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Infrared representations of free Bose fields (*)

by

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ABSTRACT. — We describe translation covariant and positive energy representations of a free massless scalar field which are normal relative to the vacuum on each forward light cone and generate globally the A. F. D. factor of type II_∞ or III_λ for each $0 < \lambda < 1$. Every properly infinite A. F. D. von Neumann algebra appears as the weak closure of some positive energy representation of the model.

These results extens to Bosons theorems previously known only for Fermions.

RÉSUMÉ. — On décrit des représentations covariantes par translation et d'énergie positive pour un champ scalaire libre sans masse, qui sont normales par rapport au vide sur chaque cône de lumière avancé et qui engendrent globalement les facteurs A. F. D. de type II_∞ ou III_λ pour tout λ, $0 < \lambda < 1$. Toute algèbre de von Neumann A. F. D. et proprement infinie apparaît comme la fermeture faible d'une représentation à énergie positive de ce modèle. Ces résultats étendent au cas des Bosons des théorèmes connus précédemment pour les Fermions seulement.

1. INTRODUCTION

By considering a variant of models studied in [*1*], it has been shown in [*2*] that it is possible to construct positive energy representations of

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the algebra of the quasi-local observables associated to a free massless Majorana field, generating a von Neumann algebra which is an approximately finite dimensional (A. F. D.) type II_∞ or type III_λ factor, for all values of λ, $0 < \lambda < 1$. (Actually every properly infinite A. F. D. von Neumann algebra appears in this way).

The representations above can also be chosen to be normal relative to the vacuum representation when restricted to each forward light cone, and disjoint from it when restricted to each past light cone, or conversely [*3*]. These examples are related to the notion of charge class in Quantum Electrodynamics introduced in [*4*] (see also [*3*] [*5*]).

In this note we extend the above results to the case of a free massless scalar field. The algebra of quasi-local observables is generated by (the bounded functions of) the field itself, whereas in the case of Fermions one must consider the even polynomials in the field operators (On the other hand, in the present case the field operators are not bounded).

The basic idea, as in the previous case, is to consider a sequence of one particle wave functions $\{ x_n \}$ with momentum supports which are mutually disjoint and contained respectively in the balls of radius ε_n around the origin (the upper bound of the energy of a particle in the mode x_n), where

$$\sum_n \varepsilon_n < \infty.$$

Then every state with at most $\nu(n)$ particles in the mode x_n, $n = 1, 2, \ldots$, and no particle in all the orthogonal modes, will generate a positive energy representation provided $\sum_n \varepsilon_n \nu(n) < \infty$. For simplicity we will choose here $\nu(n) = 1$.

In particular, by choosing our state so that in each mode x_n there is a particle with probability λ and no particle with probability $1 - \lambda$, we obtain positive energy representations generating an A. F. D. factor of type II_∞ if $\lambda = \frac{1}{2}$ or type III_μ, $\mu = \min\left(\frac{\lambda}{1-\lambda}, \frac{1-\lambda}{\lambda}\right)$, if $0<\lambda<1$, $\lambda \neq \frac{1}{2}$. By choosing the wave functions x_n approximately localized in the past light cone, we can obtain that these states are locally normal on each future light cone (and disjoint from the vacuum on each past light cone). Similar results hold for free massless particles with any spin.

It can be shown [*5*] that a similar class of representations exists for any interacting theory with neutral massless particles fulfilling asymptotic completeness below some mass threshold [*4*]. A representation in this class is obtained specifying an outgoing collision state, which describes an infrared cloud of massless particles of a given type, and is defined over the algebra of the outgoing free massless fields as in the present paper or in [*3*].

In section 2 we discuss our mathematical tools. We introduce a class of representations of the C. C. R. over a separable pre-Hilbert space K_1, associated to a given orthonormal sequence in K_1. Relative to a norm stronger than the pre-Hilbert norm, K_1 is a Banach space. Our representations are generated by states where each mode in the given orthonormal sequence is occupied by at most one particle. Every A. F. D. properly infinite von Neumann algebra appears as the weak closure of a representation of this class. Of special interest is the case where we consider the product state, over the different modes, of the mixture with constant coefficients of the one particle state and the no particle state. These special examples generate the Powers factors; the generating state is not a quasi free state. In the general case, our examples provide representations of the Weyl relations over K_1 which are continuous from the Banach space K_1 to the strong operator topology.

In section 3, we specialize K_1 to be dense in the Hilbert space of state vectors for a scalar massless particle, and the reference orthonormal sequence x_n to be chosen as described earlier on. With these choices, if $f \to \varphi(f)$ denotes the free massless scalar field and Ω its vacuum state vector, the maps $f \in \mathbb{R}^4) \to \varphi(f)\Omega$, $f \in \mathscr{S}(\mathbb{R}^4) \to \varphi(f)^*\Omega$ are continuous from $\mathscr{S}(\mathbb{R}^4)$ to K_1. Therefore we can apply the results of section 2 together with simple modifications of the arguments given in the preceding papers to obtain the desired results.

2. A CLASS OF CONTINUOUS REPRESENTATIONS OF CCR GENERATING ALL PROPERLY INFINITE AFD VON NEUMANN ALGEBRAS

With K a separable complex pre-Hilbert space, let $(\Gamma(K), \Omega, \mathrm{W})$ be the Fock representation of the Weyl relations on K, i. e. the cyclic unitary representation of the relations (2.2) below acting on the Fock space $\Gamma(K)$ generated by the functional ω_F (The Fock state) [*6*] [*7*]:

$$(2.1) \qquad \omega_F(\mathrm{W}(x)) = (\Omega, \mathrm{W}(x)\Omega) = \exp\left\{ -\frac{1}{4} \| x \|^2 \right\}, \qquad x \in K$$

$$(2.2) \qquad \mathrm{W}(x)\mathrm{W}(y) = \exp\left\{ \frac{i}{2} \operatorname{Im}(x, y) \right\} \mathrm{W}(x + y), \qquad x, y \in K.$$

It is well known that the Fock state ω_F is pure, i. e. the C* algebra A(K) generated by $\{ \mathrm{W}(x); x \in K \}$ is irreducible. By continuity and by Stone's theorem,

$$\mathrm{W}(x) = \exp \{ i\Phi(x) \}$$

with $\Phi(x)$ a selfadjoint operator, real linearly dependent upon $x \in K$. With $a(x)$ the closure of $\frac{1}{\sqrt{2}}(\Phi(x) + i\Phi(ix))$, $a(x)$ is the destruction operator $a(x)^*$ is the creation operator, and the correspondence

$$x_1, x_2, \ldots, x_n \rightarrow (n!)^{-1/2} a(x_1)^* \ldots a(x_n)^* \Omega$$

extends to a linear isometry of the symmetrized tensor product of n copies of K, $\frac{1}{n!}\Sigma PK \otimes K \otimes \ldots \otimes K$, onto the « n-particle subspace » K_n of $\Gamma(K)$.

With $K_0 = \mathbb{C}\Omega$ (the multiples of the Fock vacuum state vector), $\Gamma(K)$ is the closed orthogonal sum of K_n, $n = 0, 1, 2, \ldots$

Moreover,

$$\begin{aligned} \| a(x) |_{K_n} \| &= \sqrt{n} \| x \| \\ \| a(x)^* |_{K_n} \| &= (n+1)^{1/2} \| x \| . \end{aligned} \tag{2.3}$$

If $K = \mathbb{C}$, $K_n = \mathbb{C}\Omega_n$, where Ω_n is the n-th excited state vector of the harmonic oscillator. Let $N_0 \subset \Gamma(\mathbb{C})$ be the subspace generated by Ω_0, Ω_1, and $P_0 = [N_0]$ (¹). By (2.1), (2.2) one easily checks that

$$P_0 W(z) P_0 |_{N_0} = (I + D(z)) \exp\left\{ -\frac{1}{4} | z |^2 \right\}, \qquad z \in \mathbb{C}, \tag{2.4}$$

where

$$\{ (\Omega_j, D(z)\Omega_k) \}_{j,k=0,1} = \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}}\bar{z} \\ -\frac{i}{\sqrt{2}} z & -\frac{1}{2}| z |^2 \end{pmatrix}. \tag{2.5}$$

Note for future use that, if $| z | \leqslant 1$,

$$\| D(z) \| \leqslant (\operatorname{Tr} D(z)^* D(z))^{1/2} \leqslant \frac{\sqrt{5}}{2} | z | . \tag{2.6}$$

Since $A(\mathbb{C})$ is irreducible, note that

$$P_0 A(\mathbb{C}) P_0 |_{N_0} = \mathscr{L}(N_0) \sim M_2 , \tag{2.7}$$

where M_2 denotes the full 2×2 complex matrix algebra. With $\mathscr{K}$ a Hilbert space, $e \in \mathscr{K}$ a unit vector and V a unitary of $\Gamma(\mathbb{C})$ onto $\mathbb{C}^2 \otimes \mathscr{K}$ s. t. $VN_0 = \mathbb{C}^2 \otimes [e]$, we have

$$\begin{aligned} VP_0 A(\mathbb{C}) P_0 V^{-1} &= M_2 \otimes [e] \\ VA(\mathbb{C})'' V^{-1} &= M_2 \otimes \mathscr{B}(\mathscr{K}) . \end{aligned} \tag{2.7}$$

(¹) Here and in the following $[V]$ denotes the orthogonal projection onto the closed subspace of a Hilbert space generated by a subset V.

Let now K be an infinite dimensional Hilbert space, and $x_1, x_2, \ldots$ an orthonormal sequence in K, generating the closed subspace $\mathrm{M} \subset K$.
By equations (2.1), (2.2), the decomposition

$$K = \mathrm{M}^{\perp} \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \ldots$$

induces a factorization

$$\Gamma(K) = \Gamma(\mathrm{M}^{\perp}) \otimes \left\{ \bigotimes_{i=1}^{\infty}{}^{\{\Omega_0\}} \Gamma(\mathbb{C}) \right\} \tag{2.8}$$

where the incomplete infinite tensor product is defined by the constant sequence $\{ \Omega_0, \Omega_0, \ldots \}$ and, as above, Ω_0 is the ground state of the harmonic oscillator.

Let K_0, resp. K_1, be the linear subspace of K of the vectors x s. t. $(x_i, x) = 0$ but for finitely many indices, resp. s. t.

$$\| x \|_1 \equiv \sum_{i=1}^{\infty} | (x_i, x) | < \infty .$$

Equipped with the norm

$$||| x ||| = \| x \| + \| x \|_1 , \qquad x \in K_1 \text{ (}^2\text{)},$$

K_1 is a Banach space and K_0 is dense in K_1. We shall denote by $\mathrm{A}(V)$ the C* algebra generated by $\{ \mathrm{W}(x), x \in V \}$ for any subspace $V \subset K$.

With $\Omega_{\mathrm{M}^{\perp}}$ the Fock vacuum vector in $\Gamma(\mathrm{M}^{\perp})$, let $\mathrm{N} \subset \Gamma(K)$ be the closed subspace

$$\mathrm{N} = \Omega_{\mathrm{M}^{\perp}} \otimes \mathrm{N}_0 \otimes \mathrm{N}_0 \otimes \ldots \tag{2.9}$$

If $\mathrm{P} = [\mathrm{N}]$ and $\mathrm{Q} = [\mathbb{C}\Omega_{\mathrm{M}^{\perp}}]$, the relation:

$$\mathrm{PAP} = \mathrm{Q} \otimes \sigma(\mathrm{A}), \qquad \mathrm{A} \in \mathrm{A}(K) \tag{2.10}$$

defines a linear completely positive map σ of $\mathrm{A}(K)$ into the bounded operators on $\bigotimes_{i=1}^{\infty}{}^{\{\Omega_0\}} \mathrm{N}_0$. Let $\mathscr{A}_{\{2^n\}} = \mathrm{M}_2 \otimes \mathrm{M}_2 \otimes \ldots$ be the U.H.F. algebra generated by the sequence $\{ 2^n \}$ represented on $\bigotimes_{i=1}^{\infty}{}^{\{\Omega_0\}} \mathrm{N}_0$ in the natural way. By the remarks above the range of σ contains the full algebraic tensor product and is dense in $\mathscr{A}_{\{2^n\}}$.

By the next lemma, the transpose of σ maps states ω over $\mathscr{A}_{\{2^n\}}$ to states $\omega \circ \sigma$ over $\mathrm{A}(K_1)$. Such states possess continuity properties useful for our further applications.

2.1. LEMMA. — The application σ maps $\mathrm{A}(K_1)$ into $\mathscr{A}_{\{2^n\}}$. Let ω be a state on the U. H. F. algebra $\mathscr{A}_{\{2^n\}}$; with $\mathscr{H}$ a separable Hilbert space, we have:

(2) With $x_{\mathrm{M}^{\perp}}$ the component of x orthogonal to M, an equivalent norm is

$$||| x |||' = (\| x_{\mathrm{M}^{\perp}} \|^2 + \| x \|_1^2)^{1/2}$$

i) $\pi_{\omega\circ\sigma}(\mathrm{A}(K_1))'' \cong \pi_\omega(\mathcal{A}_{\{2^n\}})'' \overline{\otimes} \mathcal{B}(\mathcal{K})$

ii) the map $x \in K_1 \to \pi_{\omega\circ\sigma}(\mathrm{W}(x))$ is continuous from the $|||\cdot|||$ norm to the strong operator topology.

Proof. — We first prove (*ii*). By a standard argument it suffices to show that $x \in K_1 \to \omega\circ\sigma(\mathrm{W}(x))$ is continuous; to this end we will show that $x \in K_1 \to \sigma(\mathrm{W}(x))$ is norm continuous, which will also prove the first assertion. Using the Weyl relations (2.1) and the C* identity, we have

$$\| \mathrm{P}(\mathrm{W}(x) - \mathrm{W}(y))\mathrm{P} \| \leqslant \| (\mathrm{W}(x) - \mathrm{W}(y))\mathrm{P} \| = \left\| \left(\mathrm{I} - \exp\left\{ \frac{i}{2} \operatorname{Im}(y, x) \right\} \mathrm{W}(y - x) \right)\mathrm{P} \right\| ;$$

$$\| (\mathrm{I} - \mathrm{W}(x))\mathrm{P} \|^2 = \| \mathrm{P}(2\mathrm{I} - \mathrm{W}(x) - \mathrm{W}(x)^*)\mathrm{P} \| \leqslant 2 \| \mathrm{P}(\mathrm{I} - \mathrm{W}(x))\mathrm{P} \| ;$$

therefore, since the exponential is continuous on $K \times K$, it suffices to show

$$\| \mathrm{P}(\mathrm{I} - \mathrm{W}(x))\mathrm{P} \| \to 0 \quad \text{if} \quad |||x||| \to 0,\ x \in K_1 . \tag{2.11}$$

By equations (2.4), (2.9), we have

$$\mathrm{PW}(x)\mathrm{P}\,|_{\mathrm{N}} = \omega_{\mathrm{F}}(\mathrm{W}(x)) \bigotimes_{i=1}^{\infty} (\mathrm{I} + \mathrm{D}((x_i, x))),$$

If $\| x \| < 1$, $|(x_i, x)| < 1$ and by (2.6) we have

$$q \equiv \sum_{i=1}^{\infty} \| \mathrm{D}((x_i, x)) \| \leqslant \frac{\sqrt{5}}{2} \sum_{i=1}^{\infty} |(x_i, x)| = \frac{\sqrt{5}}{2} \| x \|_1 .$$

Therefore, for $|||x||| < \frac{1}{\sqrt{5}}$, $x \in K_1$, we have

$$\left\| \mathrm{I} - \bigotimes_{i=1}^{\infty} (\mathrm{I} + \mathrm{D}((x_i, x))) \right\| \leqslant \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} \| \mathrm{D}((x_i, x)) \| \right)^n = \frac{q}{1-q} \leqslant \sqrt{5} \| x \|_1 \leqslant \sqrt{5}\, |||x||| .$$

Since $x \in K \to \omega_{\mathrm{F}}(\mathrm{W}(x))$ is continuous, equation (2.11) and assertion (*ii*) follow.

By (*ii*) it follows that

$$\pi_{\omega\circ\sigma}(\mathrm{A}(K_1))'' = \pi_{\omega\circ\sigma}(\mathrm{A}(K_0))'' . \tag{2.12}$$

Note that $\mathrm{A}(K_0)$ is the norm closed ascending union of the C* subalgebras $\mathrm{A}_n = \mathrm{A}(\mathrm{M}^\perp \oplus \mathbb{C}x_1 \oplus \ldots \oplus \mathbb{C}x_n)$, i. e.

$$\mathrm{A}_n \equiv \mathrm{A}(\mathrm{M}^\perp) \otimes \underbrace{\mathrm{A}(\mathbb{C}) \otimes \ldots \otimes \mathrm{A}(\mathbb{C})}_{n \text{ times}} \otimes \mathrm{I} \otimes \mathrm{I} \otimes \ldots \tag{2.13}$$

where the tensor product refers to the factorization (2.8) of $\Gamma(K)$. For any $\omega \in \mathscr{S}(\mathscr{A}_{\{2^n\}})$, if $A \in \mathrm{A}(\mathrm{M}^\perp)$, $B \in \mathrm{A}(\mathrm{M} \cap K_0)$,

$$\omega \circ \sigma(AB) = \omega_{\mathrm{F}}(A)\omega \circ \sigma(B),$$

therefore, $\omega \circ \sigma$ extends to a locally normal state $\hat{\omega}$ of the C* algebra $\tilde{\mathrm{A}}$, defined as the uniform closure of the union of the von Neumann algebras A_n'', where:

$$\mathrm{A}_n'' \equiv \mathscr{B}(\Gamma(\mathrm{M}^\perp)) \,\overline{\otimes}\, \underbrace{\mathscr{B}(\Gamma(\mathbb{C})) \,\overline{\otimes}\, \mathscr{B}(\Gamma(\mathbb{C})) \,\overline{\otimes}\, \dots \,\overline{\otimes}\, \mathscr{B}(\Gamma(\mathbb{C}))}_{n \text{ times}} \otimes \mathrm{I} \otimes \mathrm{I} \otimes \dots$$

Denote by $\zeta_0 \otimes e$ the vector $\mathrm{V}\Omega_0$ (cf. equation (2.7)). The unitary obtained by composing the map

$$x \otimes \zeta_1 \otimes \zeta_2 \otimes \dots \in \Gamma(\mathrm{M}^\perp) \otimes \left(\bigotimes_{i=1}^{\infty} {}^{\{\Omega_0\}} \Gamma(\mathbb{C}) \right) \to x \otimes \mathrm{V}\zeta_1 \otimes \mathrm{V}\zeta_2 \otimes \dots$$

with the permutation

$$\Gamma(\mathrm{M}^\perp) \otimes \left(\bigotimes_{i=1}^{\infty} {}^{\{\zeta_0 \otimes e\}} \mathbb{C}^2 \otimes \mathscr{K} \right) \to \left(\bigotimes_{i=1}^{\infty} {}^{\{\Omega_0\}} \mathbb{C}^2 \right) \otimes \Gamma(\mathrm{M}^\perp) \otimes \left(\bigotimes_{i=1}^{\infty} {}^{\{e\}} \mathscr{K} \right)$$

induces a $*$-isomorphism ρ of $\tilde{A}$ onto the C* inductive limit of

$$A_{\{2^n\}} \otimes \mathscr{B}(\Gamma(\mathrm{M}^\perp)) \,\overline{\otimes}\, \mathscr{B}(\mathscr{K}) \,\overline{\otimes}\, \dots \,\overline{\otimes}\, \mathscr{B}(\mathscr{K}) \otimes \mathrm{I} \otimes \mathrm{I} \otimes \dots$$

Furthermore, by the definition of σ and (2.9),

$$\hat{\omega} \circ \rho^{-1} = \omega \otimes \omega_{\mathrm{F}} \otimes \varphi \otimes \varphi \otimes \varphi \otimes \dots,$$

where $\varphi(\mathrm{B}) = (e, \mathrm{B}e)$, $\mathrm{B} \in \mathscr{B}(\mathscr{K})$, and ω_{F} is the vector state of $\mathscr{B}(\Gamma(\mathrm{M}^\perp))$ induced by the Fock vacuum vector. Therefore,

$$\pi_{\omega\circ\sigma}(\mathrm{A}(K_0))'' \cong \pi_\omega(\mathscr{A}_{\{2^n\}})'' \,\overline{\otimes}\, \mathscr{B}(\mathscr{K})$$

and, by (2.12), (*i*) follows. □

2.2. THEOREM. — Let $\mathscr{R}$ be any properly infinite A. F. D. von Neumann algebra acting on a separable Hilbert space, and M, K_1, K as above. There is a cyclic representation of the Weyl relations on K_1, $x \in K_1 \to \hat{\mathrm{W}}(x)$, with cyclic vector $\hat{\Omega}$, generated by a state $\hat{\omega}(\mathrm{W}(x)) \equiv (\hat{\Omega}, \hat{\mathrm{W}}(x)\hat{\Omega})$, such that

i) $\mathscr{R} = \{ \hat{\mathrm{W}}(x), x \in K_1 \}''$;

ii) $x \in K_1 \to \hat{\mathrm{W}}(x)$ is continuous from the Banach space topology of K_1 to the strong operator topology;

iii) $\hat{\omega}(\mathrm{W}(x)) = \omega_{\mathrm{F}}(\mathrm{W}(x)) = \exp\left\{ -\frac{1}{4} \| x \|^2 \right\}$ if $x \in \mathrm{M}^\perp$; (in the state $\hat{\omega}$, each mode orthogonal to M is empty)

iv) in the state $\hat{\omega}$ each of the modes $x_1, x_2, \dots$ is occupied by at most one particle.

Proof. — By our assumption, $\mathcal{R}$ has a cyclic vector [8] and by a theorem of O. Marechal [9] there is a state ω on $\mathcal{A}_{\{2^n\}}$ such that $\pi_\omega(\mathcal{A}_{\{2^n\}})'' \cong \mathcal{R}$. By using this state for the construction in Lemma 2.1, we get a representation $x \in K_1 \to \pi_{\omega\circ\sigma}(\mathrm{W}(x))$ of the Weyl relations fulfilling *ii)* and generating a von Neumann algebra spatially isomorphic to $\mathcal{R} \overline{\otimes} \mathcal{B}(\mathcal{K})$, hence to $\mathcal{R}$ [8] Therefore there is an unitary operator U s. t.

$$x \in K_1 \to \mathrm{U}\pi_{\omega\circ\sigma}(\mathrm{W}(x))\mathrm{U}^{-1} \equiv \hat{\mathrm{W}}(x)$$

generates $\mathcal{R}$. With $\hat{\Omega}$ the image under U of the G. N. S. vector of $\omega \circ \sigma$, (*ii*) and (*iii*) are evident. Also (*iv*) follows from the definition of σ and, by the construction in the preceding Lemma, $\hat{\mathrm{W}}$ defines a normal representation of each $\mathrm{A}(\mathbb{C}x_i)$, $i = 1, 2, \ldots$ □

We end this section by discussing a bit more closely the case where $\mathcal{R}$ is a Power factor. With $0 \leqslant \lambda \leqslant 1$ let us choose the state ω of Lemma 2.1 as $\omega_\lambda = \varphi_\lambda \otimes \varphi_\lambda \otimes \ldots$, where

$$\varphi_\lambda\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (1 - \lambda)a + \lambda d. \tag{2.14}$$

Then $\pi_{\omega_\lambda}(\mathcal{A}_{\{2^n\}})''$ is the hyperfinite II_1 factor for $\lambda = \frac{1}{2}$; the III_μ A. F. D. factor for $\mu = \min\left(\frac{\lambda}{1-\lambda}, \frac{1-\lambda}{\lambda}\right), \lambda \neq \frac{1}{2}, \ 0 < \lambda < 1$; the I_∞ A. F. D. factor for $\lambda = 0$ or 1. By Lemma 2.1 $\pi_{\omega_\lambda\circ\sigma}(\mathrm{A}(K_1))''$ is the A. F. D. factor resp. of type II_∞, III_μ, I_∞ [10] [11].

By equations (2.4), (2.5), (2.10), (2.14),

$$\omega_\lambda \circ \sigma(\mathrm{W}(x)) = \omega_\mathrm{F}(\mathrm{W}(x)) \prod_{i=1}^{\infty} \left(1 - \frac{\lambda}{2} | (x_i, x) |^2\right), \qquad x \in K. \tag{2.15}$$

By arguments similar to those in the proof of Lemma 2.1, it follows easily that $x \in K \to \omega_\lambda \circ \sigma(\mathrm{W}(x))$ is continuous in the Hilbert space topology for each $0 \leqslant \lambda \leqslant 1$. It should be noted that the states (2.15) are not quasi-free (see [12] and ref. therein); they are the product states, over the decomposition (2.13) of $\mathrm{A}(K_0)$, of the Fock vacuum over $\mathrm{A}(\mathrm{M}^\perp)$ and of the mixtures with coefficients $1 - \lambda$, λ of the no particle, resp 1 particle states, in the modes x_i, $i = 1, 2, \ldots$ For each $0 < \lambda < 1$ we can describe such states by their purification as follows (cf. [13, 2] for the Fermi case).

Let $y_i(\lambda) \in K \oplus K$, $y_i(\lambda) = \sqrt{\lambda}x_i \oplus (1 - \lambda)^{1/2}x_i$, $i = 1, 2, \ldots$ and ρ_λ the state over $\mathrm{A}(K \oplus K) = \mathrm{A}(K) \otimes \mathrm{A}(K)$ defined as in equation (2.15)

$$\rho_\lambda(\mathrm{W}(y)) = \omega_\mathrm{F}(\mathrm{W}(y)) \prod_{i=1}^{\infty} \left(1 - \frac{1}{2} | (y_i(\lambda), y) |^2\right), \qquad y \in K \oplus K.$$

Clearly ρ_λ is the pure state where each mode $y_i(\lambda)$, $i = 1, 2, \ldots$ is occupied by exactly one particle and each orthogonal mode in $K \oplus K$ is empty. The restriction of ρ_λ to $A(K) \otimes I$ gives back $\omega_\lambda \circ \sigma$, i. e.

$$\omega_\lambda \circ \sigma(W(x)) = \rho_\lambda(W(x \oplus 0)), \qquad x \in K.$$

With Ω the Fock vacuum vector in $\Gamma(K \oplus K)$, and

$$\zeta_n \equiv a(y_n(\lambda))^* \ldots a(y_1(\lambda))^*\Omega$$

the sequence of vector states ω_{ζ_n} converges $*$-weakly to ρ_λ on $A(K \oplus K)$ as $n \to \infty$.

3. INFRARED REPRESENTATIONS

In this section we specify K to be the Hilbert space $L^2\left(\mathbb{R}^3, \frac{d^3k}{2|k|}\right)$ of the one particle state vectors for a massless scalar Boson. The scalar massless free field is defined as (cf. e. g. [7])

$$f \in \mathcal{S}(\mathbb{R}^4) \to \varphi(f) \equiv \frac{1}{\sqrt{2}}(a(T\bar{f}) + a(Tf)^*) \tag{3.1}$$

or $\varphi(f) = \Phi(T \operatorname{Re} f) + i\Phi(T \operatorname{Im} f)$, where

$$Tf = (2\pi)^{1/2}\hat{f}\,|_{\partial V^+}, \qquad f \in \mathcal{S}(\mathbb{R}^4) \tag{3.2}$$

i. e. $(Tf)(k) = (2\pi)^{1/2}\hat{f}(+|\vec{k}|, \vec{k})$, $\hat{f}$ denoting the Fourier transform of f. As in section 2, a, a^* denote the destruction and creation operators.

The orthonormal sequence $x_1, x_2, \ldots \in K$ is chosen here to fulfill the conditions:

i) $\operatorname{supp} x_n \subset \{ \vec{k} / |\vec{k}| \leqslant \varepsilon_n \}$; $\sum_n \varepsilon_n < \infty$

ii) $\operatorname{supp} x_n \cap \operatorname{supp} x_m = \emptyset$, $n \neq m$.

Let $K_1 \subset K$ be the corresponding Banach space defined in Section 2.

With $A(K)$ the C*-algebra generated by the Fock representation of the Weyl relations over K, define, for each double cone $\mathcal{O}$ in Minkowski space (3), the local algebra of observables associated to $\mathcal{O}$ as

$$\mathcal{A}(\mathcal{O}) = C^* \{ W(Tf) ; f \in \mathcal{S}_R(\mathbb{R}^4), \operatorname{supp} f \subset \mathcal{O} \} \tag{3.3}$$

Let:

$$\mathcal{A} = \overline{\cup \{ \mathcal{A}(\mathcal{O}) ; \mathcal{O} \in K \}} \subset A(K)$$

be the C*-algebra of quasilocal observables in our model. It is well known

(3) i. e. $\mathcal{O} = \{ x \in \mathbb{R}^4 ; x - x_1, x_2 - x \text{ are timelike} \}$ for some timelike separated x_1, x_2. Let K be the set of all double cones.

(cf. e. g. [*12*]) that $\mathscr{A}$ is irreducible on $\Gamma(K)$, $\mathscr{A}(\mathcal{O}_1)$ and $\mathscr{A}(\mathcal{O}_2)$ commute if $\mathcal{O}_1$, $\mathcal{O}_2 \in K$ and $\mathcal{O}_1 - \mathcal{O}_2$ contains no light like vectors (since the free massless field commutator has support on $x^2 = 0$). The second quantization $\Gamma(u(\mathrm{L})) \equiv \mathrm{U}(\mathrm{L})$ of the zero mass and spin irreducible representation u of the Poincaré group $\mathscr{P}$ naturally acting on K, leaves Ω invariant and induces automorphisms α_{L} of $\mathscr{A}$ s. t.

$$\alpha_{\mathrm{L}}(e^{i\varphi(f)}) = e^{i\varphi(f_{\mathrm{L}})}, \qquad f_{\mathrm{L}}(x) = f(\mathrm{L}^{-1}x),\ f \in \mathscr{S}_{\mathbb{R}}(\mathbb{R}^4)$$
$$\alpha_{\mathrm{L}}(\mathscr{A}(\mathcal{O})) = \mathscr{A}(\mathrm{L}\mathcal{O})\,; \qquad \mathrm{L} \in \mathscr{P},\ \mathcal{O} \in K\,.$$

3.1. THEOREM. — With the above specifications, $\mathscr{A} \subset \mathrm{A}(K_1)$. For each properly infinite A. F. D. Von Neumann algebra $\mathscr{R}$ acting on a separable Hilbert space, the representation of the Weyl relations over K_1 constructed in Theorem 2.2, defines a representation π of $\mathscr{A}$ s. t.

1) π is covariant for the space time translation group and obeys the spectrum condition.
2) $\pi(\mathscr{A})'' \cong \mathscr{R}$.

Proof. — First we show that T maps $\mathscr{S}(\mathbb{R}^4)$ onto a dense subspace of K_1. For each n, $x_n \in K$ and has compact support, hence $x_n \in \mathrm{L}^1\left(\mathbb{R}^3, \dfrac{d^3\vec{k}}{2\,|\,\vec{k}\,|}\right)$ and for each $f \in \mathscr{S}(\mathbb{R}^4)$,

$$|\,(\mathrm{T}f, x_n)\,| \leqslant \|\,\mathrm{T}f\,\|_\infty \,\|\,x_n\,\|_{\mathrm{L}^1}\,.$$

By the Schwartz inequality, and by (*i*), we have

$$\|\,x_n\,\|_{\mathrm{L}^1} \leqslant \|\,\chi_{\mathrm{supp}\,x_n}\,\|\,\|\,x_n\,\| = \|\,\chi_{\mathrm{supp}\,x_n}\,\|$$
$$\|\,\chi_{\mathrm{supp}\,x_n}\,\| \leqslant \left(\int_{|k|<\varepsilon_n} \frac{d^3k}{2\,|\,\vec{k}\,|}\right)^{1/2} = \sqrt{\pi}\varepsilon_n\,.$$

Therefore we have

$$\|\,\mathrm{T}f\,\|_1 \equiv \sum_{n=1}^{\infty} |\,\mathrm{T}f, x_n)\,| \leqslant \left(\sqrt{\pi}\sum_{n=1}^{\infty} \varepsilon_n\right) \|\,\mathrm{T}f\,\|_\infty\,,$$

and the range of T is contained in K_1. Since $\|\,\mathrm{T}f\,\| \leqslant \|\,\mathrm{T}f\,\|_\infty \cdot \|\,\chi_{\mathrm{supp}\,\mathrm{T}f}\,\|$, whenever $\mathrm{T}f$ has compact support, for each compact set V there is a constant C_V s. t., for all $f \in \mathscr{S}(\mathbb{R}^4)$ with supp $\mathrm{T}f$ in V, we have

$$|||\,\mathrm{T}f\,||| < \mathrm{C}_V \,\|\,\mathrm{T}f\,\|_\infty\,.$$

By the Stone-Weierstrass theorem, the closure of $\mathrm{T}\mathscr{S}(\mathbb{R}^4)$ in K_1 contains all continuous function with compact support, hence coincides with K_1.

By Theorem 2.2 it follows that

$$\pi(\mathscr{A})'' = \pi(\mathrm{A}(K_1))'' \cong \mathscr{R}\,.$$

The rest of the argument follows closely [*3*] [*5*].

By equations (2.9), (2.10), (2.12), (2.13) and the subsequent comments, we have

$$P\tilde{A}P = Q \otimes \sigma(A(K)) = \mathscr{A}_{\{2^n\}}.$$

By a theorem of Glimm [14] [15] each state ω of $\mathscr{A}_{\{2^n\}}$ is the w^*-limit of a sequence of vector states of any given irreducible representation. Let $\tilde{\omega}$ be the locally normal extension of $\omega \circ \sigma$ to $\tilde{A}$; there is a sequence of unit vectors $\zeta_n \in N = P(\Gamma(K))$ such that $\omega_{\zeta_n}|_{\tilde{A}}$ converges $*$-weakly to $\tilde{\omega}$. With $K_n \subset K$ the subspace of the functions vanishing for $|\vec{k}| \leqslant \frac{1}{n}$, the net $A(K_n)''$ is contained in $\tilde{A}$. With $\tilde{\pi} = \pi_\omega$ the locally normal extension of π to $\tilde{A}$ (cf. [16]), by continuity we have:

$$\tilde{\pi}(\tilde{A})'' = \pi(\mathscr{A})'' = \bigvee_n \tilde{\pi}(A(K_n)''). \tag{3.4}$$

The C*-subalgebra $\mathscr{B}_n$ of the elements in $A(K_n)''$ which have a norm continuous orbit under $x \in \mathbb{R}^4 \to \alpha_x$, is strongly dense in $A(K_n)''$. If $\mathscr{B}$ denotes the C*-algebra generated by $\mathscr{B}_n$, $n = 1, 2, \ldots$, by (3.4) we have also

$$\tilde{\pi}(\mathscr{B})'' = \pi(\mathscr{A})'' = \pi_{\tilde{\omega}}|_{\mathscr{B}}(\mathscr{B})''. \tag{3.5}$$

Now $\tilde{\omega} \mid \mathscr{B}$ is the $*$-weak limit of $\omega_{\zeta_n}|_{\mathscr{B}}$, as $n \to \infty$. By construction, and by condition (*i*) on the sequence $x_1, x_2, \ldots$, N is included in the spectral subspace of the energy operator relative to $[0, \Sigma\varepsilon_n]$, and $\zeta_n \in N$. Moreover the action $x \in \mathbb{R}^4 \to \alpha_x|_{\mathscr{B}}$ is strongly continuous. By a theorem of Borchers [17], $\pi_{\tilde{\omega}}|_{\mathscr{B}}$ is covariant and fullfills the spectrum condition. By (3.5) and the continuity properties of π, it follows that also π is covariant and obeys the spectrum condition. □

Note that by [18] [19] each of the representations π described in theorem 3.1 is locally normal.

In particular, the states given by equation (2.15) induces covariant positive energy representations of $\mathscr{A}$ which generate all Powers factors for $0 < \lambda < 1$, $\lambda \neq \frac{1}{2}$, and is type II_∞ for $\lambda = \frac{1}{2}$, irreducible for $\lambda = 0$ or 1.

As in the case of Fermions [3], a further restriction on the sequence of modes $x_1, x_2, \ldots, \in K$ implies that our representations are also locally normal on each future light cone. Namely, in addition to (*i*), (*ii*) assume

iii) for each n, there is a function $f_n \in \mathscr{S}(R^4)$, supp $f_n \subset V^-$, such that

$$\left\| \frac{1}{\sqrt{2}} Tf_n \oplus T\bar{f}_n - x_n \oplus 0 \right\| < \frac{2^{-n}}{2(n+1)^{1/2}}.$$

since the map $f \in \mathscr{D}(R^4) \to Tf \oplus T\bar{f}$ has dense range in $K \oplus K$, from any cnoice $x'_1, x'_2, \ldots$, fulfilling (*i*), (*ii*), we can obtain a choice $x_1, x_2, \ldots$, fulfilling (*i*), (*ii*), (*iii*) acting on each x'_n by a suitable time translation.

3.2. THEOREM. — With $x_1, x_2, \ldots \in K$ a sequence of one particle modes obeying the conditions the conditions (*i*), (*ii*), (*iii*), the representations π_λ of the quasilocal algebra of the free massless field defined by the states (2.15) are covariant, fulfill the spectrum condition, and, for each $a \in \mathbb{R}^4$,

$$\pi_\lambda|_{\mathscr{A}(V_a^+)} \cong \pi_0|_{\mathscr{A}(V_a^+)}, \tag{3.6}$$

$$\pi_\lambda|_{\mathscr{A}(V_a^-)} \not\approx \pi_0|_{\mathscr{A}(V_a^-)}, \quad \lambda \neq 0. \tag{3.7}$$

Proof. — By means of the purification described at the end of Section 2, we can see as in [*3*] that it suffices to prove the theorem for $\lambda = 1$. Covariance and spectrum condition follow from Theorem 3.1. Like in ref. [*2*], an independent explicit argument can be given, by the results of [*20*] or by adapting the argument of [*1*] along the lines below.

Since π_1 is the G. N. S. representation generated by the weak $*$ limit of the sequence of states ω_{ζ_n} of $\mathscr{A}$, $\zeta_n = \frac{1}{\sqrt{n!}} a(x_n)^* \ldots a(x_1)^*\Omega$, to prove (3.6) it suffices to show that $\omega_{\zeta_n}|_{\mathscr{A}(V^+)}$ converges in norm as $n \to \infty$. By covariance, (3.6) will follow for all $a \in \mathbb{R}^4$. Consider the following identity:

$$\begin{aligned}(\omega_{\zeta_n} - \omega_{\zeta_{n-1}})(B) &= (\zeta_n, [B, a(x_n)^*]\zeta_{n-1}) \\ &= (\zeta_n, [B, \varphi(f)]\zeta_{n-1}) + (\zeta_n, [B, (a(x_n)^* - \varphi(f))]\zeta_{n-1}).\end{aligned}$$

If we choose $B \in \mathscr{A}(V^+)$ and supp $f \subset V^-$, $f \in \mathscr{S}(\mathbb{R}^4)$, the first term vanishes by the timelike commutativity of the free massless field over the four dimensional Minkowski space. By (2.3), (3.1) we then have (recalling that $a(y)|_{K_n}$ and $a(z)^*|_{K_n}$ have orthogonal ranges):

$$\begin{aligned}|(\omega_{\zeta_n} - \omega_{\zeta_{n-1}})(B)| &\leqslant \|B\| \cdot \{ \|(a(x_n)^* - \varphi(f))|_{K_{n-1}}\| + \|(a(x_n) - \varphi(f)^*)|_{K_n}\| \} \\ &\leqslant 2(n+1)^{1/2}\|B\| \left\| \frac{1}{\sqrt{2}} Tf \oplus T\bar{f} - x_n \oplus 0 \right\|, \qquad B \in \mathscr{A}(V^+).\end{aligned}$$

By choosing $f = f_n$ as in condition (*iii*) we get

$$\|(\omega_{\zeta_n} - \omega_{\zeta_{n-1}})|_{\mathscr{A}(V^+)}\| \leqslant \frac{1}{2^n};$$

therefore ω_{ζ_n} is norm convergent on $\mathscr{A}(V^+)$ and (3.6) follows.

To prove (3.7) it suffices to exhibit a sequence of unitaries $U_n \in \mathscr{A}(V^-)$ s. t.

$$\operatorname*{weak\,lim}_{n \to \infty} (\pi_\lambda(U_n) - \omega_\lambda(U_n) \cdot I) = 0, \qquad 0 \leqslant \lambda \leqslant 1; \tag{3.8}$$

$$\lim_{n \to \infty} \omega_0(U_n) \neq \lim_{n \to \infty} \omega_\lambda(U_n), \qquad 0 < \lambda \leqslant 1. \tag{3.9}$$

With f_n as in condition (*iii*) choose

$$U_n = \exp\{ i\varphi(f_n + \bar{f}_n) \}.$$

Since

$$\left\| x_n - \frac{1}{\sqrt{2}} T(f_n + \bar{f}_n) \right\| \leqslant \sqrt{2} \left\| x_n \oplus 0 - \frac{1}{\sqrt{2}} Tf_n \oplus T\bar{f}_n \right\|,$$

by (*iii*) and the strong continuity of $x \in K \to W(x)$ we have

$$\text{(3.10)} \qquad \operatorname*{strong\,lim}_{n\to\infty} (W(x_n) - U_n) = 0.$$

Since $x_1, x_2, \ldots$ is an infinite orthonormal set and π_λ is factorial, by the Weyl relations (2.2) the sequence $W(x_n)$ fulfills (3.8). By (3.10) so does the sequence U_n. By the continuity of $x \in K \to \omega_\lambda(W(x))$ and by (2.15), for $0 \leqslant \lambda \leqslant 1$ we have

$$\lim_{n\to\infty} \omega_\lambda(U_n) = \lim_{n\to\infty} \omega_\lambda(W(x_n)) = \left(1 - \frac{\lambda}{2}\right) e^{-1/4}$$

and (3.7) follows. □

Note that the representations $\pi_\lambda |_{\mathscr{A}(V^-)}$ are mutually disjoint for distinct values of λ, $0 \leqslant \lambda \leqslant 1$.

It is clear that by an appropriate choice of the sequence $x_1, x_2, \ldots$ we can arrange that all our representations π_λ are also rotation covariant (but not Lorentz covariant unless $\lambda = 0$).

Similar results can be obtained for the algebra of quasi-local observables associated to the free massless field of any integer (resp. by the method of [*2*] [*3*], half integer) spin. One identifies the one particle Hilbert space for a massless particle of spin $j > 0$ with a subspace

$$L^2(\mathbb{R}^3, (2|k|)^{2j-1} d^3\vec{k}) \otimes \mathbb{C}^{2(2j+1)},$$

consisting of the solutions of the wave equation i. e. corresponding to the values $\pm$ of the helicity. The appropriate generalization of our method proceeds as here or in [*3*]. The bound of $\| \chi_{\operatorname{supp} x_n} \|$ (cf. the proof of Theorem 3.1 or [*3*, Appendix]) modifies in an obvious way so that

$$\sum_n \| \chi_{\operatorname{supp} x_n} \| \leqslant \text{const} \left(\sup_i \varepsilon_i^j\right) \sum_n \varepsilon_n < \infty$$

for any spin.

In particular we can construct representations of the free electromagnetic field $F_{\mu\nu}$ fulfilling all the properties above.

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