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Rotation numbers for diffeomorphisms and flows

by

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ABSTRACT. — A rotation number is defined for fairly general diffeomorphisms in 2 dimensions and flows in 3 dimensions. An extension to higher dimensionality is sometimes also possible (in particular for Hamiltonian flows).

RÉSUMÉ. — On définit un nombre de rotation pour des difféomorphismes à deux dimensions et des flots à trois dimensions assez généraux. Une extension à des dimensions supérieures est également possible dans certains cas (en particulier pour des flots hamiltoniens).

The present note was written in 1982, but not published at the time. It was felt that the results were relatively trivial, and piecewise essentially known already to the experts (as resulted from discussions with J. Moser and B. Kostant). Recent conversations have convinced me that it would be useful to publish the note after all, without claim that the results contained in it are very deep or original.

If ρ is an ergodic probability measure for a diffeomorphism or flow, it is possible to define *characteristic exponents* $\lambda_i(\rho)$ by use of the multiplicative ergodic theorem (1). We shall exhibit low dimensional situations where it is also possible to define a rotation number $\underset{\cdot}{R}(\rho)$ (2). This definition is based on an ergodic theorem for products of 2×2 real matrices, which we first discuss (3).

(1) See Oseledec [*1*], Raghunathan [*2*], Ruelle [*3*].

(2) The original definition of a rotation number (by H. Poincaré) was for orientation preserving homeomorphisms of the circle.

(3) An extension to higher dimension is given in the Appendix.

1. AN ERGODIC THEOREM FOR THE COVERING GROUP OF $SL_2(\mathbb{R})$

Let $G = SL_2(\mathbb{R})$, $\tilde{G}$ be the universal covering group of G, and $p : \tilde{G} \mapsto G$ be the canonical map. We identify the positive matrices with a subset of $\tilde{G}$. If $A \in \tilde{G}$, we write $A = U(\Theta(A)) \mid pA \mid$, where $\mid \cdot \mid$ is the absolute value and $U(\Theta)$ is the rotation by $\Theta \in \mathbb{R}$.

THEOREM. — *Let* (M, ρ) *be a probability space, and* $f : M \mapsto M$ *a map preserving* ρ. *Let also* $T : M \mapsto \tilde{G}$ *be measurable and such that* $\Theta(T(\cdot)) \in L^1(\rho)$.
If we use the notation

$$T_x^N = T(f^{N-1}x) \dots T(fx)T(x)$$

then

$$\lim_{N \to \infty} \frac{1}{N} \Theta(T_x^N) = \omega(x) \tag{1}$$

exists for ρ*-almost all* x, *and is* f*-invariant. Writing also*

$$\omega(\rho) = \int \rho(dx)\omega(x)$$

we have

$$\omega(\rho) = \lim_{N \to \infty} \frac{1}{N} \int \rho(dx)\Theta(T_x^N) \tag{2}$$

If ρ *is ergodic, then* ω *is* ρ*-almost everywhere equal to* $\omega(\rho)$.
When M *is compact and* f, T *continuous,* $\omega(\rho)$ *depends continuously on* ρ *for the vague topology, and on* f, T *for the uniform topologies.*

The above results follow readily from the ordinary ergodic theorem. Notice that if $A, B, \in \tilde{G}$, then

$$\mid \Theta(BA) - \Theta(A) - \Theta(B) \mid < \pi$$

[This is because $BA = U(\Theta(A) + \Theta(B))[U(-\Theta(A)) \mid pB \mid U(\Theta(A))] \mid pA \mid$ and $\mid \Theta(QP) \mid < \pi$ if P, Q are positive]. Therefore, if $N = km + r$, with $m > 0$, $k > 0$, and $0 \leq r < m$,

$$\mid \Theta(T_x^n) - \Theta(T_x^m) - \Theta(T_{f^m x}^m) - \dots - \Theta(T_{f^{km}x}^r) \mid < k\pi$$

The ergodic theorem shows that, for ρ-almost all x,

$$\frac{1}{km} [\Theta(T_x^m) + \dots + \Theta(T_{f^{(k-1)m}x}^m)]$$

has a limit (for each m). Since $\Theta(T^r) \in L^1(\rho)$, one also has, for ρ-almost all x,

$$\frac{1}{N} \Theta(T_{f^{km}x}^r) \to 0$$

$\left[\text{if } \Phi \in \mathrm{L}^1, \ \lim_{k\to\infty} \frac{1}{k} (\Phi(x) + \ldots + \Phi(f^{k-1}x)) = \lim_{k\to\infty} \frac{1}{k} (\Phi(x) + \ldots + \Phi(f^k x))\right.$ so that $\left.\lim_{k\to\infty} \frac{1}{k} \Phi(f^k x) = 0\right]$. From these facts (1) and (2) follow. The f-invariance of the limit is immediate, and also $\omega(\cdot) = \omega(\rho)$ ρ-almost everywhere if ρ is ergodic.

Write $\Theta_N = \int \rho(dx)\Theta(R_x^N)$. Then $|\Theta_{m+N} - \Theta_m - \Theta_N| < \pi$ so that $\frac{1}{N}\Theta_N \to \omega(\rho)$ uniformly with respect to ρ and T when

$$\sup_{x\in M} \| T(x) \|$$

is bounded. Therefore $\omega(\rho)$ depends continuously on ρ and T in the topological case.

Remark. — Even though the above theorem deals with matrices, it is of an easier nature than the multiplicative ergodic theorem. On the other hand we can show that $\omega(\rho)$ depends continuously on ρ, f, T, while the characteristic exponents are in general discontinuous.

2. ROTATION NUMBERS FOR 2-DIMENSIONAL DIFFEOMORPHISMS

Let f be a diffeomorphism of a two dimensional manifold M and ρ an f-invariant probability measure with compact support. If f is isotopic to the identity on the support of ρ [4] and if supp ρ has a parallelizable neighbourhood $\mathcal{N}$ [5], there is a natural definition of T such that $p\mathrm{T}(x)$ is the matrix representing the tangent map $\mathrm{T}f(x)$ in a trivialization of $\mathrm{T}\mathcal{N}$. The rotation number $\omega(\rho)$ is then defined by the theorem, and is continuous with respect to ρ for the vague topology and f for the C^1 topology.

Remarks. — *a*) If $M = \mathbb{R}^2$, the trivialization of M is unique. In general there may be several different trivializations, leading to different rotation numbers (that is the case for instance if M is an annulus).

b) There is also some non uniqueness in the choice of the isotopy between the identity and f. If supp ρ is connected, the ambiguity is an additive $k.2\pi$ in $\omega(\rho)$, and this may be eliminated by reducing $\omega(\rho)$ mod 2π.

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(4) We assume that there is a map $\alpha \to f_\alpha$ defined on $[0, 1]$, continuous for the C^1 topology of f_α, such that $T_x f_\alpha$ is invertible for $x \in \operatorname{supp} \rho$, and f_0 = identity, $f_1 = f$.

(5) This means that the tangent bundle is trivial: $\mathrm{T}\mathcal{N} \simeq \mathcal{N} \times \mathbb{R}^2$. This is true in particular if $\mathcal{N}$ is a disk or an annulus.

c) Suppose that we only know that f is a differentiable map, and that Tf is invertible outside a set of zero measure. It is still possible then to define $\omega(\rho)$ by making a discontinuous choice of T. (The continuity properties of $\omega(\rho)$ are then lost).

d) The usual concept of rotation number of a map F of the circle is recovered by extending F to a map of the annulus.

e) We may define a rotation number for the map $x \mapsto ax(1-x)$ of $[0, 1]$ so that $T(x)=0$ if $x < 1/2$ and $T(x) = \pi$ if $x > 1/2$ (according to (*c*) we assume that $\rho(\{1/2\}) = 0$).

f) If ρ is ergodic, with characteristic exponents $\lambda_1 > 0$, $\lambda_2 > 0$, one may interpret $\omega(\rho)$ as the average rotation angle of the stable or unstable direction along an orbit. (This fact may be used to define $\omega(\rho)$, but the continuity properties are then not obvious).

3. ROTATION NUMBERS FOR 3-DIMENSIONAL FLOWS

Let (f^t) be a flow without fixed point on the three dimensional manifold M, and ρ an (f^t)-invariant probability measure with compact support. Assume that supp ρ has a neighborhood $\mathcal{N}$ with parallelization compatible with the flow. By this we mean that the tangent bundle has a trivialization $T\mathcal{N} \simeq \mathcal{N} \times (\mathbb{R}^2 \times \mathbb{R})$ which sends the direction of the flow at x into $\{0\} \times \mathbb{R}$. There is then a natural definition of T^t such that $pT^t(x)$ is obtained from the matrix representing Tf^tx in the trivialization $\mathbb{R}^2 \times \mathbb{R}$ by taking the quotient by the factor $\mathbb{R}$. The *rotation number* $\omega(\rho)$ is then defined by the theorem, with (1) replaced by the continuous time version

$$\lim_{t\to\infty} \frac{1}{t}\,\Theta(T_x^t) = \omega(x) \tag{3}$$

This rotation number is continuous with respect to ρ for the vague topology and f the C^1 topology. As in the case of diffeomorphisms one can define $\omega(\rho)$ in more general situations, at the expense of loosing the continuity properties.

4. HIGHER DIMENSION

The above discussion of diffeomorphisms in 2 dimensions and flows in 3 dimensions does not extend straightforwardly to higher dimensionality. An extension is possible, however, if Tf is symplectic, or symplectic times a scalar. This includes the cases discussed above, and also the case of a Hamiltonian flow when there is a trivialization of the tangent bundle compatible with the flow. We give the abstract theorem in the appendix, and leave the details of application to diffeomorphisms and flows to the reader.

APPENDIX

ROTATION NUMBER FOR SYMPLECTIC MATRICES

A real matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ acting on $\mathbb{R}^n \oplus \mathbb{R}^n$ is symplectic if $M^T J M = J$, where $J = \begin{pmatrix} & I \\ -I & \end{pmatrix}$. This can be rewritten

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} & I \\ -I & \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} & I \\ -I & \end{pmatrix}$$

or

$$A^T C - C^T A = 0, \qquad A^T D - C^T B = I, \qquad B^T D - D^T B = 0 .$$

The symplectic matrices constitute the real symplectic group $Sp(n, \mathbb{R})$. Identifying $\mathbb{R}^n \oplus \mathbb{R}^n$ with $\mathbb{C}^n$ by $x \oplus y \to z = x + iy$, we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix}^T \begin{pmatrix} & I \\ -I & \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x' \cdot y - y' \cdot x$$
$$= \operatorname{Im}(z'^* \cdot z)$$

In particular, if $U = V + iW$ is unitary on $\mathbb{C}^n$, then $V^T V + W^T W = I$, $V^T W - W^T V = 0$ and $\begin{pmatrix} V & -W \\ W & V \end{pmatrix}$ is symplectic.

If M is symplectic, then M^T and $R = M^T M$ are also symplectic, so that $JR = R^{-1} J$ and $JP(R) = P(R^{-1})J$ for every polynomial P. Letting $P \to \sqrt{\ }$, we find that $S = (M^T M)^{1/2}$ is again symplectic. In the polar decomposition $M = US$, the matrices S, U are thus symplectic, S is positive and U corresponds to a unitary matrix on $\mathbb{C}^N$.

Let $\tilde{G}$ be the universal covering group of $Sp(n, \mathbb{R})$ and $p : \tilde{G} \mapsto Sp(n, \mathbb{R})$ the canonical projection. Positive symplectic matrices can be identified naturally to elements of $\tilde{G}$ because $t \mapsto S_t = S^t$ is continuous on $[0, 1]$, with $S_0 = I$ and $S_1 = S$. Therefore, the elements of G have a polar decomposition

$$\tilde{M} = \tilde{U} S$$

such that $(p\tilde{M}) = (p\tilde{U})S$ is the polar decomposition of $p\tilde{M}$. Since $\tilde{G}$ is simply connected, there is a unique continuous function $\Theta : \tilde{G} \to \mathbb{R}$ such that

$$\Theta(I) = 0$$

and

$$\det p\tilde{U} = e^{i\Theta(M)}$$

where $p\tilde{U}$ is considered as $n \times n$ complex matrix. Let S_1 and S_2 be positive symplectic matrices, and $S_2 S_1 = \tilde{M} = \tilde{U} S$. If u is an eigenvector of $\tilde{U}$, the transformation $\tilde{U} = S_2 S_1 S^{-1}$ rotates u by less than $3\pi/2$ (because $(v, Sv) > 0$ for any vector v and positive S). Therefore

$$| \Theta(S_2 S_1) | < \frac{3n\pi}{2} .$$

If $\tilde{M}_1, \tilde{M}_2 \in \tilde{G}$ we can write

$$\tilde{M}_2 \tilde{M}_1 = \tilde{U}_2 S_2 \tilde{U}_1 S_1 = \tilde{U}_2 \tilde{U}_1 (\tilde{U}_1^{-1} S_2 \tilde{U}_1) S_1$$

where $\tilde{U}_1^{-1}S_2\tilde{U}_1$ is positive, so that

$$(\tilde{U}_1^{-1}S_2\tilde{U}_1)S_1 = \tilde{U}S$$

with $|\,\Theta(\tilde{U})\,| < \dfrac{3n\pi}{2}$. Therefore

$$\Theta(\tilde{M}_2\tilde{M}_1) = \Theta(\tilde{U}_2\tilde{U}_1\tilde{U})$$
$$= \Theta(\tilde{M}_2) + \Theta(\tilde{M}_1) + \Theta(\tilde{U})$$

and finally

$$|\,\Theta(\tilde{M}_2\tilde{M}_1) - \Theta(\tilde{M}_2) - \Theta(\tilde{M}_1)\,| < \frac{3n\pi}{2} \tag{4}$$

PROPOSITION. — *Let (Ω, ρ) be a probability space, $f: \Omega \mapsto \Omega$ a measure preserving transformation, and $T: \Omega \mapsto \tilde{G}$ a measurable function such that*

$$\int |\,\Theta(T(x))\,|\, dx < \infty \tag{5}$$

Define $T_x^N = T(f^{N-1}x) \dots T(fx)T(x)$ then

$$\omega(x) = \lim_{N\to\infty} \frac{1}{N}\Theta(T_x^N) \tag{6}$$

exists ρ-almost everywhere and is f-invariant. Writing also

$$\omega(\rho) = \int \rho(dx)\omega(x)$$

we have

$$\left|\frac{1}{N}\int \rho(dx)\Theta(T_x^N) - \omega(\rho)\right| < \frac{1}{N}\frac{3n\pi}{2} \tag{7}$$

If ρ is ergodic, then ω is ρ-almost everywhere equal to $\omega(\rho)$.

When Ω is compact and f, T continuous, $\omega(\rho)$ depends continuously on ρ for the vague topology, and on f, T for the uniform topologies.

We shall derive this result from the ordinary ergodic theorem. If $N = km + r$, with $m > 0$, $k > 0$, and $0 \le r < m$, (4) yields

$$|\,\Theta(T_x^N) - \Theta(T^m(x)) - \Theta(T^m(f^m x)) - \dots - \Theta(T^r(f^{km}x))\,| < \frac{3n\pi}{2}$$

where we have written $T_x^m = T^m(x)$ for typographical reasons. The ergodic theorem shows that, for ρ-almost all x,

$$\frac{1}{km}[\Theta(T^m(x)) + \dots + \Theta(T^m(f^{(k-1)m}x))]$$

has a limit (for each m). Since $\Theta(T^r(.)) \in L^1(\rho)$ by (5) and (4) we have, for ρ-almost all x,

$$\frac{1}{N}\Theta(T^r(f^{km}x)) \to 0$$

$\Big[$if $\Phi \in L^1$, $\lim\limits_{k\to\infty}\dfrac{1}{k}(\Phi(x) + \dots + \Phi(f^{k-1}x)) = \lim\limits_{k\to\infty}\dfrac{1}{k}(\Phi(x) + \dots + \Phi(f^k x))$ so that $\lim\limits_{k\to\infty}\dfrac{1}{k}\Phi(f^k x) = 0\Big]$. From these facts (6) follows. The f-invariance of the limit is immediate.

Write $\Theta_N = \int \rho(dx)\Theta(T_x^N)$, so that $N^{-1}\Theta \to \omega(\rho)$. We have $|\,Q_{P-Q} - \Theta_P - \Theta_Q\,| < 3n\pi/2$, and therefore also $\left| \frac{1}{\mathrm{NP}} \Theta_{\mathrm{NP}} - \frac{1}{\mathrm{N}} \Theta_{\mathrm{N}} \right| < \frac{\mathrm{P}-1}{\mathrm{NP}} \frac{3n\pi}{2}$, and (7) follows. The uniformity of the convergence implies that $\omega(\rho)$ depends continuously on ρ, f, T in the topological case.

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