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Comparison of exact and approximate causal solutions of a model curved-space wave equation (*)

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ABSTRACT. — A retarded Green function is constructed for a model curved space-time scalar wave equation and used to find the solution to the equation for a pure-frequency point source. This solution is shown to be unique and causal. It is then given an asymptotic expansion in a small parameter and compared with the result obtained by applying singular perturbation methods to the same problem. The aim is to show that such perturbative solutions are asymptotic to exact solutions.

RÉSUMÉ. — On construit une fonction de Green retardée pour un modèle d'équation d'onde scalaire dans un espace-temps courbe et on l'utilise pour trouver la solution de l'équation dans le cas d'une source ponctuelle avec une seule fréquence. On montre que cette solution est unique et causale. On donne ensuite un développement asymptotique en un paramètre petit, et on le compare avec le résultat obtenu en appliquant des méthodes de perturbation singulière au même problème. Le but est de montrer que de telles solutions perturbatives sont asymptotiques à des solutions exactes.

INTRODUCTION

One of the impediments to progress in general relativity is the nature of the field equations themselves; their highly nonlinear character generally

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precludes the finding of exact solutions. Therefore, in order to describe the behavior of an isolated two-body system such as the binary pulsar PSR 1913+16, it is necessary to resort to some sort of approximation scheme. When one does this, however, other problems arise. In particular, one would like to be sure that the approximate solution is at least asymptotic to an exact one, which is unique and causal. Unfortunately, there are few guidelines in general relativity (G.R.) to aid in establishing such results. The next best thing is to test out the approximation techniques on a model problem which has some of the features of G. R. but which is simple enough to have an exact solution of its own. One then takes an asymptotic expansion of this solution and compares it with the perturbation expansion. If the two results agree, one gains confidence in the utility of the approximation technique for solving problems in G.R.

Such an investigation is the aim of this paper. We consider a model problem first devised by one of the authors (J.L.A.) and L. Kegeles [*1*] in order to demonstrate the usefulness of singular perturbation techniques in G.R. We construct an exact, causal Green function for this problem and use it to find the solution for the case of a pure-frequency point-source. We then perform an asymptotic expansion on this solution and compare the results with those of Anderson and Kegeles, who also considered the pure-frequency case.

Before doing all this, however, we must briefly discuss the properties of the solutions we are trying to approximate. Thus the outline of this paper is as follows: In Section I we will discuss what we mean by a unique, causal field. In Section II we construct the exact Green function. In Section III we use this function to find the solution for the case of a monofrequency point-source located at the origin. We demonstrate that this solution has the properties outlined in Section I; we conclude the third section by carrying out an asymptotic expansion of the exact solution and comparing it with the result of the Anderson-Kegeles (A-K) paper.

SECTION I

THE PROPERTIES OF EXACT SOLUTIONS

As mentioned in the Introduction, one must resort to a perturbation scheme in order to describe the field associated with an evolving, isolated system in G.R. The weak-field, slow-motion approximation is one commonly employed in this connection [*2*]. If it is to be of any use, this approximate solution must be asymptotic to an exact one which should have the following properties [*3*]:

1) *Causality:* By this we mean that if the source is quiescent prior to some fixed retarded time u_0, evolving in time after that, the field is static prior to u_0 and dynamic after that; points distant from the source learn of its « switching on » only at retarded times $> u_0$.

2) *Uniqueness:* One would like the behavior of the fields to be determined solely by the activity of the source. Consequently, one desires a solution which vanishes whenever the source does, and which has no radiation coming in from infinity [4]. Imposing such conditions eliminates homogeneous solutions which represent incoming or outgoing waves; the only homogeneous solution allowed is the trivial one.

3) *Outgoing Radiation:* In a curved-space situation one will have at any finite distance from the source a combination of incoming and outgoing waves due to the effects of backscatter. In the case of isolated systems, however, one would expect the field to tend toward flat-space behavior at future null infinity. This behavior would represent an outgoing wave; consequently, we impose the condition that the field represents a purely outgoing wave in the limit of going of Ω^+.

Imposing these conditions, especially those of causality and uniqueness, can be difficult in G. R. In linear field theories, such as flat-space electromagnetism, these conditions are assured by the presence of a retarded Green function; this not only ensures causality, but uniqueness as well, through the Kirchoff identity [5]. In G.R., however, one cannot construct a Green function because of the nonlinear nature of the field equations, yet one would like to be able to build causality and uniqueness in from the start. The most practical way to do this is to employ an iterative perturbation scheme which introduces uniqueness and causality at each step.

This has been done by Anderson and Kegeles [6] for the case of a model problem representing a scalar test field in a model curved space-time:

$$\nabla^2 \phi - (1 - kf(r))^2 \frac{\partial^2 \phi}{\partial t^2} = -4\pi q \rho(\vec{r}, \varepsilon t) \qquad \text{(Eq. (1.1))}$$

Here:

$$f(r) = \begin{cases} 1, (r \leqslant 1) \\ \dfrac{1}{r}, (r > 1) \end{cases}$$

so that our metric is flat for $r \leqslant 1$, curved for $r > 1$, and continuous at $r = 1$. This modification is undertaken to avoid a Schwarzchild-type singularity in the curved-space metric at $r = k$. To ease the task of finding exact solutions we sidestep this complication by introducing a flat metric for $r \leqslant 1$. This modification is important for the material in the present paper; in the A-K paper the focus is on the $r > 1$. Region; the flat-space region is not considered.

The parameters ε, k, and q are dimensionless constants much smaller than unity and give the slowness of source variation, the gravitational source strength, and the scalar field source strength, respectively. In the A-K paper $k=0(\varepsilon^3)$ and $q=0(\varepsilon^2)$, in analogy with a gravitationally bound system; we will assume the same in the present paper, though for our purposes we need only assume that the parameters are small.

The authors give the scalar field an asymptotic expansion in the weak-field parameter k. They specialize immediately to the spherically symmetric ($l = 0$) case for simplicity, and solve for the field to $0(k)$. At each order of the perturbation the field is expressed in terms of an integral formula which is causal in the sense discussed above.

The authors further specialize their solution to the case of an harmonically varying monofrequency source. The results is [7]:

$$\phi \sim e^{i\Omega u}\left(1 + ik\Omega e^{2i\Omega r}\left[-\operatorname{Ci}(2\Omega r) + i\operatorname{Si}(2\Omega r) - \frac{i\pi}{2}\right] - i\Omega kb\right) + 0(k^2) \qquad \text{(Eq. (1.2))}$$

where $\operatorname{Si}(x)$, $\operatorname{Ci}(x)$ are the sine and cosine integrals [8], respectively, and $u = t - r + k \ln r$ is the retarded null coordinate for the curved-space region. The variable $v = u + 2r$ and is asymptotic to the true ingoing null coordinate $v' = t + r - k \ln r$. Also:

$$b = -\left(\gamma + \ln 2\Omega + \frac{i\pi}{2}\right)$$

where γ is the Euler constant.

We now pose the question: Is the approximate solution, which is causal, asymptotic to an exact solution, which is itself unique and causal? In order to answer this question, we will construct an exact causal Green function for this model problem, use it to find the solution for the case of a pure-frequency point-source located at the origin, and show that this solution is unique. We then expand this solution for small values of the curvature parameter k and compare the result with that of Anderson and Kegeles.

SECTION II

THE MODEL PROBLEM AND ITS GREEN FUNCTION

In this section we construct the Green function for the Anderson-Kegeles model problem. We wish to solve:

$$\nabla^2 G(\vec{r}, \vec{r}', t, t') - (1 - kf(r))^2 \frac{\partial^2 G}{\partial t^2}(\vec{r}, \vec{r}', t, t') = -4\pi\delta(\vec{r} - \vec{r}')\delta(t - t') \qquad \text{(Eq. (2.1))}$$

where $r' \leqslant 1$ so that the point-source of the Green function is confined to the flat-space region, for simplicity. After Fourier analyzing with respect to time we have:

$$\nabla^2 G_{<1}(\vec{r}, \vec{r}', \omega) + (1-k)^2\omega^2 G(\vec{r}, \vec{r}', \omega) = -4\pi\delta(\vec{r} - \vec{r}')$$

and:

$$\nabla^2 G_{>1}(\vec{r}, \vec{r}', \omega) + \left(1 - \frac{k}{r}\right)^2 G(\vec{r}, \vec{r}', \omega) = 0$$

Notice that we have introduced the notation:

$$G_{\substack{<1\\>1}}(\vec{r}, \vec{r}', \omega)$$

for the Green function for $r \lesseqgtr 1$, respectively.

In both regions we perform the usual decomposition in terms of spherical harmonics:

$$G(\vec{r}, \vec{r}', \omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_{l,m}(\vec{r}, \vec{r}', \omega) Y^*_{lm}(\theta', \phi') Y_{lm}(\theta, \phi)$$

and solve for the radial part $g_{l,m}$.

In the flat-space region we impose the usual conditions of finiteness at the origin, continuity at $r = r'$, and the jump condition on the first derivative of $g_{l,m}$ at $r = r'$ [9]; we then get, for $r \leqslant 1$:

$$g_{l,m} = -i(1-k)\omega j_l((1-k)\omega r_<)h_l^{(2)}((1-k)\omega r_>) + b_l(r_<)h_l^{(1)}((1-k)\omega r_>)$$

where as usual $r_<$ and $r_>$ are the smaller and larger, respectively, of r, r'; j_l, $h_l^{(1)}$, and $h_l^{(2)}$ are the spherical Bessel and Hankel functions, respectively. We see that we have the familiar outgoing wave part represented by the spherical Hankel function of the second kind. However, we also have a second, incoming term, represented by $h_l^{(1)}$, indicating that waves emitted by the source are partially reflected by the inhomogeneity in the metric at $r = 1$. The coefficient b_l will be determined from boundary conditions at this point.

Turning now to the curved-space region, we find that the radial equation becomes:

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + \omega^2\left(1 - \frac{k}{r}\right)^2\right] g_{l,m} = 0$$

Introducing the variable $z = 2i\omega r$ we get:

$$\left[\frac{d^2}{dz^2} + \frac{2}{z}\frac{d}{dz} + \left[-\frac{1}{4} + \frac{i\kappa}{z} + \frac{\frac{1}{4} - u^2}{z}\right]\right] g_{l,m} = 0$$

where:

$$\mu^2 = \left(l + \frac{1}{2}\right)^2 - k^2\omega^2 ; \qquad \kappa = i\omega k$$

The solution to this equation is of the form [*10*]:

$$g_{\mu,\kappa}(z) = A_{\mu,\kappa} \frac{\mathscr{M}_{u,\kappa}(z)}{z} + B_{\mu,\kappa} \frac{\mathscr{W}_{u,k}(z)}{z}$$

where $\mathscr{M}$ and $\mathscr{W}$ are the Whittaker functions, which may be written in terms of the Kummer functions $M(a, b, z)$ and $U(a, b, z)$:

$$\mathscr{M}_{\mu,\kappa}(z) = e^{-\frac{1}{2}z} z^{\mu+\frac{1}{2}} M(a, b, z)$$
$$\mathscr{W}_{\mu,\kappa}(z) = e^{-\frac{1}{2}z} z^{\mu+\frac{1}{2}} U(a, b, z)$$

where:

$$a = \frac{1}{2} + \mu - i\omega k$$
$$b = 2\mu + 1$$

The coefficients $A_{\mu,\kappa}$ and $B_{\mu,\kappa}$ are determined by the conditions at $r = 1$ and the outgoing radiation condition. Considering the latter, we look at the asymptotic expansions of the Kummer functions for large r [*11*]:

$$M(a, b, z) \sim e^{\pm i\pi a} z^{-a} \frac{\Gamma(b)}{\Gamma(b-a)} \times \left(\sum_{n=0}^{R-1} \frac{(a)_n(1+a-b)_n}{n!} (-z)^{-n} + 0(|z|^{-k}) \right) + e^z z^{a-b} \frac{\Gamma(b)}{\Gamma(a)} \times \left(\sum_{n=0}^{S-1} \frac{(b-a)_n(1-a)_n}{n!} (-z)^{-n} + 0(|z|^{-s}) \right);$$

$$U(a, b, z) \sim z^{-a} \left(\sum_{n=0}^{R-1} \frac{(a)_n(1+a-b)_n}{n!} (-z)^{-n} + 0(|z|^{-R}) \right)$$

Here the (a)s are the Pochhammer symbols, defined by [*12*]:

$$(a)_n = a(a+1)(a+2) \ldots (a+n-1)$$

We see that in order to satisfy the outgoing-wave condition we must take

$A_{\mu,\kappa} = 0$. Imposing this condition eliminates the $M(a, b, z)$ term, whose expansion contains the ingoing-wave expression:

$$e^{i\omega(u+2r)}$$

We now determine the remaining coefficients b_l and $B_{\mu,\kappa}$ by imposing continuity of the Green function and its first derivative at $r = 1$.

We get:

$$b_l = i(1-k)\omega j_l((1-k)\omega r') \times \left[\frac{h_l^{(2)}((1-k)\omega)\mathscr{W}'_{\mu,\kappa}(2i\omega) - h_l^{(2)\prime}((1-k)\omega)\mathscr{W}_{\mu,\kappa}(2i\omega)}{h_l^{(1)}((1-k)\omega)\mathscr{W}'_{\mu,\kappa}(2i\omega) - h_l^{(1)\prime}((1-k)\omega)\mathscr{W}_{\mu,\kappa}(2i\omega)}\right],$$

$$B_{\mu,\kappa} = i(1-k)\omega j_l((1-k)\omega r') \times \left[\frac{h_l^{(2)}((1-k)\omega)h_l^{(1)\prime}((1-k)\omega) - h_l^{(2)\prime}((1-k)\omega)h_l^{(1)}((1-k)\omega)}{h_l^{(1)}((1-k)\omega)\mathscr{W}'_{\mu,\kappa}(2i\omega) - h_l^{(1)\prime}((1-k)\omega)\mathscr{W}_{\mu,\kappa}(2i\omega)}\right],$$

$$\left(\mathscr{W}'_{\mu,\kappa}(zi\omega) = \frac{d\mathscr{W}_{\mu,\kappa}}{dr}(2i\omega r)\bigg|_{r=1}, \quad \text{etc.}\right)$$

Or:

$$b_l = i(1-k)\omega j_l((1-k)\omega r')R_l$$
$$B_{\mu,\kappa} = i(1-k)\omega j_l((1-k)\omega r')T_{\mu,\kappa}$$

Here we have introduced an economy of notation by renaming the expressions in the brackets R_l and $T_{\mu,\kappa}$, respectively. This notation reflects the fact that these expressions are coefficients of reflected and transmitted waves, respectively.

Thus, we have for the Green function:

$$G_{<1}(\vec{r}, \vec{r}', \omega) = -4\pi i(1-k)\omega\left[\sum_{l=0}^{\infty} j_l((1-k)\omega r_<) \times \left(h_l^{(2)}((1-k)\omega r_>) - R_l h_l^{(1)}((1-k)\omega r_>)\right) \times \sum_{m=-l}^{l} Y^*_{lm}(\theta', \phi')Y_{lm}(\theta, \phi)\right],$$

$$G_{>1}(\vec{r}, \vec{r}', \omega) = 4\pi i(1-k)\omega \sum_{l=0}^{\infty} T_{\mu,\kappa} j_l((1-k)\omega r_<) \times \mathscr{W}_{\mu,\kappa}(2i\omega r) \sum_{m=-l}^{l} Y^*_{lm}(\theta, \phi)Y_{lm}(\theta, \phi) \qquad \text{(Eq. (2.2))}$$

Having found the Green function, we will now employ it to solve a simple illustrative case which will give us a check on the A-K results.

SECTION III

THE PURE-FREQUENCY $l = 0$ SOLUTION

Now that we have constructed our causal Green function we will use it to find the solution associated with a monofrequency point-source located at the origin. We consider the source function:

$$\rho(\vec{r}, t) = q e^{i\Omega t}\delta(\vec{r})$$

in order to make a connection with the A-K results. To solve for the field we use the formula:

$$\phi(\vec{r}, t) = \int_{-\infty}^{\infty} d\omega \int d\vec{r}' \mathrm{G}(\vec{r}, \vec{r}', \omega)\rho(\vec{r}, \omega)e^{i\omega t}$$

Where:

$$\rho(\vec{r}, \omega) = q\delta(\omega - \Omega)\delta(\vec{r})$$

We then get:

$$\phi_{<1}(\vec{r}, t) = q\left[\frac{e^{i\Omega(t-(1-k)r)}}{r} + \mathrm{R}_0 \frac{e^{i\Omega(t+(1-k)r)}}{r}\right], \qquad \text{(Eq. (3.1 } a))$$

$$\phi_{>1}(\vec{r}, t) = q e^{i\Omega(t-r)}(i(1-k)\Omega)\mathrm{T}_{\mu_0,\kappa_0} \times (2i\Omega r)^{\mu_0 - \frac{1}{2}}\mathrm{U}(a_0, b_0, 2i\Omega r) \qquad \text{(Eq. (3.1 } b))$$

where

$$\mu_0^2 = \frac{1}{4} - k^2\Omega^2$$
$$\kappa_0 = i\Omega k$$
$$a_0 = \frac{1}{2} + \mu_0 - \kappa_0$$
$$b_0 = 2\mu_0 + 1$$

We now demonstrate that our solution has the properties outlined in the first section:

1) **Outgoing waves at future null infinity.**

We consider the behavior of our solution in the limit:

$$\lim_{\substack{u = \text{const.} \\ r \to \infty}} \phi$$

where $u = t - r + k \ln r$ is the outgoing null coordinate for the $r > 1$

region. Considering our solution for this region and using again the asymptotic expansion for fixed u, large r, we get:

$$\phi_{>1} \sim \frac{e^{i\Omega u}}{2i\Omega r} \sum_{n=0}^{\infty} \frac{(a_0)_n(1 + a_0 - b_0)_n}{n!} \left(\frac{-1}{2i\Omega r}\right)^n$$

We see that the solution has the form of a purely outgoing wave as one approaches $\mathscr{I}^+$.

2) **Causality.**

The causality of our solution is assured if our Green function is retarded. We now demonstrate the causal nature of our Green function. We consider the Fourier frequency integral:

$$G(\vec{r}, \vec{r}', t, t') \doteq \int_{-\infty}^{\infty} d\omega G(\vec{r}, \vec{r}', \omega) e^{i\omega\, t-t')}$$

and show that for times greater than the pulse propagation time the contour must be closed in the upper half-plane, giving a nonzero contribution. For times less than this, the contour is closed in the lower half-plane, giving zero.

The outgoing wave part of the $r \leqslant 1$ solution is just the familiar flat-space retarded Green function; its causal properties are well-known. We therefore concentrate on the curved-space part and the reflected-wave flat-space part.

In examining the Fourier integral of the curved-space solution, we consider the case of fixed, finite 1, r, r' and $|\omega| \to \infty$. Apart from angular factors the asymptotic form [13] of the integrand is:

$$\frac{e^{i\omega(t-t'+(1-k)r'-r)}}{\omega^{l+2}} (2i\omega r)^{i\omega k - \frac{1}{2}} \Upsilon_{\mu_0,\kappa_0} S_a(r)$$

where:

$$\Upsilon^{-1}_{\mu_0,\kappa_0} = \left[h_0^{(1)}((1-k)\omega)\mathscr{W}'_{\mu_0\kappa_0}(2i\omega) - h_0^{(1)\prime}((1-k)\omega)\mathscr{W}_{\mu_0\kappa_0}(2i\omega)\right]$$

$$\sim \frac{e^{i(1-k)\omega}}{\omega} e^{-i\omega}(2i\omega)^{i\omega k - \frac{1}{2}} S_a(1)$$

$$\times \left[\frac{1}{2}\frac{1}{1-k} e^{-i\left(\frac{l+1}{2}\right)\pi} + \frac{(1-k)i}{2} e^{-i\left(\frac{l+2}{2}\right)\pi}\right]$$

and:

$$S_a(x) = \sum_{n=0}^{\infty} (a)_n \frac{x^n}{k^n}; \qquad a \cong \frac{1}{2}$$

So we get:

$$\frac{e^{i\omega[t-t'-(r-1-k\ln\ r+(1-k)(1-r')]}}{\omega^{l+1}} \times \frac{S_a(r)}{S_a(1)} \frac{1}{\sqrt{r}}$$

In order to evalute the ω-plane integral by means of the Cauchy theorem we see that if:

$$t - t' > (r - 1) - k \ln r + (1 - k)(1 - r')$$

we close the contour in the upper half-plane, otherwise, we close it in the lower half-plane.

The singularities present in our integrand are a pole at $\omega = 0$, and branch points at $\omega = 0$ and $\omega = \pm\left(l + \frac{1}{2}\right)/k$. The cuts are taken along the negative real axis [14]. These singularities can be avoided in the lower half-plane by deforming the contour. The expression:

$$\Upsilon_{\mu_0,\kappa_0} = \left[h_0^{(1)}((1-k)\omega)\mathscr{W}'_{\mu_0\kappa_0}(2i\omega) - h_0^{(1)\prime}((1-k)\omega) \times \mathscr{W}_{\mu_0\kappa_0}(2i\omega)\right]^{-1} \equiv \left[W(h_0^{(1)}, \mathscr{W}_{\mu_0\kappa_0})\right]^{-1}$$

has no singular points since its denominator has no zeroes. To see this, note that the denominator may be written as:

$$W(jl, \mathscr{W}_{\mu_0,\kappa_0}) + iW(n_l, \mathscr{W}_{\mu_0,\kappa_0})$$

This expression can vanish if:

$$W(j_l, \mathscr{W}_{\mu_0,\kappa_0}) = W(n_l, \mathscr{W}_{\mu_0,\kappa_0}) = 0$$

or:

$$W(j_l, \mathscr{W}_{\mu_0,\kappa_0}) = -iW(n_l, \mathscr{W}_{\mu_0,\kappa_0})$$

But this cannot occur since j_l and n_l are linearly independent functions. Therefore the denominator does not vanish, and the expression Ψ_{μ_0,κ_0} has only branch points, whose cuts may again be avoided by deforming the contour. Thus, by Cauchy's theorem we have:

$$G(\vec{r}, \vec{r}', t, t') = 0$$

when:

$$t - t' < r - 1 - k \ln r + (1 - k)(1 - r')$$

Note that this latter expression is simply the distance travelled by a pulse in going from the flat-space region point r' to a point $r > 1$. When the time interval is less than this distance we close the contour in the upper half-plane, where the singularities give a nonzero contribution. Thus we see that the curved-space part of our Green function is indeed causal.

If we consider the reflected-wave flat-space part for fixed, finite l, r, r' and large $|\omega|$ we have:

$$e^{i\omega[t-t'-(1-k)r+(1-k)r'-2(1-k)]} \qquad (r < r')$$

when closing the contour in the upper half-plane. We must have:

$$t - t' - (1 - k)[r - r' + 2] \geq 0$$

Since $| r - r' | \leqslant 1$ we have $t > t'$ and therefore this part of the Green function is also causal in the sense that reflected waves reach a point r after they have been emitted from r', where both $r, r' \leqslant 1$.

3) Uniqueness.

As mentioned in Section I, one proves uniqueness by demonstrating that the only allowed homogeneous solution is the trivial one. This is done in the flat-space case by using the Kirchoff identity to give an integral equation representation of the homogeneous solution. One then shows that if the field and its first derivatives have a certain asymptotic behavior for large r, then the Kirchoff integral vanishes and the solution is a trivial one.

We will employ a similar argument for our case. Since we are dealing with a static, curved space-time, the Kirchoff formula is not applicable. However, in our case, which resembles that of a flat-space wave-equation in an inhomogeneous medium, one may use a generalization of the Kirchoff formula, known as the Sobolev formula [*15*] [*16*]:

$$\phi(\vec{r}, t) = \frac{1}{4\pi}\oint_S \left(\sigma(\vec{r}, \vec{r}')\left[\frac{\partial\phi}{\partial n}(\vec{r}', t')\right] - [\phi(\vec{r}', t')]\frac{\partial\sigma}{\partial n}(\vec{r}, \vec{r}') + \sigma(\vec{r}, \vec{r}')\frac{\partial \Upsilon}{\partial n} \times \left[\frac{\partial\phi}{\partial t'}(\vec{r}', t')\right]\right) dS' + \frac{1}{4\pi}\int_V [\phi(\vec{r}', t')]\nabla^2_{r'}\sigma(\vec{r}, \vec{r}')dV' \quad \text{(Eq. (3.2))}$$

where:

$$\Upsilon = | \vec{r} - \vec{r}' | - k \ln | \vec{r} - \vec{r}' | = R - k \ln R = R^*$$

$$\sigma(R) = \left(\frac{1}{R^2 - kR}\right)^{1/2}$$

$$[\phi(\vec{r}', t')] = \phi(\vec{r}', t')|_{t'=t-R^*}, \qquad \text{etc.}$$

We see that Υ is the retardation factor for the curved-space region and σ is an intensity factor for this region, replacing the flat-space 1/R term in the Kirchoff formula. Note that if $k = 0$ we regain the latter expression.

In our case the identity becomes:

$$\phi(\vec{r}, t) = \frac{1}{4\pi}\oint_S \frac{1}{R}\left(1 - \frac{k}{R}\right)^{1/2}\left(\left[\frac{\partial\phi}{\partial R^*}\right] + \left[\frac{\partial\phi}{\partial t}\right]\right) + [\phi]\frac{R - \frac{k}{2}}{(R^2 - kR)^{3/2}} R^2 d\Omega' + \frac{1}{4\pi}\int_V [\phi]\nabla^2_{r'}\sigma dV' \quad \text{(Eq. (3.3))}$$

In order to have the surface integral vanish we must have:

$$\lim_{\substack{v=\text{const.}\\ R\to\infty}} D\phi = 0\left(\frac{1}{R}\right) \qquad \text{(Eq. (3.4 }a))$$

$$\lim_{\substack{v=\text{const.}\\ k\to\infty}} \phi = 0(1) \qquad \text{(Eq. (3.4 }b))$$

where:

$$D = \frac{\partial}{\partial R^*} + \frac{\partial}{\partial t},$$

$$v = t + R^*$$

We are left with the volume integral term, which is a homogeneous Volterra equation of the second kind. This type of equation has no nontrivial solutions; therefore this term also vanishes [*17*]. Consequently Eqs. (3.3 *a*) and (3.3 *b*) are the conditions which ensure uniqueness.

Applying these results to our example, we may replace the operator D by the flat-space operator:

$$D_0 = \frac{\partial}{\partial r} + \frac{\partial}{\partial t}$$

since if the first derivatives of ϕ are bounded the difference between the two operators vanishes as $r \to \infty$. We now consider the limits:

$$\lim_{\substack{v=\text{const.}\\ r\to\infty}} D_0\phi_{>1}\,;$$

$$\lim_{\substack{v=\text{const.}\\ r\to\infty}} \phi_{>1}$$

These give, respectively:

$$\lim_{\substack{v=\text{const.}\\ r\to\infty}} D_0\phi_{>1} \sim \frac{e^{i\Omega v}}{r^2} e^{-2i\Omega r}[(a_0)_0(1+a_0-b_0)_0 - 2i\Omega a_0(a_0+1)_0(1+a_0-b_0)_0] + 0(r^{-3}) \qquad \text{(Eq. (3.5 }a)),$$

$$\lim_{\substack{v=\text{const.}\\ r\to\infty}} \phi_{>1} \sim \frac{e^{i\Omega v}}{r} e^{-2i\Omega r}(a_0)_0(1+a_0-b_0)_0 + 0(r^{-2}) \qquad \text{(Eq. (3.5 }b))$$

demonstrating that our uniqueness conditions are met. Here we have used $u = v - 2r$ and the large r asymptotic expansion of $U(a, b, z)$.

We have now obtained an exact solution for the pure-frequency case and demonstrated that it is unique and causal. Our remaining task is to show that our results are consistent with those of Anderson and Kegeles. To do this we perform a weak-field expansion, that is, an asymptotic expansion on our solution for small k, the gravitational field strength parameter.

With reference to Eq. (3.1 *b*), the $r > 1$ solution may be written as:

$$\phi_{>1}(\vec{r}, t) = -2qe^{i\Omega u}e^{-i\Omega k \ln r}(2i\Omega r)^{\mu_0 - \frac{1}{2}} \times \Upsilon_{\mu_0,\kappa_0} \mathrm{U}(a_0, b_0, 2; \Omega r),$$

where, following the practice of Anderson and Kegeles, we have expressed the solution in terms of the null coordinate u in order to avoid nonuniform terms of the form $\ln r/r$.

A straightforward asymptotic expansion of this expression up to $0(k)$ gives:

$$\phi_{>1}(\vec{r}, t) \to q\frac{e^{i\Omega u}}{r}\left(1 - i\Omega kb + i\Omega k \times e^{2i\Omega r}\left[-\mathrm{C}i(2\Omega r) + i\mathrm{S}i(2\Omega r) - \frac{i\pi}{2}\right] + \Omega^2 kf(\Omega)\right) + 0(k^2) \quad \text{(Eq. (3.6))}$$

This expression can be derived by using the integral representation for $\mathrm{U}(a, b, z)$ [*18*]:

$$\mathrm{U}(a, b, z) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^{a-1}(t+1)^{b-a-1}dt$$

and performing an asymptotic expansion on it for small k, while noting that:

$$\Upsilon_{\mu_0,\kappa_0} \to -i\Omega + \Omega^2 kf(\Omega);$$

$$f(\Omega) \to ib \sin \Omega h_0^{(1)}(\Omega) - \frac{i\Omega}{2} h_0^{(1)\prime}(\Omega)h_0^{(2)}(\Omega) - \frac{1}{2\Omega^2} + h_0^{(1)}(\Omega)\left[-i\Omega b \sin \Omega + ib \sin \Omega - b\Omega^2 h_0^{(1)}(\Omega) + \frac{i\Omega}{2} h_0^{(2)}(\Omega)\right]$$

We see that the expression in Eq. (3.6), with the exception of the last term, agrees with the Anderson-Kegeles result. This last term is not present in their result because in their work the concern is with the $r > 1$ solutions and no attempt was made to connect this with solutions for $r < 1$. In our model the transmission factor $\Upsilon_{\mu_0,\kappa_0}$ arises, as we have seen, from just this process and this extra term is a result of expanding this factor in k.

SUMMARY AND CONCLUSIONS

We have constructed a unique, causal Green function for a model problem which has features similar to those found in G.R. We used this Green function to find a simple solution, which we demonstrated to be unique. An asymptotic expansion of this solution in a weak-field parameter gave a result which essentially agreed with one obtained by Anderson and Kegeles by applying singular perturbation techniques to the same problem. Therefore, the Anderson-Kegeles technique gives our model problem a perturbative solution which is asymptotic to the unique, causal exact solution. On the basis of this result we are optimistic that the Anderson-

Kegeles technique will give an appropriate perturbative solution to general relativistic problems involving fields of isolated systems; by appropriate we mean that the perturbative solution is unique and causal, and asymptotic to the exact one, also unique and causal.

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