H. Roos

A note on Jordan automorphisms

<http://www.numdam.org/item?id=AIHPA_1985__42_1_17_0>

A Note on Jordan Automorphisms

by

H. ROOS

Institut für Theoretische Physik
der Universität Göttingen, Bunsenstrasse 9,
D-3400 Göttingen, Federal Republic of Germany

ABSTRACT. — Some statements about Jordan automorphisms including a structure theorem are proved, and a continuous one-parameter group of Jordan automorphisms is constructed which is not a group of *-automorphisms.

RÉSUMÉ. — On démontre quelques résultats sur les automorphismes de Jordan, en particulier un théorème de structure, et on construit un groupe continu à un paramètre d'automorphismes de Jordan qui n'est pas un groupe de *-automorphismes.

I

A Jordan automorphism α of a C*-algebra $\mathscr{A}$ is a bijective *-preserving linear map of $\mathscr{A}$ onto itself which respects anticommutators:

$$\alpha(xy + yx) = \alpha(x)\alpha(y) + \alpha(y)\alpha(x).$$

Groups of Jordan automorphisms show up in physics when in an algebraic framework the time evolution of a quantum system is considered in the Schrödinger picture [1, 2, 3]. In these cases $\mathscr{A}$ is a von Neumann algebra. For any Jordan automorphism α of a von Neumann algebra acting on a Hilbert space $\mathscr{H}$, there is a projection E such that

$$\alpha(AB)(1 - E) = \alpha(B)\alpha(A)(1 - E),$$

i. e. α looks like an antimorphism on $(1 - E)\mathscr{H}$. The composition of two antimorphisms is a morphism; thus one might be inclined to conclude that any one-dimensional group $\{\alpha_t\}_{t\in\mathbb{R}}$ of Jordan automorphisms is actually a group of common *-automorphisms, because $\alpha_t = \alpha_{t/2} \circ \alpha_{t/2}$. This is not true in general, although it holds in many cases of interest in physics. Actually, a group of Jordan automorphisms is exhibited in [4] although in a somewhat implicit form. Evidently, this is not well known among physicists.

The aim of this note is to prove some statements (including a structure theorem) about Jordan automorphisms and—based upon these results— give an explicit construction of a continuous one-parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ of Jordan automorphisms which are not *-automorphisms.

Section II contains some statements on Jordan automorphisms which are needed in the sequel and might have some interest in their own. Section III is concerned with the structure of Jordan automorphisms. In Section IV a genuine group of Jordan automorphisms is exhibited. It is conjectured that this construction can be generalized to yield groups of Jordan automorphisms of fairly arbitrary von Neumann algebras.

II

Let us recall some known results on Jordan automorphisms [1, 5].

By J-Aut $(\mathscr{A})$ we denote the set of all Jordan automorphisms of $\mathscr{A}$. Since we are mainly interested in Jordan automorphisms of von Neumann algebras we shall not give the most general statements.

PROPOSITION 2.1. — Let $\mathscr{M}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$, and $\alpha \in$ J-Aut$(\mathscr{M})$.

   i) α is order preserving.

   ii) α is an isometry.

   iii) There is a projection E in the centre $\mathscr{Z}(\mathscr{M})$ of $\mathscr{M}$ such that for all A, B $\in \mathscr{M}$

$$\alpha(AB) = \alpha(A)\alpha(B)E + \alpha(B)\alpha(A)(1 - E). \tag{1}$$

The proof can be found e. g. in [5, Th. 3.2.3]. A projection $E \in \mathscr{Z}(\mathscr{M})$ satisfying (1) will be called a decomposing projection of α. Note that $\alpha \in$ J-Aut$(\mathscr{M})$ implies

$$\alpha^{-1} \in \text{J-Aut}(\mathscr{M}). \tag{2}$$

COROLLARY 2.2. — The mapping $A \in \mathscr{M} \mapsto \alpha(A)$ is continuous with respect to the ultraweak topology.

This is a consequence of Prop. 2.1 (i) and of (2): α is a positive normal map, hence ultraweakly continuous [6, Th. I.4.2].

We shall now prove some minor results concerning decomposing projections of Jordan automorphisms, which will be used subsequently.

Let the von Neumann algebra $\mathcal{M}$ act on the Hilbert space $\mathcal{H}$. Then we define the subspace $\mathcal{H}_0$ and the projection E_c onto $\mathcal{H}_0$ by

$$\mathcal{H}_0 = E_c\mathcal{H} := \{ \Psi \in \mathcal{H} ; [A, B]\Psi = 0 \quad \text{for all} \quad A, B \in \mathcal{M} \}. \qquad (3)$$

$[A, B] = AB - BA$ denotes the commutator, E_c is a central projection: clearly, $\mathcal{M}'\mathcal{H}_0 \subset \mathcal{H}_0$; furthermore, we have for all $C \in \mathcal{M}$ and $\Psi \in \mathcal{H}_0$

$$[A, B]C\Psi = A[B, C]\Psi + [AC, B]\Psi = 0,$$

hence $\mathcal{M}\mathcal{H}_0 \subset \mathcal{H}_0$ and thus

$$E_c \in \mathscr{Z}(\mathcal{M}). \qquad (4)$$

LEMMA 2.3. — Consider $\alpha \in$ J-Aut($\mathcal{M}$).

i) $\mathscr{Z}(\mathcal{M})$ is invariant under α.

ii) There exists a unique maximal decomposing projection $E_m \equiv E_m[\alpha]$; a decomposing projection E of α is maximal if and only if $E_c \leq E$.

iii) $\alpha(E_c) = E_c$.

iv) If α and β are Jordan automorphisms of $\mathcal{M}$ with decomposing projections E_α and E_β, respectively, then $\alpha \circ \beta \in$ J-Aut($\mathcal{M}$) with decomposing projection

$$E_{\alpha \circ \beta} = E_\alpha \alpha(E_\beta) + (1 - E_\alpha)\alpha(1 - E_\beta), \qquad (5)$$

and maximality of E_α and E_β implies maximality of $E_{\alpha \circ \beta}$.

As is easily seen, $\alpha^{-1}(E_\alpha)$ is a decomposing projection for α^{-1}; and, as a consequence of (iv), we have

$$E_m[\alpha^{-1}] = \alpha^{-1}(E_m[\alpha]). \qquad (6)$$

Note that in general $\mathscr{Z}(\mathcal{M})$ is not pointwise invariant.

Proof. — Part i) is Lemma (14) of [2].

ii) Let E be any decomposing projection of α. On $E_c\mathcal{H}$, a morphism cannot be distinguished from an antimorphism, hence

$$E_m[\alpha] := E + E_c - EE_c \in \mathscr{Z}(\mathcal{M}) \qquad (7)$$

defines a decomposing projection for α. Now assume that we could find a decomposing projection E' which is not majorized by $E_m[\alpha]$. Then there is a $\Psi \in E'(1 - E_m)\mathcal{H}, \Psi \neq 0$, and for all $A, B \in \mathcal{M}$ we have

$$\alpha(AB)\Psi = \alpha(AB)(1 - E_m)\Psi = \alpha(B)\alpha(A)\Psi$$

and

$$\alpha(AB)\Psi = \alpha(AB)E'\Psi = \alpha(A)\alpha(B)\Psi.$$

Since α maps $\mathcal{M}$ onto $\mathcal{M}$, it follows that Ψ is in $\mathcal{H}_0$, thus $E'(1 - E_m) \le E_c$ in contradiction to $1 - E_m \le 1 - E_c$ which holds by definition (7). Hence E_m is maximal and unique. By the same reasoning one shows that any decomposing projection E' containing E_c has to be maximal, i. e. equal to E_m; otherwise $E_m(1 - E')\mathcal{H}$ would contain a $\Psi \ne 0$ giving rise to a contradiction.

iii) According to (2), we have $\alpha^{-1}([A, B])E_c = [\alpha^{-1}(A), \alpha^{-1}(B)]E_c = 0$ for all A, $B \in \mathcal{M}$, hence $0 = \alpha(\alpha^{-1}([A, B]E_c)) = [A, B]\alpha(E_c)$ (note that $\alpha(AC) = \alpha(A)\alpha(C)$ if $C \in \mathcal{Z}$), and thence $\alpha(E_c) \le E_c$. Using α^{-1} instead of α we get $\alpha^{-1}(E_c) \le E_c$ and therefore $\alpha(E_c) = E_c$ because α is order preserving.

iv) Obviously, $\alpha \circ \beta$ is linear, *-preserving and bijective. $E_{\alpha \circ \beta}$ as defined above is a decomposing projection of $\alpha \circ \beta$ as is seen by straightforward calculation. According to (*ii*), E_c is contained in $E_m[\alpha]$ and in $E_m[\beta]$; therefore, due to (*iii*), $E_c = \alpha(E_c) \le \alpha(E_m[\beta])$ and $E_c \le E_m[\alpha]\alpha(E_m[\beta])$. Since $E_c(1 - E_m[\alpha]) = 0$ it follows that $E_c \le$ r.h.s. of (5), and, again using (*ii*), we get (5). $\square$

Now let $t \mapsto \alpha_t$ be a map of the additive group $\mathbb{R}$ into J-Aut($\mathcal{M}$), and assume that the map $t \mapsto \alpha_t(A)$ is weakly continuous. It is possible to choose decomposing projections E_t of α_t, $t \in \mathbb{R}$, such that E_t is ultrastrongly continuous in t. This holds especially for $E_m[\alpha_t]$:

LEMMA 2.4. — If $t \to \alpha_t(A)$, $\alpha_t \in$ J-Aut($\mathcal{M}$), is weakly continuous, then $t \mapsto E_m[\alpha_t]$ is ultrastrongly continuous.

Proof. — See Lemma (22) of [2]; the group property is not needed. $\square$

In the sequel we shall always choose $E_t = E_m[\alpha_t]$; $\{\alpha_t\}_{t \in \mathbb{R}}$ will always denote a one-parameter group of Jordan automorphisms. Applying Lemma 2.3 (*iv*) we then get

$$E_{t+t'} = E_t\alpha_t(E_{t'}) + (1 - E_t)\alpha_t(1 - E_{t'}). \tag{8}$$

The decomposing projections E_t cannot be independent of t lest the α_t be common *-automorphisms:

LEMMA 2.5. — Assume that the decomposing projections E_α of a group G of Jordan automorphisms do not depend on $\alpha \in$ G, for all $\alpha \ne id$. If G contains at least three elements then G is a group of *-automorphisms.

This is intuitively clear; but reasoning by intuition, as referred to in the introduction, should be confined to this case.

A formal proof can be easily given by application of Lemma 2.3 (*iv*).

<div align="center">III</div>

There are theorems by Connes [7] and Haagerup [8], see also [5, Th. 3.2.15], and by Bratteli and Robinson [5, Th. 3.2.18] clarifying

the structure of Jordan automorphisms of a von Neumann algebra with a cyclic and separating vector. A related result is the following structure theorem, which applies to von Neumann algebras with an α-invariant vector state given by a cyclic vector which, however, need not be separating. It also allows to deal with C*-algebras with an invariant state.

THEOREM 3.1. — Let $\mathcal{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$, $\alpha \in$ J-Aut($\mathcal{M}$). Assume that $\Omega \in \mathcal{H}$ is cyclic for $\mathcal{M}$ and defines an α-invariant state:

$$(\Omega, \alpha(A)\Omega) = (\Omega, A\Omega) \quad \text{for all} \quad A \in \mathcal{M}. \tag{9}$$

Let E be a decomposing projection of α. Then $\check{E} := \alpha^{-1}(E)$ is a decomposing projection for α^{-1}; and there exist partial isometries $\overset{1}{V}$ and $\overset{2}{V}$, where $\overset{1}{V}$ is linear, $\overset{2}{V}$ is antilinear, such that

$$\overset{1}{V}\overset{1}{V}{}^* = E, \qquad \overset{2}{V}\overset{2}{V}{}^* = 1 - E, \tag{10}$$

$$\overset{1}{V}{}^*\overset{1}{V} = \check{E}, \qquad \overset{2}{V}{}^*\overset{2}{V} = 1 - \check{E}, \tag{11}$$

and for all $A \in \mathcal{M}$

$$\alpha(A) = \overset{1}{V}A\overset{1}{V}{}^* + \overset{2}{V}A^*\overset{2}{V}{}^*, \tag{12}$$

$$\alpha^{-1}(A) = \overset{1}{V}{}^*A\overset{1}{V} + \overset{2}{V}{}^*A^*\overset{2}{V}. \tag{13}$$

Furthermore,

$$\overset{1}{V}\Omega = E\Omega, \qquad \overset{2}{V}\Omega = (1 - E)\Omega. \tag{14}$$

The case of a group of Jordan automorphisms is dealt with by the following

PROPOSITION 3.2. — Consider a one-parameter group $\{\alpha_t\}_{t\in\mathbb{R}} \subset$ J-Aut ($\mathcal{M}$) for which the cyclic vector Ω defines an invariant state of $\mathcal{M}$, then the isometries $\overset{1}{V}_t$ and $\overset{2}{V}_t$ which, according to Theorem 3.1, exist for all $t \in \mathbb{R}$, fulfil the relations

$$\overset{1}{V}_t^* = \overset{1}{V}_{-t}, \, \overset{2}{V}_t^* = \overset{2}{V}_{-t}, \tag{15}$$

$$\overset{1}{V}_t\overset{1}{V}_{t'} + \overset{2}{V}_t\overset{2}{V}_{t'} = \overset{1}{V}_{t+t'}, \tag{16}$$

$$\overset{1}{V}_t\overset{2}{V}_{t'} + \overset{2}{V}_t\overset{1}{V}_{t'} = \overset{2}{V}_{t+t'}, \tag{17}$$

provided the E_t are chosen to be maximal.

The maximality of E_t implies $E_0 = 1$, $\overset{1}{V}_0 = 1$ and $\overset{2}{V}_0 = 0$. There is a partial converse to Theorem 3.1:

THEOREM 3.3. — Given two partial isometries $\overset{1}{V}$ and $\overset{2}{V}$ on a Hilbert space $\mathscr{H}$, $\overset{1}{V}$ linear, $\overset{2}{V}$ antilinear, which fulfil

$$\overset{1}{V}\overset{1}{V}{}^* + \overset{2}{V}\overset{2}{V}{}^* = 1, \qquad \overset{1}{V}{}^*\overset{1}{V} + \overset{2}{V}{}^*\overset{2}{V} = 1. \qquad (18)$$

Let $\mathscr{B}$ (resp. $\check{\mathscr{B}}$) denote the von Neumann algebra of all operators on $\mathscr{H}$ commuting with the final (resp. initial) projection $E = \overset{1}{V}\overset{1}{V}{}^*$ (resp. $\check{E} = \overset{1}{V}{}^*\overset{1}{V}$) of $\overset{1}{V} : \mathscr{B} = \{ E \}'$, $\check{\mathscr{B}} = \{ \check{E} \}'$.
Then

$$\gamma(A) := \overset{1}{V}A\overset{1}{V}{}^* + \overset{2}{V}A^*\overset{2}{V}{}^*, \qquad A \in \check{\mathscr{B}} \qquad (19)$$

and

$$\check{\gamma}(A) := \overset{1}{V}{}^*A\overset{1}{V} + \overset{2}{V}{}^*A^*\overset{2}{V}, \qquad A \in \mathscr{B} \qquad (19')$$

define Jordan isomorphisms from $\check{\mathscr{B}}$ (resp. $\mathscr{B}$) onto $\mathscr{B}$ (resp. $\check{\mathscr{B}}$) with decomposing projections $E \in \mathscr{Z}(\mathscr{B})$ (resp. $\check{E} \in \mathscr{Z}(\check{\mathscr{B}})$) and

$$\gamma^{-1} = \check{\gamma}, \qquad (20)$$
$$E = \gamma(\check{E}), \qquad \check{E} = \check{\gamma}(E). \qquad (21)$$

If $\mathscr{M} \subset \mathscr{B} \cap \check{\mathscr{B}}$ is a von Neumann algebra which is mapped onto itself by application of γ, then $\alpha := \gamma \restriction_{\mathscr{M}}$ is a Jordan automorphism of $\mathscr{M}$. If furthermore eqs. (14) are fulfilled for a vector $\Omega \in \mathscr{H}$, then Ω defines an α-invariant state.

With the above definitions of E and $\check{E}$, condition (18) is evidently equivalent to eqs. (10) and (11).
There is also an analogue to Proposition 3.2:

PROPOSITION 3.4. — Given a family $\{ \overset{1}{V_t}, \overset{2}{V_t} \}_{t \in \mathbb{R}}$ of isometries, $\overset{1}{V_t}$ linear, $\overset{2}{V_t}$ antilinear, such that

$$\overset{1}{V_t}\overset{1}{V_t}{}^* + \overset{2}{V_t}\overset{2}{V_t}{}^* = 1, \qquad \overset{1}{V_t}{}^*\overset{1}{V_t} + \overset{2}{V_t}{}^*\overset{2}{V_t} = 1, \qquad (22)$$

and such that eqs. (15) to (17) hold.
Assume that for arbitrary $t \in \mathbb{R}$

$$\alpha_t(A) := \overset{1}{V_t}A\overset{1}{V_t}{}^* + \overset{2}{V_t}A^*\overset{2}{V_t}{}^* \qquad (23)$$

defines a map of a von Neumann algebra $\mathscr{M}$ onto itself. Then the projections $E_t := \overset{1}{V_t}\overset{1}{V_t}{}^*$ fulfil eq. (8), and $\{ \alpha_t \}_{t \in \mathbb{R}}$ is a one-parameter group of Jordan automorphisms of $\mathscr{M}$.

Proof of Theorem 3.1. — We define $\overset{1}{V}$ and $\overset{2}{V}$ by

$$\overset{1}{V}A\Omega = \alpha(A)E\Omega, \qquad A \in \mathcal{M}, \tag{24}$$

$$\overset{2}{V}A\Omega = \alpha(A^*)(1-E)\Omega, \qquad A \in \mathcal{M}. \tag{25}$$

$\overset{1}{V}$ (resp. $\overset{2}{V}$) is a well-defined linear (resp. antilinear) operator because $A\Omega = 0$ implies $\alpha(A)E\Omega = 0$ and $\alpha(A^*)(1-E) = 0$:

$$\| \alpha(A)E\Omega \|^2 = (E\Omega, \alpha(A^*A)E\Omega) = (\Omega, \alpha(A^*A\alpha^{-1}(E))\Omega) = (\Omega, A^*A\alpha^{-1}(E)\Omega) = 0$$

if $A\Omega = 0$. Here we need assumption (9). Analogously:

$$\| \alpha(A^*)(1-E)\Omega \|^2 = (\Omega, \alpha(A)\alpha(A^*)(1-E)\Omega) = (\Omega, \alpha(A^*A)(1-E)\Omega) = \ldots = 0.$$

Eqs. (24) and (25) define $\overset{1}{V}$ and $\overset{2}{V}$ on $\mathscr{D} = \mathcal{M}\Omega$. According to eq. (9), we have
$\| A\Omega \|^2 = (\Omega, \alpha(A^*A)\Omega) = (\Omega, \alpha(A)^*\alpha(A)E\Omega) + (\Omega, \alpha(A)\alpha(A)^*(1-E)\Omega)$, hence

$$\| A\Omega \|^2 = \| \overset{1}{V}A\Omega \|^2 + \| \overset{2}{V}A\Omega \|^2. \tag{26}$$

Thus $\overset{1}{V}$ and $\overset{2}{V}$ are bounded and can be extended to $\overline{\mathscr{D}} = \mathscr{H}$ (Ω is assumed to be cyclic !). Next we want to show

$$\overset{1}{V}{}^*B\Omega = \alpha^{-1}(B)\check{E}\Omega, \qquad B \in \mathcal{M}, \tag{27}$$

$$\overset{2}{V}{}^*B\Omega = \alpha^{-1}(B^*)(1 - \check{E})\Omega, \qquad B \in \mathcal{M}. \tag{28}$$

Again we need the invariance assumption (9), the cyclicity of Ω, and $\check{E} = \alpha^{-1}(E) \in \mathscr{Z}(\mathcal{M})$:

$$\begin{aligned}
(B\Omega, \overset{1}{V}A\Omega) &= (\Omega, B^*\alpha(A)E\Omega) \\
&= (\Omega, \alpha(\alpha^{-1}(B^*)A)E\Omega) = (\Omega, \alpha(\alpha^{-1}(B^*)A\alpha^{-1}(E))\Omega) \\
&= (\Omega, \alpha^{-1}(B^*)A\check{E}\Omega) = (\alpha^{-1}(B)\check{E}\Omega, A\Omega),
\end{aligned}$$

hence (27). Analogously, eq. (28) is proved.

By combining eqs. (24), (25), (27) and (28) it is a straightforward task to show eqs. (10) and (11).

In the same way, we demonstrate (12) and (13): Applying (27) and (24) we get $\overset{1}{V}A\overset{1}{V}{}^*B\Omega = \overset{1}{V}A\alpha^{-1}(B)\check{E}\Omega = \alpha(A\alpha^{-1}(B)\check{E})E\Omega = \alpha(A)B\alpha(\check{E})E\Omega = \alpha(A)EB\Omega$, where we used $E = \alpha(\check{E}) \in \mathscr{Z}(\mathcal{M})$; and therefore,

$$\overset{1}{V}A\overset{1}{V}{}^* = \alpha(A)E. \tag{29}$$

From (28) and (25) it follows that

$$\overset{2}{V}A\overset{2}{V}{}^* = \alpha(A^*)(1 - E). \tag{30}$$

Eqs. (29) and (30) imply (12). Similarly, we get

$$\overset{1}{V}{}^*A\overset{1}{V} = \alpha^{-1}(A)\check{E}, \qquad \overset{2}{V}{}^*A\overset{2}{V} = \alpha^{-1}(A^*)(1 - E)$$

and hence (13).

Eqs. (14) are special cases of (24) resp. (25).   □

Proof of Proposition 3.2. — We use $E_t = \alpha_t(E_{-t})$, which holds for maximal E_t, see eq. (8). Since $\overset{1}{V}_tA\Omega = \alpha_t(A)E_t\Omega$, we have

$$(B\Omega, \overset{1}{V}_tA\Omega) = (\Omega, \alpha_t(\alpha_{-t}(B^*))\alpha_t(A)E_t\Omega)$$
$$= (\Omega, \alpha_t(\alpha_{-t}(B^*)A)E_t, \Omega) = (\Omega, \alpha_t(\alpha_{-t}(B^*)AE_{-t})\Omega).$$

Due to the invariance of $(\Omega, \cdot\Omega)$, we get

$$(B\Omega, \overset{1}{V}_tA\Omega) = (\alpha_{-t}(B)E_{-t}\Omega, A\Omega) = (\overset{1}{V}_{-t}B\Omega, A\Omega)$$

and thus $\overset{1}{V}_t{}^* = \overset{1}{V}_{-t}$. In the same manner, we find $(B\Omega, \overset{2}{V}_tA\Omega) = (A\Omega, \overset{2}{V}_{-t}B\Omega)$ and therefore $\overset{2}{V}_t{}^* = \overset{2}{V}_{-t}$ (remember that $\overset{2}{V}_t$ is antilinear).

Next we calculate

$$\overset{1}{V}_t\overset{1}{V}_{t'}A\Omega = \overset{1}{V}_t\alpha_{t'}(A)E_{t'}\Omega = \alpha_t(\alpha_{t'}(A)E_{t'})E_t\Omega = \alpha_{t+t'}(A)\alpha_t(E_{t'})E_t\Omega, \qquad (31)$$

and analogously

$$\overset{2}{V}_t\overset{2}{V}_{t'}A\Omega = \alpha_{t+t'}(A)\alpha_t(1 - E_{t'})(1 - E_t)\Omega. \qquad (32)$$

Adding (31) and (32) and using eq. (8) we arrive at

$$(\overset{1}{V}_t\overset{1}{V}_{t'} + \overset{2}{V}_t\overset{2}{V}_{t'})A\Omega = \alpha_{t+t'}(A)E_{t+t'}\Omega = V_{t+t'}A\Omega,$$

hence eq. (16). The proof of (17) is left to the reader.   □

We need two further relations for $\overset{1}{V}$ and $\overset{2}{V}$:

LEMMA 3.5. — *i*) Eqs. (10) and (11) imply

$$\overset{1}{V}A\overset{2}{V}{}^* = 0 \qquad \text{for all} \qquad A \in \check{\mathscr{B}} = \{\check{E}\}',$$
$$\overset{1}{V}{}^*A\overset{2}{V} = 0 \qquad \text{for all} \qquad A \in \mathscr{B} = \{E\}'.$$

In the setting of Theorem 3.1, both relations hold for all $A \in \mathscr{M}$.

ii) Eqs. (10), (11) and (14) imply

$$\overset{1}{V}{}^*\Omega = \check{E}\Omega, \qquad \overset{2}{V}{}^*\Omega = (1 - \check{E})\Omega.$$

Proof. — Trivial, since eqs. (10) and (11) imply

$$\overset{1}{V} = \overset{1}{V}\check{E} \qquad \text{and} \qquad \overset{2}{V}{}^* = (1 - \check{E})\overset{2}{V}{}^*. \qquad (33)$$

Proof of Theorem 3.3. — We start noting that eqs. (21) are easily derived from (10) and (11) resp. (33) with the help of Lemma 3.5 (*i*) with A = 1.

It is obvious that γ and $\check{\gamma}$ are linear *-preserving maps which also preserve anticommutators. γ and $\check{\gamma}$ are injective: again using Lemma (3.5) (*i*) one sees that $\check{\gamma}(\gamma(A)) = \check{E}A\check{E} + (1 - \check{E})A(1 - \check{E})$, hence $\check{\gamma}(\gamma(A)) = A$ if $A \in \{ \check{E} \}' = \check{\mathscr{B}}$,

$$\gamma : \check{\mathscr{B}} \rightarrow \gamma(\check{\mathscr{B}}) \quad \text{is injective,} \quad \text{and} \quad \check{\gamma} = \gamma^{-1} \quad \text{on} \quad \gamma(\check{\mathscr{B}}). \quad (34)$$

Similarly,

$$\check{\gamma} : \mathscr{B} \rightarrow \check{\gamma}(\mathscr{B}) \quad \text{is injective,} \quad \check{\gamma}^{-1} = \gamma \quad \text{on} \quad \check{\gamma}(\mathscr{B}). \quad (35)$$

As is easily seen, E (resp. $\check{E}$) commutes with all elements of $\gamma(\check{\mathscr{B}})$ (resp. $\check{\gamma}(\mathscr{B})$), that is, we have

$$\gamma(\check{\mathscr{B}}) \subset \{ E \}' = \mathscr{B}, \qquad \check{\gamma}(\mathscr{B}) \subset \{ \check{E} \}' = \check{\mathscr{B}}. \quad (36)$$

Eqs. (34) to (36) imply that $\gamma(\check{\mathscr{B}}) = \mathscr{B}$, and that γ is a bijective map from $\check{\mathscr{B}}$ onto $\mathscr{B}$, hence a Jordan isomorphism. Clearly, $E \in \{ E \}' = \mathscr{B}$; and $E \in \gamma(\check{\mathscr{B}})' = \mathscr{B}'$ according to (36), hence $E \in \mathscr{Z}(\mathscr{B})$. Similarly one shows that $\check{E} \in \mathscr{Z}(\check{\mathscr{B}})$. By direct computation it follows that E (resp. $\check{E}$) are decomposing projections of γ (resp. $\check{\gamma}$).

It is now clear that $\gamma\!\restriction_{\mathscr{M}} \in$ J-Aut($\mathscr{M}$) if γ maps $\mathscr{M} \subset \mathscr{B} \cap \check{\mathscr{B}}$ onto itself. E need not be in the center of $\mathscr{M}$, but it is in $\mathscr{B}' \subset \mathscr{M}'$.

Lemma 3.5 (*ii*) suffices to demonstrate the last assertion of the theorem.   □

Proof of Proposition 3.4. — It is clear from Theorem 3.3 that $\alpha_t \in$ J-Aut($\mathscr{M}$) for fixed t. Eqs. (15) and (16) imply $\overset{1}{V_0} = 1$, $\overset{2}{V_0} = 0$, hence $\alpha_{t=0} = id.$; it remains to be shown that the E_t fulfil (8) and that $\alpha_t \circ \alpha_{t'} = \alpha_{t+t'}$. This can be verified by direct calculation with the help of Lemma 3.5 (*i*).   □

As already mentioned, mainly the case of a von Neumann algebra is of interest in physical applications of Jordan automorphisms. Nevertheless let us briefly consider the case of a C*-algebra $\mathscr{A}$ with a Jordan automorphism α.

PROPOSITION 3.6. — Let $\mathscr{A}$ be a C*-algebra, $\alpha \in$ J-Aut($\mathscr{A}$). Assume that there is an α-invariant state ω on $\mathscr{A}$. Then one can define a Jordan automorphism α^ω on the von Neumann algebra $\mathscr{B}_\omega = \pi_\omega(\mathscr{A})''$ generated by the representation π_ω associated with ω such that

$$\alpha^\omega(\pi_\omega(x)) = \pi_\omega(\alpha(x)) \qquad \text{for all} \qquad x \in \mathscr{A}. \quad (37)$$

Subsequently, the above structure theorem can be applied.

Proof. — Consider the universal representation π_u of $\mathscr{A}$. π_u is faithful, hence

$$\alpha_u : \pi_u(x) \rightarrow \pi_u(\alpha(x)) \quad (38)$$

is well-defined, and $\alpha_u \in$ J-Aut($\pi_u(\mathscr{A})$). It can be extended to a Jordan automorphism on $\mathscr{B} = \pi_u(\mathscr{A})''$ which will also be denoted by α_u (see e. g. Lemma (2.4) of [1] and the remarks proceeding it). The state ω gives rise to a corresponding vector state φ_Ω on $\mathscr{B}$, $\omega(A) = \varphi_\Omega(\pi_u(A))$, and clearly, φ_Ω is α_u-invariant. Define E_Ω to be the projection onto $\overline{\mathscr{B}\Omega}$, and let F_Ω be its central support. Due to the invariance of φ_Ω, F_Ω is α_u-invariant (Lemma (15) of [2]). Hence we have $\alpha_u(AF_\Omega) = \alpha_u(A)F_\Omega$, $A \in \mathscr{B}$, thence $\alpha_u \upharpoonright_{\mathscr{B}F} \in$ J-Aut($\mathscr{B}F_\Omega$); by the isomorphism $\mathscr{B}F_\Omega \simeq \mathscr{B}_\omega$, there is a corresponding Jordan automorphism α^ω of $\mathscr{B}_\omega$. Eq. (37) is then a consequence of (38). $\square$

IV

The aim of this section is to give an explicit construction of a group of genuine Jordan automorphisms.

Theorem 3.3 suggests a way of generating a Jordan isomorphism: choose a projection E, a unitary operator U and a conjugation J (i. e. $\exists$ antilinear, $J = J^*$, $J^2 = 1$) such that E and J commute. Then $\overset{1}{V} := EU$ and $\overset{2}{V} := (1 - E)JU$ fulfil the assumption (18) of Theorem 3.3. If moreover E is in the centre of a von Neumann algebra $\mathscr{M}$, and $U\mathscr{M}U^* = \mathscr{M}$, $J\mathscr{M}J \subset \mathscr{M}$, then $\alpha(A) = \overset{1}{V}A\overset{1}{V}^* + \overset{2}{V}A^*\overset{2}{V}^*$ defines a Jordan automorphism of $\mathscr{M}$ which is not a *-automorphism if $E_c \leq E \neq 1$, where E_c is defined by eq. (3).

In order to find a group of Jordan automorphisms one has to choose central projections E_t for all t in such a way that in addition eq. (8) is fulfilled:

$$E_{t+t'} = E_t\alpha_t(E_{t'}) + (1 - E_t)\alpha_t(1 - E_{t'}). \tag{8}$$

This is of course the essential difficulty. We shall proceed as follows:

1. Choose an abelian von Neumann algebra $\mathscr{Z}$, a family $\{E_t\}_{t\in\mathbb{R}}$ of projections in $\mathscr{Z}$, and a group $\{\alpha_t^0\}_{t\in\mathbb{R}} \subset$ Aut($\mathscr{Z}$) such that (8) is fulfilled with α_t^0 in place of α_t.

2. Find a von Neumann algebra $\mathscr{M}$ such that $\mathscr{Z} = \mathscr{Z}(\mathscr{M})$ and such that α_t^0 extends to $\alpha_t^U \in$ Aut($\mathscr{M}$), where α_t^U is unitarily implemented by U_t; $U_{t=0} = 1$, and $U_{t+t'} = U_tU_{t'}$.

3. Find a conjugation J fulfilling

$$JE_t = E_tJ, \qquad JU_t = U_tJ, \qquad J\mathscr{M}J \subset \mathscr{M}. \tag{39 a, b, c}$$

4. Define

$$\overset{1}{V}_t := E_tU_t, \qquad \overset{2}{V}_t := (1 - E_t)JU_t \tag{40}$$

and

$$\alpha_t(A) = \overset{1}{V}_tA\overset{1}{V}_t^* + \overset{2}{V}_tA^*\overset{2}{V}_t^*, \qquad A \in \mathscr{M}; \tag{41}$$

finally, make sure, that

$$E_c \le E_t \ne 1 \tag{42}$$

for at least one value of t.

We claim that $\{\alpha_t\}_{t \in \mathbb{R}}$ is a non-trivial group of Jordan automorphisms of $\mathcal{M}$. Assume we have accomplished steps 1. to 4. In order to show that $\{\alpha_t\}_{t \in \mathbb{R}}$ is a group of Jordan automorphisms we shall use Proposition 3.4. It is then clear that not all α_t can be automorphisms due to (42) and Lemma 2.3 (ii). Assumption (22) of Prop. 3.4 is readily checked for $\overset{1}{V}_t$ and $\overset{2}{V}_t$ defined above. Because of (39 a) we can write eq. (8) as

$$E_{t+t'} = E_t U_t E_{t'} U_t^* + (1 - E_t) U_t (1 - E_{t'}) U_t^* . \tag{43}$$

Since $U_0 = 1$, it follows that $E_0 = 1$, and thus $\overset{1}{V}_0 = E_0 U_0 = 1$, $\overset{2}{V}_0 = 0$. With $t' = -t$, eq. (43) reads

$$1 = E_t U_t E_{-t} U_t^* + (1 - E_t) U_t (1 - E_{-t}) U_t^* .$$

Multiplication from the left (resp. from the right) by $U_t^* E_t$ (resp. $U_t E_{-t}$) yields $U_t^* E_t = U_t^* (E_t U_t E_{-t}) U_t^*$ and $U_t E_{-t} = E_t U_t E_{-t}$. Inserting the second equation into the first one and using $U_t^* = U_{-t}$ we arrive at $\overset{1}{V}_t^* = U_t^* E_t = E_{-t} U_t^* = \overset{1}{V}_{-t}$. Similarly, it follows that $\overset{2}{V}_t^* = \overset{2}{V}_{-t}$; thus eqs. (15) hold. It is straightforward to check the validity of (16) and (17) using (39 a, b) and (43). Thus Proposition 3.4 is applicable. It remains to make sure that α_t maps $\mathcal{M}$ onto itself. But this is clear: since $U_t A U_t^* = \alpha_t^U(A)$, $\alpha_t^U \in \mathrm{Aut}(\mathcal{M})$, $E_t \in \mathcal{Z}(\mathcal{M})$ and $J \mathcal{M} J \subset \mathcal{M}$, α_t maps $\mathcal{M}$ into itself, and due to the group property which follows from (16) and (17), we have $\alpha_t^{-1} = \alpha_{-t}$, thus α_t is bijective.

Now let us present a concrete example. It will be evident from the definitions that E_t and U_t and hence α_t depend continuously on t.

1. Let $\mathcal{Z} = \{ T_f ; f \in \mathscr{L}^\infty(\mathbb{R}, dx) \}$ denote the set of multiplication operators on the complex Hilbert space $\mathscr{H}^0 = \mathscr{L}^2(\mathbb{R}, dx)$: $(T_f \varphi)(x) = f(x)\varphi(x)$. $\mathcal{Z}$ is a maximal abelian algebra, i. e. $\mathcal{Z}' = \mathcal{Z}$. The translation operators $U_t^0 : \varphi(x) \in \mathscr{H}^0 \mapsto \varphi(x - t)$ are unitary, they implement an automorphism group $\{\alpha_t^0\}$ of $\mathcal{Z}$. The projections E_t are defined as follows:

$$1 - E_t = T_{\chi_{I(0,t)}} ,$$

where χ_I denotes the characteristic function of the intervall I, and

$$I(t_1, t_2) = \{ x \in \mathbb{R} ; \min (t_1, t_2) < x < \max (t_1, t_2) \}$$

It is easy although somewhat lengthy to calculate

$$E_t \alpha_t^0 (1 - E_{t'}) = T_{\chi_{J(t,t')}} \quad \text{and} \quad (1 - E_t) \alpha_t^0 (1 - E_{t'}) = T_{\chi_{K(t,t')}} ,$$

and check that

$$\chi_{I(0,t)} + \chi_{J(t,t')} - \chi_{K(t,t')} = \chi_{I(0,t+t')}$$

holds, hence

$$1 - E_{t+t'} = (1 - E_t) + E_t\alpha_t^0(1 - E_{t'}) - (1 - E_t)\alpha_t^0(1 - E_{t'})$$

which is equivalent to eq. (8).

2. Now we consider $\mathscr{H} = \mathscr{H}^0 \otimes \mathscr{H}^1$, $\mathscr{H}^1 = \mathscr{L}^2(\mathbb{R}, dy)$ and define $\mathscr{M} := \mathscr{L} \otimes \mathscr{B}(\mathscr{H}^1)$. Consequently, $\mathscr{M}' = \mathscr{L}' \otimes 1 = \mathscr{L} \otimes 1$, and $\mathscr{L}(\mathscr{M}) \simeq \mathscr{L}$.
We define $U_t := U_t^0 \otimes 1$, $\alpha_t^U(A) := U_t A U_t^*$, $A \in \mathscr{M}$, thus α_t^U is — up to an isomorphism — an extension of α_t^0, and $\alpha_t^U \in \mathrm{Aut}(\mathscr{M})$.

3. The conjugation J is taken to be $J : \psi(x, y) \to (J\psi)(x, y) = \overline{\psi(x, y)}$, where the bar denotes complex conjugation. It is evident that the conditions (39, a, b, c) are satisfied.

4. $\{ \alpha_t \}$, as defined by (40) and (41) is the desired group of Jordan automorphisms. We claim that

$$E_c = 0 ; \qquad (44)$$

hence the E_t are maximal; and since $E_t \neq 1$ if $t \neq 0$, α_t is not an automorphism. Finally, let us demonstrate (44): Any vector $\Psi \in \mathscr{H}$ can be written as $\Psi = \sum_{i,j} c_{ij}\varphi_j^0 \otimes \varphi_j^1$, where $\{ \varphi_i^0 \}_{i \in \mathbb{N}}$ (resp. $\{ \varphi_j^1 \}_{j \in \mathbb{N}}$) is an orthonormal basis of $\mathscr{H}^0$ (resp. $\mathscr{H}^1$). If $\Psi \neq 0$ there is a $c_{kl} = (\varphi_k^0 \otimes \varphi_l^1, \Psi) \neq 0$. Let $P_l \in \mathscr{B}(\mathscr{H}^1)$ be the projection onto φ_l^1. Since $\dim \mathscr{H}^1 = \infty$, there is an isometry V with initial projection 1 and final projection $1 - P_l$, hence $[V^*, V] = V^*V - VV^* = P_l$ and thus $1 \otimes P_l = [1 \otimes V^*, 1 \otimes V]$ is an element of the commutator set $\mathscr{C}(\mathscr{M}) := \{ [A, B]; A, B \in \mathscr{M} \}$. Due to our choice of k and l, $(\varphi_k^0 \otimes \varphi_l^1, 1 \otimes P_l\Psi) \neq 0$, therefore $\Psi \notin E_c\mathscr{H}$ (remember that $\Psi \in E_c\mathscr{H}$ if and only if $C\Psi = 0$ for all $C \in \mathscr{C}(\mathscr{M})$). Since Ψ was arbitrarily chosen, E_c must vanish.

It is rather plausibel that the above construction can be generalized in several ways. Having in mind the central decomposition of a von Neumann algebra on a separable Hilbert space we are led to the conjecture that any such von Neumann algebra which is neither abelian nor a factor and which has a sufficiently large (non-discrete) centre admits a group of genuine Jordan automorphisms.

After having finished this paper I learned that a rather similar construction was given by G. A. Raggio [9]. (In his paper the maximality of the decomposing projections is not proven). To my knowledge, it has not been published.

REFERENCES

[1] R. V. Kadison, *Topology*, t. **3**, Suppl. 2, 1965, p. 177.

[2] H. Roos, *Physica*, t. **100 A**, 1980, p. 183.

[3] G. L. Sewell, *Lett. Math. Phys.*, t. **6**, 1982, p. 209.

[4] O. Bratteli and D. W. Robinson, *Ann. Inst. H. Poincaré*, t. **25** (A), 1976, p. 139.

[5] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics.* New York, Springer Verlag, 1979.

[6] J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien.* Paris, Gauthier-Villars, 1969.

[7] A. Connes, *Ann. Inst. Fourier*, Grenoble, t. **24**, 1974, p. 121.

[8] U. Haagerup, *Math. Scand.*, t. **37**, 1975, p. 271 and Preprint Copenhagen n° 15, 1973.

[9] G. A. Raggio, *Groups of Jordan- *-Automorphisms on W*-Algebras.* Preprint Zürich, 1980.