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Space-times admitting a covariantly constant spinor field

by

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ABSTRACT. — It is shown that if a space-time V admits a covariantly constant spinor field ψ_A , and hence a covariantly constant null vector field l_μ determined by ψ_A , its Ricci tensor is proportional to the tensor product of l_μ by itself. Further, the conformal tensor of V is shown to be of Petrov-Penrose type N. That is the four index symmetric spinor determined by the conformal tensor of V is proportional to the spinor product of ψ_A with itself. The Bianchi identities are used to show that empty asymptotically flat space-times with infinitely extendible null geodesics tangent to l_μ are flat.

RÉSUMÉ. — On montre que, si un espace temps V admet un champ spinoriel constant par covariance ψ_A , et par suite un champ de vecteurs isotrope constant par covariance l_μ déterminé par ψ_A , son tenseur de Ricci est proportionnel au tenseur produit de l_μ par lui-même. En outre, on montre que le tenseur conforme de V est du type N de Petrov-Penrose, c'est-à-dire que le spineur symétrique à quatre indices déterminé par le tenseur conforme de V est proportionnel au produit spinoriel de ψ_A par lui-même. On utilise les identités de Bianchi pour montrer que des espaces temps vides et asymptotiquement plats, dont les géodésiques isotropes tangentes à l_μ sont prolongeables indéfiniment, sont plats.

1. INTRODUCTION

It is the purpose of this paper to characterize those space-times that admit a covariantly constant two component spinor field. In addition, we shall prove that asymptotically flat empty space-times that contain such a spinor field are flat.

We shall use the notation and results of reference [1] where spinors ϕ^A and ψ^A satisfying

$$\phi^A \psi_A = 1 ; \quad (1.1)$$

are introduced. Spinor indices are manipulated with the asymmetric Levi-Civita tensor density ε_{AB} and ε^{AB} by the rules

$$\phi_A = \varepsilon_{AB} \phi^B$$

and

$$\phi^A = \phi_B \varepsilon^{BA}$$

Equation (1.1) is equivalent to

$$\phi^A \psi_B - \psi^A \phi_B = \delta_B^A \quad (1.2)$$

In terms of these the Newman-Penrose spin coefficients are defined as

$$\begin{aligned} A_{0v} &= \phi^A \phi_{A;v} \\ A_{1v} &= \phi^A \psi_{A;v} = \psi^A \phi_{A;v} \\ A_{2v} &= \psi^A \psi_{A;v} \end{aligned} \quad (1.3)$$

where the semi-colon denotes the spinor covariant derivative with respect to the metric of the space-time V.

The spin-coefficients define three two forms by the equation

$$R_{i\mu\nu} = A_{i\mu;v} - A_{i\mu;v} - \sqrt{2} E_{ijk} A_\mu^j A_\nu^k \quad i, j, k = (1, 2, 3) \quad (1.4)$$

where

$$E_{ijk} = \sqrt{a} \varepsilon_{ijk}$$

with ε_{ijk} the Levi-Civita alternating tensor density and

$$\begin{aligned} a &= \det \| a_{ij} \| \\ \| a_{ij} \| &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1/2 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \| a^{ij} \| = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ a_{ij} a^{jk} &= \delta_i^k \end{aligned}$$

Lower case latin indices are manipulated with the a_{ij} and a^{ij} .

The quantities X_{iAB} are defined by

$$\begin{aligned} X_{0AB} &= \phi_A \phi_B & &= X_{AB}^2 \\ X_{1AB} &= \frac{1}{2}(\phi_A \phi_B + \psi_A \psi_B) & &= -\frac{1}{2} X_{AB}^1 \\ X_{2AB} &= \psi_A \psi_B & &= X_{AB}^0 \end{aligned}$$

and satisfy

$$X_{iAB} X^{iCD} = \frac{1}{2}(\delta_A^C \delta_B^D + \delta_B^C \delta_A^D)$$

The $R_{i\mu\nu}$ are related to the spinor curvature two-forms $R_{AB\mu\nu}$ by the equations

$$R_{i\mu\nu} = X_i^{AB} R_{AB\mu\nu} = \frac{1}{4} X_i^{AB} p_{AB}^{\rho\sigma} R_{\rho\sigma\mu\nu}$$

where $R_{\rho\sigma\mu\nu}$ are the components of the Riemann curvature tensor of V whose metric is $g_{\mu\nu}$. This metric determines 2×2 matrices

$$q_\mu = \|\gamma_\mu^{\dot{A}}{}_B\|$$

such that

$$\bar{q}_\mu q_\nu + \bar{q}_\nu q_\mu = 2g_{\mu\nu} 1_2$$

when the bar over a quantity denotes its complex conjugate. Matrices $\|p_{\mu\nu}^{\dot{A}}{}_B\|$ are defined by the equations

$$2p_{\mu\nu} = \bar{q}_\mu q_\nu - \bar{q}_\nu q_\mu.$$

Greek indices are manipulated by $g_{\mu\nu}$ and $g^{\mu\nu}$. Various algebraic properties of the matrices $p_{\mu\nu}$ are given in the appendix.

We may also write

$$R_{i\mu\nu} = \frac{1}{8} \left((C_{ij} - \frac{R}{3} a_{ij}) p_{\mu\nu}^j + S_{i\bar{j}} \bar{P}_{\mu\nu}^j \right)$$

where R is the scalar curvature of the space-time,

$$C_{ij} = \frac{1}{4} p_{AB}^{\rho\sigma} C_{\rho\sigma\mu\nu} p_{CD}^{\mu\nu} X_i^{AB} X_j^{CD} = R_{i\mu\nu} p_j^{\mu\nu}$$

where $C_{\rho\sigma\mu\nu}$ is the conformal tensor of V , and

$$S_{i\bar{j}} = \frac{1}{4} p_{AB}^{\rho\sigma} R_{\rho\sigma\mu\nu} \bar{p}_{CD}^{\mu\nu} X_i^{AB} \bar{X}_j^{CD}.$$

When we write

$$C_{ABCD} = X_{AB}^i C_{ij} X_{CD}^j$$

we have

$$\begin{aligned} C_{ABCD} &= C_0 \phi_A \phi_B \phi_C \phi_D + 4C_1 (\phi_A \phi_B \phi_C \psi_D) \\ &\quad + 6C_2 (\phi_A \phi_B \psi_C \psi_D) + 4C_3 (\psi_A \psi_B \psi_C \psi_D) + C_4 \psi_A \psi_B \psi_C \psi_D \end{aligned}$$

where the parentheses denote the symmetric sum and

$$\| C_{ij} \| = \left\| \begin{array}{ccc} C_4 & -C_3 & C_2 \\ -C_3 & C_2 & -C_1 \\ C_2 & -C_1 & C_0 \end{array} \right\|$$

2. THE COVARIANTLY CONSTANT CONDITION

This condition may be expressed as the equations

$$\phi_{A;\mu} = 0 \quad (2.1)$$

The integrability conditions of these equations are

$$\phi_{A;\mu\nu} - \phi_{A;\nu\mu} = \phi_{C R A \mu\nu} = 0 \quad (2.2)$$

In this section we shall discuss consequences of equations (2.1). It follows from these equations and equations (1.3) that

$$A_{0\mu} = A_{1\mu} = 0 \quad (2.3)$$

Equations (1.4) then imply that

$$R_{i\mu\nu} = \delta_i^2 (A_{2\nu;\mu} - A_{2\mu;\nu}) \quad (2.4)$$

Since the quantities $p_{\mu\nu}^i$ (and $\bar{p}_{\mu\nu}^i$) provide a basis for self-dual (and anti-self-dual) two forms in V , we may write

$$A_{2\nu;\mu} - A_{2\mu;\nu} = \frac{1}{8} (Z^i p_{i\mu\nu} + \zeta^i \bar{p}_{i\mu\nu})$$

where

$$\begin{aligned} Z^i &= p^{i\mu\nu} (A_{2\nu;\mu} - A_{2\mu;\nu}) \\ \zeta^i &= \bar{p}^{i\mu\nu} (A_{2\nu;\mu} - A_{2\mu;\nu}) \end{aligned}$$

Hence

$$R_{i\mu\nu} = \frac{1}{8} (\delta_i^2 Z_j p_{\mu\nu}^j + \delta_i^2 \zeta_j \bar{p}_{\mu\nu}^j)$$

Then

$$C_{ij} - \frac{R}{3} A_{ij} = \delta_i^2 Z_j$$

Since the left hand side of this equation is symmetric in i and j we must have

$$Z_j = \delta_j^2 Z$$

Similarly we have

$$S_{i\bar{j}} = \delta_i^2 \zeta_{\bar{j}} = \bar{S}_{j\bar{2}}$$

Hence

$$\zeta_i = \zeta \delta_i^2$$

with

$$\zeta = \bar{\zeta}.$$

Thus

$$C_{ij} - \frac{1}{3} R a_{ij} = Z \delta_i^2 \delta_j^2.$$

Since

$$a^{ij} C_{ij} = 0$$

we have

$$-R = Z a^{22} = 0$$

and

$$C_{ij} = Z \delta_i^2 \delta_j^2.$$

Hence

$$\begin{aligned} C_0 &= Z \\ C_1 &= C_2 = C_3 = C_4 = 0 \end{aligned}$$

and

$$C_{ABCD} = Z \phi_A \phi_B \phi_C \phi_D \quad (2.5)$$

Since

$$R_{AB\dot{C}\dot{D}} = S_{i\bar{j}} X_{AB}^i \bar{X}_{CD}^{\bar{j}} = - \left(R_{\mu\rho} - \frac{R}{4} g_{\mu\rho} \right) \gamma_{CA}^\rho \gamma_{DB}^\mu$$

it follows from the expression for S_{ij} that

$$\zeta \phi_A \phi_B \bar{\phi}_C \bar{\phi}_D = - \left(R_{\mu\rho} - \frac{R}{4} g_{\mu\rho} \right) \gamma_{CA}^\rho \gamma_{DB}^\mu$$

The spinors ϕ_A and ψ_A define a null tetrad of vectors by the equations

$$\begin{aligned} l_\mu &= \frac{1}{\sqrt{2}} \gamma_{\mu\dot{A}\dot{B}} \bar{\phi}_A \phi_B \\ n_\mu &= \frac{1}{\sqrt{2}} \gamma_{\mu\dot{A}\dot{B}} \bar{\psi}_A \psi_B \\ m_\mu &= \frac{1}{\sqrt{2}} \gamma_{\mu\dot{A}\dot{B}} \bar{\psi}_A \phi_B \\ \bar{m}_\mu &= \frac{1}{\sqrt{2}} \gamma_{\mu\dot{A}\dot{B}} \bar{\phi}_A \psi_B \end{aligned}$$

On using the fact that

$$\gamma_{CA}^\rho \gamma_\sigma^{\dot{C}\dot{A}} = -2\delta_\sigma^\rho$$

it follows from the above expression that

$$R_{\sigma\tau} = -\frac{1}{2} \zeta l_\sigma l_\tau. \quad (2.6)$$

Where l_σ is the covariantly constant null vector associated with the covariantly constant spinor ϕ_A .

Note that since

$$R_{AB\mu\nu} = X_{AB}^i R_{i\mu\nu} = 1/8\phi_A \phi_B (Zp_{\mu\nu}^2 + \zeta \bar{p}_{\mu\nu}^2).$$

Hence equations (2.2), the integrability equations of equations (2.1) are satisfied.

3. THE BIANCHI IDENTITIES

These identities imply that

$$C_{\lambda\mu\nu;\kappa}^\kappa = -\frac{1}{2} \left[\left(R_{\lambda\nu} - \frac{R}{6} g_{\lambda\nu} \right)_{;\mu} - \left(R_{\lambda\mu} - \frac{R}{6} g_{\lambda\mu} \right)_{;\nu} \right]. \tag{3.1}$$

The spinor

$$C_{ABCD} = \frac{1}{4} P_{AB}^{\rho\sigma} C_{\rho\sigma\mu\nu} P_{CD}^{\mu\nu}.$$

Hence

$$\begin{aligned} \gamma^{\tau\dot{E}A} C_{ABCD;\tau} &= -\frac{1}{4} \gamma^{\tau\dot{E}}{}_A P^{\rho\sigma A}{}_B C_{\rho\sigma\mu\nu;\tau} P_{CD}^{\mu\nu} \\ &= -C_{\alpha\mu\nu;\tau}^\tau \gamma^{\alpha\dot{E}}{}_B P_{CD}^{\mu\nu} \end{aligned} \tag{3.2}$$

as follows from the fact that

$$\gamma^{\tau\dot{E}A} P^{\sigma\rho A}{}_B = -[E^{\alpha\tau\rho\sigma} + g^{\alpha\sigma} g^{\tau\rho} - g^{\alpha\rho} g^{\tau\sigma}] \gamma_{\alpha B}^{\dot{E}}$$

and the fact that the conformal tensor is equal to its double dual.

Substituting from equations (2.5), (2.6) and (3.1) into equations (3.2) we obtain

$$\gamma^{\tau\dot{E}A} Z_{,\tau} \phi_A \phi_B \phi_C \phi_D = -\frac{1}{2} l_\alpha \gamma^{\alpha\dot{E}A} \psi_B \zeta_{,\mu} l_\nu P_{CD}^{\mu\nu}. \tag{3.3}$$

On multiplying these equations by $\psi^B \psi^C \psi^D$ and summing one finds that

$$\gamma^{\tau\dot{E}A} Z_{,\tau} \phi_A = l_\alpha \gamma^{\alpha\dot{E}A} \psi_A \zeta_{,\mu} \bar{m}^\mu$$

on using the fact that

$$P_{CD}^{\mu\nu} \psi^C \psi^D = 2(n^\mu \bar{m}^\nu - n^\nu \bar{m}^\mu)$$

Hence

$$l^\tau Z_{,\tau} = 0 \tag{3.4}$$

$$m^\tau Z_{,\tau} = -\zeta_{,\tau} m^\tau \tag{3.5}$$

since

$$l_\alpha m^\alpha = l_\alpha n^\alpha + 1 = 0.$$

Equations (3.4) and (3.5) are equivalent to equations (3.3).

It follows from equation (3.3) that Z is a constant along the null geodesic with tangent l_μ . If the space-time is asymptotically flat and such a null geodesic reaches null infinity Z must vanish there. Therefore it must vanish all along the null geodesic. Hence Z must vanish at all events

of space-time which lie on the infinitely extendible null geodesics with covariantly constant tangent vectors l_μ . At these events of V the conformal tensor of V vanishes. The Ricci tensor of V is given by equation (2.6) with ζ satisfying

$$\bar{m}^\tau \zeta_{,\tau} = 0$$

at these events of V . The space time V is flat if all events of V are of the type described above and $\zeta = 0$, that is the space time is empty.

APPENDIX

Algebraic Properties of the Spinors $\gamma^{\mu\dot{A}}_B$ and $p^{\mu\nu C}_D$.

The former spinors are defined as follows:

$$\|\gamma^{\mu\dot{A}}_B\| = q^\mu$$

where

$$\bar{q}^\mu q^\nu + \bar{q}^\nu q^\mu = 2g^{\mu\nu}1_2,$$

the bar over a quantity denoting its complex conjugate. That is,

$$\bar{\gamma}^{\mu\dot{A}}_B \gamma^{\nu\dot{B}}_C + \bar{\gamma}^{\nu\dot{A}}_B \gamma^{\mu\dot{B}}_C = 2g^{\mu\nu}\delta^{\dot{A}}_{\dot{C}} \quad (\text{A. 1})$$

Hence

$$\gamma^{\mu\dot{A}}_B \gamma^{\nu\dot{B}}_A = 2g^{\mu\nu} \quad (\text{A. 2})$$

The spinors

$$\gamma^{\mu\dot{A}}_B = \bar{e}_{AC} \gamma^{\mu\dot{C}}_B \quad (\text{A. 3})$$

satisfy

$$\bar{\gamma}^{\mu\dot{A}}_B = \gamma^{\mu\dot{B}}_A \quad (\text{A. 4})$$

that is form Hermitian matrices. Equations (A.2) are thus equivalent to

$$\gamma^{\mu\dot{A}}_B \gamma_{\nu\dot{A}B} = -2\delta^\mu_\nu \quad (\text{A. 5})$$

with

$$\begin{aligned} \gamma_{\nu\dot{A}B} &= g_{\nu\rho} \gamma^{\rho\dot{A}}_B \\ \gamma_{\nu\dot{A}B} &= \gamma_{\nu\dot{C}B} \epsilon^{\dot{C}\dot{A}} \end{aligned}$$

Since the four matrices $\gamma^{\mu\dot{A}}_B$ form a basis for 2×2 Heimitian matrices $H_{\dot{A}B}$ we may write

$$H_{\dot{A}B} = h_\sigma \gamma^{\dot{A}\sigma}_B$$

with

$$h_\sigma = -\frac{1}{2} H_{\dot{A}B} \gamma^{\dot{A}\sigma}_B = \bar{h}_\sigma.$$

Since the mapping from $H_{\dot{A}B}$ to h^σ is one to one, we must have

$$\gamma^{\dot{A}\sigma}_B \gamma^{\dot{C}\sigma}_D = -2\delta^{\dot{A}\dot{C}} \delta^B_D \quad (\text{A. 6})$$

The spinor $p_{\mu\nu}{}^A_B$ defined by the equations

$$\|p_{\mu\nu}{}^A_B\| = p_{\mu\nu} = \frac{1}{2} (\bar{q}_\mu q_\nu - \bar{q}_\nu q_\mu) = -p_{\nu\mu} \quad (\text{A. 7})$$

satisfies

$$\text{trace } p_{\mu\nu} = p_{\mu\nu}{}^A_A = 0 \quad (\text{A. 8})$$

$$p^{\mu\nu} = \frac{1}{2} E^{\mu\nu\sigma\tau} p_{\sigma\tau} = g^{\mu\sigma} g^{\nu\tau} p_{\sigma\tau} \quad (\text{A. 9})$$

$$\bar{p}^{\mu\nu} = \frac{1}{2} E^{\mu\nu\sigma\tau} \bar{p}_{\sigma\tau} = -g^{\mu\sigma} g^{\nu\tau} p_{\sigma\tau} \quad (\text{A. 10})$$

where

$$E^{\sigma\tau\mu\nu} = \frac{1}{\sqrt{g}} \epsilon^{\sigma\tau\mu\nu}; \quad E_{\sigma\tau\mu\nu} = \sqrt{g} \epsilon_{\sigma\tau\mu\nu}$$

$\varepsilon^{\sigma\mu\nu}$ is the Levi-Civita tensor density, g is the determinant of the metric tensor $g_{\mu\nu}$ and hence is a pure imaginary tensor. The $p_{\mu\nu}$ form a (redundant) basis for all traceless 2×2 matrices.

It follows from equations (A.1) and (A.7) that

$$\bar{\gamma}^{\mu\dot{A}}_{\dot{B}} \gamma^{\dot{B}}_{\dot{C}} = g^{\mu\nu} \delta^{\dot{A}}_{\dot{C}} + p^{\mu\nu\dot{A}}_{\dot{C}} \tag{A.11}$$

It may be verified by choosing a particular coordinate system in V and in the spin space that

$$p_{\lambda\mu} p_{\sigma\tau} = - (E_{\lambda\mu\sigma\tau} - g_{\lambda\tau} g_{\mu\sigma} + g_{\lambda\sigma} g_{\mu\tau}) 1_2 \tag{A.12}$$

Hence

$$\| \gamma^{\dot{A}}_{\dot{B}} p^{\dot{B}}_{\dot{C}} \| = q_{\mu} p_{\sigma\tau} = - (E_{\nu\mu\sigma\tau} - g_{\nu\tau} g_{\mu\sigma} + g_{\nu\sigma} g_{\mu\tau}) q^{\nu} \tag{A.13}$$

$$\| p^{\dot{A}}_{\dot{B}} \gamma^{\dot{B}}_{\dot{C}} \| = p_{\sigma\tau} \bar{q}_{\mu} = (E_{\nu\mu\sigma\tau} - g_{\nu\tau} g_{\mu\sigma} + g_{\nu\sigma} g_{\mu\tau}) \bar{q}^{\nu} \tag{A.14}$$

It follows from (A.12) that

$$p_{\lambda\mu\dot{A}\dot{B}} p^{\dot{A}\dot{B}}_{\sigma\tau} = - p^{\dot{A}}_{\dot{B}} p^{\dot{B}}_{\sigma\tau\dot{A}} = 2(E_{\lambda\mu\sigma\tau} + g_{\lambda\sigma} g_{\mu\tau} - g_{\lambda\tau} g_{\mu\sigma}) \tag{A.15}$$

It may be verified from the definition of $p_{\mu\nu\dot{A}\dot{B}}$ and equation (A.6) that

$$p^{\sigma\tau}_{\dot{C}\dot{D}} p^{\dot{A}\dot{B}}_{\sigma\tau} = 4(\delta^{\dot{A}}_{\dot{C}} \delta^{\dot{B}}_{\dot{D}} + \delta^{\dot{A}}_{\dot{D}} \delta^{\dot{B}}_{\dot{C}}) \tag{A.16}$$

$$\gamma^{\sigma\dot{C}\dot{D}} p_{\sigma\tau\dot{A}\dot{B}} = - (\gamma^{\dot{C}}_{\dot{A}} \delta^{\dot{D}}_{\dot{B}} + \gamma^{\dot{D}}_{\dot{B}} \delta^{\dot{C}}_{\dot{A}}) \tag{A.17}$$

$$p_{\mu\nu}{}^{\dot{A}\dot{B}} \bar{p}^{\dot{C}\dot{D}}_{\sigma} = - 2\gamma_{\mu}{}^{\dot{C}\dot{A}} \gamma_{\sigma}{}^{\dot{D}\dot{B}} - g_{\mu\sigma} \varepsilon^{\dot{A}\dot{B}} \varepsilon^{\dot{C}\dot{D}} - p_{\mu\sigma}{}^{\dot{A}\dot{B}} \varepsilon^{\dot{C}\dot{D}} - \bar{p}_{\mu\sigma}{}^{\dot{C}\dot{D}} \varepsilon^{\dot{A}\dot{B}} \tag{A.18}$$

If $\phi^{\dot{A}}$ and $\psi^{\dot{A}}$ are a pair of spinors satisfying

$$\phi^{\dot{A}} \psi_{\dot{A}} = 1 \tag{A.19}$$

then

$$\phi^{\dot{A}} \psi_{\dot{B}} - \psi^{\dot{A}} \phi_{\dot{B}} = \delta^{\dot{A}}_{\dot{B}}. \tag{A.20}$$

The vectors

$$\begin{aligned} l_{\mu} &= \frac{1}{\sqrt{2}} \gamma_{\mu\dot{A}\dot{B}} \bar{\phi}^{\dot{A}} \phi^{\dot{B}} \\ n_{\mu} &= \frac{1}{\sqrt{2}} \gamma_{\mu\dot{A}\dot{B}} \bar{\psi}^{\dot{A}} \psi^{\dot{B}} \\ m_{\mu} &= \frac{1}{\sqrt{2}} \gamma_{\mu\dot{A}\dot{B}} \bar{\psi}^{\dot{A}} \phi^{\dot{B}} \\ \bar{m}_{\mu} &= \frac{1}{\sqrt{2}} \gamma_{\mu\dot{A}\dot{B}} \bar{\phi}^{\dot{A}} \psi^{\dot{B}} \end{aligned} \tag{A.21}$$

form a null tetrad such that

$$g_{\mu\nu} = - l_{\mu} n_{\nu} - n_{\mu} l_{\nu} + m_{\mu} \bar{m}_{\nu} + \bar{m}_{\mu} m_{\nu} \tag{A.22}$$

The quantities

$$\begin{aligned} p_0{}^{\mu\nu} &= p^{\mu\nu}_{\dot{A}\dot{B}} \phi^{\dot{A}} \phi^{\dot{B}} = 2(m^{\mu} l^{\nu} - m^{\nu} l^{\mu}) = \dot{p}_0{}^{\mu\nu} \\ p_1{}^{\mu\nu} &= p^{\mu\nu}_{\dot{A}\dot{B}} \phi^{\dot{A}} \psi^{\dot{B}} = (m^{\mu} \bar{m}^{\nu} - \bar{m}^{\mu} m^{\nu} + n^{\mu} l^{\nu} - n^{\nu} l^{\mu}) = \dot{p}_1{}^{\mu\nu} \\ p_2{}^{\mu\nu} &= p^{\mu\nu}_{\dot{A}\dot{B}} \psi^{\dot{A}} \psi^{\dot{B}} = 2(n^{\mu} \bar{m}^{\nu} - n^{\nu} \bar{m}^{\mu}) = p_2{}^{\mu\nu} \end{aligned}$$

where

$$\dot{p}_i{}^{\mu\nu} = \frac{1}{2} E^{\mu\nu\sigma\tau} g_{\sigma\alpha} g_{\tau\beta} p_i{}^{\alpha\beta}.$$

That is the $p_i^{\mu\nu}$ are three linearly independent self-dual anti-symmetric tensors. Their complex conjugates are anti-self-dual anti-symmetric tensors.

It may be verified that

$$\begin{aligned} p_i^{\mu\nu} p_{j\mu\nu} &= 8a_{ij} \\ p_i^{\mu\nu} \bar{p}_{j\mu\nu} &= 0 \end{aligned}$$

where

$$\| a_{ij} \| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 1 & 0 & 0 \end{vmatrix}.$$

REFERENCE

- [1] A. H. TAUB, « Curvature Invariants, Characteristic Classes and Petrov Classification of Space-Times », in *Differential Geometry and Relativity* Cahen & Flato (eds.), D. Reidel Publishing Co., Dordrecht-Holland, 1976.

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