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Space-times admitting a covariantly constant spinor field

by

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ABSTRACT. — It is shown that if a space-time V admits a covariantly constant spinor field ψ_A, and hence a covariantly constant null vector field l_μ determined by ψ_A, its Ricci tensor is proportional to the tensor product of l_μ by itself. Further, the conformal tensor of V is shown to be of Petrov-Penrose type N. That is the four index symmetric spinor determined by the conformal tensor of V is proportional to the spinor product of ψ_A with itself. The Bianchi identities are used to show that empty asymptotically flat space-times with infinitely extendible null geodesics tangent to l_μ are flat.

RÉSUMÉ. — On montre que, si un espace temps V admet un champ spinoriel constant par covariance ψ_A, et par suite un champ de vecteurs isotrope constant par covariance l_μ déterminé par ψ_A, son tenseur de Ricci est proportionnel au tenseur produit de l_μ par lui-même. En outre, on montre que le tenseur conforme de V est du type N de Petrov-Penrose, c'est-à-dire que le spineur symétrique à quatre indices déterminé par le tenseur conforme de V est proportionnel au produit spinoriel de ψ_A par lui-même. On utilise les identités de Bianchi pour montrer que des espaces temps vides et asymptotiquement plats, dont les géodésiques isotropes tangentes à l_μ sont prolongeables indéfiniment, sont plats.

1. INTRODUCTION

It is the purpose of this paper to characterize those space-times that admit a covariantly constant two component spinor field. In addition, we shall prove that asymptotically flat empty space-times that contain such a spinor field are flat.

We shall use the notation and results of reference [1] where spinors ϕ^{A} and ψ^{A} satisfying

$$\phi^{A}\psi_{A} = 1 \; ; \tag{1.1}$$

are introduced. Spinor indices are manipulated with the asymmetric Levi-Civita tensor density ε_{AB} and ε^{AB} by the rules

$$\phi_{A} = \varepsilon_{AB}\phi^{B}$$

and

$$\phi^{A} = \phi_{B}\varepsilon^{BA}$$

Equation (1.1) is equivalent to

$$\phi^{A}\psi_{B} - \psi^{A}\phi_{B} = \delta^{A}_{B} \tag{1.2}$$

In terms of these the Newman-Penrose spin coefficients are defined as

$$\begin{aligned} A_{0\nu} &= \phi^{A}\phi_{A;\nu} \\ A_{1\nu} &= \phi^{A}\psi_{A;\nu} = \psi^{A}\phi_{A;\nu} \\ A_{2\nu} &= \psi^{A}\psi_{A;\nu} \end{aligned} \tag{1.3}$$

where the semi-colon denotes the spinor covariant derivative with respect to the metric of the space-time V.

The spin-coefficients define three two forms by the equation

$$R_{i\mu\nu} = A_{i\mu;\nu} - A_{i\mu;\nu} - \sqrt{2}E_{ijk}A^{j}_{\mu}A^{k}_{\nu} \qquad i, j, k = (1, 2, 3) \tag{1.4}$$

where

$$E_{ijk} = \sqrt{a}\varepsilon_{ijk}$$

with ε_{ijk} the Levi-Civita alternating tensor density and

$$a = \det \| a_{ij} \|$$

$$\| a_{ij} \| = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1/2 & 0 \\ 1 & 0 & 0 \end{pmatrix} ; \qquad \| a^{ij} \| = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$a_{ij}a^{jk} = \delta^{k}_{i}$$

Lower case latin indices are manipulated with the a_{ij} and a^{ij}.

The quantities X_{iAB} are defined by

$$X_{0AB} = \phi_A \phi_B \qquad = X^2_{AB}$$

$$X_{1AB} = \frac{1}{2}(\phi_A \phi_B + \psi_A \phi_B) = -\frac{1}{2} X^1_{AB}$$

$$X_{2AB} = \psi_A \psi_B \qquad = X^0_{AB}$$

and satisfy

$$X_{iAB} X^{iCD} = \frac{1}{2}(\delta^C_A \delta^D_B + \delta^C_B \delta^D_A)$$

The $R_{i\mu\nu}$ are related to the spinor curvature two-forms $R_{AB\mu\nu}$ by the equations

$$R_{i\mu\nu} = X_i^{AB} R_{AB\mu\nu} = \frac{1}{4} X_i^{AB} p^{\rho\sigma}_{AB} R_{\rho\sigma\mu\nu}$$

where $R_{\rho\sigma\mu\nu}$ are the components of the Riemann curvature tensor of V whose metric is $g_{\mu\nu}$. This metric determines 2×2 matrices

$$q_\mu = \| \gamma_\mu{}^{\dot{A}}{}_B \|$$

such that

$$\bar{q}_\mu q_\nu + \bar{q}_\nu q_\mu = 2 g_{\mu\nu} 1_2$$

when the bar over a quantity denotes its complex conjugate. Matrices $\| p_{\mu\nu}{}^A{}_B \|$ are defined by the equations

$$2 p_{\mu\nu} = \bar{q}_\mu q_\nu - \bar{q}_\nu q_\mu .$$

Greek indices are manipulated by $g_{\mu\nu}$ and $g^{\mu\nu}$. Various algebraic properties of the matrices $p_{\mu\nu}$ are given in the appendix.

We may also write

$$R_{i\mu\nu} = \frac{1}{8}\left((C_{ij} - \frac{R}{3} a_{ij}) p^j_{\mu\nu} + S_{i\bar{j}} \bar{P}^j_{\mu\nu} \right)$$

where R is the scalar curvature of the space-time,

$$C_{ij} = \frac{1}{4} p^{\rho\sigma}_{AB} C_{\rho\sigma\mu\nu} p^{\mu\nu}_{CD} X_i^{AB} X_j^{CD} = R_{i\mu\nu} p_j^{\mu\nu}$$

where $C_{\rho\sigma\mu\nu}$ is the conformal tensor of V, and

$$S_{i\bar{j}} = \frac{1}{4} p^{\rho\sigma}_{AB} R_{\rho\sigma\mu\nu} \bar{p}^{\mu\nu}_{CD} X_i^{AB} \bar{X}_j^{CD} .$$

When we write

$$C_{ABCD} = X^i_{AB} C_{ij} X^j_{CD}$$

we have

$$C_{ABCD} = C_0 \phi_A \phi_B \phi_C \phi_D + 4 C_1 \phi_{(A} \phi_B \phi_C \psi_{D)} + 6 C_2 \phi_{(A} \phi_B \psi_C \psi_{D)} + 4 C_3 \psi_{(A} \psi_B \psi_C \phi_{D)} + C_4 \psi_A \psi_B \psi_C \psi_D$$

where the parentheses denote the symmetric sum and

$$\| C_{ij} \| = \left\| \begin{matrix} C_4 & -C_3 & C_2 \\ -C_3 & C_2 & -C_1 \\ C_2 & -C_1 & C_0 \end{matrix} \right\|$$

2. THE COVARIANTLY CONSTANT CONDITION

This condition may be expressed as the equations

$$\phi_{A;\mu} = 0 \tag{2.1}$$

The integrability conditions of these equations are

$$\phi_{A;\mu\nu} - \phi_{A;\nu\mu} = \phi_C R^C_{A\mu\nu} = 0 \tag{2.2}$$

In this section we shall discuss consequences of equations (2.1). It follows from these equations and equations (1.3) that

$$A_{0\mu} = A_{1\mu} = 0 \tag{2.3}$$

Equations (1.4) then imply that

$$R_{i\mu\nu} = \delta_i^2(A_{2\nu;\mu} - A_{2\mu,\nu}) \tag{2.4}$$

Since the quantities $p^i_{\mu\nu}$ (and $\bar{p}^i_{\mu\nu}$) provide a basis for self-dual (and anti-self-dual) two forms in V, we may write

$$A_{2\nu;\mu} - A_{2\mu;\nu} = \frac{1}{8}(Z^i p_{i\mu\nu} + \zeta^i \bar{p}_{i\mu\nu})$$

where

$$Z^i = p^{i\mu\nu}(A_{2\nu;\mu} - A_{2\mu;\nu})$$
$$\zeta^i = \bar{p}^{i\mu\nu}(A_{2\nu;\mu} - A_{2\mu;\nu})$$

Hence

$$R_{i\mu\nu} = \frac{1}{8}(\delta_i^2 Z_j p^j_{\mu\nu} + \delta_i^2 \zeta_j \bar{p}^j_{\mu\nu})$$

Then

$$C_{ij} - \frac{R}{3} A_{ij} = \delta_i^2 Z_j$$

Since the left hand side of this equation is symmetric in i and j we must have

$$Z_j = \delta_j^2 Z$$

Similarly we have

$$S_{i\bar{j}} = \delta_i^2 \zeta_j = \bar{S}_{j\bar{2}}$$

Hence

$$\zeta_i = \zeta \delta_i^2$$

with

$$\zeta = \bar{\zeta}.$$

Thus

$$C_{ij} - \frac{1}{3} R a_{ij} = Z\delta_i^2\delta_j^2 .$$

Since

$$a^{ij}C_{ij} = 0$$

we have

$$- R = Za^{22} = 0$$

and

$$C_{ij} = Z\delta_i^2\delta_j^2 .$$

Hence

$$C_0 = Z$$
$$C_1 = C_2 = C_3 = C_4 = 0$$

and

$$C_{ABCD} = Z\phi_A\phi_B\phi_C\phi_D \tag{2.5}$$

Since

$$R_{AB\dot{C}\dot{D}} = S_{i\bar{j}}X^i_{AB}\overline{X}^j_{CD} = -\left(R_{\mu\rho} - \frac{R}{4} g_{\mu\rho}\right)\gamma^\rho_{\dot{C}A}\gamma^\mu_{\dot{D}B}$$

it follows from the expression for S_{ij} that

$$\zeta\phi_A\phi_B\overline{\phi}_C\overline{\phi}_D = -\left(R_{\mu\rho} - \frac{R}{4} g_{\mu\rho}\right)\gamma^\rho_{\dot{C}A}\gamma^\mu_{\dot{D}B}$$

The spinors ϕ_A and ψ_A define a null tetrad of vectors by the equations

$$l_\mu = \frac{1}{\sqrt{2}}\gamma_\mu{}^{\dot{A}B}\overline{\phi}_A\phi_B$$

$$n_\mu = \frac{1}{\sqrt{2}}\gamma_\mu{}^{\dot{A}B}\overline{\psi}_A\psi_B$$

$$m_\mu = \frac{1}{\sqrt{2}}\gamma_\mu{}^{\dot{A}B}\overline{\psi}_A\psi_B$$

$$\overline{m}_\mu = \frac{1}{\sqrt{2}}\gamma_\mu{}^{\dot{A}B}\overline{\phi}_A\psi_B$$

On using the fact that

$$\gamma^\rho_{\dot{C}A}\gamma_\sigma^{\dot{C}A} = - 2\delta^\rho_\sigma$$

it follows from the above expression that

$$R_{\sigma\tau} = -\frac{1}{2}\zeta l_\sigma l_\tau . \tag{2.6}$$

Where l_σ is the covariantly constant null vector associated with the covariantly constant spinor ϕ_A.

Note that since

$$R_{AB\mu\nu} = X^i_{AB}R_{i\mu\nu} = 1/8\phi_A\phi_B(Zp^2_{\mu\nu} + \zeta\bar{p}^2_{\mu\nu}).$$

Hence equations (2.2), the integrability equations of equations (2.1) are satisfied.

3. THE BIANCHI IDENTITIES

These identities imply that

$$C^{\kappa}_{\lambda\mu\nu;\kappa} = -\frac{1}{2}\left[\left(R_{\lambda\nu} - \frac{R}{6}g_{\lambda\nu}\right)_{;\mu} - \left(R_{\lambda\mu} - \frac{R}{6}g_{\lambda\mu}\right)_{;\nu}\right]. \tag{3.1}$$

The spinor

$$C_{ABCD} = \frac{1}{4}p^{\rho\sigma}_{AB}C_{\rho\sigma\mu\nu}p^{\mu\nu}_{CD}.$$

Hence

$$\gamma^{\tau\dot{E}A}C_{ABCD;\tau} = -\frac{1}{4}\gamma^{\tau\dot{E}}{}_{A}p^{\rho\sigma A}{}_{B}C_{\rho\sigma\mu\nu;\tau}p^{\mu\nu}_{CD}$$
$$= -C^{\tau}{}_{\alpha\mu\nu;\tau}\gamma^{\alpha\dot{E}}{}_{B}p^{\mu\nu}_{CD} \tag{3.2}$$

as follows from the fact that

$$\gamma^{\tau\dot{E}}{}_{A}p^{\sigma\rho A}{}_{B} = -[E^{\alpha\tau\sigma\rho} + g^{\alpha\sigma}g^{\tau\rho} - g^{\alpha\rho}g^{\tau\sigma}]\gamma^{\dot{E}}_{\alpha B}$$

and the fact that the conformal tensor is equal to its double dual.

Substituting from equations (2.5), (2.6) and (3.1) into equations (3.2) we obtain

$$\gamma^{\tau\dot{E}A}Z_{,\tau}\phi_A\phi_B\phi_C\phi_D = -\frac{1}{2}l_\alpha\gamma^{\alpha\dot{E}}{}_{B}\zeta,_{\mu}l_\nu p^{\mu\nu}_{CD}. \tag{3.3}$$

On multiplying these equations by $\psi^B\psi^C\psi^D$ and summing one finds that

$$\gamma^{\tau\dot{E}A}Z,_{\tau}\phi_A = l_\alpha\gamma^{\alpha\dot{E}A}\psi_A\zeta,_{\mu}\overline{m}^{\mu}$$

on using the fact that

$$p^{\mu\nu}_{CD}\psi^C\psi^D = 2(n^\mu\overline{m}^\nu - n^\nu\overline{m}^\mu)$$

Hence

$$l^\tau Z,_{\tau} = 0 \tag{3.4}$$
$$m^\tau Z,_{\tau} = -\zeta,_{\tau}m^\tau \tag{3.5}$$

since

$$l_\alpha m^\alpha = l_\alpha n^\alpha + 1 = 0.$$

Equations (3.4) and (3.5) are equivalent to equations (3.3).

It follows from equation (3.3) that Z is a constant along the null geodesic with tangent l_μ. If the space-time is asymptotically flat and such a null geodesic reaches null infinity Z must vanish there. Therefore it must vanish all along the null geodesic. Hence Z must vanish at all events of space-time which lie on the infinitely extendible null geodesics with covariantly constant tangent vectors l_μ. At these events of V the conformal tensor of V vanishes. The Ricci tensor of V is given by equation (2.6) with ζ satisfying

$$\overline{m}^\tau \zeta_{,\tau} = 0$$

at these events of V. The space time V is flat if all events of V are of the type described above and $\zeta = 0$, that is the space time is empty.

APPENDIX

Algebraic Properties of the Spinors $\gamma^{\mu\dot{A}}{}_{B}$ and $p^{\mu\nu C}{}_{D}$.

The former spinors are defined as follows:

$$\| \gamma^{\mu\dot{A}}{}_{B} \| = q^{\mu}$$

where

$$\overline{q}^{\mu} q^{\nu} + \overline{q}^{\nu} q^{\mu} = 2 g^{\mu\nu} 1_{2},$$

the bar over a quantity denoting its complex conjugate. That is,

$$\overline{\gamma}^{\mu\dot{A}}{}_{B} \gamma^{\nu\dot{B}}{}_{C} + \overline{\gamma}^{\nu\dot{A}}{}_{B} \gamma^{\mu\dot{B}}{}_{C} = 2 g^{\mu\nu} \delta^{\dot{A}}_{\dot{C}} \tag{A.1}$$

Hence

$$\gamma^{\mu\dot{A}}{}_{B} \gamma^{\nu\dot{B}}{}_{A} = 2 g^{\mu\nu} \tag{A.2}$$

The spinors

$$\gamma^{\mu}{}_{\dot{A}B} = \overline{\varepsilon}_{AC} \gamma^{\mu\dot{C}}{}_{B} \tag{A.3}$$

satisfy

$$\overline{\gamma}^{\mu}{}_{\dot{A}B} = \gamma^{\mu}{}_{\dot{B}A} \tag{A.4}$$

that is form Hermitian matrices. Equations (A.2) are thus equivalent to

$$\gamma^{\mu\dot{A}B} \gamma_{\nu\dot{A}B} = - 2 \delta^{\mu}_{\nu} \tag{A.5}$$

with

$$\gamma_{\nu\dot{A}B} = g_{\nu\rho} \gamma^{\rho}_{\dot{A}B}$$

$$\gamma_{\nu}{}^{\dot{A}B} = \gamma_{\nu}{}^{\dot{A}}{}_{C} \varepsilon^{CB}$$

Since the four matrices $\gamma^{\mu}{}_{AB}$ form a basis for 2 × 2 Heimitian matrices $H_{\dot{A}B}$ we may write

$$H_{\dot{A}B} = h_{\sigma} \gamma^{\sigma}_{\dot{A}B}$$

with

$$h_{\sigma} = - \frac{1}{2} H_{\dot{A}B} \gamma_{\sigma}^{\dot{A}B} = \overline{h}_{\sigma}.$$

Since the mapping from $H_{\dot{A}B}$ to h^{σ} is one to one, we must have

$$\gamma_{\sigma}^{\dot{A}B} \gamma^{\sigma}{}_{\dot{C}D} = - 2 \delta^{\dot{A}}_{\dot{C}} \delta^{B}_{D} \tag{A.6}$$

The spinor $p_{\mu\nu}{}^{A}{}_{B}$ defined by the equations

$$\| p_{\mu\nu}{}^{A}{}_{B} \| = p_{\mu\nu} = \frac{1}{2} (\overline{q}_{\mu} q_{\nu} - \overline{q}_{\nu} q_{\mu}) = - p_{\nu\mu} \tag{A.7}$$

satisfies

$$\text{trace } p_{\mu\nu} = p_{\mu\nu}{}^{A}{}_{A} = 0 \tag{A.8}$$

$$\overset{*}{p}{}^{\mu\nu} = \frac{1}{2} E^{\mu\nu\sigma\tau} p_{\sigma\tau} = g^{\mu\sigma} g^{\nu\tau} p_{\sigma\tau} \tag{A.9}$$

$$\overline{p}^{\mu\nu} = \frac{1}{2} E^{\mu\nu\sigma\tau} \overline{p}_{\sigma\tau} = - g^{\mu\tau} g^{\nu\tau} p_{\sigma\tau} \tag{A.10}$$

where

$$E^{\sigma\tau\mu\nu} = \frac{1}{\sqrt{g}} \varepsilon^{\sigma\tau\mu\nu}; \qquad E_{\sigma\tau\mu\nu} = \sqrt{g} \varepsilon_{\sigma\tau\mu\nu}$$

$\varepsilon^{\sigma\tau\mu\nu}$ is the Levi-Civita tensor density, g is the determinent of the metric tensor $g_{\mu\nu}$ and hence is a pure imaginary tensor. The $p_{\mu\nu}$ form a (redundant) basis for all traceless 2 × 2 matrices.

It follows from equations (A.1) and (A.7) that

$$\bar{\gamma}^{\mu\dot{A}}{}_{B}\gamma^{\nu\dot{B}}{}_{C} = g^{\mu\nu}\delta^{A}_{C} + p^{\mu\nu A}{}_{C} \tag{A.11}$$

It may be verified by choosing a particular coordinate system in V and in the spin space that

$$p_{\lambda\mu}p_{\sigma\tau} = -(E_{\lambda\mu\sigma\tau} - g_{\lambda\tau}g_{\mu\sigma} + g_{\lambda\sigma}g_{\mu\tau})1_2 + g_{\lambda\tau}p_{\mu\sigma} - g_{\mu\tau}p_{\lambda\sigma} - g_{\lambda\sigma}p_{\mu\tau} + g_{\mu\sigma}p_{\lambda\tau}. \tag{A.12}$$

Hence

$$\| \gamma^{\dot{A}}_{\mu B}p^{B}_{\sigma\tau C} \| = q_{\mu}p_{\sigma\tau} = -(E_{\nu\mu\sigma\tau} - g_{\nu\tau}g_{\mu\sigma} + g_{\nu\sigma}g_{\mu\tau})q^{\nu} \tag{A.13}$$

$$\| p^{A}_{\sigma\tau B}\gamma^{\dot{B}}{}_{C} \| = p_{\sigma\tau}\bar{q}_{\mu} = (E_{\nu\mu\sigma\tau} - g_{\nu\tau}g_{\mu\sigma} + g_{\nu\sigma}g_{\mu\tau})\bar{q}^{\nu} \tag{A.14}$$

It follows from (A.12) that

$$p_{\lambda\mu AB}p^{AB}_{\sigma\tau} = -p^{A}_{\lambda\mu B}p^{B}_{\sigma\tau A} = 2(E_{\lambda\mu\sigma\tau} + g_{\lambda\sigma}g_{\mu\tau} - g_{\lambda\tau}g_{\mu\sigma}) \tag{A.15}$$

It may be verified from the definition of $p_{\mu\nu AB}$ and equation (A.6) that

$$p^{\sigma\tau}_{CD}p^{AB}_{\sigma\tau} = 4(\delta^{A}_{C}\delta^{B}_{D} + \delta^{A}_{D}\delta^{B}_{C}) \tag{A.16}$$

$$\gamma^{\sigma\dot{C}D}p_{\sigma\tau AB} = -(\gamma^{\dot{C}}_{\tau A}\delta^{D}_{B} + \gamma_{\tau B}^{\dot{C}}\delta^{D}_{A}) \tag{A.17}$$

$$p_{\mu\nu}{}^{AB}\bar{p}^{\nu}{}_{\sigma}{}^{\dot{C}\dot{D}} = -2\gamma_{\mu}{}^{\dot{C}A}\gamma_{\sigma}{}^{\dot{D}B} - g_{\mu\sigma}\varepsilon^{AB}\varepsilon^{\dot{C}\dot{D}} - p_{\mu\sigma}{}^{AB}\varepsilon^{\dot{C}\dot{D}} - \bar{p}_{\mu\sigma}{}^{\dot{C}\dot{D}}\varepsilon^{AB} \tag{A.18}$$

If ϕ^{A} and ψ^{A} are a pair of spinors satisfying

$$\phi^{A}\psi_{A} = 1 \tag{A.19}$$

then

$$\phi^{A}\psi_{B} - \psi^{A}\phi_{B} = \delta^{A}{}_{B}. \tag{A.20}$$

The vectors

$$\begin{aligned} l_{\mu} &= \frac{1}{\sqrt{2}}\gamma_{\mu\dot{A}B}\bar{\phi}^{A}\phi^{B} \\ n_{\mu} &= \frac{1}{\sqrt{2}}\gamma_{\mu\dot{A}B}\bar{\psi}^{A}\psi^{B} \\ m_{\mu} &= \frac{1}{\sqrt{2}}\gamma_{\mu\dot{A}B}\bar{\psi}^{A}\phi^{B} \\ \bar{m}_{\mu} &= \frac{1}{\sqrt{2}}\gamma_{\mu\dot{A}B}\bar{\phi}^{A}\psi^{B} \quad p^{\nu}{}_{\sigma} \end{aligned} \tag{A.21}$$

form a null tetrad such that

$$g_{\mu\nu} = -l_{\mu}n_{\nu} - n_{\mu}l_{\nu} + m_{\mu}\bar{m}_{\nu} + \bar{m}_{\mu}m_{\nu} \tag{A.22}$$

The quantities

$$\begin{aligned} p_0{}^{\mu\nu} &= p^{\mu\nu}_{AB}\phi^{A}\phi^{B} = 2(m^{\mu}l^{\nu} - m^{\nu}l^{\mu}) = \overset{*}{p}_0{}^{\mu\nu} \\ p_1{}^{\mu\nu} &= p^{\mu\nu}_{AB}\phi^{A}\phi^{B} = (m^{\mu}\bar{m}^{\nu} - \bar{m}^{\mu}m^{\nu} + n^{\mu}l^{\nu} - n^{\nu}l^{\mu}) = \overset{*}{p}_1{}^{\mu\nu} \\ p_2{}^{\mu\nu} &= p^{\mu\nu}_{AB}\psi^{A}\psi^{B} = 2(n^{\mu}\bar{m}^{\nu} - n^{\nu}\bar{m}^{\mu}) = p_2{}^{\nu\mu\nu} \end{aligned}$$

where

$$\overset{*}{p}{}^{\mu\nu}_{i} = \frac{1}{2}E^{\mu\nu\sigma\tau}g_{\sigma\alpha}g_{\tau\beta}p_{i}^{\alpha\beta}.$$

That is the $p_i^{\mu\nu}$ are three linearly independent self-dual anti-symmetric tensors. Their complex conjugates are anti-self-dual anti-symmetric tensors.

It may be verified that

$$p_i^{\mu\nu} p_{j\mu\nu} = 8a_{ij}$$
$$p_i^{\mu\nu} \overline{p}_{j\mu\nu} = 0$$

where

$$\| a_{ij} \| \quad \left\| \begin{matrix} 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 1 & 0 & 0 \end{matrix} \right\| .$$

REFERENCE

[1] A. H. TAUB, « Curvature Invariants, Characteristic Classes and Petrov Classification of Space-Times », in *Differential Geometry and Relativity* Cahen & Flato (eds.), D. Reidel Publishing Co., Dordrecth-Holland, 1976.

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