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DAVID ELWORTHY
AUBREY TRUMAN

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Feynman maps, Cameron-Martin formulae and anharmonic oscillators

by

David ELWORTHY and **Aubrey TRUMAN**

Maths Institute, University of Warwick, Coventry CV4 7AL

Maths Department, University College of Swansea, Singleton Park, Swansea SA2 8PP

ABSTRACT. — A Cameron-Martin formula is established for the Feynman map definition of the Feynman integral in non-relativistic quantum mechanics. This formula involves the Morse or Maslov indices and shows the connection with the Feynman-Hibbs definition of the path integral and the Fresnel integrals of Albeverio and Hoegh-Krohn. Applications are given to the Feynman-Kac-Itô formula for anharmonic oscillator potentials.

RÉSUMÉ. — On établit une formule de Cameron-Martin pour l'intégrale de Feynman en mécanique quantique non relativiste. Cette formule fait intervenir les indices de Morse et de Maslov et montre la relation avec la définition de Feynman-Hibbs de l'intégrale de chemins et les intégrales de Fresnel considérées par Albeverio et Hoegh-Krohn. On donne des applications à la formule de Feynman-Kac-Itô pour des potentiels d'oscillateurs anharmoniques.

1. INTRODUCTION

In this paper, as announced in Ref. (5), we derive a Cameron-Martin formula for Feynman integrals in non-relativistic quantum mechanics (Theorem 3 C). A special case of this formula explains how Feynman integrals transform under linear changes of integration variables. Here the transformation law is very similar to the well-known Cameron-Martin formula for Wiener integrals [3]. The more general result which we prove here involves Morse (or Maslov) indices [10] and shows clearly the connection between the Feynman-Hibbs definition of the path integral and Albeverio and Hoegh-Krohn's Fresnel integrals relative to non-singular quadratic forms [1] [7].

Our simple derivation depends rather heavily upon the use of oscillatory integrals introduced by Hörmander [9]. This treatment seems to make the proofs technically easy because it is equivalent to working with the physicists' convention « $\exp(\pm i\infty) = 0$ » [6]. It is known that this convention gives physically correct answers in a wide variety of circumstances (see Ref. (6)). Thus, oscillatory integrals seem natural in this context [8].

We give a simple application of the Cameron-Martin formula to establishing the Feynman-Kac-Itô formula for a class of anharmonic oscillator potentials by proving the L^2-convergence of Feynman integrals (see Sections 4 A, B, C). Applications to quasiclassical expansions were already given in Refs. (5 *b*) and (11). Our Cameron-Martin result also highlights a potential difficulty in defining Feynman integrals by general approximation schemes as we explain in Corollary 3 C. We also discuss other connections with the Wiener integral such as the transformation properties under translations of the path-space (see Sections 5 A, B). Again the proofs here are not involved if one works with oscillatory integrals.

2. FINITE DIMENSIONAL INTEGRALS

A. Oscillatory Integrals.

Let $B : \mathbb{R}^m \to \mathbb{R}^m$ be linear and self-adjoint with respect to $\langle \cdot, \cdot \rangle$, the Euclidean inner product on $\mathbb{R}^m$, so that $B^* = B$. Set $T = 1 + B$. Suppose $\phi \in \mathscr{S}(\mathbb{R}^m)$, the space of rapidly decreasing functions $\phi : \mathbb{R}^m \to \mathbb{C}$ and take $\varepsilon > 0$. For a measurable map $f : \mathbb{R}^m \to E$ into a complex Banach space E set

$$l_\phi^\varepsilon(f) = \int_{\mathbb{R}^m} (2\pi i)^{-m/2} \exp\left\{ \frac{i}{2} \langle Tx, x \rangle \right\} \phi(\varepsilon x) f(x) dx ,$$

where $z^{-m/2}$ is chosen so that for $\operatorname{Re}(z) \geq 0$ it is continuous and equals 1 at 1.

If $l_\phi(f) = \lim_{\varepsilon \downarrow 0} l_\phi^\varepsilon(f)$ exists for all $\phi \in \mathscr{S}(\mathbb{R}^m)$ with $\phi(0) = 1$ and has a value independent of ϕ we say that $f \in \mathscr{I}_B(\mathbb{R}^m ; E)$ and we write

$$\int_{\mathbb{R}^m}^{0} (2\pi i)^{-m/2} \exp\left\{ \frac{i}{2} \langle Tx, x \rangle \right\} f(x) dx = l_\phi(f) ,$$

where $\int^0$ denotes the « oscillatory » integral. We are usually concerned with $E = \mathbb{C}$ and will any way assume that E is separable.

B. A Simple Cameron-Martin Formula.

By a measure of bounded variation we will mean one of the form $\mu = h(\cdot)\,|\,\mu\,|$, where h is a complex-valued function with $|\,h(x)\,| = 1$, all x, and where $|\,\mu\,|$ is a finite (positive) measure. Set $\|\,\mu\,\| = \int 1d\,|\,\mu\,| \geq 0$.

DEFINITION. — For a Hilbert space H, let $\mathcal{M}(\mathrm{H})$ denote the space of such measures on H, equipped with its Borel σ field, and let $\mathcal{F}(\mathrm{H})$ denote the space of their Fourier transforms: if $f \in \mathcal{F}(\mathrm{H})$ then $f : \mathrm{H} \to \mathbb{C}$ and

$$f(\alpha) = \int_{\mathrm{H}} \exp\{\, i(\alpha, x)\,\}\, d\mu_f(x),$$

for some $\mu_f \in \mathcal{M}(\mathrm{H})$, (,) being the Hilbert space inner product.

It is not easy to characterise the functions in $\mathcal{F}(\mathrm{H})$ even for $\mathrm{H} = \mathbb{R}^m$. However, since each $\mu \in \mathcal{M}(\mathrm{H})$ is a finite, linear, complex combination of positive measures, every $f \in \mathcal{F}(\mathrm{H})$ is a $\mathbb{C}$-linear combination of functions of positive type [see Ref. (12 b)]. In particular, it is bounded and uniformly continuous on H. For infinite dimensional H there is the extra necessary condition that it must be continuous in the Sazanov topology [see Ref. (14)]. Since the Fourier transform maps $\mathcal{S}(\mathbb{R}^m)$ into itself, we have the trivial result that $\mathcal{S}(\mathbb{R}^m) \subset \mathcal{F}(\mathbb{R}^m)$.

Our first formula is a slight extension of the well-known basis for the principle of stationary phase. [See Ref. (9), p. 145]. A proof is given here to show how it is a finite dimensional « Cameron-Martin formula », as well as showing where the extra phase factor arises. The index, Ind T, of a self-adjoint linear automorphism $\mathrm{T} : \mathbb{R}^m \to \mathbb{R}^m$ is the dimension of the negative eigenspace. We need the following lemma.

LEMMA 2B. — Let $\mathrm{T} : \mathbb{R}^m \to \mathbb{R}^m$ be a self-adjoint linear isomorphism. Then, for $h = fg$, $f \in \mathcal{S}(\mathbb{R}^m)$, $g \in \mathcal{F}(\mathbb{R}^m)$ and $\rho > 0$,

$$(2\pi i\rho)^{-m/2} \int_{\mathbb{R}^m} \exp\left\{ \frac{i}{2\rho} \langle \mathrm{T}x, x \rangle \right\} h(x)dx = \exp\left\{ \frac{-\pi i}{2} \operatorname{Ind} \mathrm{T} \right\} |\det \mathrm{T}|^{-\frac{1}{2}} \int_{\mathbb{R}^m \times \mathbb{R}^m} \exp\left\{ \frac{-i\rho}{2} \langle \mathrm{T}^{-1}x, x \rangle \right\} \tilde{f}(x-p)d\mu_g(p)dx,$$

$\tilde{}$ being inverse Fourier transform.

Proof. — First suppose $g \equiv 1 \in \mathcal{F}(\mathbb{R}^m)$.

i) If T is positive definite by a simple change of scale the above is equivalent to

$$(2\pi i\rho)^{-m/2}\int \exp\left\{\frac{i}{2\rho}\langle x, x\rangle\right\} f(x)dx = \int \exp\left\{\frac{-i\rho}{2}\langle \alpha, \alpha\rangle\right\} d\mu_f(\alpha),$$

where $d\mu_f(\alpha) = \tilde{f}(\alpha)d\alpha$. This is a standard result [see e. g. Albeverio and Hoegh-Krohn].

ii) The formula follows for T negative definite, Ind T $= m$, by replacing ρ by $-\rho$ and observing that for $\rho > 0$

$$(2\pi(-i\rho))^{-m/2} = (2\pi\rho)^{-m/2}\exp\left(\frac{m\pi i}{4}\right) = (2\pi i\rho)^{-m/2}\exp\left(\frac{m\pi i}{2}\right).$$

iii) For arbitrary T write $\mathbb{R}^m$ as the product of the positive and negative eigenspaces, $\mathbb{R}^m = E^+ \times E^-$. Then write T as $T = T^+ \times T^-$, for $T^+ : E^+ \to E^+$ positive definite, $T^- : E^- \to E^-$ negative definite. The formula now follows from (*i*) and (*ii*) above for $f: E^+ \times E^- \to \mathbb{C}$ of the form $f(u, v) = f^+(u)f^-(v), f^\pm \in \mathcal{S}(E^\pm)$. It therefore follows for f a linear combination of such functions i. e. for $f \in \mathcal{S}(E^+) \otimes \mathcal{S}(E^-)$. Since $\mathcal{S}(E^+) \otimes \mathcal{S}(E^-)$ is dense in $\mathcal{S}(\mathbb{R}^m)$, as can be seen by using Hermite polynomials, the result now follows by a simple continuity argument.

We now prove the result for $h = fg$, $f \in \mathcal{S}(\mathbb{R}^m)$ and general $g \in \mathcal{F}(\mathbb{R}^m)$. Let $I_{rs} : \mathbb{R}^m \to \mathbb{R}^m$ be a linear operator with r eigenvalues (-1) and s eigenvalues $(+1)$, $r+s=m$ and $\rho > 0$. Then we show that

$$(2\pi i\rho)^{-m/2}\int \exp\left\{\frac{i}{2\rho}\langle y, I_{rs}y\rangle\right\} h(y)dy$$
$$= \exp\left(-\frac{i\pi r}{2}\right)\int \exp\left\{-\frac{i\rho}{2}\langle x, I_{rs}x\rangle\right\} \tilde{f}(x-p)d\mu_g(p)dx,$$

$\tilde{}$ being Fourier transform. For substituting $g(y) = \int \exp\{i\langle y, p\rangle\} d\mu_g(p)$ and using Fubini's theorem, gives

$$\int \exp\left\{\frac{i}{2\rho}\langle y, I_{rs}y\rangle\right\} h(y)dy$$
$$= \int d\mu_g(p)\left\{\int \exp\left\{\frac{i}{2\rho}\langle y, I_{rs}y\rangle\right\}\exp\{i\langle y, p\rangle\} f(y)dy\right\}.$$

Since $I_{rs}^2 = I_{om} = I$, we see that

$$\text{r.h.s.} = \int d\mu_g(p)\exp\left\{-\frac{i\rho}{2}\langle p, I_{rs}p\rangle\right\}\left\{\int \exp\left\{\frac{i}{2\rho}\langle u, I_{rs}u\rangle\right\} f(u - I_{rs}\rho p)du\right\}.$$

But $f \in \mathscr{S}(\mathbb{R}^m)$ and so we can use the above to evaluate the bracketed term giving after a little calculation

$$\text{r. h. s.} = \exp\left(-\frac{i\pi r}{2}\right)\int d\mu_g(p)\tilde{f}(x-p)\exp\left\{-\frac{i\rho}{2}\langle x, \mathrm{I}_{rs}x\rangle\right\}dx.$$

A simple change of scale then gives finally for $h=fg$, $f \in \mathscr{S}(\mathbb{R}^m)$, $g \in \mathscr{F}(\mathbb{R}^m)$, T as above and $\rho > 0$

$$(2\pi i\rho)^{-m/2}\int \exp\left\{\frac{i}{2\rho}\langle y, \mathrm{T}y\rangle\right\}h(y)dy = \exp\left(-\frac{\pi i}{2}\operatorname{Ind}\mathrm{T}\right)|\det \mathrm{T}|^{-\frac{1}{2}}$$

$$\int \exp\left\{-\frac{i\rho}{2}\langle x, \mathrm{T}^{-1}x\rangle\right\}\tilde{f}(x-p)d\mu_g(p)dx$$

as required. □

PROPOSITION 2 B. — Suppose $\mathrm{T} = 1 + \mathrm{B}$ is a self-adjoint linear bijection $\mathrm{T}: \mathbb{R}^m \to \mathbb{R}^m$. Then $\mathscr{F}(\mathbb{R}^m) \subset \mathscr{I}_{\mathrm{B}}(\mathbb{R}^m; \mathbb{C})$ and for $g \in \mathscr{F}(\mathbb{R}^m)$

$$(2\pi i\rho)^{-m/2}\int_{\mathbb{R}^m}^{0} \exp\left\{\frac{i}{2\rho}\langle \mathrm{T}x, x\rangle\right\}g(x)dx = \exp\left\{-\frac{\pi i}{2}\operatorname{Ind}\mathrm{T}\right\}|\det \mathrm{T}|^{-\frac{1}{2}}$$

$$\int_{\mathbb{R}^m} \exp\left\{-\frac{i\rho}{2}\langle \mathrm{T}^{-1}x, x\rangle\right\}d\mu_g(x).$$

Proof. — Define f in the last lemma by $f(x)=\phi(\varepsilon x)$, $\varepsilon > 0$, $\phi \in \mathscr{S}(\mathbb{R}^m)$. Then we obtain

$$(2\pi i\rho)^{-m/2}\int \exp\left\{\frac{i}{2\rho}\langle y, \mathrm{T}y\rangle\right\}g(y)\phi(\varepsilon y)dy = \exp\left\{-\frac{\pi i}{2}\operatorname{Ind}\mathrm{T}\right\}|\det \mathrm{T}|^{-\frac{1}{2}}$$

$$\exp\left\{-\frac{i\rho}{2}\langle x, \mathrm{T}^{-1}x\rangle\right\}\varepsilon^{-m}\tilde{\phi}\left(\frac{x-p}{\varepsilon}\right)d\mu_g(p)dx$$

$$= \exp\left\{-\frac{\pi i}{2}\operatorname{Ind}\mathrm{T}\right\}|\det \mathrm{T}|^{-\frac{1}{2}}$$

$$\int \exp\left\{-\frac{i\rho}{2}\langle p+\varepsilon v, \mathrm{T}^{-1}(p+\varepsilon v)\rangle\,\tilde{\phi}(v)d\mu_g(p)dv\right\}.$$

Since $\int \tilde{\phi}(v)dv = \phi(0) = 1$, letting $\varepsilon \downarrow 0$ proves the proposition for any $g \in \mathscr{F}(\mathbb{R}^m)$. □

C. Integrable Functions.

We will not need this section later. It shows that there is a wide class of functions in $\mathscr{I}_{\mathrm{B}}(\mathbb{R}^m, \mathbb{C})$ and is essentially a simple special case from Hörmander [Ref. (9)].

Let $\mathrm{T}: \mathbb{R}^m \to \mathbb{R}^m$ be a linear isomorphism and $\chi: \mathbb{R}^m \to \mathbb{R}$ a C^∞ function with compact support which is identically 1 near the origin. Define $a: \mathbb{R}^m \to \mathbb{R}^m$ and $c: \mathbb{R}^m \to \mathbb{C}$ by

$$a(x) = (1 - \chi(x)) \| Tx \|^{-2} T(x)$$

and $c = \chi + i \operatorname{div} a$, $\| \quad \|$ being Euclidean norm.

Let L denote the differential operator on functions $f: \mathbb{R}^m \to \mathbb{C}$

$$L(f)(x) = i Df(x)(a(x)) + c(x)$$

i. e. $L = i \sum_j a^j \frac{\partial}{\partial x^j} + c$ for $a(x) = (a^1(x), \ldots, a^m(x))$, and ${}^tL$ the operator given by

$${}^tL(f)(x) = - i \sum_j a^j \frac{\partial}{\partial x^j} + \chi .$$

Then

$${}^tL\left(\exp\left\{ \frac{i}{2} \langle T(\cdot), (\cdot) \rangle \right\} \right) = \exp\left\{ \frac{i}{2} \langle T(\cdot), (\cdot) \rangle \right\}.$$

DEFINITION. — For real numbers n, λ with $0 \leq \lambda \leq 1$ we will say that a C^∞ map $f: \mathbb{R}^m \to \mathbb{C}$ is in $S^n_\lambda(\mathbb{R}^m)$ if for all $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$ there is a constant C_α such that

$$\left| \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_m}}{\partial x_m^{\alpha_m}} f(x) \right| \leq C_\alpha (1 + \| x \|)^{n - \lambda |\alpha|}, \qquad \text{all} \quad x \in \mathbb{R}^m,$$

where $|\alpha| = |\alpha_1| + |\alpha_2| + \ldots + |\alpha_m|$.

For example if f is C^∞ and satisfies $f(tx) = t^n f(x)$, for large $\| x \|$ and $t > 0$ then $f \in S^n_1$.

Hörmander shows that $S^n_\lambda(\mathbb{R}^m)$ is a Fréchet space under the topology defined by taking as seminorms the best constants C_α. The space increases as n increases and λ decreases. If $f \in S^n_\lambda(\mathbb{R}^m)$ and $g \in S^{n'}_\lambda(\mathbb{R}^m)$ then the product (fg) is in $S^{n+n'}_\lambda(\mathbb{R}^m)$. The details of this and of the proof of the following proposition of Hörmander can be extracted from [9]. The example afterwards, worked from first principles, may be helpful. (The bound on k is not best possible.)

PROPOSITION 2 C. — If $T = 1 + B$ is an isomorphism and $0 < \lambda$ then

$$S^\infty_\lambda(\mathbb{R}^m) \equiv \bigcup_n S^n_\lambda(\mathbb{R}^m) \subset \mathscr{I}_B(\mathbb{R}^m; \mathbb{C}).$$

In fact, if $f \in S^n_\lambda(\mathbb{R}^m)$, then for $k > (n + m + 1)/\lambda$

$$(2\pi i)^{-m/2} \int^0 \exp\left\{ \frac{i}{2} \langle Tx, x \rangle \right\} f(x) dx = (2\pi i)^{-m/2} \int_{\mathbb{R}^m} \exp\left\{ \frac{i}{2} \langle Tx, x \rangle \right\} L^k(f)(x) dx .$$

Proof. — Let $\phi \in \mathcal{S}(\mathbb{R}^m)$ with $\phi(0) = 1$ and take $\varepsilon > 0$. For $f \in S^n_\lambda(\mathbb{R}^m)$ set $f_\varepsilon(x) = \phi(\varepsilon x)f(x)$. Then

$$\begin{aligned} l^\varepsilon_\phi(f) &= (2\pi i)^{-m/2} \int \exp\left\{\frac{i}{2}\langle \mathrm{T}x, x\rangle\right\} f_\varepsilon(x)dx \\ &= (2\pi i)^{-m/2} \int {}^t\mathrm{L}\left(\exp\left\{\frac{i}{2}\langle \mathrm{T}x, x\rangle\right\}\right) f_\varepsilon(x)dx \\ &= (2\pi i)^{-m/2} \int \exp\left\{\frac{i}{2}\langle \mathrm{T}x, x\rangle\right\} \mathrm{L}(f_\varepsilon)(x)dx \\ &= \dots \\ &= (2\pi i)^{-m/2} \int \exp\left\{\frac{i}{2}\langle \mathrm{T}x, x\rangle\right\} \mathrm{L}^k(f_\varepsilon)(x)dx. \end{aligned} \tag{*}$$

Now $f_\varepsilon \in S^{n+1}_\lambda(\mathbb{R}^m)$ so $\mathrm{L}^k(f_\varepsilon) \in S^{n+1-\lambda k}_\lambda(\mathbb{R}^m)$. Also, if $n + 1 - k\lambda < -m$, then $S^{n+1-\lambda k}_\lambda(\mathbb{R}^m) \subset \mathrm{L}^1(\mathbb{R}^m)$ with a continuous inclusion. Therefore, if $\lambda k > n + 1 + m$, r. h. s. of (*) determines a continuous function of $f_\varepsilon \in S^{n+1}_\lambda$. However, $f_\varepsilon \to f$ in S^{n+1}_λ as $\varepsilon \to 0$. Thus, with $\rho = 1$,

$$\lim_{\varepsilon \to 0} l^\varepsilon_\phi(f) = (2\pi i)^{-m/2} \int_{\mathbb{R}^m} \exp\left\{\frac{i}{2}\langle \mathrm{T}x, x\rangle\right\} \mathrm{L}^k(f)(x)dx,$$

for $\lambda k > n + m + 1$. □

EXAMPLE 2 C. — Take $m = 1$ and $f(x) = x^2$. Integrating by parts, using $\phi \in \mathcal{S}(\mathbb{R})$ and $\phi(0) = 1$, gives

$$\int x^2 \phi(\varepsilon x) \exp\{ix^2/2\}\, dx = i \int \phi(\varepsilon x) \exp\{ix^2/2\}\, dx - \varepsilon^2 \int \phi''(\varepsilon x) \exp\{ix^2/2\}\, dx \to i(2\pi i)^{\frac{1}{2}}, \quad \text{as } \varepsilon \to 0.$$

Hence we see that $\int^0 x^2 \exp(ix^2/2)dx = i(2\pi i)^{\frac{1}{2}}$, while $\int x^2 \exp(ix^2/2)dx$ does not exist as an improper (Riemann or Lebesgue) integral.

D. Analyticity.

One approach to Feynman integration is by analytic continuation to Gaussian integrals, although for this our linear map T will need to be positive. First note that we can, in exactly the same way, define oscillatory integrals $\mathcal{I}^\rho_{\mathrm{B}}(f)$, $\rho \in \mathbb{R}$, by

$$\mathcal{I}^\rho_{\mathrm{B}}(f) = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^m} (2\pi i\rho)^{-m/2} \exp\left\{\frac{i}{2\rho}\langle \mathrm{T}x, x\rangle\right\} \phi(\varepsilon x) f(x)dx$$

$\rho \neq 0$ and

$$\mathcal{I}_{\mathrm{B}}^{0}(f) = |\det \mathrm{T}|^{-\frac{1}{2}} f(0).$$

When T is positive definite and f has at most polynomial growth we can define $\mathcal{I}_{\mathrm{B}}^{z}(f)$ for $im\, z < 0$ by

$$\mathcal{I}_{\mathrm{B}}^{z}(f) = (2\pi i z)^{-m/2} \int_{\mathbb{R}^m} \exp\left\{ \frac{i}{2z} \langle \mathrm{T}x, x \rangle \right\} f(x) dx .$$

PROPOSITION 2 D. — Suppose T is positive definite and $f \in \mathrm{S}_{\lambda}^{\infty}(\mathbb{R}^m)$ some $0 < \lambda \leq 1$ or $f \in \mathcal{F}(\mathbb{R}^m)$. Then $\mathcal{I}_{\mathrm{B}}^{z}(f)$ is analytic in $im\, z < 0$ and continuous on $im\, z \leq 0$.

Proof. — The case $f \in \mathcal{F}(\mathbb{R}^m)$ is known: it follows from the analogue of Lemma [2 B]. Assume therefore that $f \in \mathrm{S}_{\lambda}^{\infty}(\mathbb{R}^m)$.

For χ and $a : \mathbb{R}^m \to \mathbb{R}^m$ as in § C set

$$c_z = \chi + iz \operatorname{div} a$$

and define the differential operator L_z by

$$\mathrm{L}_z = iz \sum_j a^j \frac{\partial}{\partial x^j} + c_z .$$

Then for

$${}^t\mathrm{L}_z = -iz \sum_j a^j \frac{\partial}{\partial x^j} + \chi ,$$

$${}^t\mathrm{L}_z \exp\left\{ \frac{i}{2z} \langle \mathrm{T}(\cdot), (\cdot) \rangle \right\} = \exp\left\{ \frac{i}{2z} \langle \mathrm{T}(\cdot), (\cdot) \rangle \right\} \qquad z \neq 0 .$$

Consequently as in Proposition 2 C, for $im\, z \leq 0$, $z \neq 0$

$$\mathcal{I}_{\mathrm{B}}^{z}(f) = (2\pi i z)^{-m/2} \int_{\mathbb{R}^m} \exp\left\{ \frac{i}{2z} \langle \mathrm{T}x, x \rangle \right\} \mathrm{L}_z^k(f)(x) dx$$

for sufficiently large k, independent of z.

Since r. h. s. is analytic on $im\, z < 0$ and continuous on $im\, z \leq 0$, $z \neq 0$, the result follows after some standard manipulation for the case $z = 0$. □

EXAMPLE 2 D. — For $im\, z \leq 0$

$$(2\pi i z)^{-\frac{1}{2}} \int_{\mathbb{R}}^{\mathrm{O}} x^n \exp\left\{ \frac{ix^2}{2z} \right\} dx = \begin{cases} 0, \ n \text{ odd}, \ n > 0 \\ \dfrac{(2p)!}{2^p p!}(iz)^p, \quad n = 2p, \quad p = 1, 2, \ldots \end{cases}$$

The result is standard for $z = -i\sigma$, $\sigma > 0$, and if $f(x) = x^n$ then $f \in \mathrm{S}_1^{\infty}(\mathbb{R})$ and so we can apply the proposition (cf. Ref. (18)).

3. FEYNMAN MAPS

A. Hilbert Space of Paths.

We consider here a quantum mechanical system with Hamiltonian $H = -2^{-1}\Delta + V$, Δ being the Laplacian on $\mathbb{R}^d$, V some real-valued potential on $\mathbb{R}^d$, whose precise properties we specify later. For simplicity we take units so that Planck's constant h is equal to 2π and particle mass m is unity.

DEFINITION. — The Hilbert space of paths H is the space of continuous functions $\gamma : [0, t] \to \mathbb{R}^d$, satisfying $\gamma(t)=0$, with $\gamma(\tau)=(\gamma^1(\tau), \gamma^2(\tau), \ldots, \gamma^d(\tau))$, $\tau \in [0, t]$, γ^i absolutely continuous, $\dfrac{d\gamma^i}{d\tau} \in L^2[0, t]$, for $i = 1, 2, \ldots, d$.

H is then a real separable Hilbert space with inner product $(,)$

$$(\gamma', \gamma) = \int_0^t \left\langle \frac{d\gamma'}{d\tau}(\tau), \frac{d\gamma}{d\tau}(\tau) \right\rangle d\tau$$

$\langle , \rangle$ being the Euclidean inner product in $\mathbb{R}^d$.

Let $\pi = \{ 0 = t_0 < t_1 < t_2 \ldots < t_{m(\pi)+1} = t \}$ be a finite partition of $[0, t]$. Define the piecewise linear approximation p_π by

$$(P_\pi \gamma)(s) = \gamma(t_j) + (s - t_j)[\gamma(t_{j+1}) - \gamma(t_j)][t_{j+1} - t_j]^{-1},$$

$t_j \le s \le t_{j+1}$, $j = 0, 1, 2, \ldots, m(\pi)$. Then the following lemma is valid:

LEMMA 3 A. — $P_\pi : H \to H$ is a projection and $P_\pi \xrightarrow{s} 1$ as $\delta(\pi) \to 0$, where $\delta(\pi) = \max\limits_{j=0,1,2,\ldots,m(\pi)} | t_{j+1} - t_j |$.

Proof. — See for example Ref. (19). □

We shall require the complex Gaussian $e_z : H \to \mathbb{C}$ defined by

$$e_z(\gamma) = \exp\left\{ \frac{i}{2z}(\gamma, \gamma) \right\}, \qquad im\, z \le 0, \qquad z \ne 0.$$

For the complex-valued functional $f : H \to \mathbb{C}$ we now define the Feynman integral $\mathscr{F}^z(f)$.

DEFINITION. — We define $\mathscr{F}^z_\pi$ according to

$$\mathscr{F}^z_\pi(f) = \int_{P_\pi H}^0 (fe_z)|_{P_\pi H} \Big/ \int_{P_\pi H}^0 e_z|_{P_\pi H}, \qquad im\, z \le 0, \qquad z \ne 0.$$

Then, if $\lim\limits_{\delta(\pi)\to 0} \mathscr{F}^z_\pi(f)$ exists we say that f is $\mathscr{F}^z$-integrable and write

$$\mathscr{F}^z(f) = \lim_{\delta(\pi)\to 0} \mathscr{F}^z_\pi(f).$$

When $z = 1$, $\mathscr{F}^z$ is just the Feynman integral of Feynman and Hibbs suitably abstracted for suitable integrands [7]. When $z = -i$, $\mathscr{F}^z$ reduces to the Wiener integral. This explains why we write $\mathscr{F}^1$ as $\mathscr{F}$ and $\mathscr{F}^{-i}$ as $\mathbb{E}$.

B. The Index.

If $L \in L(H, H)$ is compact and self-adjoint it has a complete set of eigenvectors, with eigenvalues of finite multiplicity and with 0 as their only possible limit point. It follows that if $T = (1 + L)$ is invertible then the index of T,

$\operatorname{Ind} T = \#$ of $-ve$ eigenvalues of T counted according to multiplicity, is the number of eigenvalues of L less than -1, taking multiplicities into account. In particular $\operatorname{Ind} T < \infty$. In general the index is defined to be the maximum dimension of those subspaces on which T is negative definite. We require the lemma:

LEMMA 3 B. — For any net $\{ P_\alpha, \alpha \in a \}$ of orthogonal projections strongly convergent to the identity and for any compact self-adjoint L

$$\lim_\alpha \operatorname{Ind}(1 + P_\alpha L P_\alpha) = \operatorname{Ind}(1 + L).$$

Proof. — Suppose $\operatorname{Ind}(1 + L) = p$ and $\lambda_1, \lambda_2, \ldots, \lambda_p$ are eigenvalues of L each less than -1, with corresponding orthonormal eigenvectors e_1, $e_2, \ldots, e_p$ in H. Then

$$((1+L)e_i, e_i) = \lambda_i + 1 < 0$$

and so $\lim_\alpha ((1+L)P_\alpha e_i, P_\alpha e_i) = \lambda_i + 1$ and in particular for sufficiently large α

$$((1+L)P_\alpha e_i, P_\alpha e_i) < 0.$$

But $((1+P_\alpha L P_\alpha)P_\alpha e_i, P_\alpha e_i) = ((1+L)P_\alpha e_i, P_\alpha e_i)$ and therefore $\{ P_\alpha e_1, \ldots, P_\alpha e_p \}$ lie in a subspace in which $(1 + P_\alpha L P_\alpha)$ is negative definite. Since $P_\alpha e_1, \ldots, P_\alpha e_p$ are linearly independent for sufficiently large α, $\lim_\alpha (P_\alpha e_i, e_j) = (e_i, e_j) = \delta_{ij}$, this gives

$$\underline{\lim}_\alpha \operatorname{Ind}(1 + P_\alpha L P_\alpha) \geq \operatorname{Ind}(1 + L).$$

On the other hand if K_α is the subspace spanned by the negative eigenvalue eigenfunctions of $(1 + P_\alpha L P_\alpha)$ then K_α lies in $P_\alpha H$ and so if $v \in K_\alpha$, with $v \neq 0$,

$$((1+L)v, v) = ((1+L)P_\alpha v, P_\alpha v) = ((1+P_\alpha L P_\alpha)v, v) < 0.$$

Thus $\operatorname{Ind}(1+L) \geq \operatorname{Ind}(1+P_\alpha L P_\alpha)$, for all $\alpha \in a$. □

C. The Cameron-Martin Formula.

We now give our most useful result.

THEOREM 3 C. — Let $L : H \to H$ be trace-class and self-adjoint with $(1 + L): H \to H$ a bijection. Let $g : H \to \mathbb{C}$ be defined by

$$g(\gamma) = \exp\left\{\frac{i}{2}(\gamma, L\gamma)\right\} f(\gamma), \qquad \gamma \in H,$$

where $f \in \mathcal{F}(H)$. Define Ind $(1 + L)$ as above. Then g is $\mathcal{F}$-integrable and

$$\mathcal{F}(g) = |\det(1 + L)|^{-\frac{1}{2}} \exp\left\{-\frac{\pi i}{2}\operatorname{Ind}(1 + L)\right\} \int_H d\mu_f(\gamma) \exp\left\{-\frac{i}{2}(\gamma, (1 + L)^{-1}\gamma)\right\},$$

det being the Fredholm determinant.

Proof. — The proof is technically simple but involves a certain amount of computation. To avoid a vast number of subscripts and superscripts, we give the proof here for the case $d = 1$. We write $\gamma_j = \gamma(t_j)$ for the partition $\pi = \{0 = t_0 < t_1 < t_2 \ldots < t_{m(\pi)+1} = t\}$, $\Delta\gamma_j = \gamma_{j+1} - \gamma_j$, $\Delta t_j = t_{j+1} - t_j$. $\{e_j\}_{j=0}^{m(\pi)}$ denotes the orthonormal basis of $P_\pi H$,

$$e_j(\cdot) = [G(t_{j+1}, .) - G(t_j, .)][t_{j+1} - t_j]^{-\frac{1}{2}},$$

G being the reproducing kernel $G(\sigma, \tau) = t - \sigma \vee \tau$, $e_j(s) = t_j - t_{j+1}$, $s < t_j$, $e_j(s) = s - t_{j+1}$, $t_j \leq s \leq t_{j+1}$, $e_j(s) = 0$, $s > t_{j+1}$. For all $\gamma \in H^1$, we have $(P_\pi\gamma)(\tau) = \sum_{j=0}^{m(\pi)} \Delta\gamma_j e_j(\tau) \Delta t_j^{-\frac{1}{2}}$. For g as above

$$\mathcal{F}_\pi(g) = \prod_{j=0}^{m(\pi)} (2\pi i \Delta t_j)^{-\frac{1}{2}} \int^0 \prod_{j=0}^{m(\pi)} d(\Delta\gamma_j) \exp\left\{\frac{i}{2}(\gamma, P_\pi(1 + L)P_\pi\gamma)\right\} \times \left[\int \exp\{i(P_\pi\gamma, P_\pi\gamma')\} d\mu_f(\gamma')\right],$$

$\int^0$ being the oscillatory integral.

The above equation and Proposition 2 B now give after a little calculation

$$\mathcal{F}_\pi(g) = |\det(1 + P_\pi L P_\pi)|^{-\frac{1}{2}} \exp\left\{-\frac{i\pi}{2}\operatorname{Ind}(1 + P_\pi L P_\pi)\right\} \int_H d\mu_f(\gamma') \exp\left\{-\frac{i}{2}(\gamma', P_\pi(1 + P_\pi L P_\pi)^{-1} P_\pi \gamma')\right\}$$

det being the trace-class continuous Fredholm determinant. Since $P_\pi \xrightarrow{s} 1$ as $\delta(\pi) \to 0$ and since $P_\pi L P_\pi \to L$ in trace-norm as $\delta(\pi) \to 0$, the final result follows from Lemmas 3 A and 3 B by letting $\delta(\pi) \to 0$ in the above identity. □

To see that the above result is just an extended Cameron-Martin formula, consider the case in which $(1+L)>0$, $(1+L)=(1+K)^2$, $K : H \to H$ being self-adjoint. Defining $f(\cdot)=h((1+K)\cdot)$, the above yields, for $h\in\mathscr{F}(H)$,

$$\mathscr{F}\left[\exp\left\{\frac{i}{2}(K.,K.)+i(K.,.)\right\}h((1+K).)\right]=|\det(1+K)|^{-1}\mathscr{F}[h].$$

This is just a Cameron-Martin formula for the Feynman integral [*19*].

We have extended the above result to include the possibility that $(1+L)$ has negative eigenvalues. This gives rise to the Morse or Maslov indices in the second factor of the above expression for $\mathscr{F}(g)$. The first factor is of course just a Jacobian determinant. The third term is precisely a Fresnel integral relative to a non-singular quadratic form [*1*].

In the literature other families of projections are often used to define the Feynman integral.

Let $\mathscr{P}$ be a family of projections on H strongly convergent to 1 e. g. $\mathscr{P}$ might be the sequence of projections obtained by using truncated Fourier series for the paths $\gamma\in H$. Let $\mathscr{F}_{\mathscr{P}}$ be the Feynman integral obtained by replacing $\{P_\pi\}$ by $\mathscr{P}$ in our definition. Then a consequence of the results of the last section and the above method of proof is our next corollary.

COROLLARY 3 C. — Let $(1+L): H \to H$ be a bijection, for self-adjoint, Hilbert-Schmidt L. Let $g: H \to \mathbb{C}$ be defined by $g(\gamma)=\exp\left\{\frac{i}{2}(\gamma, L\gamma)\right\}f(\gamma)$, $f\in\mathscr{F}(H)$. Then

$$\mathscr{F}_{\mathscr{P}}(g)=\exp\left\{-\frac{\pi i}{2}\operatorname{Ind}(1+L)\right\}|\det_2(1+L)|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}\lim_{P\in\mathscr{P}}\operatorname{trace}(PLP)\right\}$$
$$\times\int_H e^{-i/2(\gamma,(1+L)^{-1}\gamma)}d\mu_f(\gamma),$$

whenever the limit exists.

Proof. — Let $\{\lambda_i\}_{i=1}^{\infty}$ be the real eigenvalues of the self-adjoint Hilbert-Schmidt L arranged in ascending order. The regularised Fredholm-Carlman determinant $\det_2$ is defined by

$$\det_2(1-\lambda L)=\prod_{i=1}^{\infty}(1-\lambda\lambda_i)e^{\lambda\lambda_i}\quad(\lambda\in\mathbb{R})$$

This infinite product converges for Hilbert-Schmidt L since $(1-\lambda\lambda_i)e^{\lambda\lambda_i}=1+0(\lambda_i^2)$ and $\Sigma\lambda_i^2<\infty$ for Hilbert-Schmidt L. Moreover trivially for a projection $P\in\mathscr{P}$

$$\det(1+PLP)=\det_2(1+PLP)e^{\operatorname{trace}(PLP)}.$$

The final result follows by referring to the last proof using the fact that $\det_2(1+.)$ is Hilbert-Schmidt continuous [16]. □

In the last corollary it should be borne in mind that if L is not trace-class by varying $\mathscr{P}$ *any* non zero value can be obtained for the third factor. Thus one cannot be too cavalier in evaluating Feynman integrals with different families of projections. Note that, more generally, we have shown that if L is compact and self-adjoint then $\mathscr{F}_{\mathscr{P}}(g)$ exists if and only if $\lim_{\mathscr{P}} \det(1+\mathrm{PLP})$ exists and in particular if L is Hilbert-Schmidt then $\mathscr{F}_{\mathscr{P}}(g)$ exists and is independent of $\mathscr{P}$ if and only if L is trace-class, see also Blattner [2].

4. APPLICATION TO ANHARMONIC OSCILLATORS

We show in this section how the Feynman-Kac-Itô formula for anharmonic oscillator potentials can be deduced from the above. Namely we shall prove:

THEOREM 4. — The solution of the Schrödinger equation

$$i\frac{\partial\psi}{\partial t}(x,t) = -2^{-1}\Delta_x\psi(x,t) + \mathrm{V}(x)\psi(x,t)$$

with Cauchy data $\psi(x,0) = \phi(x) \in \mathscr{F} \cap \mathrm{L}^2(\mathbb{R}^d)$, for the real anharmonic potential $\mathrm{V}(x) = 2^{-1}x\Omega^2x + \mathrm{V}_0(x)$, $\mathrm{V}_0 \in \mathscr{F}(\mathbb{R}^d)$, Ω a positive definite quadratic form, is given by the Feynman integral

$$\psi(x,t) = \mathscr{F}\left[\exp\left\{-i\int_0^t \mathrm{V}(\gamma(\tau)+x)d\tau\right\}\phi(\gamma(0)+x)\right]$$

$t \neq \left(n+\frac{1}{2}\right)\pi/\Omega_j$, Ω_j^2 an eigenvalue of Ω^2, $\Omega_j > 0$, $n = 0, 1, 2, \ldots$

A. The Evaluation of a Feynman Integral.

We shall be considering the anharmonic oscillator potential

$$\mathrm{V}(x) = 2^{-1}x\Omega^2x + \mathrm{V}_0(x),$$

where $\mathrm{V}_0 \in \mathscr{F}(\mathbb{R}^d)$ and $.\Omega^2.$ is a positive definite quadratic form on $\mathbb{R}^d$. We define $\Omega : \mathbb{R}^d \to \mathbb{R}^d$ the real symmetric positive definite operator with the property

$$x\Omega^2x = \langle x, \Omega^2x \rangle, \qquad x \in \mathbb{R}^d,$$

$\langle\ ,\ \rangle$ being the Euclidean inner product on $\mathbb{R}^d$.

Corresponding to Ω define $L : H \to H$ by

$$\frac{d^2}{ds^2}(L\gamma)(s) = \Omega^2\gamma(s),$$

$\frac{d}{ds}(L\gamma)(s)|_{s=0} = 0$, $(L\gamma)(t) = 0$, for all $\gamma \in H$, so that explicitly for $s \in [0, t]$

$$(L\gamma)(s) = -\int_s^t ds' \int_0^{s'} (\Omega^2\gamma)(s'')ds'', \qquad \gamma \in H.$$

Observe that for (,) the Hilbert space inner product and for $\delta, \gamma \in H$

$$(\delta, L\gamma) = \int_0^t \langle \dot{\delta}(s), (\dot{L\gamma})(s) \rangle ds = -\int_0^t \langle \delta(s), (\ddot{L\gamma})(s) \rangle ds$$
$$= -\int_0^t \langle \delta(s), \Omega^2\gamma(s) \rangle ds,$$

showing that L is self-adjoint with respect to the H inner product. Using the condition $\frac{d}{ds}(L\gamma)(s)|_{s=0} = 0$, a simple but tedious calculation gives for $t \neq \left(n + \frac{1}{2}\right)\pi/\Omega_j$, $n \in \mathbb{Z}$, Ω_j any eigenvalue of Ω,

$$(1+L)^{-1}\gamma(s) = \gamma(s) - \Omega\int_s^t \sin[\Omega(s'-s)]\gamma(s')ds' + \sin[\Omega(t-s)]$$
$$\int_0^t [\cos\Omega t]^{-1}\Omega\cos(\Omega s')\gamma(s')ds'$$

Moreover the following lemma is valid.

LEMMA 4 A. — The self-adjoint $L : H \to H$ is trace-class. If

$$t \neq \left(n + \frac{1}{2}\right)\pi/\Omega_j, \qquad n \in \mathbb{Z},$$

Ω_j an eigenvalue of Ω

$$\text{ind}(1 + L) = \sum_{j=1}^{d}\left[\frac{\Omega_j t}{\pi} + \frac{1}{2}\right],$$

$[\cdot]$ being the integer part, $\Omega_1, \Omega_2, \ldots, \Omega_d$ being the eigenvalues of Ω repeated according to multiplicity. And finally

$$\det(1 + L) = \det(\cos(\Omega t)).$$

Proof. — There is no loss of generality if we assume that Ω is diagonal. We do this to simplify the algebra. We look for eigenvalues of L in the form $(-p^2)$ so that $L\gamma = -p^2\gamma$. Since $\frac{d^2}{ds^2}(L\gamma)(s) = \Omega^2\gamma(s)$, this leads to

$\gamma(s)=(\gamma^1(s), \gamma^2(s), \ldots, \gamma^d(s))$, where $\gamma^i(s)=A_i \sin[\Omega_i(s-\varepsilon_i)/p]$, for constants A_i, ε_i, $i = 1, 2, \ldots, d$. Substituting these into the explicit expression for $(L\gamma)$ above gives that necessarily $\Omega_i\varepsilon/p = \left(m_i + \frac{1}{2}\right)\pi$, $\Omega_i t/p = \left(n_i + \frac{1}{2}\right)\pi$; m_i, $n_i \in \mathbb{Z}$. Hence the only possible p are of the form $\Omega_i t \Big/ \left(n_i + \frac{1}{2}\right)\pi$, $n_i \in \mathbb{Z}$. Using the fact that $\sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{-2} < \infty$, it follows that $L: H \to H$ is trace-class. For the eigenvalues $p = \Omega_i t \Big/ \left(n_i + \frac{1}{2}\right)\pi$ to contribute to the index of $(1+L)$ we require $p > 1$ and so $n_i = 0, 1, 2, \ldots, \left[\frac{\Omega_i t}{\pi} - \frac{1}{2}\right]$, [ ] being integer part, and if the eigenvalues Ω_i are distinct the corresponding (non-degenerate) eigenfunction is $\left(0, 0, \ldots, 0, \cos\left[\left(n_i + \frac{1}{2}\right)s\pi/t\right], 0, \ldots, 0\right)$ i^{th} entry being non-zero. The case of non-distinct Ω_i is dealt with similarly. To complete the proof we merely observe that

$$\cos(x) = \prod_{n=0}^{\infty} \left[1 - \frac{x^2}{\left(n + \frac{1}{2}\right)^2 \pi^2}\right]. \quad \square$$

COROLLARY 4 A. — Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then

$$\mathcal{F}\left[\exp\left\{-\frac{i}{2}\int_0^t [x+\gamma(s)]\Omega^2[x+\gamma(s)]ds\right\}\phi(x+\gamma(0))\right] = \exp(-it H_0)\phi(x),$$

where $H_0 = -2^{-1}\Delta_x + 2^{-1}x\Omega^2 x$.

Proof. — The proof is no more complicated for $d > 1$ than it is for $d=1$. To avoid having to use an elaborate notation we give here the proof in the case $d=1$. Let $G(\sigma, \tau) = t - \sigma \vee \tau$ be the reproducing kernel and define $b \in H$ by $b(x, \alpha)(\cdot) = \alpha G(0, .) - x\Omega^2 \int_0^t G(s, .)ds$, for $\alpha, x \in \mathbb{R}$. Then, for $\phi(x) = \int \exp(i\alpha x)\tilde{\phi}(\alpha)d\alpha$, $\tilde{\phi} \in \mathcal{S}(\mathbb{R})$,

$$\mathcal{F}\left[\exp\left\{-\frac{i}{2}\int_0^t [x+\gamma(s)]\Omega^2[x+\gamma(s)]ds\right\}\phi(x+\gamma(0))\right]$$
$$= \mathcal{F}\left[\exp\left\{\frac{i}{2}(\gamma, L\gamma)\right\}\int_{\mathbb{R}} d\alpha \exp\left\{i(b(x, \alpha), \gamma) - \frac{it}{2}x\Omega^2 x + i\alpha x\right\}\tilde{\phi}(\alpha)\right].$$

A straight-forward application of the last lemma and Theorem 3C gives

$$\text{r.h.s.} = |\det(1 + L_t)|^{-\frac{1}{2}} \exp\left\{ -\frac{i\pi}{2} \operatorname{ind}(1 + L_t) \right\}$$

$$\int_{\mathbb{R}} \exp\left\{ -\frac{i}{2}(b(x, \alpha), (1 + L_t)^{-1} b(x, \alpha)) \right\} \exp\left\{ -\frac{it}{2} x\Omega^2 x + i\alpha x \right\} \tilde{\phi}(\alpha) d\alpha$$

and after a tedious calculation we obtain

$$\text{r.h.s.} = |\cos(\Omega t)|^{-\frac{1}{2}} \exp\left\{ -\frac{i\pi}{2} \operatorname{ind}(1 + L_t) \right\} \exp\left\{ \frac{-ix\Omega \tan \Omega t x}{2} \right\}$$

$$\times \int_{\mathbb{R}} \exp\left\{ i\langle \alpha, \sec(\Omega t) x \rangle - \frac{i}{2}\langle \alpha, \Omega^{-1} \tan(\Omega t) \alpha \rangle \right\} \tilde{\phi}(\alpha) d\alpha$$

$$= [\Omega^{-1} \tan(\Omega t)]^{-\frac{1}{2}} |\cos(\Omega t)|^{-\frac{1}{2}} \exp\left\{ -\frac{i\pi}{2} \operatorname{ind}(1 + L_t) \right\}$$

$$\times (2\pi i)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp\left\{ \frac{i}{2} [x\Omega \cot(\Omega t) x + y\Omega \cot(\Omega t) y - 2x\Omega \operatorname{cosec}(\Omega t) y] \right\} \phi(y) dy,$$

where in $[\Omega^{-1} \tan(\Omega t)]^{-\frac{1}{2}}$ the same branch of square root is being taken. Hence, using the above notation,

$$\mathscr{F}\left[\exp\left\{ -\frac{i}{2} \int_0^t [x + \gamma(s)] \Omega^2 [x + \gamma(s)] ds \right\} \phi(x + \gamma(0)) \right]$$

$$= (2\pi i)^{-\frac{1}{2}} [\Omega^{-1} \sin(\Omega t)]^{-\frac{1}{2}} \int_{\mathbb{R}} dy\, \tilde{\phi}(y) \exp\left\{ \frac{i}{2} [x\Omega \cot(\Omega t) x + y\Omega \cot(\Omega t) y - 2x\Omega \operatorname{cosec}(\Omega t) y] \right\}.$$

We now recognise r. h. s. as being equal to $\exp(-itH_0)\phi(x)$, for $\phi \in \mathscr{S}(\mathbb{R})$. This completes the proof of the lemma. □

B. The Proof of the Feynman-Kac-Itô Formula.

The proof here is based on the second proof given in Simon [see Ref. (15), p. 50] for the diffusion equation. It avoids the lengthy computations of the proof given in Ref. (1) (see also Refs. (13), (17)).

First some notation. For $x \in \mathbb{R}^d$ and $t \geq 0$ let H_t be the space of paths which we have denoted by H and let $\mathscr{F}^{x,t}(g)$ be the Feynman integral

$$\mathscr{F}^{x,t}(g) = \mathscr{F}(g(x + \gamma)),$$

where $\gamma \mapsto g(\gamma)$ is a suitable function of paths on $\mathbb{R}^d$ and $(x + \gamma)$ refers to the path $s \mapsto x + \gamma(s)$. Also, for $0 \leq u \leq t$ let $\mu_u\{V_0, x\}$, $\nu_u^t\{V_0, x\}$

and $\lambda_u^t\{x\}$ be the measures on H_t whose Fourier transforms when evaluated at $\gamma \in H_t$ are $V_0(x+\gamma(u))$,

$$\exp\left\{-i\int_u^t V_0(x+\gamma(s))ds\right\} \quad \text{and} \quad \exp\left\{-i\int_u^t \langle \Omega^2 x, \gamma(s)\rangle ds\right\}$$

respectively. We shall often write

$$\mu_u\{V_0\} \equiv \mu_u\{V_0, x\},$$
$$\nu_u^t \equiv \nu_u^t\{V_0, x\},$$
and
$$\lambda_u^t \equiv \lambda_u^t\{x\},$$

and give $\mu_u\{\phi\} = \mu_u\{\phi, x\}$ the corresponding meaning. We shall find it convenient to use the notation « $\mu(d\gamma)$ » for « $d\mu(\gamma)$ » in integrals.

If $\{\mu_u : a \leq u \leq b\}$ is a family in $\mathscr{M}(H)$, we shall let $\int_a^b \lambda_u du$ denote the measure on H given by

$$f \mapsto \int_a^b \left\{\int_H f(\gamma)\lambda_u(d\gamma)\right\} du$$

whenever it exists.

Then, since for any continuous path γ

$$\exp\left\{-i\int_0^t V_0(\gamma(s))ds\right\} = 1 - i\int_0^t V_0(\gamma(u)) \exp\left\{-i\int_u^t V_0(\gamma(s))ds\right\} du, \quad (a)$$

we have, by Fubini's theorem,

$$\nu_0^t = \delta_0 - i\int_0^t (\mu_u\{V_0\} * \nu_u^t)du \quad (b)$$

where δ_0 is the Dirac measure at $0 \in H_t$. Since $(\delta, L\gamma) = -\int_0^t \langle \delta(s), \Omega^2\gamma(s)\rangle ds$ for all $\delta, \gamma \in H_t$, we have

$$2^{-1}\int_0^t \langle x+\gamma(s), \Omega^2(x+\gamma(s))\rangle ds = 2^{-1}\langle x, \Omega^2 x\rangle t + \int_0^t \langle \Omega^2 x, \gamma(s)\rangle ds - 2^{-1}(\gamma, L\gamma) \quad (c)$$

Therefore, if we set

$$U_t\phi(x) = \mathscr{F}^{x,t}\left[\exp\left\{-i\int_0^t V(\gamma(s))ds\right\}\phi(\gamma(0))\right] \quad (d)$$

the Cameron-Martin formula, Theorem 3 C, assures us that $(U_t\phi) : \mathbb{R}^d \to \mathbb{C}$ exists and $U_t\phi(x) = N_t \int_{H_t} \exp\left\{-\frac{i}{2}(\gamma, (1+L)^{-1}\gamma)\right\}(\lambda_0^t * \nu_0^t * \mu_0(\phi))(d\gamma)$ for $N_t = \exp\left\{-\frac{\pi i}{2}\operatorname{Ind}(1+L_t)\right\}|\det(1+L_t)|^{-\frac{1}{2}}\exp\left\{-\frac{i}{2}\langle x, \Omega^2 x\rangle t\right\}$, where the subscript t reminds us of the t dependence.

Applying (*b*) we obtain

$$U_t\phi(x) = I_t - i\int_0^t \alpha_u du\,,$$

where

$$I_t = N_t\int_{H_t}\exp\left\{-\frac{i}{2}(\gamma,(1+L)^{-1}\gamma)\right\}(\lambda_0^t * \mu_0\{\phi\})(d\gamma)$$

and

$$\alpha_u = N_t\int_{H_t}\exp\left\{-\frac{i}{2}(\gamma,(1+L)^{-1}\gamma)\right\}(\lambda_0^t * \mu_u\{V_0\} * \nu_u^t * \mu_0\{\phi\})(d\gamma)\,.$$

We can now use the Cameron-Martin formula in the other direction to get

$$I_t = \mathscr{F}^{x,t}\left\{\exp\left\{-i\int_0^t \langle \gamma(s),\Omega^2\gamma(s)\rangle ds\right\}\phi(\gamma(0))\right\} = \exp(-iH_0t)\phi(x)\,,$$

by Corollary 4 A, for $\phi \in \mathscr{S}(\mathbb{R}^d)$, and

$$\alpha_u = \mathscr{F}^{x,t}\left[\exp\left\{-\frac{i}{2}\int_0^t \langle \gamma(s),\Omega^2\gamma(s)\rangle ds - i\int_u^t V_0(\gamma(s))ds\right\}V_0(\gamma(u))\,\phi(\gamma(0))\right].$$

Let π_1 and π_2 be partitions of $[0, u]$ and $[0, t-u]$ respectively and let π be the partition of $[0, t]$ whose partition points are those of π_1 up to time u and translates by u of those of π_2 thereafter. Then by Fubini's theorem, if

$$\alpha_{\pi,u} = \mathscr{F}_\pi^{x,t}\left[\exp\left\{-\frac{i}{2}\int_0^t \langle \gamma(s),\Omega^2\gamma(s)\rangle ds - i\int_u^t V_0(\gamma(s))ds\right\}V_0(\gamma(u))\phi(\gamma(0))\right],$$

we have

$$\alpha_{\pi,u} = \mathscr{F}_{\pi_2}^{x,t-u}\left[\exp\left\{-i\int_0^{t-u} V(\gamma(s))ds\right\}V_0(\gamma(0))\right.$$
$$\left.\mathscr{F}_{\pi_1}^{\gamma(0),u}\left[\exp\left\{-\frac{i}{2}\int_0^u \langle \gamma_1(s),\Omega^2\gamma_1(s)\rangle ds\right\}\phi(\gamma_1(0))\right]\right],$$

where $\gamma \in H_{t-u}$ and $\gamma_1 \in H_u$ are the integration variables. Assume now that the right hand limits are strongly convergent in $L^2(\mathbb{R}^d)$. Then r. h. s. converges to $\alpha_u = U_{t-u}(V_0 \exp(-iH_0u)\phi)$. Hence, assuming the L^2-convergence of Feynman integrals, we have proved that $(U_t\phi)$ as defined in Eq (*d*) satisfies

$$(U_t\phi)(x) = \exp(-iH_0t)\phi(x) - i\int_0^t U_{t-u}(V_0\exp(-iH_0u)\phi)(x)du$$
$$= \exp(-iH_0t)\phi(x) - i\int_0^t U_u(V_0\exp(iH_0u)\exp(-iH_0t)\phi)(x)du\,.$$

It is now a simple matter to show that the iterative solution of this integral equation in $L^2(\mathbb{R}^d)$ is just the Dyson series for $\exp(-iHt)\phi(x)$, where $H = -2^{-1}\Delta_x + 2^{-1}x\Omega^2 x + V_0(x) = (H_0 + V)$. (See Ref. (12b)).

The desired L^2-convergence of Feynman integrals is established in the next section.

C. L^2-convergence of Feynman Integrals.

In this section we establish one important inequality useful in discussing the L^2-convergence of Feynman path integrals. For the partition

$$\pi_m = \{ 0 = t_0 < t_1 < t_2 < \ldots < t_m = t \}$$

denote the corresponding Feynman path-integral over paths x ending at b by $\mathscr{F}^z_m\{ x : x(t) = b \}$. Then we shall prove:

LEMMA 4 C. — In $\mathbb{R}^d$, for $z=\rho \in \mathbb{R}\backslash\{0\}$, if $\phi_0 \in L^2(\mathbb{R}^d)$ and $V_0 \in \mathscr{F}(\mathbb{R}^d)$, with $V_0(x) = \int_{\mathbb{R}^d} \exp(i\alpha x)d\mu(\alpha)$, $\| V_0 \| = \| \mu \| = \int d|\mu| < \infty$, we have

$$\overline{\lim_{m\to\infty}} \left\| b \mapsto \mathscr{F}^z_{m\{x:x(t)=b\}}\left[\exp\left\{ -\frac{i}{\rho}\int_0^t V(x(s))ds \right\} \phi_0(x(0)) \right] \right\|_{L^2} \leq \| \phi_0 \|_{L^2} \exp\left\{ \frac{t}{|\rho|} \| V_0 \|_0 \right\},$$

where the anharmonic oscillator potential $V(x) = V_0(x) + 2^{-1}\langle Ax, x \rangle$, $A : \mathbb{R}^d \to \mathbb{R}^d$ being linear and symmetric.

Proof. — Let $F_m(b)$ be the given path integral: we must compute $\| F_m \|_{L^2}$. Take $\{ 0 = t_0 < t_1 < \ldots < t_m = t \}$ to define $\mathscr{F}^z_m$. Set $\Delta_k t = t_{k+1} - t_k$, $\Delta_k x = x_{k+1} - x_k$ with $x_k = x(t_k)$, $k = 0, 1, 2, \ldots$. Then, if

$$x_\pi(s) = [(t_{k+1}-s)x_k + (s-t_k)x_{k+1}]/\Delta_k t, \qquad t_k \leq s \leq t_{k+1},$$

$$F_m(x_m) = (2\pi i\rho)^{-dm/2} \prod_k (\Delta_k t)^{-d/2} \times \int_{\mathbb{R}^d\times\ldots\times\mathbb{R}^d} \exp\left\{ -\frac{i}{\rho}\int_0^t V(x_\pi(s))ds \right\} \prod_{k=0}^{m-1} \exp\left\{ \frac{i}{2\rho\Delta_k t} |\Delta_k x|^2 \right\} \phi_0(x_0)dx_0 \ldots dx_{m-1}.$$

For $r = 0, 1, \ldots, m$, set

$$P_r \equiv P_r(x_0, x_1, \ldots, x_r) = \exp\left\{ -\frac{i}{\rho}\int_0^{t_r} V(x_\pi(s))ds \right\} \times \prod_{k=0}^{r-1} \exp\left\{ \frac{i}{2\rho\Delta_k t} |\Delta_k x|^2 \right\} (2\pi i\rho)^{-dr/2} \prod_0^{r-1} (\Delta_k t)^{-d/2}.$$

Then

$$F_m(x_m)=(2\pi i\rho\Delta_{m-1}t)^{-d/2}\int_{\mathbb{R}^d\times\ldots\times\mathbb{R}^d}\exp\left\{-\frac{i}{\rho}\int_{t_{m-1}}^{t}V(x_\pi(s))ds+\frac{i}{2\rho\Delta_{m-1}}|\Delta_{m-1}x|^2\right\}$$

$$P_{m-1}(x_1,x_2,\ldots,x_{m-1})\phi_0(x_0)dx_0dx_1,\ldots,dx_{m-1}.$$

Note that

$$\int_{t_{m-1}}^{t_m}\langle Ax_\pi(s),x_\pi(s)\rangle ds$$

$$=(\Delta_{m-1}t)^{-2}\int_{t_{m-1}}^{t_m}\langle A\{(t_m-s)x_{m-1}+(s-t_{m-1})x_m\},(t_m-s)x_{m-1}+(s-t_{m-1})x_m\rangle ds$$

$$=\frac{1}{3}(\Delta_{m-1}t)\{|||x_{m-1}|||^2+\langle Ax_m,x_{m-1}\rangle+|||x_m|||^2\}$$

where $|||u|||=\sqrt{\langle Au,u\rangle}$.

Set $K_r^l(x_r)\equiv K_r^l(x_r)(s_1,\ldots,s_l;\alpha_1,\ldots,\alpha_l)$

$$=\exp\left\{-\frac{i}{\rho}|||x_r|||^2\frac{\Delta_r t}{6}+i\sum_j\langle\alpha_j,(t_{r+1}-s_j)x_r/\Delta_r t\rangle\right\}$$

$$\times\int_{\mathbb{R}^d\times\mathbb{R}^d\times\ldots\times\mathbb{R}^d}P_r(x_0,\ldots,x_r)\phi_0(x_0)dx_0,\ldots,dx_r.$$

Then

$$F_m(x_m)=(2\pi i\rho)^{-d/2}\times\int_{x_{m-1}\in\mathbb{R}^d}\sum_{l=0}^{\infty}\left(-\frac{i}{\rho}\right)^l\frac{1}{l!}\int_{t_{m-1}}^{t}$$

$$\ldots\int_{t_{m-1}}^{t}\int_{\mathbb{R}^d}\ldots\int_{\mathbb{R}^d}\{A_{m-1}^l\}d\mu(\alpha_1)\ldots d\mu(\alpha_l)ds_1\ldots ds_l dx_{m-1},$$

where

$$A_{m-1}^l\equiv\exp\left\{i\sum_{j=1}^{l}\left\langle\alpha_j,\frac{(s_j-t_{m-1})}{\Delta_{m-1}t}\alpha_m\right\rangle+\frac{i}{2\rho\Delta_{m-1}t}|x_m|^2-\frac{i}{6\rho}\Delta_{m-1}t\,|||x_m|||^2\right\}$$

$$\times\exp\left\{-\frac{i}{6\rho}\Delta_{m-1}t\langle Ax_m,x_{m-1}\rangle-\frac{i}{\rho\Delta_{m-1}t}\langle x_m,x_{m-1}\rangle\right\}K_{m-1}^l(x_{m-1}).$$

Thus,

$$\|F_m\|_{L^2}\le|2\pi\rho\Delta_{m-1}t|^{-d/2}\times\sum_{l=0}^{\infty}\frac{1}{l!}\frac{1}{|\rho|^l}\int_{t_{m-1}}^{t}\ldots\int_{t_{m-1}}^{t}\int_{\mathbb{R}^d}$$

$$\ldots\int_{\mathbb{R}^d}\|B_{m-1}^l\|_{L^2}d\mu(\alpha_1)\ldots d\mu(\alpha_l)ds_1\ldots ds_l,$$

where B^l_{m-1} is the function

$$x_m \mapsto \int_{\mathbb{R}^d} \exp\left\{ i\left\langle -\frac{\Delta_{m-1}t}{6\rho} A x_m - \frac{1}{\rho\Delta_{m-1}t} x_m, x_{m-1} \right\rangle \right\} K^l_{m-1}(x_{m-1}) dx_{m-1} .$$

Hence B is the Fourier transform (with strange conventions) of K^l_{m-1} evaluated at $(-\Delta_{m-1}Ax_m t/6\rho - x_m/\rho\Delta_{m-1}t)$.

Now, if $\tilde{}$ denotes Fourier transform, for sufficiently small Δt,

$$\begin{aligned}\int_{\mathbb{R}^d} | \tilde{K}^l_{m-1}([-\Delta t A/6\rho - 1/\rho\Delta t]x_m) |^2 dx_m \\ &= \int_{\mathbb{R}^d} \left| \det\left(\frac{\Delta t A}{6\rho} + \frac{1}{\rho\Delta t} \right) \right|^{-1} | \tilde{K}^l_{m-1}(y) |^2 dy \\ &= | \rho\Delta t |^d \left| \det\left(\frac{(\Delta t)^2}{2} A + 1 \right) \right|^{-1} \int_{\mathbb{R}^d} | \tilde{K}^l_{m-1}(y) |^2 dy \\ &= | \rho\Delta t |^d \left| \det\left(\frac{(\Delta t)^2}{2} A + 1 \right) \right|^{-1} (2\pi)^d \| K^l_{m-1}(\cdot) \|^2_{L^2},\end{aligned}$$

by the Plancherel theorem

Thus,

$$\begin{aligned}\| F_m \|_{L^2} \leq \sum_{l=0}^{\infty} \frac{1}{l!} \frac{1}{|\rho|^l} \\ \times \int_{t_{m-1}}^{t} \cdots \int_{t_{m-1}}^{t} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \| K^l_{m-1}(\cdot)(s_1, \ldots, s_l; \\ \alpha_1, \ldots, \alpha_l) \|_{L^2} d\mu(\alpha_1) \ldots d\mu(\alpha_l) ds_1 \ldots ds_l \\ \times \left| \det\left(\frac{(\Delta_{m-1}t)^2}{6} A + 1 \right) \right|^{-\frac{1}{2}} \leq \sum_{l=0}^{\infty} \frac{1}{l!} \frac{1}{|\rho|^l} \int_{t_{m-1}}^{t} \cdots \int_{t_{m-1}}^{t} \int_{\mathbb{R}^d} \\ \cdots \int_{\mathbb{R}^d} \| K^0_{m-1}(\cdot) \|_{L^2} d\mu(\alpha_1) \ldots d\mu(\alpha_l) ds_1 \ldots ds_l \\ \times \left| \det\left(\frac{(\Delta_{m-1}t)^2}{6} A + 1 \right) \right|^{-\frac{1}{2}},\end{aligned}$$

where $K^0_{m-1}(x_{m-1}) = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} P_{m-1}(x_0, \ldots, x_{m-1}) \phi_0(x_0) dx_0 \ldots dx_{m-2}$.

Thus,

$$\begin{aligned}\| F_m \|_{L^2} &\leq \sum_{l=0}^{\infty} \frac{1}{l!} \frac{1}{|\rho|^l} (\Delta_{m-1}t)^l \| \mu \|^l \| K^0_{m-1}(\cdot) \|_{L^2} \left| \det\left(\frac{(\Delta_{m-1}t)^2}{6} A + 1 \right) \right|^{-\frac{1}{2}} \\ &= \| K^0_{m-1}(\cdot) \|_{L^2} \exp\left\{ \frac{\Delta_{m-1}t \| \mu \|}{|\rho|} \right\} \left| \det\left(\frac{(\Delta_{m-1}t)^2}{6} A + 1 \right) \right|^{-\frac{1}{2}}.\end{aligned}$$

$$\| F_m \|_{L^2} \leq e^{t\|\mu\|/|\rho|} \| \phi_0 \|_{L^2} \prod_{r=0}^{m-1} \left| \det \left(\frac{(\Delta_{m-1}t)^2}{6} A + 1 \right) \right|^{-\frac{1}{2}}$$

Now, if $\alpha_1, \ldots, \alpha_k$ are the eigenvalues of A, for small enough $\Delta_r t$,

$$\begin{aligned}
\sum_{r=0}^{m-1} \left| \det \left(\frac{(\Delta_{m-1}t)^2}{6} A + 1 \right) \right|^{-\frac{1}{2}} &= \prod_r \prod_j \left| \frac{(\Delta_r t)^2}{6} \alpha_j + 1 \right|^{-\frac{1}{2}} \\
&\leq \prod_r \prod_j \left(1 - \frac{(\Delta_r t)^2}{6} | \alpha_j | \right)^{-\frac{1}{2}} \\
&\leq \exp \left\{ \sum_{r,j} \left[\frac{\dfrac{(\Delta_r t)^2 | \alpha_j |}{12}}{1 - \dfrac{(\Delta_r t)^2 | \alpha_j |}{6}} \right] \right\} \\
&\leq \exp \left\{ \frac{\varepsilon t \operatorname{trace} | A |}{1 - \dfrac{\varepsilon^2}{6} \| A \|} \right\}, \qquad \text{if mesh } \pi < \varepsilon,
\end{aligned}$$

and so the result follows. □

When $A = \Omega^2$, for positive definite Ω^2 as above, the results of the last section ensure the L^2-convergence of Feynman sums for initial data $\phi_0 \in \mathcal{F}(\mathbb{R}^d)$ and times $t \neq \left(n + \frac{1}{2} \right) \pi / \Omega_j$, Ω_j the eigenvalues of Ω, $n \in \mathbb{Z}$.

COROLLARY 4 C. — For the above anharmonic oscillator potentials, at all save a discrete series of times, the Feynman sums converge in L^2 for any $\phi_0 \in L^2$ (i. e. we have strong convergence.)

Proof. — Uniform boundedness theorem [see Ref. (4), p. 60] and the fact that we have L^2-convergence, if $\phi_0 \in \mathcal{S}(\mathbb{R}^d)$, at times

$$t \neq \left(n + \frac{1}{2} \right) \pi / \Omega_j, \qquad n \in \mathbb{Z}. \qquad \square$$

5. SOME CONNECTIONS WITH THE WIENER INTEGRALS

In this section we spell out some of the connections between the Feynman maps and the Wiener integral.

A. Analytic continuation.

Let $\dot{\mathbb{R}}^d$ be the one-point compactification of $\mathbb{R}^d$. Identifying a typical path $\gamma : [0, t] \to \mathbb{R}^d \cup \{\infty\}$ $(\gamma(t) = 0)$, with an element of $\Gamma = \prod_{[0,t]} \dot{\mathbb{R}}^d$, in the weak product topology, gives, according to Tychonoff's theorem, a compact Hausforff model for the path-space Γ (see Ref. (12*b*), p.277]. $C(\Gamma)$ denotes the space of continuous functions defined on Γ. When $f \in C(\Gamma)$ is such that $f|_H \in \mathscr{F}(H)$, we say that $f \in C_0(\Gamma)$.

LEMMA 5 A. — $C_0(\Gamma)$ is dense in $C(\Gamma)$.

Proof. — The point here is that $C_0(\Gamma)$ is a subalgebra of $C(\Gamma)$ containing 1, the identity, which is the Fourier transform of δ_0 the Dirac measure concentrated at $0 \in H$. $C_0(\Gamma)$ is closed under multiplication because $\mathscr{M}(H)$ is closed under convolution $*$ and $\mu_{fg} = \mu_f * \mu_g$. Clearly $\bar{f} \in C_0(\Gamma) \Rightarrow f \in C_0(\Gamma)$, – being complex conjugate with

$$\mu_{\bar{f}}(A) = \overline{(\mu_f(-A))}$$

for each Borel $A \subset H$. Finally, $C_0(\Gamma)$ separates points of Γ, because $\gamma \neq \gamma'$; $\gamma, \gamma' \in \Gamma \Rightarrow \exists \sigma \in [0, t)$ such that $\gamma(\sigma) \neq \gamma'(\sigma)$ and so $\exists\ \alpha \in \mathbb{R}$ with

$$\alpha[\gamma(\sigma) - \gamma'(\sigma)] \not\equiv 0 \ (\mathrm{mod}\ 2\pi)$$

giving $e^{i\alpha\gamma(\sigma) - \gamma^2(\sigma)} \neq e^{i\alpha\gamma'(\sigma) - \gamma'^2(\sigma)}$ for $\gamma \mapsto e^{i\alpha\gamma(\sigma) - \gamma^2(\sigma)} \in C_0(\Gamma)$. □

By inspection, if $f(\gamma) \geq 0$, $\gamma \in \Gamma$, $f \in C_0(\Gamma)$, then $\mathscr{F}^{-i}(f) \geq 0$ and so for real-valued $f \in C_0(\Gamma)$, using $\mathscr{F}^{-i}(1) = 1$,

$$\mathscr{F}^{-i}(f) \leq \sup_{\gamma \in \Gamma} |f(\gamma)|.$$

It follows easily from this that for complex-valued $f \in C_0(\Gamma)$

$$|\mathscr{F}^{-i}(f)| \leq \sup_{\gamma \in \Gamma} |f(\gamma)|.$$

Corresponding to the unique continuous extension $\mathbb{E}$ of $\mathscr{F}^{-i}$ to $C(\Gamma)$, according to the Riesz-Markov theorem, $\exists$ a unique regular Borel measure μ on Γ with

$$\mathbb{E}(f) = \int_\Gamma f(\gamma) d\mu(\gamma).$$

Since $\mathscr{F}_\pi^{-i}(f) = \int_{C_0[0,t]} (f \circ P_\pi)(\gamma) d\mu_w(\gamma)$, for μ_w Wiener measure supported on $C_0[0, t]$, we see that, in fact,

$$\mathbb{E}(f) = \int_{C_0[0,t]} f(\gamma) d\mu_w(\gamma).$$

Hence, for $f \in C_0(\Gamma)$, we have proved the infinite dimensional Parseval identity

$$\int_{C_0[0,t]} f(\gamma) d\mu_w(\gamma) = \int_H \exp\left\{ -\frac{1}{2} \| \gamma \|^2 \right\} d\mu_f(\gamma),$$

which by the denseness of $C_0(\Gamma)$ is sufficient to determine μ_w. We summarize our results in the next proposition.

PROPOSITION 5 A. — Let $f \in \mathscr{F}(H)$, the Banach algebra of Fourier transforms of measures of bounded absolute variation on H, with $\| \ \|_0$ defined by $\| f \|_0 = \int_H d | \mu_f |$, $f(\gamma) = \int_H \exp\left\{ i(\gamma', \gamma) \right\} d\mu_f(\gamma')$. Then

$$\mathscr{F}^s(f) = \int_H \exp\left\{ -\frac{is}{2} \| \gamma \|^2 \right\} d\mu_f(\gamma),$$

is a regular analytic function of s in $ims < 0$, continuous in $ims \leqslant 0$. Further, if $f : C_0[0, t] \to \mathbb{C}$ is a continuous bounded function, with $f|_H \in \mathscr{F}(H)$, then the Feynman integral $\mathscr{F}$ and the Wiener integral satisfy

$$\mathscr{F}(f) = \lim_{\varepsilon \to 0+} \mathscr{F}^{1-i\varepsilon}(f), \qquad \mathbb{E}(f) = \mathscr{F}^{-i}(f).$$

Interpolation gives for $s > 0$

$$| \mathscr{F}^{se^{-i\alpha}}(f) | \leq \| f \|_0^{1-\frac{2\alpha}{\pi}} \| f \|_\infty^{\frac{2\alpha}{\pi}}, \qquad 0 \leq \alpha \leq \frac{\pi}{2},$$

where

$$\| f \|_\infty = \sup_{\gamma \in C_0[0,t]} | f(\gamma) | .$$

Proof. — The first part of the theorem follows from

$$\mathscr{F}_\pi^s(f) = \int_H \exp\left\{ -\frac{is}{2} \| P_\pi \gamma \|^2 \right\} d\mu_f(\gamma),$$

the dominated convergence theorem and Morera's theorem. The second part of the theorem follows from Hadamard's three line theorem by considering $\phi(z) = \mathscr{F}^s(f)$, for fixed f as above, for $s = e^{-\frac{\pi i z}{2}}$. [See Ref. (12 *b*), p. 33.] □

B. Wick's Theorem and the Translation Formula.

In this section by exploiting the similarities with the Wiener integral we extend the class of $\mathscr{F}$-integrable functionals. We shall conclude this section with a version of Wick's theorem. We begin with a proposition which represents the simplest version of a translation formula. (See also Refs. (1), (18), (19)).

PROPOSITION 5 B. — For fixed $a \in H$, define $g_a : H \to \mathbb{C}$ by $g_a(\gamma) = g(\gamma + a)$, $g \in \mathscr{F}(H)$. Then, if $im\, s \leq 0$, $\exp\left\{\frac{i}{s}(a,.)\right\} g_a(\cdot)$ is $\mathscr{F}^s$-integrable and

$$\mathscr{F}^s\left(\exp\left\{\frac{i}{s}(a,.)\right\} g_a(\cdot)\right) = \exp\left\{-\frac{i}{2s}\|a\|^2\right\}\mathscr{F}^s(g).$$

Proof. — Define $f : H \to \mathbb{C}$ by $f(\gamma) = \exp\left\{+\frac{i}{s}(a,\gamma)\right\} g_a(\gamma)$, for fixed $a \in H$. Then for a partition π a simple computation gives

$$\mathscr{F}^s_\pi(f) = \exp\left\{-\frac{i}{2s}\|P_\pi a\|^2\right\}\prod_{j=0}^{m(\pi)}(2\pi i \Delta t_j)^{-d/2}$$
$$\int^0 \exp\left\{\frac{i}{2s}\|P_\pi(\gamma + a)\|^2\right\} g_a(P_\pi \gamma)\prod_{j=0}^{m(\pi)} d^d(\Delta\gamma_j).$$

Since $g_a(P_\pi\gamma) = g_a(P_\pi(\gamma + a) - (P_\pi a))$ and $g_a(\tilde{\gamma} - P_\pi a) = g(\tilde{\gamma} + (1 - P_\pi)a)$, we obtain

$$g_a(\tilde{\gamma} - P_\pi a) = \int \exp\{i(\tilde{\gamma}, \gamma'')\}\exp\{i((1 - P_\pi)a, \gamma'')\}\, d\mu_g(\gamma'').$$

Hence,

$$\mathscr{F}^s_\pi(f) = \exp\left\{-\frac{i}{2s}\|P_\pi a\|^2\right\}\prod_{j=0}^{m(\pi)}(2\pi i \Delta t_j)^{-d/2}$$
$$\times \lim_{\varepsilon \to 0}\int \phi(\varepsilon\Delta\gamma'_0 - \varepsilon\Delta a_0, \varepsilon\Delta\gamma'_1 - \varepsilon\Delta a_1, \ldots, \varepsilon\Delta\gamma_{m(\pi)} - \varepsilon\Delta a_{m(\pi)})$$
$$\exp\left\{\frac{i}{2s}\prod_{j=0}^{m(\pi)}(\Delta\gamma'_j, \Delta\gamma'_j)\Delta t_j^{-1}\right\}$$
$$\exp\left\{i\sum_{j=0}^{m(\pi)}(\Delta\gamma'_j, \Delta\gamma''_j)\Delta t_j^{-1}\right\}\prod_{j=0}^{m(\pi)} d^d(\Delta\gamma'_j)\exp\{i((1 - P_\pi)a, \gamma'')\}\, d\mu_g(\gamma''),$$

for $\phi \in \mathscr{S}(\mathbb{R}^{d(m(\pi)+1)})$, with $\phi(0) = 1$.

Writing $\phi(p) = \int \exp\left\{ i \sum_{0}^{m(\pi)} \langle p_j, q_j \rangle \right\} \tilde{\phi}(q_0, \ldots, q_{m(\pi)})dq$ and using Lemma 2B gives

$$\mathscr{F}_\pi^s(f) = \exp\left\{ -\frac{i}{2s} \| P_\pi a \|^2 \right\}$$

$$\times \lim_{\varepsilon \to 0} \iint dq \exp\left\{ -i\varepsilon \sum_{j=0}^{m(\pi)} \langle \Delta a_j, q_j \rangle - \frac{is}{2} \sum_{j=0}^{m(\pi)} \langle \varepsilon q_j + \Delta\gamma_j'' \Delta t_j^{-1}, \varepsilon q_j + \Delta\gamma_j'' \Delta t_j^{-1} \rangle \Delta t_j \right\} \tilde{\phi}(q)$$

$$\exp\{ i((1 - P_\pi)a, \gamma'')d\mu_g(\gamma'') .$$

And so, since $\int \tilde{\phi}(q)dq = \phi(0) = 1$, using dominated convergence for $im\, s < 0$,

$$\mathscr{F}_\pi^s(f) = \exp\left\{ -\frac{i}{2s} \| P_\pi a \|^2 \right\} \int_H \exp\left\{ -\frac{is}{2} (P_\pi \gamma'', \gamma'') \right\}$$

$$\exp\{ i((1 - P_\pi)a, \gamma'') \} d\mu_g(\gamma'') .$$

The final result follows letting $\delta(\pi) \to 0$ so that $P_\pi \xrightarrow{s} 1$. □

COROLLARY. — Let $\mathbb{E}$ denote expectation with respect to the Wiener measure μ_w. Then, for fixed $a \in H$, with $\dot{a}_j$ of bounded absolute variation, $a = (a_1, \ldots, a_d)$, each $j = 1, 2, \ldots, d$,

$$\mathbb{E}\{ \exp\{ -(a, .)g_a(\cdot) \} \} = \exp\left\{ \frac{\| a \|^2}{2} \right\} \mathbb{E}(g) ,$$

when $g : C_0[0, t] \to \mathbb{C}$ is a continuous bounded function.

Proof. — This is a simple application of the dominated convergence theorem, using the facts that, if $\dot{a}_j$ is of bounded absolute variation, $j = 1, 2, \ldots, d$,

$$| (P_\pi a, \gamma) | \leq C \| \gamma \|_\infty ,$$

$$\| \gamma \|_\infty = \sup_{\substack{s \in [0,t] \\ i = 1,2,\ldots,d}} | \gamma_i(t) |, \quad \text{and} \quad \exp\{ C \| \gamma \|_\infty \} \in L^1 [C_0(0, t), d\mu_w] ,$$

while $(P_\pi a, \gamma) \to (a, \gamma)$ a. e. w. r. t. μ_w, a satisfying above. □

We conclude with a final result. (See also Ref. (18)).

EXAMPLE 5 B. — Suppose $f \in \mathscr{F}(H)$ and $\beta_1, \ldots, \beta_k \in H$ with

$$\int_H (1 + | x |)^k | \mu | (dx) < \infty .$$

Then, if $im\, z \leq 0$

$$\mathscr{F}^z(\langle \beta_1, x \rangle \ldots \langle \beta_k, x \rangle f(x)) = i^k \int_H e^{-\frac{iz}{2}|x|^2} H_k(\beta_1, \ldots, \beta_k; z)(x) \mu(dx)$$

where $H_k(\beta_1, \ldots, \beta_k; z): H \to \mathbb{C}$ is the Hermite polynomial

$$H_k(\beta_1, \ldots, \beta_k; z)(x) = e^{-\frac{iz}{2}|x|^2} D_{\beta_1} \ldots D_{\beta_k} \left\{ e^{+\frac{iz}{2}|x|^2} \right\}.$$

For $H = \mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n) \cap \mathscr{F}(\mathbb{R}^n)$ with $im\, z < 0$ the result is immediate from the Plancherel theorem and the definition of H_k. It follows for $H = \mathbb{R}^n$ by continuity in z and then by mollifying f. For infinite dimensional H it then follows by dominated convergence.

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