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Logarithmic asymptotic behaviour of the renormalized G-convolution product in four-dimensional Euclidean space

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ABSTRACT. — We give an asymptotic logarithmic behaviour in r-dimensional Euclidean momentum space of the renormalized G-convolution product H_G^{ren} associated with a general graph G. This study is an extension of previous result which contained only the power law asymptotic behaviour with respect to external momenta.

RÉSUMÉ. — On obtient un comportement asymptotique logarithmique dans l'espace Euclidien à r dimensions des impulsions pour le produit de G-convolution renormalisé H_G^{ren} associé à un graphe général G. Cette étude est une extension de résultats précédents qui contenaient seulement le comportement asymptotique en puissances des impulsions externes.

INTRODUCTION

In [*1*] [*2*], Weinberg functional classes have been introduced to prove convergence of the (Euclidean) renormalized G-convolution product H_G^{ren} associated with a general graph G. In [*3*], an asymptotic behaviour of H_G^{ren} in momentum space has been proved in terms of external r-momenta.

In view of the procedure used in [3], it appeared clearly that a more accurate asymptotic estimate including logarithmic behaviour could be easily derived in an analogous way. Moreover, some recent studies of equations of motion concerning Φ_4^4-coupling models ([4]) require this logarithmic increase.

The aim of this paper is then to give a precise logarithmic asymptotic behaviour of the Euclidean renormalized G-convolution product H_G^{ren} in r (and in particular in 4)-dimensions, using the general notion of Weinberg class given in [5], and so produce an extension of the results of [3].

We just mention a work by Fink [6], giving some logarithmic estimates for particular self-energy graphs.

After a brief recall of the main properties of Weinberg's functional class and of the integrability criterium, including logarithmic behaviour, we define the class of symbols $\Sigma^{\mu,\nu}$ (resp. the admissible Weinberg's class $\mathscr{A}^{\alpha,\beta,\sigma,\omega}$) which is a straightforward extension of Σ^{μ} (resp. of $\mathscr{A}^{\alpha,\sigma,\omega}$) introduced in [1], the index ν (resp. β) denoting the logarithmic contribution.

Then we consider a graph G, and we associate to each vertex v with n_v incoming lines (resp. to each line i) of G, a general n_v-point function (resp. a two-point function) H^{n_v} (resp. $H_i^{(2)}$). We use the recursive definition of the euclidean renormalized integrand R_G defined in [1] to prove that R_G belongs to a definite Weinberg class as soon as H^{n_v} and $H_i^{(2)}$ belong to suitable symbol class Σ^{μ_v,ν_v}, Σ^{μ_i,ν_i}. Therefore, a direct use of an analog of Weinberg's theorem gives us the requested asymptotic behaviour of the corresponding renormalized G-convolution product H_G^{ren}.

For conciseness, we have omitted the proof of a technical result (see lemma 2.4, infra), which will appear elsewhere [7].

1. PRELIMINARY RESULTS

1.1. Statement of Weinberg's theorem [5].

1.1.1. *Weinberg's functional classes.*

Let $f : E = \mathbb{R}^n \to \mathbb{C} \cdot f$ is said to be an element of $A_n^{\alpha,\beta}$ if and only if, for each subspace $S \subset \mathbb{R}^n$, there exists two coefficients $\alpha(S)$ and $\beta(S)$ such that, for any choice of $m \leqslant n$ independant vectors $L_1, L_2, \ldots L_m$, and any bounded region $W \subset \mathbb{R}^n$, we have:

$$f(L_1\eta_1\eta_2 \ldots y_m + L_2\eta_2 \ldots \eta_m + \ldots + L_m y_m + C)$$
$$= O(\eta_1^{\alpha\{L_1\}} (\text{Log}\,\eta_1)^{\beta\{L_1\}} \ldots \eta_m^{\alpha\{L_1,\ldots,L_m\}} (\text{Log}\,\eta_m)^{\beta\{L_1,\ldots,L_m\}})$$

when $\eta_k \to \infty, C \in W$.

That is to say, if there exists a set of numbers $b_1, \ldots, b_m \geqslant 1$, and a constant $M > 0$ (depending on $L_1, \ldots, L_m$ and W) such that:

$$\left| f\left(\sum_{j=1}^{m} L_j \eta_j \ldots \eta_m + C\right)\right| \leq M \prod_{j=1}^{m} \eta_j^{\alpha(\overline{\{L_1,\ldots,L_j\}})} (\text{Log}\, \eta_j)^{\beta(\overline{\{L_1,\ldots,L_j\}})} \quad (1.1)$$

when the real variables $\eta_j (j = 1, \ldots, m)$ belong to the region $\{ \eta_j \geq b_j \}$; in (1.1) $\{ \overline{L_1, \ldots, L_j} \}$ denotes the linear closure of the set $\{ L_1, \ldots, L_j \}$.

The functions α and β are assumed to be bounded real-valued function on the set of the linear subspaces of E, and are called asymptotic indicatrices of $A_n^{\alpha,\beta}$.

We then can obtain by the above definitions, the following:

PROPOSITION 1.1.

a) $A_n^{\alpha,\beta}$ is a vector space on $\mathbb{R}$ or $\mathbb{C}$.

b) if $f_1 \in A_n^{\alpha_1,\beta_1}$, $f_2 \in A_n^{\alpha_2,\beta_2}$, then $f_1 f_2 \in A_n^{\alpha_1+\alpha_2,\beta_1+\beta_2}$

c) if $\alpha < \alpha'$, $A_n^{\alpha,\beta} \subset A_n^{\alpha',\beta'}$, $\forall \beta, \beta'$

if $\alpha = \alpha'$ and $\beta < \beta'$: $A_n^{\alpha,\beta} \subset A_n^{\alpha',\beta'}$

N. B. — In the following, A_n^{α} denotes the class $A_n^{\alpha,0}$.

1.1.2. *Weinberg's integrability criterium (case $\beta = 0$).*

Let I be a subspace of $\mathbb{R}^n$ spanned by $L'_1, \ldots, L'_k$, and consider the integral:

$$f_I(P) = \int_{\mathbb{R}^k} dy_1 \ldots dy_k f(P + L'_1 y_1 + \ldots + L'_k y_k)$$

$$= \int_I f(P + P') d^k P' \quad (1.2)$$

THEOREM 1.1. — *Suppose that* $f \in A_m^{\alpha} \cap L^1_{loc}(\mathbb{R}^n)$ (1),

let: $$D_I = \max_{S' \subset I} \{ \alpha(S') + \dim S' \}$$

If $D_I < 0$, *then*:

i) $f_I(P)$ *exists*

ii) $f_I(P) \in A_{n-k}^{\alpha_I}$, *with asymptotic coefficient* $\alpha_I(S)$ *for* $S \subset E$ (*where* $R^n = E \oplus I$) *given by*:

$$\alpha_I(S) = \max_{\Lambda(I)S'=S} \{ \alpha(S') + \dim S' - \dim S \} \quad (1.3)$$

where $\Lambda(I)$ *is the projection along* I *and the* max *is taken on all subspaces* S' *which project on* S (*cf.* [5]).

(1) $L^1_{loc}(\mathbb{R}^n)$denotes the usual lebesgue space of locally integrable classes of functions in $\mathbb{R}^n$.

We note that the logarithmic behaviour has no influence on the convergence criterium, it is therefore requested for the asymptotic behaviour.

1.2. Logarithmic behaviour.

We consider: $f: \mathscr{E}^{N}_{(K,k)} = \mathscr{E}^{n}_{(K)} \times E^{m}_{(k)} \to \mathbb{C}$.

We suppose that f belongs to the Weinberg class $A^{\alpha,\beta}_{N}$ on $\mathscr{E}^{N}_{(K,k)}$, and we use the notations:

— χ is the canonical projection of $\mathscr{E}^{N}_{(K,k)}$ on $\mathscr{E}^{n}_{(K)}$

— $\mathscr{M} = \{ S' \subset \mathscr{E}^{N}_{(K',k)} : \quad \chi(S') = S, \quad \dim S' = \dim S \}$

— $\mathscr{M}' = \{ S' \subset \mathscr{E}^{N}_{(K,k)} : \quad \chi(S') = S, \quad \dim S' \neq \dim S \}$

— $\alpha_{\mathscr{M}}(S) = \max_{S' \in \mathscr{M}} \alpha(S')$; $\quad \beta_{\mathscr{M}}(S) = \max_{S' \in \mathscr{M}} \beta(S')$, and same notations for $\alpha_{\mathscr{M}'}(S)$ and $\beta_{\mathscr{M}'}(S)$.

Then we have:

THEOREM 1.2. — *Suppose that* $f \in A^{\alpha,\beta}_{N}$ *and* $\max_{S \subset E^{rm}_{(k)}} (\alpha(S) + \dim S) < 0$. *Then*

— *the integral* $f_I(K) = \int_{E^{rm}_{(k)}} f(K, k) \cdot d^{rm}k$ *converges absolutely.*

— $f_I \in A^{\alpha_I,\beta_I}_{n}$, *with the coefficients*: $\forall S \subset \mathscr{E}^{n}_{(K)}$:

$$\alpha_I(S) = \max_{\chi(S')=S} (\alpha(S') + \dim S' - \dim S)$$

$$\beta_I(S) = \begin{cases} \beta_{\mathscr{M}}(S) & \text{if} \quad \alpha_I(S) = \alpha_{I,\mathscr{M}}(S); \quad \alpha_{I,\mathscr{M}}(S) \neq \alpha_{I,\mathscr{M}'}(S) \\ \beta_{\mathscr{M}'}(S) & \text{if} \quad \alpha_I(S) = \alpha_{I,\mathscr{M}'}(S); \quad \alpha_{I,\mathscr{M}}(S) \neq \alpha_{I,\mathscr{M}'}(S) \\ 1 + \beta_{\mathscr{M}}(S) + \beta_{\mathscr{M}'}(S) & \text{if} \quad \alpha_{I,\mathscr{M}}(S) = \alpha_{I,\mathscr{M}'}(S) \end{cases}$$

Proof.— A direct derivation of Weinberg's estimate in [*5*].

2. SOME NEW FUNCTIONAL CLASSES

2.1. The classes Σ^{μ_p,μ_l}_{n}.

In order to take full account of a logarithmic behaviour, we need to slightly modify the class of symbols Σ^{μ}_{n} introduced in [*3*]. We define then:

DÉFINITION 2.1. — Let μ_p, μ_l arbitrary real numbers. A function f on the vector space $(\mathscr{E}_n, \| \cdot \|)$ is said to belong to the class Σ^{μ_p,μ_l}_{n} if it belongs to $C^\infty(\mathscr{E}_n)$ and if, for every $\nu \in \mathbb{N}$ and every homogeneous polynomial $P_\nu(D)$, there is a constant C_{ν,μ_p,μ_l} such that:

$$| P_\nu(D) f(K) | \leq C_{\nu,\mu_p,\mu_l} ||| P_\nu ||| \cdot (1 + \| K \|)^{\mu_p - \nu} (\mathrm{Log}(1 + \| K \|))^{\mu_l - \nu_l} \quad (2.1)$$

where $|||\cdot|||$ is a certain norm of P_ν in $\mathscr{E}_n^{\overset{\otimes}{s}\nu}$ (2), and ν_l is *one* if $\mu_p \in \mathbb{N}$ and $\nu > \mu_p$, *zero* otherwise.

We have then the following connection between $\Sigma_n^{\mu_p,\mu_l}$, and the Weinberg classes :

Let E_N denote a N-dimensional vector space and $\underline{\lambda}$ a linear mapping from E_N to $\mathscr{E}_n$, We have:

LEMMA 2.1. — *For every function f on $\mathscr{E}_n$ which belongs to $\Sigma_n^{\mu_p,\mu_l}$ the inverse image $\underline{\lambda}^*f$ belongs to the Weinberg-class $A_N^{\alpha^\mu,\beta^\mu}$ on E_N, the asymptotic indicatrices of which are given by*:

$$\begin{cases} \alpha^\mu(S) = 0 & \text{if} \quad S \subset \operatorname{Ker} \underline{\lambda} \\ \alpha^\mu(S) = \mu_p & \text{if} \quad S \not\subset \operatorname{Ker} \underline{\lambda} \end{cases} \tag{2.2}$$

$$\begin{cases} \beta^\mu(S) = 0 & \text{if} \quad S \subset \operatorname{Ker} \underline{\lambda} \\ \beta^\mu(S) = \mu_l & \text{if} \quad S \not\subset \operatorname{Ker} \underline{\lambda} \end{cases} \tag{2.3}$$

Moreover, for every integer $\nu > 0$, and every homogeneous polynomial $Q_\nu(D)$ of degree ν on E_N, the function $Q_\nu(D)\underline{\lambda}^*f$ belongs to $A_N^{\alpha',\beta'}$ with:

$$\begin{cases} \alpha' = \alpha^\mu - \nu \\ \beta' = \beta^\mu - \lambda(\alpha^\mu, \beta^\mu, \nu) \end{cases} \tag{2.4}$$

where λ is an integer function defined by:

$$\begin{aligned} &\lambda(\alpha, \beta, \nu) = 1 \quad \text{if} \quad \alpha \in \mathbb{N}, \quad \beta \neq 0, \quad \nu \geqslant \alpha + 1 \\ &\lambda(\alpha, \beta, \nu) = 0 \quad \text{otherwise}. \end{aligned} \tag{2.5}$$

Remark. — In the following, and when there is no ambiguity, we write λ_ν instead of $\lambda(\alpha, \beta, \nu)$.

Proof. — Let $(L_1, \ldots, L_m)$ an arbitrary set of independant vectors ($m \leqslant N$) and W a bounded region in E.

Let $J \leq m$ the integer such that:

$$\forall j \leqslant J \quad \underline{\lambda}(L_j) = \{0\}$$
$$\underline{\lambda}(L_{J+1}) \neq \{0\}$$

If $J = m$

$$\left| (\underline{\lambda}^* f)\left(\sum_{j=1}^{m} L_j \eta_j \ldots \eta_m + C \right) \right| = |f(\underline{\lambda}(C))| \leqslant M$$

with $M = \sup_{C \in W} |f(\underline{\lambda}(C))|$.

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(2) $\mathscr{E}_n^{\overset{\otimes}{s}\nu}$ is the ν^{th} symmetrized tensor product of $\mathscr{E}_n$.

If $J < m$

$$\left|(\underline{\lambda}^* f)\left(\sum_{j=1}^{m} L_j\eta_j \dots \eta_m + C\right)\right| = \left|f\left(\sum_{j=J+1}^{m} \underline{\lambda}(L_j)\eta_j \dots \eta_m + C\right)\right|.$$

Then, with the assumption: $\forall j \leqslant m,\ \eta_j \geqslant 1,\ C \in W$, we have:

$$\|K\| = \left\|\sum_{J+1}^{m} \underline{\lambda}(L_j)\eta_j \dots \eta_m + \underline{\lambda}(C)\right\|$$

$$\leqslant \left(\sum_{J+1}^{m} \|\underline{\lambda}(L_j)\| + \sup_W \|\underline{\lambda}(C)\|\right)\prod_{J+1}^{m} \eta_j$$

so :

$$(1 + \|K\|)^{\mu_p} \leqslant M \prod_{j=1}^{m} \eta_j^{\alpha\mu\{\overline{L_1,\dots,L_j}\}}$$

with α^μ given by 2.2, and the notation:

$$M = C_0\left(1 + \sum_{J+1}^{m} \|\lambda(L_j)\| + \sup_W |\lambda(C)|\right)$$

For the log part, we have:

$$\text{Log}\,(1 + \|K\|) \leqslant \text{Log}\left(M\prod_{J+1}^{m} \eta_j\right)$$

$$\leqslant C\prod_{J+1}^{m} \text{Log}\,\eta_j$$

with suitable constant $C > 0$. Then:

$$\text{Log}\,(1 + \|K\|)^{\mu_l} \leqslant C\prod_{j=1}^{m} (\text{Log}\,\eta_j)^{\beta\mu\{\overline{L_1,\dots,L_j}\}}$$

with β^μ given by 2.3.

The second part of the lemma is easily derived if we take $P_\nu(D)f$ instead of f in the preceding arguments, if we notice that:

$$\begin{aligned} D(x^\alpha (\text{Log}\,x)^\beta) &\in A^{\alpha-1,\beta} \quad \text{if} \quad \alpha \neq 0 \\ &\in A^{\alpha-1,\beta-1} \quad \text{if} \quad \alpha = 0 \end{aligned}$$

2.2. The Weinberg admissible classes.

In the following, we consider the vector space:

$$\mathscr{E}^{rN}_{(K,k)} = E^{rm}_{(k)} \times \mathscr{E}^{r(n-1)}_{(K)}$$

and the canonical projectors χ (resp. π) of $\mathscr{E}^{rN}_{(K,k)}$ on $\mathscr{E}^{r(n-1)}_{(K)}$ (resp. $E^{rm}_{(k)}$). We are going to extend the definition of admissibility given in [3].

We denote by $t^{d}_{(K)}f$ the Taylor expansion of degree d of f with respect to K at K = 0.

DÉFINITION 2.2. — A couple of sets of subspaces (σ, ω), with $\sigma \subset E^{rm}_{(k)}$ and $\omega \in \mathscr{E}^{rN}_{(K,k)}$ is called « admissible » if it satisfies the following properties:

a) $$\sigma \subset \omega$$

b) $$\forall S \subset \omega, \qquad \pi(S) \in \sigma$$

c) $$S \in \omega \quad \text{and} \quad S' \supset S \quad \text{imply} \quad S' \subset \omega$$

d) $$\{0\} \notin \sigma, \qquad \{0\} \notin \omega .$$

Let α, β asymptotic indicatrices on $\mathscr{E}^{rN}_{(K,k)}$ such that for every subspace $S \in \omega$ one has:

$$\alpha(S) = \alpha(\pi(S))$$
$$\beta(S) = \beta(\pi(S))$$

We associate with α, β, σ, ω a class $\mathscr{A}^{\alpha,\beta,\sigma,\omega}_{rN}$ of admissible Weinberg functions $f(K, k)$ by the conditions:

i) $$f \in A^{\alpha,\beta}_{rN}$$

ii) For every homogeneous derivative polynomial P_v, $P_v(D_K)f$ belongs to the class $A^{\alpha_v,\beta_v}_{rN}$ defined as follows:

$$\begin{array}{ll} \forall S \in \omega & \alpha_v(\pi(S)) = \alpha_v(S) = \alpha(S) - v \\ \forall S \notin \omega & \alpha_v(S) \quad = \alpha(S) \\ \forall S \in \omega & \beta_v(\pi(S)) = \beta_v(S) = \beta(S) - \lambda_v \\ \forall S \notin \omega & \beta_v(S) \quad = \beta(S) \end{array}$$

LEMMA 2.2. — *Let (σ, ω) be an admissible couple in $\mathscr{E}^{rN}_{(K,k)}$; let $f(K, k)$ an admissible Weinberg function in $\mathscr{A}^{\alpha,\beta,\sigma,\omega}_{rN}$ and let $h(K, k) = t^{d}_{(K)}f(K, k)$. Then for every admissible couple (σ', ω') in $\mathscr{E}^{rN}_{(K,k)}$ such that $\sigma' \supset \sigma$, there exists a class $\mathscr{A}^{\alpha',\beta',\sigma',\omega'}_{rN}$ which contains h and which satisfies the following properties:*

i) $$\forall S \subset \mathscr{E}^{rN}_{(K,k)}$$

a) $$\begin{array}{lll} \alpha'(S) = \alpha(\pi(S)) & \textit{if} & \pi(S) \in \sigma \\ \beta'(S) = \beta(\pi(S)) - \lambda_d & \textit{if} & \pi(S) \in \sigma \end{array}$$

b) $$\begin{array}{llll} \alpha'(S) = \alpha(\pi(S)) + d & \textit{if} & \pi(S) \notin \sigma & \pi(S) \in \sigma' \\ \beta'(S) = \beta(\pi(S)) & \textit{if} & \pi(S) \in \sigma & \pi(S) \in \sigma' \end{array}$$

ii) $\forall S \subset E^{rm}_{(k)}$ *with* $S \notin \sigma'$

$$\alpha'(S) = \alpha(S)$$
$$\beta'(S) = \beta(S)$$

Proof. — See [*1*] for the power asymptotic indicatrix. The β' behaviour is easily derived from the lemma 2.1:

We show only the situation for $\pi(S) \notin \sigma$.

We have:

$$h(K, k) = \sum_{0 \leqslant |\nu| \leqslant d} \frac{K^\nu}{\nu!} D^\nu_K f(0, k)$$

where ν is a multi-index.

We find that:

for $D^\nu_K f(0, k)$, the logarithmic indicatrix is:

$$\beta_{|\nu|}(S) = \beta(S) - \lambda_{|\nu|} \quad \text{if} \quad S \in \sigma$$
$$\beta_{|\nu|}(S) = \beta(S) \quad \text{if} \quad S \notin \sigma$$

So, for every admissible couple (σ', ω') in $\mathscr{E}^{rN}_{(K,k)}$, $(\sigma' \supset \sigma)$ the function $\pi^*(D^\nu_K f|_{K=0})(K, k) = D^\nu_K f(0, k)$ belongs to $\mathscr{A}^{\hat{\alpha}_\nu, \hat{\beta}_\nu, \sigma', \omega'}$ with:

$$\hat{\beta}_\nu(S) = \beta(\pi(S)) - \lambda_{|\nu|} \quad \text{if} \quad \pi(S) \in \sigma$$
$$\hat{\beta}_\nu(S) = \beta(\pi(S)) \quad \text{if} \quad \pi(S) \notin \sigma$$

Then, in each case:

$$\hat{\beta}_\nu(S) \leq \beta(\pi(S)) - \lambda_d$$
$$\hat{\beta}_\nu(S) = \beta(\pi(S))$$

So, the β' behaviour is that described in *i*), *a*), *b*).

We have also the following result (analogous to lemma 2.2 of [*3*]):

LEMMA 2.3. — *Let* $f(K, k)$ *an admissible Weinberg function in* $\mathscr{A}^{\alpha,\beta,\sigma,\omega}_{rN}$ *and* $g(K, k)$ *the Taylor rest of order d of* f: $g = (1 - t^d_{(K)})f$.

Then for every admissible couple (σ', ω') *in* $\mathscr{E}^{rN}_{(K,k)}$ *with* $\sigma' \subset \sigma$, $\omega' \subset \omega$ *there exists a class* $\mathscr{A}^{\alpha',\beta',\sigma',\omega'}_{rN}$ *which contains g and satisfies the following properties*:

a) $\forall S \in \omega'$:

$$\alpha'(S) = \alpha'(\pi(S)) = \alpha(S)$$
$$\beta'(S) = \beta'(\pi(S)) = \beta(S)$$

b) $\forall S \notin \omega'$, $S \in \omega$, $S \not\subset E^{rm}_{(k)}$:

$$\alpha'(S) = \alpha(S)$$
$$\beta'(S) = \beta(S)$$

c) $\forall S \notin \omega$, $S \not\subset E^{rm}_{(k)}$, $\pi(S) \in \sigma$:

$$\alpha'(S) = \sup(\alpha(S), \alpha(\pi(S)))$$
$$\beta'(S) = \sup(\beta(S), \beta(\pi(S)))$$

d) $\forall S \notin \omega$, $S \not\subset E^{rm}_{(k)}$, $\pi(S) \notin \sigma$:

$$\alpha'(S) = \sup(\alpha(S), \alpha(\pi(S)) + d)$$
$$\beta'(S) = \sup(\beta(S), \beta(\pi(S)))$$

e) $\forall S \subset E^{rm}_{(k)}$, $S \in \sigma$, $S \notin \sigma'$:

$$\alpha'(S) = \alpha(S) - d - 1$$
$$\beta'(S) = \beta(S) - \lambda_{d+1}$$

f) $\forall S \subset E^{rm}_{(k)}$, $S \notin \sigma$:

$$\alpha'(S) = \alpha(S)$$
$$\beta'(S) = \beta(S)$$

The proof is a direct application of lemma A.2 for the logarithmic behaviour, and is given in [*3*] for the power-law asymptotic behaviour.
We have then the following lemma giving the « graded » behaviour for Taylor rests of Weinberg function, which is a direct consequence of lemma 2.3 and of a technical result given in [7].

LEMMA 2.4. — *Let $f(K, k)$ an admissible Weinberg function belonging to $\mathscr{A}^{\alpha,\beta,\sigma,\omega}$ and let $g(K, k)$ be the Taylor rest of order d of f: $g=(1-t^{d}_{(K)})f$. Then $\forall n \geqslant 0$, there exists a class $A^{\alpha_n,\beta_n}_{rN}$ of Weinberg functions which contains every derivative of order n of g, and satisfying the following properties, $\forall S \in \mathscr{E}^{rN}_{(K,k)}$:*

a) If $S \subset E^{rm}_{(k)}$ *and* $S \in \omega$:

$$\begin{cases} \underline{\alpha}(S) = \alpha(S) - n \\ \underline{\beta}_n(S) = \beta(S) - \lambda_n \end{cases}$$

b) If $S \subset E^{rm}_{(k)}$ *and* $S \in \sigma$:

$$\begin{cases} \underline{\alpha}_n(S) = \alpha(S) - \sup(n, d+1) \\ \underline{\beta}_n(S) = \beta(S) - \sup(\lambda_n, \lambda_{d+1}) \end{cases}$$

c) If $S \not\subset E^{rm}_{(k)}$ *and* $S \notin \omega$, $\pi(S) \in \sigma$:

$$n \leqslant d: \begin{cases} \underline{\alpha}_n(S) = \sup\{\alpha(S), \alpha(\pi(S)) - n\} \\ \underline{\beta}_n(S) = \sup\{\beta(S), \beta(\pi(S)) - \lambda_n\} \end{cases}$$
$$n > d: \begin{cases} \underline{\alpha}_n(S) = \alpha(S) \\ \underline{\beta}_n(S) = \beta(S) \end{cases}$$

d) *If* $S \subset E^{rm}_{(k)}$ *and* $S \notin \omega$, $\pi(S) \notin \sigma$:

$$n \leqslant d: \begin{cases} \underline{\alpha}_n(S) = \sup \{ \alpha(S), \alpha(\pi(S)) + d - n \} \\ \underline{\beta}_n(S) = \sup \{ \beta(S), \beta(\pi(S)) - \lambda_n \} \end{cases}$$

$$n > d: \begin{cases} \underline{\alpha}_n(S) = \alpha(S) \\ \underline{\beta}_n(S) = \beta(S) \end{cases}$$

e) *If* $S \subset E^{rm}_{(k)}$ *and* $S \notin \sigma$:

$$\begin{cases} \underline{\alpha}_n(S) = \alpha(S) \\ \underline{\beta}_n(S) = \beta(S) \end{cases}$$

3. ASYMPTOTIC BEHAVIOUR OF THE RENORMALIZED G-CONVOLUTION PRODUCT

We consider a general connected graph G with n external lines and m independant loops. We follow then the definition 2.*b* of [*3*]: with each vertex $v \in \mathcal{N}$ (resp. line $i \in \mathcal{L}$) we associate a completely amputated n_v(point (resp. 2 point) function $H^{n_v}(K^v)$ (resp. $H^{(2)}(l_i)$) on the space $\mathbb{C}^{r(n_v - 1)}$ (resp. $\mathbb{C}^r$) of the set

$$K^v = \left\{ K^v_a \subset \mathbb{R}^{r-1} + i\mathbb{R}, \qquad 1 \leqslant a \leqslant n_v, \qquad \sum_a K^v_a = 0 \right\}$$

(resp. of $l_i \in \mathbb{R}^{r-1} + i\mathbb{R}$) of the momenta associated with the vertex (resp. the momentum associated with the line i).

We assume the analogous of hypothesis H.1 of [*3*], with the following modification:

Hypothesis H.1 *bis*

$$\begin{matrix} H^{n_v}(K^v) \in \Sigma^{\mu_v, \mu^l_v}_{r(n_v - 1)} \\ H^{(2)}(l_i) \in \Sigma^{\mu^p_i, \mu^l_i}_r \end{matrix} ; \qquad \mu^p_v,\ \mu^l_v,\ \mu^p_i,\ \mu^l_i \text{ integers}.$$

We have then, following the definitions 2.4, 2.5 of [*1*]:

LEMMA 3.1. — *The non-renormalized integrand associated with* G, *defined by*:

$$I_G(K, k) = \prod_{v \in \mathcal{N}} H^{n_v}(K^v(K, k)) \cdot \prod_{i \in \mathcal{L}} H^{(2)}(l_i(K, k)) \tag{3.1}$$

belongs to a class of admissible Weinberg functions $\mathcal{A}^{\alpha_G, \beta_G, \sigma_G, \omega_G}_{rN}$ *with the properties*:

$$\sigma_G = \{ S \in E^{rm}_{(k)} : \quad S \not\subset \text{Ker } \lambda_i, \forall i \in \mathcal{L} \} \tag{3.2}$$

$$\omega_G = \{ S \in \mathscr{E}^{rN}_{(K,k)} : S \not\subset \text{Ker } \lambda_i, \forall i \in \mathcal{L} ; \pi(S) \in \sigma_G \} \tag{3.3}$$

$\forall S \in \mathscr{E}^{rN}_{(K,k)}$;

$$\alpha_G = \sum_{\substack{v \in \mathscr{N} \\ S \not\subset \operatorname{Ker} \lambda_v}} \mu_v^p + \sum_{\substack{i \in \mathscr{L} \\ S \not\subset \operatorname{Ker} \lambda_i}} \mu_i^p \tag{3.4}$$

$$\beta_G = \sum_{\substack{v \in \mathscr{N} \\ S \not\subset \operatorname{Ker} \lambda_v}} \mu_v^l + \sum_{\substack{i \in \mathscr{L} \\ S \not\subset \operatorname{Ker} \lambda_i}} \mu_i^l \tag{3.5}$$

Proof. — A simple derivation of lemma 2.2 of [1] for power-law asymptotic behaviour and a strictly analogous argument for the logarithmic one, give the proof.

Following definition 2.*c* of [3], we have an analogous result for reduced subgraphs:

We consider subgraphs and forests U(G) of G. For every subgraph $\gamma \subset G$ with n_γ external lines and $m(\gamma)$ independent loops and given a forest U, we consider the functions I_γ (resp. $I_{\bar{\gamma}(U)}$) defined on $\mathscr{E}^{rN_\gamma}_{(K^\gamma,k)} = \mathscr{E}^{r(n_\gamma - 1)}_{(K^\gamma)} \times \mathscr{E}^{rm(\gamma)}_{(k)}$ with $N_\gamma = n_\gamma - 1 + m(\gamma)$, of the set of external and internal variables of γ by:

$$I_{\bar{\gamma}(U)}(K^\gamma, k) = \prod_{v \in \mathscr{N}_{\bar{\gamma}}} H^{n_v}(K^v(K^\gamma, k)) \cdot \prod_{i \in \mathscr{L}_{\bar{\gamma}}} H_i^{(2)}(l_i(K^\gamma, k))$$

and analogous representation for I_γ.

(We denote by $\mathscr{N}_{\bar{\gamma}}$ (resp. $\mathscr{L}_{\bar{\gamma}}$) the set of vertices (resp. internal lines) of the reduced graph.).

LEMMA 3.2. — *$I_{\bar{\gamma}(U)}(K^\gamma, k)$ belongs to the Weinberg admissible class $\mathscr{A}^{\alpha_{\bar{\gamma}}, \beta_{\bar{\gamma}}, \sigma_{\bar{\gamma}}, \omega_{\bar{\gamma}}}$ with:*

$$\sigma_{\bar{\gamma}} = \{ S \not\subset E^{rm}_{(k)} : S \subset \operatorname{Ker} \lambda_i^\gamma, \forall i \in \mathscr{L}_{\bar{\gamma}} \} \tag{3.6}$$

$$\omega_{\bar{\gamma}} = \{ S_\gamma \subset \mathscr{E}^{rN_\gamma}_{(K^\gamma,k)} : S_\gamma \not\subset \operatorname{Ker} \lambda_i^\gamma, \forall i \in \mathscr{L}_{\bar{\gamma}}, \pi(S_\gamma) \in \sigma_{\bar{\gamma}} \} \tag{3.7}$$

$\forall S_\gamma \subset \mathscr{E}^{rN_\gamma}_{(K^\gamma,k)}$:

$$\alpha_{\bar{\gamma}}(S_\gamma) = \sum_{\substack{i \in \mathscr{L}_\gamma \\ S_\gamma \not\subset \operatorname{Ker} \lambda_i^\gamma}} \mu_i^p + \sum_{\substack{v \in \mathscr{N}_\gamma \\ S_\gamma \not\subset \operatorname{Ker} \lambda_v}} \mu_v^p \tag{3.8}$$

$$\beta_{\bar{\gamma}}(S_\gamma) = \sum_{\substack{i \in \mathscr{L}_{\bar{\gamma}} \\ S_\gamma \not\subset \operatorname{Ker} \lambda_i^\gamma}} \mu_i^l + \sum_{\substack{v \in \mathscr{N}_{\bar{\gamma}} \\ S_\gamma \not\subset \operatorname{Ker} \lambda_v^\gamma}} \mu_v^l \tag{3.9}$$

Proof. — Same arguments reproducing those of lemma 3.1.

DÉFINITION. — *i*) For G and every subgraph $\gamma \subset G$ we define the corresponding dimension $d(G)$ and $d(\gamma)$, $d(\bar{\gamma})$ (resp. $d_l(G)$, $d_l(\gamma)$, $d_l(\bar{\gamma})$), by:

$$\left\{\begin{aligned} d(G) &= \sum_{i\in\mathscr{L}} \mu_i^p + \sum_{v\in\mathscr{N}} \mu_v^p + rm \\ d_l(G) &= \sum_{i\in\mathscr{L}} \mu_i^l + \sum_{v\in\mathscr{N}} \mu_v^l \end{aligned}\right.$$

$$\left\{\begin{aligned} d(\gamma) &= \sum_{i\in\mathscr{L}_\gamma} \mu_i^p + \sum_{v\in\mathscr{N}_\gamma} \mu_v^p + rm(\gamma) \\ d_l(\gamma) &= \sum_{i\in\mathscr{L}_\gamma} \mu_i^l + \sum_{v\in\mathscr{N}_\gamma} \mu_v^l \end{aligned}\right.$$

$$\left\{\begin{aligned} d(\bar{\gamma}) &= \sum_{i\in\mathscr{L}_{\bar{\gamma}}} \mu_i^p + \sum_{v\in\mathscr{N}_{\bar{\gamma}}} \mu_v^p + rm(\bar{\gamma}) \\ d_l(\bar{\gamma}) &= \sum_{i\in\mathscr{L}_{\bar{\gamma}}} \mu_i^l + \sum_{v\in\mathscr{N}_{\bar{\gamma}}} \mu_i^l \end{aligned}\right.$$

Remarks. — In the following, we omit the p index in μ^p, when there is no ambiguity.

We have the following identities:

$$d(\gamma) = d(\bar{\gamma}) + \sum_{1\leqslant a\leqslant c_\gamma} d(\gamma_a)$$

$$d_l(\gamma) = d_l(\bar{\gamma}) + \sum_{1\leqslant a\leqslant c_\gamma} d_l(\gamma_a)$$

for the reduced graph $\bar{\gamma}$ of γ (relative to a certain forest $U(\gamma)$), the sum holding for all $\gamma_a \in \mathscr{M}_\gamma(U)$ (maximal subgraphs cf. [*1*]).

In the following, we are going to prove (see the notations of th. 1.2):

THEOREM 3.1. — *i*) *The renormalized* G-*convolution product* $H_G^{\text{ren}}(K)$ *belongs to a class* $A_{r(n-1)}^{\alpha_H,\beta_H}$ *of Weinberg functions on* $\mathscr{E}_{(K)}^{r(n-1)}$; *the corresponding asymptotic coefficients* α_H, β_H *satisfy*:

α) $\forall S \subset \mathscr{E}_{(K)}^{r(n-1)}$

$$\alpha_H(S) = d(G) + \max_{\chi(S')=S}\left\{ - \sum_{\substack{\mu_i<0\\ S'\subset \operatorname{Ker}\lambda_i}} \mu_i - \sum_{\substack{\mu_v<0\\ S'\subset \operatorname{Ker}\lambda_v}} \mu_v - \dim \pi(S') + \dim S' - \dim S \right\} \tag{3.10}$$

$$\beta_H(S) = \begin{cases} \beta_{\mathcal{M}}(S') & \text{if} \quad \alpha_H(S) = \alpha_{H,\mathcal{M}}(S) ; \quad \alpha_{H,\mathcal{M}}(S) \neq \alpha_{H,\mathcal{M}'}(S) \\ \beta_{\mathcal{M}}\ (S) & \text{if} \quad \alpha_H(S) = \alpha_{H,\mathcal{M}'}(S) ; \quad \alpha_{H,\mathcal{M}}(S) \neq \alpha_{H,\mathcal{M}'}(S) \\ 1 + \beta_{\mathcal{M}}(S) + \beta_{\mathcal{M}'}(S) & \text{if} \quad \alpha_{H,\mathcal{M}}(S) = \alpha_{H,\mathcal{M}'}(S) \end{cases} \tag{3.11}$$

With:

$$\beta(S') = d_l(G) - \sum_{\substack{\mu_v^l < 0 \\ S' \subset \operatorname{Ker} \lambda_v}} \mu^l - \sum_{\substack{\mu_i^l < 0 \\ S' \subset \operatorname{Ker} \lambda_i}} \mu_i^l$$

β) *When* $\mu_i > 0$, $\mu_v > 0$; $\forall i \in \mathcal{L}$, $\forall v \in \mathcal{N}$ *then*:

$$\alpha_H(S) = d(G) \tag{3.12}$$

$$\beta_H(S) = 2d_l(G) + 1 \tag{3.13}$$

DÉFINITIONS. — We consider an arbitrary set of nested spaces $\hat{S}_j \subset \mathcal{E}^{rN}_{(K,k)}$, $j = 1, \ldots, L$; $L \leqslant N$ (with dim $\hat{S}_j = rj$):

$$\hat{\mathcal{F}} = \{ \hat{S}_j \subset \mathcal{E}^{rN}_{(K,k)} : \hat{S}_j \subset \hat{S}_{j+1}, 1 \leqslant j \leqslant L \} \tag{3.14}$$

and the corresponding set:

$$\mathcal{F} = \{ S^{(i)} \subset E^{rm}_{(k)} : S^{(i)} = \pi(\hat{S}), \hat{S} \in \hat{\mathcal{F}}, 1 \leqslant i \leqslant \tilde{m}, \tilde{m} \leqslant m \} \tag{3.15}$$

We call $\mathcal{M}_\mu(U) = \{ \mu_a ; 1 \leqslant a \leqslant c_\mu \}$ the set of all subgraphs $\mu_a \in U(\mu)$ maximal in μ, with respect to the forest U.

We note:

$$W^j(U) = \{ \gamma \in U : \forall i \in \mathcal{L}_{\bar{\gamma}(U)}, S_\gamma = \{ K^\gamma = 0, k \in S^{(j)} \} \subset \operatorname{Ker} \lambda_i^\gamma \} \tag{3.16}$$

$$\mathcal{B}^{\mathcal{F}}(U) = \{ \gamma \in U : \exists S^{(j)} \in \mathcal{F} : \gamma \notin W^{(j)}(U) \text{ and } \gamma \in \mathcal{M}_\mu(U) \text{ for } \mu \in W^j(U) \} \tag{3.17}$$

It has been proved in [3] that the generalized renormalized integrand $R_G(K, k)$ could be defined as a sum of terms corresponding to the set $\mathcal{U}(\mathcal{F})$ of complete forests U w. r. t. $\mathcal{F}$ by the proposition:

PROPOSITION 3.1 [1 b]. — *Given any tested set* $\mathcal{F}$, *and the corresponding set of complete forests* $\mathcal{U}(\mathcal{F})$, *we have the following expression for* $R_G(K, k)$:

$$R_G(K, k) = \sum_{U \in \mathcal{U}(\mathcal{F})} (1 - t^{d(G)}) Y_G^{(U)}(K, k) \tag{3.18}$$

where $Y_G^{(U)}$ *and all auxiliary functions* $\{ Y_\gamma^{(U)} ; \gamma \in U \}$ *are defined by the recursion formula*:

$$Y_\gamma^{(U)} = I_{\gamma(U)} \prod_{\gamma_a \in \mathcal{M}_\gamma(U)} S_a^* f_a^{(U)} Y_{\gamma_a}^{(U)} \tag{3.19}$$

with $\begin{cases} f_a^{(U)} = (1 - t^{d(\gamma_a)}) & \text{if} \quad \gamma_a \in \mathscr{B}^{\mathscr{F}}(U) \\ f_a^{(U)} = - t^{d(\gamma_a)} & \text{if} \quad \gamma_a \notin \mathscr{B}^{\mathscr{F}}(U) \end{cases}$

DÉFINITIONS 3.1. —

$$\mathscr{B}_\gamma(U) = \{ \mu \in U(\gamma) \cap \mathscr{B}^{\mathscr{F}}(U) : \exists \text{ sequence } \mu_j \text{ of } U(\gamma) \cap \mathscr{B}^{\mathscr{F}}(U) : j = 1, \ldots, r ;$$
$$\mu_{j+1} \supset \mu_j,\ \mu_j \in \mathscr{M}_{j+1}(U),\ \mu_r \in \mathscr{M}_\gamma(U) \} \quad (3.20)$$

$$\hat{\sigma}_\gamma = \{ S \subset E_{(k)}^{rm} : \exists S^{(j)} \in \mathscr{F} \quad \text{s. t.} \quad \gamma \notin W^j(U) \quad \text{and} \quad S^{(j)} \subset S \} \quad (3.21)$$

$$\hat{\omega}_\gamma = \left\{ S_\gamma \subset \mathscr{E}_{(K^\gamma, k)}^{rN_\gamma} : S_\gamma \not\subset \operatorname{Ker} \lambda_i^\gamma,\ \forall i \in \mathscr{L}_{\bar{\gamma}} U \left(\bigcup_{\mu \in \mathscr{B}_\gamma(U)} \mathscr{L}_{\bar{\mu}} \right) ;\ \pi(S_\gamma) \in \hat{\sigma}_\gamma \right\} \quad (3.22)$$

$$\omega_{\gamma_a}^{(\gamma)} = \begin{cases} \{ S_{\gamma_a} \subset \mathscr{E}_{(K_a^{\gamma_a}, k)}^{rN_{\gamma_a}} ;\ \pi_a(S_{\gamma_a}) \in \hat{\sigma}_\gamma,\ S_{\gamma_a} \subset \hat{\omega}_{\gamma_a} \} & \text{if} \quad \gamma_a \in \mathscr{B}_\gamma(U) \quad (3.23) \\ \{ S_{\gamma_a} \subset \mathscr{E}_{(K_a^{\gamma_a}, k)}^{rN_{\gamma_a}} :\ \pi_a(S_{\gamma_a}) \in \hat{\sigma}_\gamma \} & \text{if} \quad \gamma_a \notin \mathscr{B}_\gamma(U) \quad (3.24) \end{cases}$$

We give then the following notation:
For every $\gamma \subset G$ we denote by $\mathscr{K}_{\bar{\gamma},p}^{(S)}$, $\mathscr{K}_{\bar{\gamma},l}^{(S)}$, the following integers:

$$\mathscr{K}_{\bar{\gamma},p}^{(S)} = - \sum_{\substack{v \in \mathscr{N}_{\bar{\gamma}} \\ \{\mu_v^p < 0 ;\, S \subset \operatorname{Ker} \lambda_v^\gamma\}}} \mu_v^p - \sum_{\substack{i \in \mathscr{L}_{\bar{\gamma}} \\ \{\mu_i^p < 0 ;\, S \subset \operatorname{Ker} \lambda_i^\gamma\}}} \mu_i^p \quad (3.25)$$

$$\mathscr{K}_{\bar{\gamma},l}^{(S)} = - \sum_{\substack{v \in \mathscr{N}_{\bar{\gamma}} \\ \{\mu_v^l < 0 ;\, S \subset \operatorname{Ker} \lambda_v^\gamma\}}} \mu_v^l - \sum_{\substack{i \in \mathscr{L}_{\bar{\gamma}} \\ \{\mu_l < 0 ;\, S \subset \operatorname{Ker} \lambda_i^\gamma\}}} \mu_i^l \quad (3.26)$$

PROPOSITION 3.2. — *For every* $\gamma \in U(G)$, $U \in \mathscr{U}(\mathscr{F})$, *the corresponding* $Y_\gamma^{(U)}$ *belongs to the class* $\mathscr{A}_{rN}^{\alpha_\gamma, \beta_\gamma, \hat{\sigma}_\gamma, \hat{\omega}_\gamma}$ *with the following properties*: *Let* $S \in \hat{\mathscr{F}}$. $\forall S_\gamma = s_\gamma^G S$:

i) *If* $S_\gamma \in \hat{\omega}_\gamma$

$$\begin{cases} \alpha_\gamma(S_\gamma) = \alpha_\gamma(\pi(S_\gamma)) \leqslant d(\gamma) + \sum_{\substack{\gamma_a \in U(\gamma) \\ \pi(S_\gamma) \in \hat{\sigma}_{\gamma_a}}} rm(\bar{\gamma}_a) & (3.27) \\ \beta_\gamma(S_\gamma) = \beta_\gamma(\pi(S_\gamma)) \leqslant d_l(\gamma) & (3.28) \end{cases}$$

ii) *If* $S_\gamma \notin \hat{\omega}_\gamma$, $S_\gamma \subset E_{(k)}^{rm}$

$$\begin{cases} \alpha_\gamma(S_\gamma) \leqslant d(\gamma) - \sum_{\substack{\gamma_a \in U(\gamma) \\ \pi(S_\gamma) \in \hat{\sigma}_{\gamma_a}}} rm(\bar{\gamma}_a) + \sum_{\mu \in \mathscr{B}_\gamma(U) \cup \{\gamma\}} \mathscr{K}_{\bar{\mu},p}^{(S)} & (3.29) \\ \beta_\gamma(S_\gamma) \leqslant d_l(\gamma) + \sum_{\mu \in \mathscr{B}_\gamma(U) \cup \{\gamma\}} \mathscr{K}_{\bar{\mu},l}^{(S)} & (3.30) \end{cases}$$

iii) If $S_\gamma \subset E^{rm}_{(k)}$, $S_\gamma \notin \hat{\sigma}_\gamma$

— *if* $\exists \gamma_a : S_\gamma \in \hat{\sigma}_{\gamma_a}$:

$$\begin{cases} \alpha_\gamma(S_\gamma) \leqslant - \sum_{\substack{\gamma_a \in U(\gamma) \\ S_\gamma \in \hat{\sigma}_{\gamma a}}} rm(\bar{\gamma}_a) - 1 & (3.31) \\ \beta_\gamma(S_\gamma) \leqslant d_l(\gamma) - \sum_{\substack{\gamma_a \in U(\gamma) \\ S_\gamma \in \hat{\sigma}_{\gamma a}}} \lambda_{d(\gamma_a)+1} & (3.32) \end{cases}$$

— *if* $\forall \gamma_a : S_\gamma \notin \hat{\sigma}_{\gamma_a}$:

$$\begin{cases} \alpha_\gamma(S_\gamma) = 0 & (3.33) \\ \beta_\gamma(S_\gamma) = 0 & (3.34) \end{cases}$$

We show first three auxiliary lemmas, using preceding definitions for $\mathscr{B}_\gamma(U)$, $\hat{\sigma}_\gamma$, $\hat{\omega}_\gamma$, $\omega_\gamma^{(\gamma_a)}$.

LEMMA 3.3. — *The function* $I_{\bar{\gamma}(U)}$ *belongs to the class* $\mathscr{A}^{\alpha_{\bar{\gamma}}, \beta_{\bar{\gamma}}, \hat{\sigma}_\gamma, \hat{\omega}_\gamma}$ *which satisfies the properties*:

Let $S \in \hat{\mathscr{F}}$; $\forall S_\gamma = s_\gamma^G S$:

If $S_\gamma \in \hat{\omega}_\gamma$:

$$\begin{cases} \alpha_{\bar{\gamma}}(S_\gamma) = \alpha_{\bar{\gamma}}(\pi(S_\gamma)) = d(\bar{\gamma}) - rm(\bar{\gamma}) \\ \beta_{\bar{\gamma}}(S_\gamma) = \beta_{\bar{\gamma}}(\pi(S_\gamma)) = d_l(\bar{\gamma}) \end{cases} \tag{3.35}$$

If $S_\gamma \notin \hat{\omega}_\gamma$, $S_\gamma \subset E^{rm}_{(k)}$:

$$\begin{cases} \alpha_{\bar{\gamma}}(S_\gamma) \leqslant d(\bar{\gamma}) - rm(\bar{\gamma}) + \mathscr{K}^{(S)}_{\bar{\gamma},p} \\ \beta_{\bar{\gamma}}(S_\gamma) \leqslant d_l(\bar{\gamma}) + \mathscr{K}^{(S)}_{\bar{\gamma},l} \end{cases} \tag{3.36}$$

If $S_\gamma \subset E^{rm}_{(k)}$, $S_\gamma \notin \hat{\sigma}_\gamma$:

$$\begin{cases} \alpha_{\bar{\gamma}}(S_\gamma) = 0 \\ \beta_{\bar{\gamma}}(S_\gamma) = 0 \end{cases} \tag{3.37}$$

Proof. — By lemma 3.2 we know that $I_{\bar{\gamma}(U)} \in \mathscr{A}^{\alpha_{\bar{\gamma}}, \beta_{\bar{\gamma}}, \sigma_{\bar{\gamma}}, \omega_{\bar{\gamma}}}$ defined by (3.6), (3.7), (3.8), (3.9). So by the lemmas (3.10) and (3.11) of [*1*] we can see that $I_{\bar{\gamma}(U)} \in \mathscr{A}^{\alpha_{\bar{\gamma}}, \beta_{\bar{\gamma}}, \hat{\sigma}_\gamma, \hat{\omega}_\gamma}$. Moreover, it is easy to verify the following lemma (analogous to lemma 4.1 of [*1*]):

LEMMA 3.4. — $\mathscr{F}$ *and* $U \in \mathscr{U}(\mathscr{F})$ *being given, the function* $I_\gamma(K^\gamma, k)$ *belongs to a class* $\mathscr{A}_{rN_\gamma}^{\alpha_{\bar{\gamma}}, \beta_{\bar{\gamma}}, \hat{\sigma}_\gamma, \hat{\omega}_\gamma}$ *of admissible Weinberg functions with the following properties*:

i) For every $S^{(j)} \in \mathscr{F}$ *s. t.* $S^{(j)} \notin \hat{\sigma}_\gamma$:

$$\begin{cases} \alpha_{\bar{\gamma}}(S^{(j)}) = 0 \\ \beta_{\bar{\gamma}}(S^{(j)}) = 0 \end{cases} \tag{3.38}$$

ii) For every $S^{(j)} \in \mathscr{F}$ *s. t.* $S^{(j)} \in \hat{\sigma}_\gamma$, *the coefficients corresponding to every* $S_\gamma \in \hat{\omega}_\gamma$ *s. t.* $\pi(S_\gamma) = S^{(j)}$, *satisfy*:

$$\begin{cases} \alpha_{\overline{\gamma}}(S_\gamma) = d(\overline{\gamma}) - rm(\overline{\gamma}) \\ \beta_{\overline{\gamma}}(S_\gamma) = d_l(\overline{\gamma}) \end{cases} \tag{3.39}$$

with:

$$d_l(\overline{\gamma}) = \sum_{v \in \mathscr{N}_{\overline{\gamma}}} \mu_v^l + \sum_{i \in \mathscr{L}_{\overline{\gamma}}} \mu_i^l$$

Then, using lemma 3.4 and notations (3.25), (3.26) ends the proof of lemma (3.3).

LEMMA 3.5. — *For every* $\gamma_a \in \mathscr{M}_\gamma(U)$ *with* $\gamma_a \in \mathscr{B}^{\mathscr{F}}(U)$, *the function* $S_a^*(1 - t^{d(\gamma_a)})Y_{\gamma_a}^{(U)}$ *belongs to the class*: $\mathscr{A}^{\alpha_\gamma^{(a)}, \beta_\gamma^{(a)}, \hat{\sigma}_\gamma, \hat{\omega}_\gamma}$ *which satisfies the following properties*: *let* $S \in \hat{\mathscr{F}}$:

a) if $S_\gamma \in \hat{\omega}_\gamma$:

$$\alpha_\gamma^{(a)}(S_\gamma) = \alpha_\gamma^{(a)}(\pi(S_\gamma)) \leqslant d(\gamma_a) - \sum_{\substack{\mu_a \in U(\gamma_a) \\ \pi(S_\gamma) \in \hat{\sigma}_{\mu_a}}} rm(\overline{\mu}_a) \tag{3.40}$$

$$\beta_\gamma^{(a)}(S_\gamma) = \beta_\gamma^{(a)}(\pi(S_\gamma)) \leqslant d_l(\gamma_a) \tag{3.41}$$

b) if $S_\gamma \notin \hat{\omega}_\gamma$, $S_\gamma \not\subset E_{(k)}^{rm}$:

$$\alpha_\gamma^{(a)}(S_\gamma) \leqslant d(\gamma_a) - \sum_{\substack{\mu_a \in U(\gamma_a) \\ \pi(S_\gamma) \in \sigma_{\mu_a}}} rm(\overline{\mu}_a) + \sum_{\gamma'_a \in \mathscr{B}_{\gamma_a} \cup \{\gamma_a\}} \mathscr{K}_{\overline{\gamma}'_a, p}^{(S)} \tag{3.42}$$

$$\beta_\gamma^{(a)}(S_\gamma) \leqslant d_l(\gamma_a) + \sum_{\gamma'_a \in \mathscr{B}_{\gamma_a} \cup \{\gamma_a\}} \mathscr{K}_{\overline{\gamma}'_a, l}^{(S)} \tag{3.43}$$

c) If $S_\gamma \subset E_{(k)}^{rm}$, $S_\gamma \notin \hat{\sigma}_\gamma$:

$$\alpha_\gamma^{(a)}(S_\gamma) \leqslant - \sum_{\substack{\mu_a \in U(\gamma_a) \\ \pi(S_\gamma) \in \hat{\sigma}_{\mu_a}}} rm(\overline{\mu}_a) - 1 \tag{3.44}$$

$$\beta_\gamma^{(a)}(S_\gamma) \leqslant d_l(\gamma_a) - \lambda_{d(\gamma_a)+1} \tag{3.45}$$

or, if $\forall \mu_a \in U(\gamma_a)$, $S_\gamma \notin \hat{\sigma}_{\mu_a}$:

$$\alpha_\gamma^{(a)}(S_\gamma) = 0 \tag{3.46}$$

$$\beta_\gamma^{(a)}(S_\gamma) = 0 \tag{3.47}$$

Proof. — We suppose that the preceding properties are true for all

$\gamma_a \in \mathscr{M}_\gamma(U)$, then we establish a recursion, in the same manner as for lemma 3.2 in [3]:

Application of lemma 2.3 shows that the function $(1 - t^{d(\gamma_a)})Y_{\gamma_a}$ belongs to $\mathscr{A}^{\tilde{\alpha}_{\gamma_a}, \tilde{\beta}_{\gamma_a}, \hat{\sigma}^{(\gamma)}_{\gamma_a}, \hat{\omega}^{(\gamma)}_{\gamma_a}}$. We have:

1) If $S_{\gamma_a} \in \omega^{(\gamma)}_{\gamma_a}$, we obtain, by lemma 2.3 *a*):

$$\tilde{\beta}_{\gamma_a}(S_{\gamma_a}) = \tilde{\beta}_{\gamma_a}(\pi(S_{\gamma_a})) = \beta_{\gamma_a}(S_{\gamma_a}) \leqslant d_l(\gamma_a) \tag{3.48}$$

2) *a*) If $S_{\gamma_a} \notin \omega^{(\gamma)}_{\gamma_a}$, $S_{\gamma_a} \in \hat{\omega}_{\gamma_a}$, $S_{\gamma_a} \not\subset E^{rm}_{(k)}$, lemma 2.3 *b*) yields:

$$\tilde{\beta}_{\gamma_a}(S_{\gamma_a}) = \beta_{\gamma_a}(S_{\gamma_a}) \leqslant d_l(\gamma_a) \tag{3.49}$$

b) If $S_{\gamma_a} \notin \omega^{(\gamma)}_{\gamma_a}$, $S_{\gamma_a} \not\subset E^{rm}_{(k)}$, $\pi(S_{\gamma_a}) \in \hat{\sigma}_{\gamma_a}$, lemma 2.3 *c*) yields:

$$\tilde{\beta}_{\gamma_a}(S_{\gamma_a}) = \sup(\beta_{\gamma_a}(S_{\gamma_a}), \beta(\pi_a(S_{\gamma_a}))) \tag{3.50}$$

Then, by (3.49), (3.50) and the recursion hypothesis:

$$\tilde{\beta}_{\gamma_a}(S_{\gamma_a}) \leqslant d_l(\gamma_a) + \sum_{\gamma'_a \in \{\gamma_a\} \cup \mathscr{B}_\gamma(U)} \mathscr{K}^{(S)}_{\gamma'_a, l} \tag{3.51}$$

c) If $S_{\gamma_a} \notin \hat{\omega}_{\gamma_a}$, $S_{\gamma_a} \not\subset E^{rm}_{(k)}$ and $\pi(S_{\gamma_a}) \notin \hat{\sigma}_{\gamma_a}$, lemma 2.3 *d*) gives:

$$\tilde{\beta}_{\gamma_a}(S_{\gamma_a}) = \sup(\beta_{\gamma_a}(S_{\gamma_a}), \beta_{\gamma_a}(\pi_a(S_{\gamma_a})))$$

Then, we get, in all cases equation (3.51).

3) *a*) If $S_{\gamma_a} \subset E^{rm}_{(k)}$, $S_{\gamma_a} \in \hat{\sigma}_{\gamma_a}$, $S_{\gamma_a} \notin \hat{\sigma}_\gamma$, property *e*) of lemma 2.3 yields:

$$\tilde{\beta}_{\gamma_a}(S_{\gamma_a}) = \beta_{\gamma_a}(S_{\gamma_a}) - \lambda_{d(\gamma_a)+1} \tag{3.52}$$

The couple $\hat{\sigma}_{\gamma_a}$, $\hat{\omega}_{\gamma_a}$ being admissible, we have $\hat{\sigma}_{\gamma_a} \subset \hat{\omega}_{\gamma_a}$, we put (3.48) in (3.52) to obtain:

$$\tilde{\beta}_{\gamma_a}(S_{\gamma_a}) \leqslant d_l(\gamma_a) - \lambda_{d(\gamma_a)+1} \tag{3.53}$$

b) If $S_{\gamma_a} \subset E^{rm}_{(k)}$, $S_{\gamma_a} \notin \hat{\sigma}_{\gamma_a}$, property *f*) of lemma 2.3 gives, with (3.47):

$$\tilde{\beta}_{\gamma_a}(S_{\gamma_a}) = \beta_{\gamma_a}(S_{\gamma_a}) \leqslant d_l(\gamma_a) - \lambda_{d(\gamma_a)+1} \tag{3.54}$$

or

$$\tilde{\beta}_{\gamma_a}(S_{\gamma_a}) = 0, \quad \text{if} \quad \forall \mu_a \in U(\gamma_a), \quad S_{\gamma_a} \notin \hat{\sigma}_{\mu_a} \tag{3.55}$$

We can see then easily that $S^*_a(1 - t^{d(\gamma_a)})Y_{\gamma_a} \in \mathscr{A}^{\alpha^{(a)}_\gamma, \beta^{(a)}_\gamma, \hat{\sigma}_\gamma, \hat{\omega}_\gamma}$ with:

$$\text{If} \quad S \in \hat{\mathscr{F}}, \quad \forall S_\gamma = s^G_\gamma S: \quad \beta^{(a)}_\gamma(S_\gamma) = \tilde{\beta}_{\gamma_a}(S_{\gamma_a}) \tag{3.56}$$

with $S_{\gamma_a} = S^\gamma_{\gamma_a} S_\gamma$.

But we have the property of the $s^{\gamma'}_\gamma$:

$$s^G_{\gamma_a} = s^G_\gamma \circ s^\gamma_{\gamma_a} \tag{3.57}$$

Moreover:

$$\pi_a(S_{\gamma_a}) = \pi(S_\gamma) \tag{3.58}$$

Then, properties (*a*), (*b*), (*c*) of lemma are obtained from (3.49), (3.51) and (3.54), (3.56).

LEMMA 3.6. — *For every* $\gamma_a \in \mathcal{M}_\gamma(U)$ *with* $\gamma_a \notin \mathcal{B}^{\mathcal{F}}(U)$, *the function* $S_a^*(-t^{d(\gamma_a)})Y_{\gamma_a}$ *belongs to the class* $\mathcal{A}_{rN_\gamma}^{\alpha_\gamma^{(a)},\beta_\gamma^{(a)},\hat{\sigma}_\gamma,\hat{\omega}_\gamma}$ *with the following properties*:

Let $S \in \hat{\mathcal{F}}$, $\forall S_\gamma = s_\gamma^G S$:

a) If $S_\gamma \in \hat{\omega}_\gamma$:

$$\begin{cases} \alpha_\gamma^{(a)}(S_\gamma) = \alpha_\gamma^{(a)}(\pi(S_\gamma)) \leqslant d(\gamma_a) - \sum\limits_{\substack{\mu_a \in U(\gamma_a) \\ \pi(S_\gamma)\in\hat{\sigma}_{\mu_a}}} rm(\bar{\mu}_a) & (3.59) \\ \beta_\gamma^{(a)}(S_\gamma) \leqslant d_l(\gamma_a) & (3.60) \end{cases}$$

b) If $S_\gamma \notin \hat{\omega}_\gamma$, $S_\gamma \not\subset E_{(k)}^{rm}$:

$$\begin{cases} \alpha_\gamma^{(a)}(S_\gamma) \leqslant d(\gamma_a) - \sum\limits_{\substack{\mu_a \in U(\gamma_a) \\ \pi(S_\gamma)\in\hat{\sigma}_{\mu_a}}} rm(\bar{\mu}_a) & (3.61) \\ \beta_\gamma^{(a)}(S_\gamma) \leqslant d_l(\gamma_a) & (3.62) \end{cases}$$

c) If $S_\gamma \subset E_{(k)}^{rm}$, $S_\gamma \notin \hat{\sigma}_\gamma$:

$$\begin{cases} \alpha_\gamma^{(a)}(S_\gamma) \leqslant -\sum\limits_{\substack{\mu_a \in U(\gamma_a) \\ \pi(S_\gamma)\in\hat{\sigma}_{\mu_a}}} rm(\bar{\mu}_a) - 1 & (3.63) \\ \beta_\gamma^{(a)}(S_\gamma) \leqslant d_l(\gamma_a) - \lambda_{d(\gamma_a)+1} & (3.64) \end{cases}$$

or, if $\forall \mu_a \in U(\gamma_a)$, $S_j \notin \hat{\sigma}_{\mu_a}$:

$$\begin{cases} \alpha_\gamma^{(a)}(S_\gamma) = 0 & (3.65) \\ \beta_\gamma^{(a)}(S_\gamma) = 0 & (3.66) \end{cases}$$

Proof. — We suppose that $\gamma_a \notin \mathcal{B}^{\mathcal{F}}(U)$. From the recurrence hypothesis, $Y_{\gamma_a}^{(U)} \in \mathcal{A}^{\alpha_{\gamma_a},\beta_{\gamma_a},\hat{\sigma}_{\gamma_a},\hat{\omega}_{\gamma_a}}$ with asymptotic coefficients given by the expression (3.40) to (3.47) with replacement $\gamma \to \gamma_a$, and for $\forall S_{\gamma_a} = s_{\gamma_a}^G S$ with $S \in \hat{\mathcal{F}}$. We apply then lemma 2.2 to the function $(-t^{d(\gamma_a)})Y_{\gamma_a}$. The roles of (σ', ω') (resp. (σ, ω)) are now played by the admissible couples $(\hat{\sigma}_\gamma, \hat{\omega}_{\gamma_a})$ (resp. $(\hat{\sigma}_{\gamma_a}, \hat{\omega}_{\gamma_a})$) in view of (3.21), (3.22), (3.23), (3.24).

1) Let $\pi(S_{\gamma_a}) \in \hat{\sigma}_\gamma$. From properties *i) a) b)* of lemma 2.2, we obtain:

$$\tilde{\beta}_{\gamma_a}(S_{\gamma_a}) = \beta_{\gamma_a}(\pi(S_{\gamma_a})) \quad \text{if} \quad \pi(S_{\gamma_a}) \in \hat{\sigma}_{\gamma_a} \quad (3.67)$$

$$\tilde{\beta}_{\gamma_a}(S_{\gamma_a}) = \beta_{\gamma_a}(\pi(S_{\gamma_a})) - \lambda_{d(\gamma_a)} \quad \text{if} \quad \pi(S_{\gamma_a}) \notin \hat{\sigma}_{\gamma_a} \quad (3.68)$$

Then, we insert (3.41) (resp. (3.45), (3.47)) into (3.67) (resp. (3.68)) to obtain:

$$\tilde{\beta}_{\gamma_a}(S_{\gamma_a}) = \tilde{\beta}_{\gamma_a}(\pi(S_{\gamma_a})) \leqslant d_l(\gamma_a) \quad (3.69)$$

2) Let $S_{\gamma_a} \not\subset E^{rm}_{(k)}$ and $\pi(S_{\gamma_a}) \notin \hat{\sigma}_\gamma$; we have the inclusion property $\hat{\sigma}_\gamma \supset \hat{\sigma}_{\gamma_a}$ so: $\pi(S_{\gamma_a}) \notin \hat{\sigma}_{\gamma_a}$, so (3.68) holds, in which we insert (3.46):

$$\tilde{\beta}_{\gamma_a}(S_{\gamma_a}) \leqslant d_l(\gamma_a) \tag{3.70}$$

3) Let $S_{\gamma_a} \subset E^{rm}_{(k)}$, $S_{\gamma_a} \notin \sigma_\gamma$; then $S_{\gamma_a} \notin \sigma_{\gamma_a}$, so we insert property *ii)* of lemma 2.2 in (3.45), (3.47):

$$\tilde{\beta}_{\gamma_a}(S_{\gamma_a}) = \beta_{\gamma_a}(S_{\gamma_a}) \leqslant d_l(\gamma_a) - \lambda_{d(\gamma_a)+1} \tag{3.71}$$

(if $\exists$ at least one $\mu_a \in U(\gamma_a)$ with $S_{\gamma_a} \in \hat{\sigma}_{\mu_a}$)

$$\tilde{\beta}_{\gamma_a}(S_{\gamma_a}) = 0 \tag{3.72}$$

(if $\forall \mu_a \in U(\gamma_a)$, $S_{\gamma_a} \notin \hat{\sigma}_{\gamma_a}$).
We apply then property of S^*_a operation, which ends the proof.

Proof of proposition 3.2. — We apply lemmas (3.3), (3.5), (3.6) to the different factors of the function $Y^{(U)}_\gamma$ in eq. (3.19). Then we use the product-stability of admissible-Weinberg-classes. We find that $Y^{(U)}_\gamma \in \mathscr{A}^{\alpha_\gamma, \beta_\gamma, \hat{\sigma}_\gamma, \hat{\omega}_\gamma}$; the asymptotic coefficients are given by the following, for all $S_\gamma \in \mathscr{E}^{rN_\gamma}_{(K,k)}$ such that $S_\gamma = s^G_\gamma S$, $S \in \hat{\mathscr{F}}$:

If $S_\gamma \in \hat{\omega}_\gamma$, by addition of (3.35), (3.41), (3.60), we have:

$$\beta_\gamma(S_\gamma) \leqslant d_l(\gamma)$$

If $S_\gamma \notin \hat{\omega}_\gamma$, $S_\gamma \not\subset E^{rm}_{(k)}$:

$$\beta_\gamma(S_\gamma) \leqslant d_l(\gamma) + \sum_{\mu \in \mathscr{B}_\gamma(U) \cup \{\gamma\}} \mathscr{K}^{(S)}_{\mu,l}$$

If $S_\gamma \subset E^{rm}_{(k)}$, $S_\gamma \in \hat{\sigma}_\gamma$:

$$\beta_\gamma(S_\gamma) \leqslant d_l(\gamma) - \sum_{\substack{\gamma_a \in U(\gamma) \\ \pi(S_\gamma) \in \hat{\sigma}_{\mu_a}}} \lambda_{d(\gamma_a)+1}$$

(if $\exists \gamma_a : S_\gamma \in \hat{\sigma}_{\gamma_a}$)

$$\beta_\gamma(S_\gamma) = 0$$

(if $\forall \gamma_a \in U(\gamma)$, $S_\gamma \in \hat{\sigma}_{\gamma_a}$).

THEOREM 3.2. — *The function* $R_G(K, k)$ *and every partial derivative* $D^l_{(K)} R_G(K, k)$ *w. r. t. the external momenta* K, *of total order* $l \geqslant 0$, *belongs to a Weinberg class* $A^{\alpha_l, \beta_l}_{rN}$ *in* $\mathscr{E}^{rN}_{(K,k)}$, *with the properties*: $\forall S \subset \mathscr{E}^{rN}_{(K,k)}$:

if $S \subset E^{rm}_{(k)}$
if $S \in \omega_G$:

$$\begin{cases} \alpha_l(S) = d(G) - \dim \pi(S) - l & (3.73) \\ \beta_l(S) = d_l(G) - \lambda_l & (3.74) \end{cases}$$

if $S \notin \omega_G$:

$$\begin{cases} \alpha_l(S) = d(G) - \dim \pi(S) - \sum\limits_{\substack{\mu_v^p < 0 \\ S \subset \mathrm{Ker}\,\lambda_v}} \mu_v^p - \sum\limits_{\substack{\mu_i^p < 0 \\ S \subset \mathrm{Ker}\,\lambda_i}} \mu_i^p & (3.75) \\ \beta_l(S) = d_l(G) - \sum\limits_{\substack{\mu_v^l < 0 \\ S \subset \mathrm{Ker}\,\lambda_v}} \mu_v^l - \sum\limits_{\substack{\mu_v^l < 0 \\ S \subset \mathrm{Ker}\,\lambda_i}} \mu_i^l & (3.76) \end{cases}$$

if $S \subset E_{(k)}^{rm}$:

$$\begin{cases} \alpha_l(S) = -\dim S - 1 & (3.77) \\ \beta_l(S) = d_l(G) - \inf \left\{ \lambda_l, \lambda_{d(G)+1}, \sum\limits_{\substack{\gamma \in U(G) \\ \pi(S) \in \hat{\sigma}_\gamma}} \lambda_{d(\gamma)+1} \right\} & (3.78) \end{cases}$$

Proof. — By application of proposition 3.2 to the case $\gamma = G$, we obtain first that $Y_G^{(U)}$ belongs to a class $\mathscr{A}^{\alpha_G, \beta_G, \hat{\sigma}_G, \hat{\omega}_G}$ which satisfies the following properties; $\forall S_j \in \hat{\mathscr{F}}$:

i) if $S_j \in \hat{\omega}_G$:

$$\beta_G(S_j) = \beta_G(\pi(S_j)) \leqslant d_l(G) \tag{3.79}$$

ii) if $S_j \notin \hat{\omega}_G$, $S_j \not\subset E_{(k)}^{rm}$:

$$\beta_G(S_j) \leqslant d_l(G) + \sum_{\mu \in \mathscr{B}(U) \cup \{G\}} \mathscr{K}_{\mu,l}^{(S_j)} \tag{3.80 a}$$

— if $\pi(S_j) \in \hat{\sigma}_G$:

$$\beta_G(\pi(S_j)) \leqslant d_l(G) \tag{3.80 b}$$

— if $\pi(S_j) \notin \hat{\sigma}_G$:

$$\beta_G(\pi(S_j)) \leqslant d_l(G) - \sum_{\substack{\gamma \in U(G) \\ \pi(S_j) \in \hat{\sigma}_\gamma}} \lambda_{d(\gamma)+1} \tag{3.80 c}$$

iii) if $S_j \subset E_{(k)}^{rm}$, $S_j \notin \hat{\sigma}_G$:

$$\beta_G(S_j) \leqslant d_l(G) - \sum_{\substack{\gamma \in U(G) \\ \pi(S_j) \in \hat{\sigma}_\gamma}} \lambda_{d(\gamma)+1} \tag{3.81}$$

or $$\beta_G(S_j) = 0 \quad \text{if} \quad S_j = \{0\}$$

Then we apply lemma 2.4 to the function $\tilde{X}^{(U)} = (1 - t^{d(G)})Y_G^{(U)}$; it follows that every partial derivative $D_{(K)}^{(l)}\tilde{X}^{(U)}$ of total order $l \geqslant 0$ of $\tilde{X}^{(U)}$ belongs to a class $A^{\alpha_l^{(U)}, \beta_l^{(U)}}$ of Weinberg functions; the corresponding asymptotic coefficients are obtained by inserting (3.79), (3.80), (3.81) inside properties a, b, c, d, e of lemma 2.4:

$S_j \notin E^{rm}_{(k)},\ S_j \in \hat{\omega}_G$:

$$\beta_l^{(U)}(S_j) = \beta_G(S_j) - \lambda_l \leqslant d_l(G) - \lambda_l \tag{3.82}$$

$S_j \subset E^{rm}_{(k)},\ S_j \in \sigma_G$:

$$\beta_l^{(U)}(S_j) = \beta_G(S_j) - \lambda_l \leqslant d_l - \lambda_l\,; \quad \text{if} \quad \lambda_l \leqslant \lambda_{d(G)+1} \tag{3.83}$$

$$\beta_l^{(U)}(S_j) = \beta_G(S_j) - \lambda_{d(G)+1} \leqslant d_l - \lambda_{d+1}\,; \quad \text{if} \quad \lambda_l > \lambda_{d(G)+1}$$

$S_j \not\subset E^{rm}_{(k)},\ S_j \notin \hat{\omega}_G,\ \pi(S_j) \in \hat{\sigma}_G$:

$$\beta_l^{(U)}(S_j) = \sup\{\beta_G(S_j), \beta_G(\pi(S_j)) - \lambda_l\} = \beta_G(S_j)$$

$$\leqslant d_l(G) + \sum_{\gamma \in \mathscr{B}_G(U) \cup \{G\}} \mathscr{K}_{\bar{\gamma},l}^{(S_j)} \text{ in all cases } (l \leqslant d+1 \text{ or } l > d+1) \tag{3.84}$$

$S_j \not\subset E^{rm}_{(k)},\ S_j \notin \hat{\omega}_G,\ \pi(S_j) \notin \hat{\sigma}_G$:

$$\beta_l^{(U)}(S_j) = \sup\{\beta_G(S_j), \beta_G(\pi(S_j)) - \lambda_n\}$$

$$\leqslant d_l(G) + \sum_{\gamma \in \mathscr{B}_G(U) \cup \{G\}} \mathscr{K}_{\bar{\gamma},l}^{(S_j)} \text{ in all cases} \tag{3.85}$$

$S_j \subset E^{rm}_{(k)},\ S_j \notin \hat{\sigma}_G$:

$$\beta_l^{(U)}(S_j) \leqslant d_l(G) - \sum_{\substack{\gamma \in U(G) \\ \pi(S_j) \in \hat{\sigma}_\gamma}} \lambda_{d(\gamma)+1}$$

$$\beta_l^{(U)}(S_j) = 0 \quad \text{if} \quad S_j = \{0\} \tag{3.86}$$

We have then the following inequality:

$$\sum_{\gamma \in \mathscr{B}_G(U) \cup \{G\}} \mathscr{K}_{\bar{\gamma},l}^{(S_j)} \leqslant - \sum_{\substack{v \in \mathscr{N} \\ \mu_v < 0 \\ S_j \subset \operatorname{Ker} \lambda_v}} \mu_v^l - \sum_{\substack{i \in \mathscr{L} \\ \mu_i < 0 \\ S_j \subset \operatorname{Ker} \lambda_i}} \mu_i^l \tag{3.87}$$

So, by combining (3.82), (3.84), (3.85) with (3.87) we get:

If $S_j \notin E^{rm}_{(k)},\ S_j \in \hat{\omega}_G$:

$$\beta_l^{(U)}(S_j) \leqslant d_l(G) - \lambda_l \tag{3.88}$$

If $S_j \in E^{rm}_{(k)},\ S_j \in \hat{\omega}_G$:

$$\beta_l^{(U)}(S_j) \leqslant d_l(G) - \sum_{\substack{v \in \mathscr{N} \\ \mu_v^l < 0 \\ S \subset \operatorname{Ker} \lambda_v}} \mu_v^l - \sum_{\substack{i \in \mathscr{L} \\ \mu_i^l < 0 \\ S \subset \operatorname{Ker} \lambda_i}} \mu_i^l \tag{3.89}$$

If $S_j \subset E_{(k)}^{rm}$:

$$\beta_l^{(U)}(S_j) \leqslant d_l(G) - \inf\left\{ \lambda_l, \lambda_{d(G)+1}, \sum_{\substack{\gamma \in U(G) \\ \pi(S_j) \in \hat{\sigma}_\gamma}} \lambda_{d(\gamma)+1} \right\} \tag{3.90}$$

For an arbitrary sequence $\{ L_1, L_2, \ldots, L_{\tilde{n}} \}$ of $\tilde{n}$ independent vectors, with $\tilde{n} \leqslant N$, and an arbitrary bounded region W in $\mathscr{E}_{(K,k)}^{rN}$, we consider the ordered set $\{ (L_1, \ldots, L_j) ; j \leqslant \tilde{n} \}$, and we associate with this set a unique nested set of subspaces: $\hat{\mathscr{F}} = \{ S_1, \ldots, S_{\tilde{n}} \}$ by the definition:

$$\forall j : 1 \leqslant j \leqslant \tilde{n} : \qquad S_j = \{ L_1, \ldots, L_j \} \tag{3.91}$$

We deduce, from the above results that, for every forest $U \in \mathscr{U}(\mathscr{F})$ there exist numbers $b_j(U) \geqslant 1$ $(1 \leqslant j \leqslant \tilde{n})$ and M_U such that the function $\tilde{X}_U^{(l)} = D_{(K)}^l (1 - t^{d(G)}) Y_G^{(U)}$ satisfies the bound:

$$\left| \tilde{X}_U^{(l)} \left(\sum_{j=1}^{n} L_j \eta_j \ldots \eta_n + C \right) \right| \leqslant M_U \prod_{j=1}^{\tilde{n}} \eta_j^{\alpha_l^{(U)}(S_j)} (\operatorname{Log} \eta_j)^{\beta_l^{(U)}(S_j)} \tag{3.92}$$

where S_j is defined in (3.91), the asymptotic coefficients $\alpha_l^{(U)}$ and $\beta_l^{(U)}$ are given by (3.73), (3.75), (3.77) (cf. [3]), and (3.88), (3.89), (3.90), provided that $\forall j = 1, \ldots, \tilde{n}$ $\eta_j \geqslant b_j(U)$ and $C \in W$. If we put:

$$M = \sum_{v \in \mathscr{U}(\mathscr{F})} M_U \qquad b_j = \sup_{U \in \mathscr{U}(\mathscr{F})} b_j(U)$$

from the expression (3.18) of R_G, we obtain:

$$\left| D_{(K)}^l R_G \left(\sum_{j=1}^{\tilde{n}} L_j \eta_j \ldots \eta_{\tilde{n}} + C \right) \right| \leqslant M \prod_{j=1}^{\tilde{n}} \eta^{\underline{\alpha}_l(S_j)} (\operatorname{Log} \eta_j)^{\underline{\beta}_l(S_j)}$$

with :

$$\underline{\alpha}_l(S_j) = \sup_{U \in \mathscr{U}(\mathscr{F})} \alpha_l^{(U)}(S_j)$$

$$\underline{\beta}_l(S_j) = \sup_{U \in \mathscr{U}(\mathscr{F})} \beta_l^{(U)}(S_j)$$

provided that $\forall j, \eta_j \geqslant b_j$ and $C \in W$. We define then the class $A_{rN}^{\alpha_l, \beta_l}$ such that: $\forall S \in \mathscr{E}_{(K,k)}^{rN}$:

If $S \subset E_{(k)}^{rm}$, $S \in \omega_G$,

$$\alpha_l(S) = d(G) - \dim \pi(S) - l$$

$$\beta_l(S) = d_l(G) - \lambda_l$$

If $S \not\subset E^{rm}_{(k)}$, $S \notin \omega_G$

$$\alpha_l(S) = d(G) - \dim \pi(S) - \sum_{\mu^p_v < 0} \mu^p_v - \sum_{\mu^p_i < 0} \mu^p_i$$

$$\beta_l(S) = d_l(G) - \sum_{\mu^l_v < 0} \mu^l_v - \sum_{\mu^l_i < 0} \mu^l_i$$

If $S \subset E^{rm}_{(k)}$

$$\alpha_l(S) = -\dim S - 1$$

$$\beta_l(S) = d_l(G) - \inf\left\{ \lambda_l, \lambda_{d(G)+1}, \sum_{\substack{\gamma \in U(G) \\ \pi(S) \in \hat{\sigma}_\gamma}} \lambda_{d(\gamma)+1} \right\}$$

We obtain then that $D^l_{(K)} R_G \in A^{\alpha_l, \beta_l}_{rN}$, and this ends the proof.

Proof of theorem 3.1. — We shall directly apply Weinberg's theorem 1.2. The asymptotic coefficient β_H for every subspace $S \subset \mathcal{E}^{r(n-1)}_{(K,k)}$ is found by inserting (3.74), (3.76), (3.78) in (1.4).

More precisely for:

$$H^{ren}(K) = \int_{E^{rm}_{(k)}} R_G(K, k) d^{rm}k$$

We have, in view of theorem 1.2

$$\beta_H(S) = \begin{cases} \beta_{\mathcal{M}}(S) & \text{if} \quad \alpha_H(S) = \alpha_{H,\mathcal{M}}(S); \quad \alpha_{H,\mathcal{M}}(S) \neq \alpha_{H,\mathcal{M}'}(S) \\ \beta_{\mathcal{M}'}(S) & \text{if} \quad \alpha_H(S) = \alpha_{H,\mathcal{M}'}(S); \quad \alpha_{H,\mathcal{M}}(S) \neq \alpha_{H,\mathcal{M}'}(S) \\ 1 + \beta_{\mathcal{M}}(S) + \beta_{\mathcal{M}'}(S) & \text{if} \quad \alpha_{H,\mathcal{M}}(S) = \alpha_{H,\mathcal{M}'}(S) \end{cases}$$

with $\beta(S)$ given by theorem 3.2.

Moreover, when all μ^l_v and μ^l_i are non negative, we find:

$$\beta_H(S) = 1 + 2d_l(G)$$

This ends the proof.

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