that the first term of (3.15) is bounded by ε in the time mean for $T > T'(\varepsilon)$. But so is the third term provided $T > \operatorname{Max}(T(\varepsilon), T'(\varepsilon))$, from lemma 3.5 (*ii*), and thus $\frac{1}{T}\int_0^T dt \, \| F(| x_\alpha | < \rho)G'(H_\alpha)e^{-itH}\psi \| < \sqrt{3}\varepsilon$, which completes the proof of proposition 3.3.

We now state and prove lemma 3.6.

LEMMA 3.6.

$$\lim_{R\to\infty} \operatorname{Sup}_{t\geq 0} \| F(| y_\alpha | < R/2)E^{-}_{\alpha,k,R}(s, s-t) \| = 0$$

The proof is an easy application of the « non-stationary phase method » ([*17*, lemma A1]) Given any $\varphi \in \mathscr{R}(G'(H_\alpha))$,

$$(e^{-itE_{\alpha,k}}E^{-}_{\alpha,k,R}(s, s-t)\varphi)(x_\alpha, y_\alpha) = \varphi_{\alpha,k}(x_\alpha) \cdot$$

$$\cdot \int_{\substack{z'_\alpha \cdot k'_\alpha < 0 \\ |z'_\alpha| > R}} dz'_\alpha dk'_\alpha \int dq_\alpha e^{iq_\alpha \cdot y_\alpha - i\omega^{-}_\alpha(s, s-t; z'_\alpha, q_\alpha)} \hat{\eta}(q_\alpha - k'_\alpha) \cdot \langle \varphi_{\alpha,k} \otimes \eta_{z'_\alpha k'_\alpha}, \varphi \rangle \tag{3.21}$$

But if $H_\alpha > \frac{4(3+\sqrt{2})}{3}a^2$, and $h_\alpha = E_{\alpha,k} \leq 0$, then $\frac{3}{4}q_\alpha^2 > \frac{4}{3}(3+\sqrt{2})a^2$, so that $\langle \psi_{\alpha,k} \otimes \eta_{z'_\alpha k'_\alpha}, \varphi \rangle = 0$ if $| k'_\alpha | < \frac{8a}{3} - \frac{2a}{3} = 2a$ because of the support property of $\hat{\eta}$. Now for s negative and $| s |$ sufficiently large, we have:

$$\left| \nabla_{q_\alpha}\omega^{-}_\alpha(s, s-t; z'_\alpha, q_\alpha) - z'_\alpha + \frac{3}{2}tq_\alpha \right| < 1$$

and on the other hand the conditions $z'_\alpha \cdot k'_\alpha < 0$, $| z'_\alpha | > R$ imply

$$\left| z'_\alpha - \frac{3}{2}tk'_\alpha \right|^2 > z'^2_\alpha + 9a^2t^2 \qquad \text{for any} \quad t > 0$$

Therefore

$$| \nabla_{q_\alpha}\omega^{-}_\alpha(s, s-t; z'_\alpha, q_\alpha) | > \frac{| z'_\alpha |}{\sqrt{2}} + 2at - at - 1 > R/\sqrt{3} + at \tag{3.22}$$

for R sufficiently large. Thus if $|y_\alpha| < R/2$, the integral over q_α is non-stationary, and therefore bounded by

$$c_n\left(\frac{|z'_\alpha|}{\sqrt{3}} - \frac{R}{2} + at\right)^{-n} \qquad \text{for any} \quad n \in \mathbb{N}$$

This easily implies that the norm of

$$F(|y_\alpha| < R/2)E^-_{\alpha,k,R}(s, s - t)G'(H_\alpha)\varphi$$

is bounded by $C'_n(R + at)^{-n} \|\varphi\|$ for any n, which completes the proof of lemma 3.6.

We now come back to the proof of lemmas 3.4 and 3.5. For $t = s$, $E^\pm_{\alpha,k,R}(s, s) = G'\left(\frac{3q_\alpha^2}{4} + E_{\alpha,k}\right)P_{\alpha,k}P^\pm_R(y_\alpha)$ so that it is a bounded operator in $\mathcal{H}$. For $t \neq s$, it is no longer obvious that $E^\pm_{\alpha,k,R}(s, t)$ defines a bounded operator in $\mathcal{H}$. We give a proof of this fact, using a result on oscillatory integrals due to Fujiwara [13] (see proposition 4.1 below). Let $\varphi \in \mathcal{H}$. Then

$$\int |\langle \eta_{z'_\alpha k'_\alpha} \otimes \varphi_{\alpha,k}, \varphi \rangle|^2 dz'_\alpha dk'_\alpha = \|\varphi\|^2 \tag{3.23}$$

so that for any fixed $k'_{\alpha'}$ the integral over z'_α in $E^\pm_{\alpha,k,R}(s, t)\varphi$ takes the form

$$\int dz'_\alpha e^{i\phi(z'_\alpha, q_\alpha)} f(z'_\alpha, k'_\alpha) \tag{3.24}$$

where $f(z'_\alpha, k'_\alpha) \equiv F(z'_\alpha \cdot k'_\alpha \gtrless 0, |z'_\alpha| > R)\langle \eta_{z'_\alpha k'_\alpha} \otimes \varphi_{\alpha,k}, \varphi \rangle$ belongs to $L^2(\mathbb{R}^\nu, dz'_\alpha)$, and where

$$\phi(z'_\alpha, q_\alpha) \equiv -\omega^\pm_\alpha(s, t; z'_\alpha, q_\alpha) - (t - s)E_{\alpha,k}$$

satisfies

$$\left|\frac{\partial^2\phi}{\partial z'_\alpha \partial q_\alpha} + 1\right| = \left|-\frac{\partial^2\omega^\pm_\alpha}{\partial z'_\alpha \partial q_\alpha} + 1\right| \leqslant C|s|^{2-m(2)} \to 0 \quad \text{as} \quad |s| \to \infty$$

so that $\left|\operatorname{Det}\left(\frac{\partial^2\phi}{\partial z'^i_\alpha \partial q^j_\alpha}\right)\right| \geqslant 1/2$ for $|s|$ sufficiently large. Therefore for $|s|$ large, the L^2 norm w. r. to q_α of (3.24) is bounded by $C\left(\int dz'_\alpha |f(z'_\alpha, k'_\alpha)|^2\right)^{1/2}$ uniformly for $t \gtrless s$. But the integration support over k'_α in (3.21) is a fixed compact K, so that

$$\begin{aligned} \|E^\pm_{\alpha,k,R}(s, t)\varphi\| &\leqslant C\int_K dk'_\alpha\left(\int dz'_\alpha |f(z'_\alpha, k'_\alpha)|^2\right)^{1/2} \\ &\leqslant C|K|^{1/2}\left(\int dk'_\alpha dz'_\alpha |f(z'_\alpha, k'_\alpha)|^2\right)^{1/2} \\ &\leqslant C|K|^{1/2}\|\varphi\| \end{aligned}$$

from (3.23).

Given that, proof of lemma 3.4 is a simple, technical adaptation of that of Hörmander's [*17*] for the existence of two-body modified wave operators; we do not give the details for this, because very similar but harder estimates are derived along the proof of lemma 3.5 (*i*). We first prove lemma 3.5 (*ii*), proceeding similarly to [*8*]: if the $T(\varepsilon)$ under investigation can be found, any $T > T(\varepsilon)$ belongs to some interval $[NT(\varepsilon), (N+1)T(\varepsilon)[$ with N integer, and the L.H. S. of (3.18) is smaller than

$$\frac{1}{NT(\varepsilon)} \sum_{j=0}^{N} \int_{jT(\varepsilon)}^{(j+1)T(\varepsilon)} dt \, \| F(|x_\alpha| < \rho)(1-P_\alpha)P_R(y_\alpha)G'(H_\alpha)e^{-itH}\psi \|$$

$$\leqslant \frac{1}{NT(\varepsilon)} \sum_{j=0}^{N} \int_0^{T(\varepsilon)} dt \, \| F(|x_\alpha| < \rho)(1-P_\alpha)P_R(y_\alpha)G'(H_\alpha)e^{-i(t+jT(\varepsilon))H}\psi \|$$

$$\leqslant \frac{2}{T(\varepsilon)} \operatorname*{Sup}_{\tau \geqslant 0} \int_0^{T(\varepsilon)} dt \, \| F(|x_\alpha| < \rho)(1-P_\alpha)P_R(y_\alpha)G'(H_\alpha)e^{-i(t+\tau)H}\psi \| \qquad (3.25)$$

Thus the proof reduces in showing that there exists $R(\varepsilon)$ such that $R > R(\varepsilon)$ implies that (3.25) is smaller than ε, or which is sufficient

$$\operatorname*{Sup}_{0 \leqslant t \leqslant T(\varepsilon)} \| F(|x_\alpha| < \rho)(1-P_\alpha)P_R(y_\alpha)G(H_\alpha)(e^{-itH} - e^{-itH_\alpha}) \| < \varepsilon/4 \qquad (3.26)$$

where $T(\varepsilon)$ is such that

$$\operatorname*{Sup}_{\tau \geqslant 0} T(\varepsilon)^{-1} \int_0^{T(\varepsilon)} dt \, \| F(|x_\alpha| < \rho)(1-P_\alpha)P_R(y_\alpha)G'(H_\alpha)\, e^{-itH_\alpha}e^{-itH}\psi \| < \varepsilon/4 \qquad (3.27)$$

((3.27) easily follows from a slight extension by Enss [*9*] of Ruelle's theorem [*29*] for h_α, using properties (P_1, P_2) of subsystems which imply that $(1-P_\alpha)$ is the projector over the spectral continuous subspace of h_α). But the L. H. S. of (3.26) is dominated by

$$\operatorname*{Sup}_{0 \leqslant s \leqslant t \leqslant T(\varepsilon)} \sum_{\beta \neq \alpha} T(\varepsilon) \, \| P_R(y_\alpha)F(|x_\alpha| < \rho)(1-P_\alpha)G'(H_\alpha)e^{-isH_\alpha}V_\beta e^{-i(t-s)H} \|$$

For $T(\varepsilon)$ fixed, $F(|x_\alpha| < \rho)(1-P_\alpha)G'(H_\alpha)e^{-isH_\alpha}V_\beta$ is a uniformly continuous (in *s*) family of compact operators, and $P_R(y_\alpha)$ strongly converges to zero as $R \to \infty$, thus (3.26) holds for $R >$ some $R(\varepsilon)$ (depending on ε and $T(\varepsilon)$). This completes the proof of part (*ii*) of lemma 3.5.

For the proof of part (*i*) we may restrict ourselves to a finite number N of bound states in P_α because:

$$\left\| F(|x_\alpha| < \rho)F(h_\alpha \leqslant 0) \sum_{k > N} P_{\alpha,k} \right\| \to 0 \quad \text{as} \quad N \to \infty$$

Thus it is enough to show that for each k

$$\lim_{R\to\infty} \operatorname{Sup}_{t>0} \| e^{-itH} G'(H_\alpha) P_R^+(y_\alpha) P_{\alpha,k} - E^+_{\alpha,k,R}(s, s+t) \| = 0 \qquad (3.28)$$

or equivalently

$$\lim_{R\to\infty} \int_0^\infty dt \left\| \left(H - i\frac{\partial}{\partial t} \right) E^+_{\alpha,k,R}(s, s+t) \right\| = 0 \qquad (3.29)$$

(because $E^+_{\alpha,k,R}(s, s) = G'(H_\alpha)P_R^+(y_\alpha)P_{\alpha,k}$

Now we show that the integrand in (3.29) is bounded by $(R + at)^{-1-\varepsilon}$ for some $\varepsilon > 0$, which implies the result. When applied to a vector φ in $\mathcal{H}$, the operator in (3.29) can be written, in X-space, as

$$(J_{\alpha,k}\varphi)(X) = \varphi_{\alpha,k}(x_\alpha) e^{-itE_{\alpha,k}} \int_{\substack{z'_\alpha . k'_\alpha > 0 \\ |z'_\alpha| > R}} dz'_\alpha dk'_\alpha \int dq_\alpha \hat{\eta}(q_\alpha - k'_\alpha) \cdot$$
$$\cdot\, e^{iq_\alpha \cdot y_\alpha - i\omega^+_\alpha (s,s+t;z'_\alpha,q_\alpha)} G\left(\frac{3q_\alpha^2}{4} + E_{\alpha,k} \right) \sum_{\beta \neq \alpha} (V'_\beta(x_\beta) - V_{\beta,t+s}(\nabla_{q_\alpha}\omega^+_\alpha)) \langle \varphi_{\alpha,k} \otimes \eta_{z'_\alpha k'_\alpha}, \varphi \rangle \qquad (3.30)$$

(we recall that V'_β includes the short range part of the potential). But, as in the proof of lemma 3.6:

$$| \nabla_{q_\alpha}\omega^+_\alpha(s, s+t; z'_\alpha, q_\alpha) | > \frac{|z'_\alpha|}{\sqrt{3}} + \left(\frac{3}{\sqrt{2}} - 1 \right) at > \left(\frac{3}{\sqrt{2}} - 1 \right) a(t+s) + \varepsilon R$$

for R large enough, some ε. Therefore $V_{\beta,t+s}(\nabla_{q_\alpha}\omega^+_\alpha) = V_\beta(\nabla_{q_\alpha}\omega^+_\alpha)$. Now using property (P_5) for the possible zero eigenvalues, together with the exponential decrease of non-threshold eigenstates, we have that

$$\lim_{R\to\infty} \int_0^\infty dt \, \| F(|x_\alpha| > b(t+s+R)) J_{\alpha,k}\varphi \| \leqslant$$
$$\leqslant C \| \varphi \| \lim_{R\to\infty} \int_0^\infty dt \, \| F(|x_\alpha| > b(t+s+R)) \varphi_{\alpha,k}(x_\alpha) \|$$
$$= C \| \varphi \| \lim_{R\to\infty} R^{1-\delta}$$

(any $b > 0$) by a simple use of proposition 4.1 below. Furthermore the contribution of $F(|y_\alpha| < a(t+s+R))J_{\alpha,k}\varphi$ obeys (3.29) by an easy « non-stationary phase » argument as in lemma 3.6 .Therefore we are left to consider

$$F(|x_\alpha| < a(t+s+R)) F(|y_\alpha| > a(t+s+R)) (J_{\alpha,k}\varphi)(X) \qquad (3.30)'$$

and we use the splitting

$$V'_\beta(x_\beta) - V_\beta(\nabla_{q_\alpha}\omega^+_\alpha) = V^s_\beta(x_\beta) + (V_\beta(x_\beta) - V_\beta(y_\alpha)) + (V_\beta(y_\alpha) - V_\beta(\nabla_{q_\alpha}\omega^+_\alpha)) \qquad (3.31)$$

We first consider the contribution of the last term in (3.31) which vanishes at the point of stationary phase defined by

$$y_\alpha = \nabla_{q_\alpha}\omega_\alpha^+(s, s+t\,; z'_\alpha, q_\alpha) \qquad (3.32)$$

Therefore, in the contribution of this term to (3.30) the q_α-integration can be expanded in a stationary phase expansion. From proposition 2.9 for ω_α^+ instead of ω^+, we deduce that, if we denote by c the value $1-\delta$ of $\mu(j+1) - \mu(j)$ for large j, the derivatives of order $\geqslant 2$ of the mapping

$$q_\alpha \mapsto t^{2c-1}\omega_\alpha^+(s, s+t\,; z'_\alpha, t^{-c}q_\alpha)$$

are uniformly bounded. Taking advantage of this fact, as in [17, proof of theorem 3.9], we can perform a change of variable $q_\alpha \mapsto t^{-c}q_\alpha$. Then using a partition of unity $\sum_g \chi(q_\alpha - g) = 1$ where g runs over lattice points of $\mathbb{R}^\nu$, we can consider contributions close to and far from the critical point $q_\alpha = \tilde{q}_\alpha(y_\alpha, z'_\alpha)$:

— for those g with

$$t^{c-1}\,|\, y_\alpha - \nabla_{q_\alpha}\omega_\alpha^+(s, s+t\,; z'_\alpha, t^{-c}g)\,| > \mathrm{K}$$

a « non-stationary phase » estimate holds as in [17] and we do not reproduce the detailed estimates leading to the bound:

$$\mathrm{C_N R}^{-m(0)}(1 + |\, z'_\alpha - y_\alpha\,|)^{-\nu-1}\,\mathrm{Min}\,(1, t^{-\mathrm{N}})$$

for the contribution of these terms to the integral over q_α; therefore, using the boundedness in $\mathrm{L}^2(\mathbb{R}^\nu)$ of the convolution by a $\mathrm{L}^1(\mathbb{R}^\nu)$ function, their contribution to $\|\, \mathrm{J}_{\alpha,k}\varphi\,\|$ is bounded in norm by $\mathrm{C_N R}^{-m(0)}\,\mathrm{Min}\,(1, t^{-\mathrm{N}})\,\|\,\varphi\,\|$ (any N)

— the number of g's such that

$$t^{c-1}\,|\, y_\alpha - \nabla_{q_\alpha}\omega_\alpha^+(s, s+t\,; z'_\alpha, gt^{-c})\,| \leqslant \mathrm{K} \qquad (3.33)$$

is obviously uniformly bounded in t, and each of the corresponding terms can be expanded, using lemma A.4 of [17] into finitely many terms in the asymptotic expansion of the stationary-phase method at $q_\alpha = \tilde{q}_\alpha$, plus an error after N steps bounded by

$$\mathrm{C} \underset{0 \leqslant k'' \leqslant k'}{\mathrm{Max}}\; t^{(1-2c)\left(-\frac{\nu}{2}-\mathrm{N}\right)+\nu(k'')-k''c} \qquad (t \geqslant 1) \qquad (3.34)$$

for some $k'' > \frac{\nu}{2} + 2\mathrm{N}$ [17, lemma A.4] (where we have used the fact that the k'^{th} Sobolev norm in q_α of

$$[\mathrm{V}_\beta(y_\alpha) - \mathrm{V}_\beta(\nabla_{q'_\alpha}\omega_{\alpha,k}^+)]\mathrm{G}'\left(\frac{3q'^2_\alpha}{4} + \mathrm{E}_{\alpha,k}\right)\tilde{\eta}(q'_\alpha - k'_\alpha)\Big|_{q'_\alpha = q_\alpha t^{-c}}$$

is bounded by C $\underset{0\leqslant k''\leqslant k'}{\text{Max}}\ t^{\nu(k'')-k''c}$ from prop. 2.9 (*iii*)). But

$$\nu(k') - k'c \leqslant \nu(2) - 2c\,, \qquad \text{and as} \quad 1 - 2c > 0\,,$$

the exponent in (3.34) is negative and can be made arbitrarily large for large N. But (3.33) implies $|\, y_\alpha - z'_\alpha \,| \leqslant \mathrm{K}' t^{1-c}$ and therefore

$$t^{-1} \leqslant \mathrm{K}'' \,|\, y_\alpha - z'_\alpha \,|^{-(1-c)-1}$$

Thus one can extract from (3.34) a factor $(1+|\, y_\alpha - z'_\alpha \,|)^{-\mathrm{N}'}$ for some $\mathrm{N}' > \nu$, which yields a bounded contribution to $\|\, \mathrm{J}_{\alpha,k}\varphi \,\|$ as above (because the convolution by $(1 + |\, z'_\alpha \,|)^{-\mathrm{N}'}$ is bounded in $\mathrm{L}^2(\mathbb{R}^\nu)$) multiplied by a factor $\|\, \varphi \,\|\, t^{-1-\varepsilon}$ with $\varepsilon > 0$ $(t \geqslant 1)$.

In order to study the various terms in the stationary-phase expansion, we come back to the original variable q_α, and we use a Morse lemma [*17*, lemma A.6]:

there exists a change of variable $q_\alpha \mapsto \omega$ such that if $t > 1$:

$$y_\alpha \cdot q_\alpha - \omega^+_\alpha(s, s + t\,; z'_\alpha, q_\alpha) - id\,(q_\alpha \mapsto \tilde{q}_\alpha) = (t/2)(\omega, \mathrm{A}\omega)$$

where A is nothing but the non-singular matrix of the second derivatives of ω^+_α/t w. r. to q_α at $q_\alpha = \tilde{q}_\alpha$. Of course this change of variable depends on y_α, $z'_{\alpha'}$ s.

We denote by $\mathscr{J}(\omega)$ the jacobian and by $\mathscr{V}(\omega)$ the image of

$$\sum_{\beta\neq\alpha} [\mathrm{V}_\beta(y_\alpha) - \mathrm{V}_\beta(\nabla_{q_\alpha}\omega^+_{\alpha,k})]\mathrm{G}'\left(\frac{3q_\alpha^2}{4} - \mathrm{E}_{\alpha,k}\right)\hat{\eta}(q_\alpha - k'_\alpha)$$

by this change of variable. The contribution to $\|\, \mathrm{J}_{\alpha,k}\varphi \,\|$ of the terms of order n in the stationary phase expansion is

$$t^{-\nu/2-n}\varphi_{\alpha,k}(x_\alpha)\int dz'_\alpha dk'_\alpha e^{i\tilde{q}_\alpha\cdot y_\alpha - i\omega^\pm_\alpha\,(s,s+t;z'_\alpha,\tilde{q}_\alpha)}\cdot\langle\, \varphi_{\alpha,k}\otimes\eta_{z'_\alpha k'_\alpha}, \varphi\,\rangle(\mathrm{L}_{2n}\mathscr{V}\mathscr{J})(0) \tag{3.35}$$

where L_{2n} is a differential operator in ω of order $2n$. As we have already noticed, n has to be $\geqslant 1$, because the 0^{th} order term vanishes. But

$$\phi(y_\alpha, z'_\alpha) \equiv t(\tilde{q}_\alpha\cdot y_\alpha - \omega^+_\alpha(s, s + t\,; z'_\alpha, \tilde{q}_\alpha)) \tag{3.36}$$

satisfies

$$|\, \mathrm{Det}\,(\partial^2\phi/\partial y^i_\alpha\partial z'^j_\alpha) \,| \geqslant \mathrm{C} \tag{3.37}$$

for some constant C independent of s, t, y_α and z'_α (where Det a_{ij} is the determinant of the $\nu \times \nu$ matrix whose elements are a_{ij}). Namely it follows from (3.32) that

$$\partial^2\phi/\partial y^i_\alpha\partial z'^j_\alpha = t\,\partial\tilde{q}^i_\alpha/\partial z'^j_\alpha \tag{3.38}$$

and therefore

$$\operatorname{Det}\left(\frac{\partial^2 \phi}{\partial y_\alpha^i \partial z_\alpha'^j}\right) = -\frac{\operatorname{Det}(\partial^2 \omega_\alpha^+ / \partial q_\alpha^i \partial z_\alpha'^j)}{\operatorname{Det}(\partial^2 \omega_\alpha^+ / t\partial q_\alpha^i \partial q_\alpha^j)} \tag{3.39}$$

so that (3.37) holds by proposition 2.9 (*ii*). Furthermore $(L_{2n}\mathscr{V}\mathscr{J})(0)$ is a $\mathscr{C}^\infty$ function of y_α and z'_α whose derivatives satisfy Lemma 3.7 below. Thus one can perform a change of variable $z'_\alpha \mapsto \sqrt{t}z'_\alpha$, $y_\alpha \mapsto \sqrt{t}y_\alpha$ so that Proposition 4.1 is applicable. It implies that there exists an integer K such that the contribution to $\| J_{\alpha,k}\varphi \|$ of (3.35) be bounded by

$$Ct^{\nu/2} \sup_{n\geq 1} t^{-\nu/2-n} \| \varphi \| \sup_{\substack{|i|,|j|\leq K \\ z'_\alpha, q_\alpha, y_\alpha : y_\alpha \cdot q_\alpha > 0}} | D^i_{z'_\alpha} D^j_{y_\alpha} (L_{2n}\mathscr{V}\mathscr{J})(0)(\sqrt{t}z'_\alpha, \sqrt{t}y_\alpha) | \tag{3.40}$$

(because the support of integration in k'_α in compact). Using lemma 3.7 below, (3.40) is bounded by

$$C \sup_{n\geq 1} t^{\nu(2n)-n} \| \varphi \| = Ct^{\nu(2)-1} \| \varphi \| \qquad \forall t \geq 1 \tag{3.41}$$

LEMMA 3.7. — $\forall i, j$ and $\forall t > 1$, there exists a constant C_{ij} s. t.

$$\sup_{z'_\alpha, q_\alpha, y_\alpha : y_\alpha \cdot q_\alpha > 0} | D^i_{z'_\alpha} D^j_{y_\alpha} (L_{2n}\mathscr{V}\mathscr{J})(0)(z'_\alpha, y_\alpha) | \leq C_{ij} t^{\nu(2n) - \frac{|i|+|j|}{2}} \tag{3.42}$$

Proof. — This will follow from the estimates:

$$| D^i_{z'_\alpha} D^j_{y_\alpha} (D^a_\omega \mathscr{V})(0) | \leq C_{ij} t^{\nu(|a|) - \frac{|i|+|j|}{2}}$$

$$| D^i_{z'_\alpha} D^j_{y_\alpha} (D^b_\omega \mathscr{J})(0) | \leq C_{ij} t^{\mu(|b|+1) - \frac{|i|+|j|}{2}} \qquad \forall t \geq 1$$

because the L. H. S. of (3.42) is then bounded by $t^{-\frac{|i|+|j|}{2}}$ times

$$\sup_{\substack{|a|+|b|=2n \\ |a|\geq 1}} C_{ij} t^{\nu(|a|)+\mu(|b|+1)} = C_{ij} t^{\nu(2n)}$$

But using [*17*, lemmas 3.6 and A.6] we easily see that it is enough to show:

$$| D^i_{z'_\alpha} D^j_{y_\alpha} D^a_{q_\alpha} V_\beta(\nabla_{q_\alpha} \omega^+_{\alpha,k}) |_{q_\alpha = \tilde{q}_\alpha} | \leq C_{ij} t^{\nu(|a|) - \frac{|i|+|j|}{2}} \tag{3.43}$$

$$| D^i_{z'_\alpha} D^j_{y_\alpha} D^b_{q_\alpha} \omega^+_\alpha (s, s+t; z'_\alpha, \tilde{q}_\alpha) | \leq C_{ij} t^{1+\mu(|b|-1) - \frac{|i|+|j|}{2}} \tag{3.44}$$

But one checks easily that for $|i| \geq 1$:

$$| D^i_{z'_\alpha y_\alpha} \tilde{q}_\alpha (z'_\alpha, y_\alpha) | \leq C |t|^{-1-\frac{|i|}{2}} \tag{3.45}$$

Thus using [*17*, lemma 3.6] again, (3.43-3.44) easily follow from proposition 2.9 (*iii*) and (3.45).

In order to complete the proof of lemma 3.5 (*i*), we split the integration support in t in (3.29) into

$$0 \leqslant t \leqslant R^{m(0)/2}$$

and
$$t \geqslant R^{m(0)/2}$$

For t in the first interval, we write

$$e^{itE_{\alpha,k}}(J_{\alpha,k}\varphi)(X) = \varphi_{\alpha,k}(x_\alpha) \int_{\substack{z'_\alpha \cdot k'_\alpha > 0 \\ |z'_\alpha| > R}} dz'_\alpha dk'_\alpha e^{i(\tilde{q}_\alpha \cdot y_\alpha - \omega_\alpha^+ \quad (s,s+t;z'_\alpha,\tilde{q}_\alpha))}$$
$$\langle \varphi_{\alpha,k} \otimes \eta_{z'_\alpha k'_\alpha}, \varphi \rangle \int dq_\alpha \sum_{\beta \neq \alpha} [V_\beta(y_\alpha) - V_\beta(\nabla_{q_\alpha}\omega_{\alpha,k}^+)] \times$$
$$\times\ e^{i(q_\alpha \cdot y_\alpha - \omega_\alpha^+ \quad (s,s+t;z'_\alpha,q_\alpha) - id(q_\alpha \to \tilde{q}_\alpha)} \hat{\eta}(q_\alpha - k'_\alpha)\, G'\left(\frac{3q_\alpha^2}{4} + E_{\alpha,k}\right)$$

and we bound the integral over q_α by $CR^{-m(0)}$ because of (3.33) which implies $|y_\alpha| > R/\sqrt{3}$ for $|s|$ large enough. Then using, proposition 4.1 again, $\| J_{\alpha,k}\varphi \| \leqslant CR^{-m(0)}$ for $t \in (0, R^{m(0)/2})$ and therefore

$$\int_0^{R^{m(0)/2}} dt \, \| J_{\alpha,k}\varphi \| \leqslant CR^{-m(0)/2}$$

which goes to zero as $R \to \infty$.

For t in the interval $(R^{m(0)/2}, \infty)$ we use the estimate (3.41) for the terms of the stationary-phase expansion, together with estimate $\| \varphi \| t^{-1-\varepsilon}$ for the rest, so that its contribution to $\int_0^\infty dt \, \| J_{\alpha,k}\varphi \|$ is bounded by

$$C \| \varphi \| \int_{R^{m(0)/2}}^\infty dt (t^{-1-\varepsilon} + t^{-1+\nu(2)})$$

which also goes to zero as $R \to \infty$ because $\nu(2) < 0$.

This completes the proof of lemma 3.5 (*i*) for the contribution of the last term in (3.31).

We now consider the contribution to (3.30)′ of the two first terms in (3.31), omitting about the factor $(1 + H_0)^{-1}$ in (3.2) that only deals with possible local singularities. Then, as operator norms, we have

$$\| V_\beta^s(x_\beta) F(|x_\alpha| < a(t+s+R)) F(|y_\alpha| > a(t+s+R)) \| \in L^1(dR)$$
$$\| F(|x_\alpha| < a(t+s+R)) F(|y_\alpha| > a(t+s+R)) (V_\beta(x_\beta) - V_\beta(y_\alpha)) P_{\alpha,k} \|$$
$$\leqslant C(t + s + R)^{-m(1)} \| x_\alpha \varphi_{\alpha,k} \|_{L^2(\mathbb{R}^\nu)}.$$

Therefore their contribution satisfies (3.29) because $m(1) > 1$, and because of properties $(P_3) \cup (P_5)$.

This completes the proof of lemma 3.5 (*i*).

The intuitive meaning of the sequences ρ_n and $\tau_n^{\pm}$ is that the particles described by the time-zero state ψ become more and more separated from each other along the time evolution labelled by $\tau_n^{\pm}$. Moreover we can get control over their relative kinetic energy, and their relative direction of flight:

PROPOSITION 3.8. — Let ψ be as in proposition 3.3 and $\tau_n^{\pm}$ be the sequences obtained in it. Then for any $f \in \mathscr{C}_0^\infty(\mathbb{R}^\nu)$ and any $\phi \in \mathscr{C}_0^\infty(\mathbb{R})$ we have:

i) $\| [\phi(\mathrm{H}) - \phi(\mathrm{H}_0)]e^{-i\tau_n^{\pm}\mathrm{H}}\psi \| \to 0 \quad \text{as} \quad n \to \infty$

ii) $\| [f(x_\alpha/\tau_n^{\pm}) - f(2p_\alpha)]e^{-i\tau_n^{\pm}\mathrm{H}}\psi \| \to 0 \quad \text{as} \quad n \to \infty$

iii) $\| [f(y_\alpha/\tau_n^{\pm}) - f(3q_\alpha/2)]e^{-i\tau_n^{\pm}\mathrm{H}}\psi \| \to 0 \quad \text{as} \quad n \to \infty$

iv) $\| [\phi(x_\alpha^2/4\tau_n^{\pm 2}) - \phi(h_\alpha)]e^{-i\tau_n^{\pm}\mathrm{H}}\psi \| \to 0 \quad \text{as} \quad n \to \infty$

v) Let $\mathrm{X} = (x_\alpha, y_\alpha)$ and $\mathrm{V} = \left(2p_\alpha, \dfrac{3q_\alpha}{2}\right)$, and let $\phi' \in \mathscr{C}_0^\infty(0, \infty)$. Then

$$\left\| \phi'\left(\left|\frac{\mathrm{X}}{\tau_n^{\pm}} - \mathrm{V}\right|\right)e^{-i\tau_n^{\pm}\mathrm{H}}\psi \right\| \to 0 \quad \text{as} \quad n \to \infty .$$

Proof. — As $| \mathrm{V}_\alpha(x_\alpha) | \leqslant \mathrm{C} | x_\alpha |^{-m(0)}$, for $| x_\alpha | \geqslant 1$ with $m(0) > 0$, we have for the increasing sequence ρ_n associated to $\tau_n^{\pm}$

$$\| (\mathrm{H}-\mathrm{H}_0)e^{-i\tau_n^{\pm}\mathrm{H}}\psi \| \leqslant \sum_\alpha (\| \mathrm{V}_\alpha \mathrm{F}(|x_\alpha| > \rho_n) \| + \mathrm{C} \| \mathrm{F}(|x_\alpha| < \rho_n)e^{-i\tau_n^{\pm}\mathrm{H}}\psi \|).$$

The second term goes to zero by proposition 3.2 and the first one is bounded by $\mathrm{C}\rho_n^{-m(0)}$ which also converges to zero as $n \to \infty$. The rest of the proof is similar to that of the short range case [*8*, see also 10] and we do not reproduce it here.

4. L²-BOUNDEDNESS OF SOME OSCILLATORY INTEGRALS

PROPOSITION 4.1. — (Fujiwara [*13*], see also [*3*]). Let $\mathscr{H}$ be an Hilbert space, with $\| . \|$ the norm in it, and $\mathscr{B}(\mathscr{H})$ the space of bounded operators on $\mathscr{H}$, and denote by $\mathrm{L}^2(\mathbb{R}^\nu, \mathscr{H})$ the space of square-integrable functions $\mathbb{R}^\nu \mapsto \mathscr{H}$, with norm $||| f ||| = (dx \| f(x) \|^2)^{1/2}$.

Let $f \in \mathrm{L}^2(\mathbb{R}^\nu, \mathscr{H})$, and $\mathrm{S} : \mathbb{R}^\nu \times \mathbb{R}^\nu \mapsto \mathscr{B}(\mathscr{H})$, $a \in \mathscr{C}^\infty(\mathbb{R}^\nu) \times \mathscr{C}^\infty(\mathbb{R}^\nu)$ be such that:

i) For any multiindices i, j, there exists a positive constant C_{ij} such that

$$|D_x^i D_y^j a(x, y)| \leqslant C_{ij}$$

ii) $S \in \mathscr{C}^\infty(\mathbb{R}^\nu \times \mathbb{R}^\nu, \mathscr{B}(\mathscr{H}))$, is selfadjoint, and $\nabla_x S$ and $\nabla_y S$ are multiples of the identity operator in $\mathscr{B}(\mathscr{H})$.

iii) $D_s(x, y) \equiv \text{Det}\left(\dfrac{\partial^2 S(x, y)}{\partial x^i \partial y^j}\right)$ is a complex valued function satisfying uniformly in $x, y \in \mathbb{R}^\nu \times \mathbb{R}^\nu$

$$|D_s(x, y)| \geqslant C > 0$$

iv) $\forall$ multiindices i and j, there exists a positive constant C_{ij} such that

$$|D_x^i D_y^j D_s(x, y)| \leqslant C_{ij}$$

Then $A(\lambda)$ defined by

$$(A(\lambda)f)(x) \equiv \int dy a(x, y) \exp [i\lambda S(x, y)] f(y)$$

is a bounded operator in $L^2(\mathbb{R}^\nu, \mathscr{H})$ with

$$|||A(\lambda)f||| \leqslant C'\lambda^{-\nu/2} |||f|||$$

$\forall f \in L^2(\mathbb{R}^\nu, \mathscr{H})$, $\lambda \geqslant 1$.

REMARK 4.1. — Fujiwara's papers [*3*] [*13*] only deal with the case $\mathscr{H} = \mathbb{C}$, but for further use we present here a slight extension where $S(x, y)$ is operator-valued in some Hilbert space $\mathscr{H}$, but its partial derivatives w. r. to x or y are complex valued.

Sketch of proof. — By a simple homogeneity argument ($x \mapsto \lambda^{1/2}x$, $y \mapsto \lambda^{1/2}y$, which yields $\lambda D_{s'} \mapsto D_s$), the case $\lambda \geqslant 1$ follows from the case $\lambda = 1$, that we consider below. Let $g_i (i \in \mathbb{Z})$ be the unit lattice points of $\mathbb{R}^\nu$, and $\{ g_i(z) \}_{i \in \mathbb{Z}}$ be a smooth partition of unity in $\mathbb{R}^\nu$ subordinate to the covering of open cubes of side 2 with center at g_i. We set:

$$a_{ij}(x, y) \equiv g_i(x) g_j(y) a(x, y) \tag{4.1}$$

$$(A_{ij} f)(x) \equiv \int dy a_{ij}(x, y) e^{iS(x,y)} f(y) \tag{4.2}$$

so that

$$A(1) \equiv A = \sum_{i,j} A_{ij} \tag{4.3}$$

Then, dut to the following lemma, it is enough to prove that

a) $$\sup_{i,j} |||A_{ij} f||| \leqslant C |||f|||$$

b) there exists a positive $h_{ij,i'j'}$ such that

$$||| A_{ij}A^*_{i'j'}f ||| \leqslant h^2_{ij,i'j'} ||| f ||| \tag{4.4}$$

$$||| A^*_{ij}A_{i'j'}f ||| \leqslant h^2_{ij,i'j'} ||| f ||| \tag{4.5}$$

$$\operatorname{Sup}_{ij} \sum_{i'j'} h_{ij,i'j'} \leqslant N < \infty \tag{4.6}$$

$$\operatorname{Sup}_{i'j'} \sum_{ij} h_{ij,i'j'} \leqslant N < \infty \tag{4.7}$$

LEMMA 4.2 (Cotlar-Stein) [4]. — Let A(z) be a weakly measurable, uniformly bounded operator-valued function $Z \mapsto \mathscr{H}$, where Z is a measure-space of measure dz, and $\mathscr{H}$ a separable Hilbert-space. Assume

$$\| A^*(z)A(z') \| \leqslant h(z, z')^2$$
$$\| A(z)A^*(z') \| \leqslant h(z, z')^2$$

where $h(z, z') \geqslant 0$ is the kernel of a bounded operator in $L^2(Z)$ of norm Z. Then

$$\left\| \int_E dz A(z) \right\| \leqslant N$$

uniformly in E, E being a finite measure subspace of Z.

As the integration support in (4.2) is fixed, up to a translation (*a*) easily follows from (*i*) and Schwarz inequality. On the other hand we have

$$\begin{aligned}(A^*_{i'j'}A_{ij})(x, y) &= \int dz \bar{a}_{i'j'}(z, x) a_{ij}(z, y) e^{i[S(z,y) - S(z,x)]} \\ &= g_{j'}(x) g_j(y) \int dz g_{i'}(z) g_i(z) \bar{a}(z, x) a(z, y) e^{i[S(z,y) - S(z,x)]}\end{aligned} \tag{4.8}$$

Let L be the following differential operator

$$L \equiv - i \frac{\nabla_z [S(z, y) - S(z\ x)] \cdot \nabla_z}{| \nabla_z [S(z, y) - S(z, x)] |^2} \tag{4.9}$$

It satisfies

$$(L - 1) \exp [i(S(z, y) - S(z, x))] = 0 \tag{4.10}$$

so that (4.8) becomes, for any entire number l :

$$\begin{aligned}(A^*_{i'j'}A_{ij})(x, y) &= g_{j'}(x) g_j(y) \int dz g_{i'}(z) g_i(z) \bar{a}(z, x) a(z, y) L^l \exp [i(S(z, y) - S(z, x))] \\ &= g_{j'}(x) g_j(y) \int dz e^{i[S(z,y) - S(z,x)]} (L^{*l})(g_{i'}(z)\, g_i(z) \bar{a}(z, x) a(z, y))\end{aligned} \tag{4.12}$$

where L* is the adjoint of the operator L:

$$L^* = L - i\left[\nabla_z, \frac{\nabla_z[S(z,y) - S(z,x)]}{|\nabla_z[S(z,y) - S(z,x)]|^2}\right] \tag{4.13}$$

But it is easy to check by induction, using (i), that

$$|(L^*)^l g_{i'}(z) g_i(z) \bar{a}(z,x) a(z,y)| \leqslant C'_l |\nabla_z[S(z,y) - S(z,x)]|^{-l}$$

where C'_l is a constant times $\sup_{x,y} \sup_{|i| \leqslant l} |D^i_x a(x,y)|$.

But under assumptions (ii) and (iv), (iii) is equivalent to the following: there exists a positive constant C such that

$$|\nabla_z[S(z,x) - S(z,y)]| \geqslant C|x-y| \tag{4.14}$$

$$|\nabla_z[S(x,z) - S(y,z)] \geqslant C|x-y| \tag{4.15}$$

uniformly in $x, y \in \mathbb{R}^\nu$ (by a simple use of the global implicit function theorem). Therefore for any integer l, there exists a positive constant C such that:

$$|(L^*)^l g_{i'}(z) g_i(z) \bar{a}(z,x) a(z,y)| \leqslant C_l |x-y|^{-l} \tag{4.16}$$

Let χ be the characteristic function of the set $|x| \leqslant 8\sqrt{n}$; then for $x \neq y$, we have, from (4.12) and (4.16):

$$\|(A^*_{i'j'} A_{ij})(x,y)\| \leqslant C_l \frac{q_{j'}(x) g_j(y)}{|x-y|^l} \chi(g_{i'} - g_i) \tag{4.17}$$

We shall now find a bound for the Hilbert-Schmidt norm of

$\|(A^*_{i'j'} A_{ij})(x,y)\|$ in $L^2(\mathbb{R}^\nu)$:

$$\int dxdy \|(A^*_{i'j'} A_{ij})(x,y)\|^2 \leqslant C_0 \chi(g_i - g_{i'}) \chi(g_j - g_{j'}) \cdot$$

$$\cdot \int_{|x-y|<4\sqrt{n}} dxdy g^2_{j'}(x) g^2_j(y) + C_l \chi(g_i - g_{i'}) \int_{|x-y|>4\sqrt{n}} dxdy g^2_{j'}(x) g^2_j(y)$$

$$\cdot 2^{2l} |g_j - g_{j'}|^{-2l} \leqslant C_l \chi(g_i - g_{i'})(1 + |g_j - g_{j'}|^{-2l}) \tag{4.18}$$

Similarly we have

$$\int \|(A_{i'j'} A^*_{ij})(x,y)\|^2 dxdy \leqslant C_l \chi(g_j - g_{j'})(1 + |g_i - g_{i'}|^{-2l}) \tag{4.19}$$

Thus we can choose

$$h_{ij,i'j'} = C_l\left(\chi(g_j - g_{j'})\sqrt{1 + |g_i - g_{i'}|^{-2l}} + (i,j) \mapsto (i',j')\right) \tag{4.20}$$

with $l > n$, which obviously satisfies (4.6-4.7). This completes the proof of proposition 4.1.

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