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On the uniqueness problem for quite full logics

by

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ABSTRACT. — We present an example of a quite full logic that possesses two distinct bounded observables not being distinguishable by their expectations. This answers the uniqueness problem presented by S. Gudder [2].

RÉSUMÉ. — On donne un exemple de logique « tout à fait complète » (cf. Définition 4) possédant deux observables bornés distincts qu'on ne peut distinguer par leurs valeurs moyennes. Cela répond au problème d'unicité posé par S. Gudder [2].

Let (L, M) be a quantum logic. In the centre of our interest there are observables, i. e. homomorphisms from the σ-algebra $\mathscr{B}(\mathrm{R})$ of the Borel subsets of a real line to L. We can consider an observable to be a « generalized random variable » on L. An observable x is bounded if there is a bounded set $\mathrm{B} \in \mathscr{B}(\mathrm{R})$ such that $x(\mathrm{B}) = 1$. The value $m(x) = \int_{\mathrm{R}} \lambda m[x(d\lambda)]$ is called an expectation of an observable x in the state $m \in \mathrm{M}$. A natural question has arisen: if two bounded observables have the same expectation in any state $m \in \mathrm{M}$, are then those observables equal? It is known [2, 5] that the answer is « yes » for two important examples of logics, viz. for Boolean σ-algebras and for lattices of closed subspaces of a Hilbert space. Of course, this question (called the uniqueness problem or the uniqueness condition) is reasonable only if M is a « relatively rich » set. It might be if

(L, M) was a quite full logic. For such logics, the problem was originally published [*2*]. Some partial results have occured that have brought several sufficient or necessary conditions on a logic (L, M) to fulfil the uniqueness condition [*3, 6, 8, 11*]. The general question of quite full logics have remained open up to now. In this paper we will present an example of a quite full logic (L, M) and two bounded observables x, y on L such that $m(x) = m(y)$ for any $m \in \mathrm{M}$. This example shows up that a quite full logic does not possess the uniqueness condition in general.

We start with basic definitions (a detailed exposition may be found in [*5*]).

DEFINITION 1. — Let $\mathrm{L} = (\mathrm{L}, \leqslant, ')$ be an orthomodular σ-orthocomplete poset (abbr. OMP), M a set of probability (σ-additive) measures on L. A pair (L, M) is called a logic, elements of M are called states on L.

DEFINITION 2. — An observable is a mapping $x : \mathscr{B}(\mathrm{R}) \to \mathrm{L}$ from the σ-algebra $\mathscr{B}(\mathrm{R})$ of the Borel subsets of the real line to L such that

i) $x(\mathrm{R}) = 1$,
ii) $x(\mathrm{R} - \mathrm{A}) = x(\mathrm{A})'$,
iii) $x\left(\bigcup_{i=1}^{\infty} \mathrm{A}_i\right) = \bigvee_{i=1}^{\infty} x(\mathrm{A}_i)$ for any collection $\{ \mathrm{A}_i \in \mathscr{B}(\mathrm{R}) \mid i \in \mathrm{N} \}$.

An observable x is called bounded if there is a bounded set $\mathrm{A} \in \mathscr{B}(\mathrm{R})$ such that $x(\mathrm{A}) = 1$.

Let (L, M) be a logic and let x be a bounded observable on L. Take a state $m \in \mathrm{M}$. Then $m_x = m \circ x$ is a probability measure on $\mathscr{B}(\mathrm{R})$ and the integral $m(x) = \int_{\mathrm{R}} \lambda \cdot m[x(d\lambda)] = \int_{\mathrm{R}} \lambda \cdot m_x(d\lambda)$ exists.

DEFINITION 3. — The value $m(x)$ is called an expectation of an observable x in the state m. We say that a logic (L, M) possesses the condition U (uniqueness) if any two bounded observables x, y are equal whenever the equality $m(x) = m(y)$ is fulfiled for any $m \in \mathrm{M}$.

There are examples of logics that allow only one state or even that do not possess any state at all [*1, 7, 12*]. The question of possessing the condition U is senseless for such logics, but they have also very little meaning as regards applications to quantum mechanics. Therefore there is usually some condition added to the definition of a logic (L, M) such that the set M is then « rich enough ». Most frequently used condition is that of a quite full logic.

DEFINITION 4. — A logic (L, M) is called quite full provided the following statement holds true for any $a, b \in \mathrm{L}$: if $m(a) = 1$ implies $m(b) = 1$ for any $m \in \mathrm{M}$ then $a \leqslant b$.

The uniqueness problem for observables was formulated at first for quite full logics as early as 1966 [*2*]. Some partial results have appeared up to now but the original problem has not been solved in spite of being repeated more times [*3*, *4*, *5*, *6*]. We would like to present the most important results along this line.

THEOREM 1. — Let x, y be observables on a quite full logic (L, M) and let there is a set $B \in \mathscr{B}(R)$ such that B has at most one limit point and $x(B) = 1$. Suppose $m(x) = m(y)$ for any $m \in M$. Then $x = y$.

Proof. — See [*2*] or [*5*].

THEOREM 2. — Let a logic (L, M) satisfies the following conditions

i) for any $a \in L$ there is $m \in M$ such that $m(a) = 1$,

ii) for any $a, b \in L$, $a \nleqslant b'$ and for any $\varepsilon \in (0, 1)$ there is $m \in M$ such that $m(a) \geqslant 1 - \varepsilon$, $m(b) \geqslant 1 - \varepsilon$.

If x, y are bounded observables on (L, M) and $m(x) = m(y)$ for any $m \in M$ then $x = y$.

Proof. — See [*11*] or (in a slightly weaker form) [*8*].

The connections between quite full logics and logics satisfying the assumption of Theorem 2 are derived in [*9*, *10*]. In [*6*] S. Gudder introduced examples resulting in the following statement: if we weaken the assumption that (L, M) is a quite full logic, then (L, M) need not possess the condition U. We give an example of a quite full logic that does not fulfil the condition U.

DEFINITION 5. — An atom in a logic (L, M) is such an element $p \in L$ that for any $q \in L$, $q \leqslant p$ it holds $q = 0$ or $q = p$.

EXAMPLE 1. — Let Z be the set of all integers. Let an OMP L consists of exactly two distinct blocks (maximal Boolean sub σ-algebras), generated by the sets of atoms $\{ a_i \mid i \in Z \}$ and $\{ b_i \mid i \in Z \}$ respectively. Let x, y be two observables on L defined by $x(1/2^i) = a_{-i}$, $x(1) = a_0$, $x(2 - 1/2^i) = a_i$, $y(1/2^i) = b_{-i}$, $y(1) = b_0$, $y(2 - 1/2^i) = b_i$, $i \in N$.

For $j \in Z$ define states m_1, m_2, m_3 as follows (a state is defined by its values on all atoms): $m_1(a_j) = m_1(b_j) = 1$, $m_2(a_j) = 1$, $m_2(b_{j+i}) = 1/2^i$, $m_3(a_j) = 1$, $m_3(b_{j-i}) = 1/2^i$, $i \in N$. These states vanish for the other atoms. It holds $m_1(x) = m_1(y)$, $m_2(x) < m_2(y)$, $m_3(x) > m_3(y)$. There exists $\varepsilon \in (0, 1/2)$ (even rational) such that if we set $m_\varepsilon = 1/2m_1 + \varepsilon m_2 + (1/2 - \varepsilon)m_3$ then $m_\varepsilon(x) = m_\varepsilon(y)$. Denote the state m_ε by m_j^a. It holds $m_j^a(a_j) = 1$, $m_j^a(a_i) = 0$ for $i \neq j$ and $m_j^a(b_i) \neq 0$, $m_j^a(b_i) \neq 1$ for any $i \in Z$. Analogously we construct m_j^b such that $m_j^b(b_j) = 1$, $m_j^b(b_i) = 0$ for $i \neq j$ and $m_j^b(a_i) \in (0, 1)$ for any $i \in Z$ and then the states m_i^a, m_i^b for any $i \in Z$. Denote $M = \{ m_i^a, m_i^b \mid i \in Z \}$.

It is obvious that $m(x) = m(y)$ for any $m \in M$. We shall show that (L, M) is a quite full logic. Let $m \in M$, $c \in L$ and $m(c) = 1$. Then either $c = 1$ or there exists an atom $d \leqslant c$ such that $m(d) = 1$. Suppose that $c_1, c_2 \in L$ and that $m(c_1) = 1$ implies $m(c_2) = 1$ for any $m \in M$. If $c_1 = 1$ then $m(c_1) = 1 = m(c_2)$ for any $m \in M$ and hence $c_2 = 1$, $c_1 = c_2$. If $c_1 \neq 1$, it holds either $c_1 = \bigvee_{i \in I} a_i$ or $c_1 = \bigvee_{i \in I} b_i$, $I \subseteq Z$. Suppose the former case (in the latter one the proof proceeds dually). Then

$$\{ m \in M \mid m(c_1) = 1 \} = \{ m_i^a \mid i \in I \} .$$

If $c_2 = 1$ then trivially $c_1 \leqslant c_2$. We may thus assume that $c_2 \neq 1$. Then $m_i^a(c_2) = 1$ implies $a_i \leqslant c_2$. As $m_i^a(c_2) = 1$ for any $i \in I$, it holds

$$c_1 = \bigvee_{i \in I} a_i \leqslant c_2 ,$$

what was to prove. We have constructed a quite full logic (L, M) that does fulfil the condition U.

If m_1, m_2 are states on L, then $m = \alpha m_1 + (1 - \alpha)m_2$, $\alpha \in (0, 1)$ is again a state on L. A state m is called pure if the expression $m = \alpha m_1 + (1 - \alpha)m_2$, $\alpha \in (0, 1)$ implies $m = m_1 = m_2$. The previous example can be improved in such a way that M is a set of pure states. First we present a lemma which may be useful in other situations as well.

LEMMA 3. — Let a rational number ε, $0 < \varepsilon < 1$ be given. Then there exists a logic (L, M) and two elements $a, b \in L$ such that

i) $m(a) + m(b) \leqslant 1 + \varepsilon$ for any $m \in M$,
ii) there is a (pure) state $m \in M$ such that $m(a) = 1$, $m(b) = \varepsilon$,
iii) if $u, v \in L$, $(u, v) \neq (a, b)$ then there is a (pure) state $m \in M$ such that $m(u) = 1 = m(v)$.

Proof. — We will construct such logics using Greechie's representation. We suppose the reader to be familiar with the interpretation of such diagrams (an exposition of their construction and interpretation can be found in [1]).

Let a rational number $\varepsilon = p/q$, $p, q \in N$, $p < q$ be given. Denote (L, M) a logic depicted in Fig. 1, let M be the set of all states on L. Then

$$\sum_{i=1}^{q} \sum_{j=1}^{p} c_{ij} \leqslant p \quad \text{and} \quad \sum_{i=1}^{q} \sum_{j=1}^{p} c_{ij} + \sum_{i=1}^{q} d_i + \sum_{i=1}^{q} e_i = q .$$

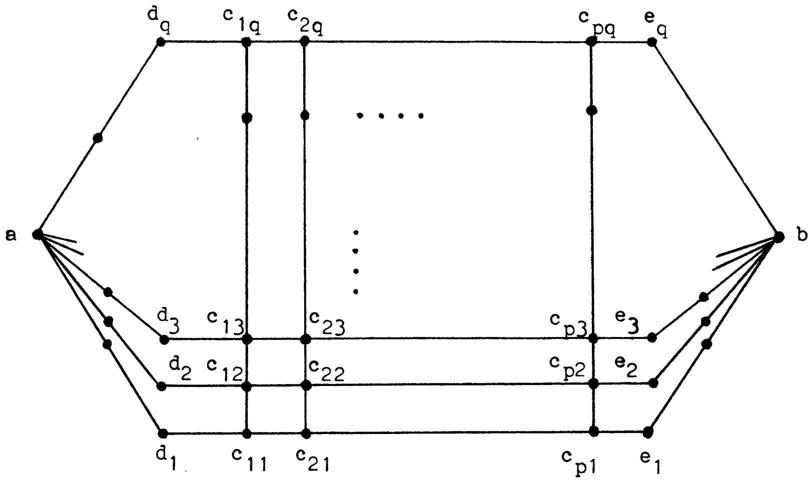

FIG. 1.

Thus $\sum_{i=1}^{q} d_i + \sum_{i=1}^{q} e_i \geqslant q - p$. Now $q \cdot m(a) \leqslant \sum_{i=1}^{q} (1 - d_i) = q - \sum_{i=1}^{q} d_i$, $q \cdot m(b) \leqslant q - \sum_{i=1}^{q} e_i$. Hence

$$q(m(a) + m(b)) \leqslant 2q - \sum_{i=1}^{q} d_i - \sum_{i=1}^{q} e_i \leqslant 2q - (q - p) = q + p$$

and

$$m(a) + m(b) \leqslant (q + p)/q = 1 + p/q = 1 + \varepsilon .$$

We have proved the assertion (*i*). Existence of states in (*ii*) and (*iii*) is obvious. The proof is finished.

We denote a logic (L, M) that meets assertions of Lemma 3 with a given rational number ε by P_ε. We are going to use logics P_ε to construct our final example. We shall refine Example 1 in such a way that M will be a set of pure states.

EXAMPLE 2. — Let (L, M) be a logic introduced in Example 1. For any $i, j \in Z$ let $\varepsilon = m_i^q(b_j)$ and denote $L_{ij} = P_\varepsilon$ and p_{ij}, $q_{ij} \in L_{ij}$ the atoms corresponding to a, b resp. in Example 1. Now we identify

$$a_i = p_{i0} = p_{i1} = p_{i,-1} = p_{i2} = p_{i,-2} = p_{i3} = p_{i,-3} = \dots \quad \text{for any} \quad i \in Z$$

and

$$b_j = q_{0j} = q_{1j} = q_{-1,j} = q_{2j} = q_{-2,j} = q_{3j} = q_{-3,j} = \dots \quad \text{for any} \quad j \in Z .$$

Denote $L_1 = L \cup \bigcup_{i,j \in Z} L_{ij}$ (previous identifications are concerned). It follows from the theory of Greechie diagrams that L_1 is an OMP.

We denote M_1 such a set of states on L_1 that $m_1 \in M_1$ if and only if

i) m_1 restricted to L equals to some $m_i^a \in M$ ($m_i^b \in M$, resp.),

ii) m_1 restricted to L_{kj} ($k, j \in Z$) equals to a state m on L_{kj} such that $m(a_k) = m_i^a(a_k)$, $m(b_j) = m_i^a(b_j)$ ($m(a_k) = m_i^b(a_k)$, $m(b_j) = m_i^b(b_j)$ resp.) and if $m = \alpha m' + (1 - \alpha)m''$ for some real number $\alpha \in (0, 1)$ and states m', m'' on L_{kj} with the same properties as m then $m = m' = m''$.

As regards Lemma 3, (L_1, M_1) is a quite full logic. Besides M_1 was defined in such a way that any state $m \in M_1$ is pure.

Let x, y be the same observables as in Example 1. Then $m(x) = m(y)$ for any $m \in M_1$ but $x \neq y$. The proof is finished.

Final remark. — Let L be an OMP and denote $\mathscr{S}(L)$ the set of all states on L. In the definition of a logic (L, M) (Definition 1) a set of states M can be a proper subset of $\mathscr{S}(L)$. A logic is often defined strictly as $(L, \mathscr{S}(L))$. We are not able to prove or disprove whether any quite full logic $(L, \mathscr{S}(L))$ possesses the condition U. As regards Example 2, there should be a counter-example. It seems improbable that such a strong condition might result only from considering the set $\mathscr{S}(L)$ instead of some its « rich enough » subset M. Although the original Gudder's problem has been answered (Example 1) it would be desirable to solve the uniqueness problem in this stronger form.

REFERENCES

[1] R. J. GREECHIE, Orthomodular lattices admitting no states, *J. Comb. Theory*, t. **10**, 1971, p. 119-132.

[2] S. GUDDER, Uniqueness and existence properties of bounded observables, *Pacific J. Math.*, t. **19**, 1966, p. 81-93, 578-589.

[3] S. GUDDER, Axiomatic Quantum Mechanics and Generalized Probability Theory, in *Probabilistic Methods in Applied Mathematics*, Vol. 2 (A. Bharucha, Reid, ed.), Academic Press, New York, 1970.

[4] S. GUDDER, *Some unsolved problems in quantum logics*, in *Mathematical Foundations of Quantum Theory* (A. R. Marlow, ed.), Academic Press, New York, 1978.

[5] S. GUDDER, *Stochastic Methods in Quantum Mechanics*, North Holland, New York, 1979.

[6] S. GUDDER, Expectation and transition probability, *Int. J. Theor. Physics*, t. **20**, 1981, p. 383-395.

[7] P. PTÁK, Logics with given centers and state spaces, *Proc. Amer. Math. Soc.*, t. **88**, 1983, p. 106-109.

[8] P. PTÁK, V. ROGALEWICZ, Regularly full logics and the uniqueness problem for observables, *Ann. Inst. H. Poincaré*, t. **38**, 1983, p. 69-74.

[9] P. PTÁK, V. ROGALEWICZ, Measures on orthomodular partially ordered sets, *J. Pure Appl. Algebra*, t. **28**, 1983, p. 75-80.

[*10*] V. ROGALEWICZ, Remarks about measures on orthomodular posets, *Časopis pro pěstování matematiky*, t. **109**, 1984, p. 93-99.

[*11*] V. ROGALEWICZ, *A note on the uniqueness problem for observables*, *Acta Polytechnica*, 1984, to appear.

[*12*] F. W. SHULTZ, A characterization of state spaces of orthomodular lattices, *J. Comb. Theory*. A **17**, 1974, p. 317-328.

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