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I. M. SIGAL

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Complex transformation method and resonances in one-body quantum systems

by

I. M. SIGAL (*)
Department of Theoretical Mathematics,
The Weizmann Institute of Science, Rehovot 76100, Israel

ABSTRACT. — We develop a new spectral deformation method in order to treat the resonance problem in one-body systems. Our result on the meromorphic continuation of matrix elements of the resolvent across the continuous spectrum overlaps considerably with an earlier result of E. Balslev [B] but our method is much simpler and more convenient, we believe, in applications. It is inspired by the local distortion technique of Nuttall-Thomas-Babbitt-Balslev (see [BB2] [J1] [J2] [N] [T]), further developed in [B] but patterned on the complex scaling method of Combes and Balslev (see [AC] [BC] [Sim1] [Hu]). The method is applicable to the multicenter problems in which each potential can be represented, roughly speaking, as a sum of exponentially decaying and dilation-analytic, spherically symmetric parts.

RÉSUMÉ. — Nous développons une nouvelle méthode de déformation spectrale pour traiter le problème des résonances dans les systèmes à un corps. Notre résultat sur le prolongement méromorphe d'éléments de matrice de la résolvante à travers le spectre continu a un recouvrement important avec un résultat antérieur de Balslev [B], mais notre méthode est beaucoup plus simple et nous semble plus commode pour les applications. Elle est inspirée par la technique de distorsion locale de Nuttal-Thomas-Babbitt-Balslev (voir [BB2] [J1] [J2] [N] [T]) développée ensuite dans [B], mais elle suit la méthode de changement d'échelle com-

(*) Charles H. Revson Senior Scientist.

plexe de Combes et Balslev (voir [AC] [BC] [Sim1] [Hu]). La méthode s'applique aux problèmes à plusieurs centres pour lesquels chaque potentiel peut être représenté, en gros, comme une somme de termes à symétrie sphérique, analytiques par dilatation et à décroissance exponentielle.

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1. INTRODUCTION

The complex scaling method, which is, essentially, due to J. M. Combes and E. Balslev ([C] [BC] [AC], see also [Sim1]) turned out to be a very effective tool in the study of resonances (see [San], especially the contribution by B. Simon [Sim2]). However the method is restricted to the dilation analytic potentials, hence its preoccupation with atomic systems. C_0^∞ and multicenter (i. e. of the form $\Sigma V_i(x - R_i)$) potentials are not treated by this method. This leaves out an essential part of the nuclear systems and the entire subject of Born-Oppenheimer moleculae. Two rigorous methods were developed in order to overcome this restriction: local distortion technique due to Babbitt and Balslev [BB2], Nuttall [N] and Thomas [T] (see also [J1] [J2]) and the exterior complex scaling method of Simon [Sim3] (we mention also non-rigorous but effective method of McCurdy and Rescigno [McCR]).

In this paper we propose the third-complex transformation-method. This method generalizes the complex scaling method in a way different from the exterior scaling method and which is close in spirit to the local distortion technique. As in the other two methods its aim is to deform the spectrum of a Hamiltonian, H, in question. It is applicable to one-particle systems with multicenter potential, produced by sums of exponentially decaying and dilation-analytic, spherically symmetric terms. It proves for these potentials that the matrix elements of the resolvent $\langle u, (H - z)^{-1} v \rangle$ on a dense set of vectors u and v have meromorphic continuations across the continuous spectrum of H (into the second Riemann sheet) with poles of these continuations occuring at the complex eigenvalues of a certain non-self-adjoint operator (and which, of course, are independent of u and v). This result overlaps considerably with one of Balslev [B] but our proof is simpler and is, we believe, more accessible to generalizations. The following problems suggest themselves naturally to the next attack:

i) Complex transformation method for many-body systems.

ii) Existence of resonances in the multicenter problem (Born-Oppenheimer molecules).

2. RESULTS

We consider a class of Schrödinger operators $H = -\Delta + V(x)$ on $L^2(\mathbb{R}^n)$. Here Δ is the Laplacian and V is a real potential. We assume that V is Δ-compact, i. e. compact as a multiplication operator from the Sobolev space $H_2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Then H is self-adjoint on $D(H) = D(\Delta) = H_2(\mathbb{R}^n)$ and $\sigma_{ess}(H) = \sigma(-\Delta) = [0, \infty)$ (see [RSII] [IV] [Sig1]).

We use the following notations: $\| \zeta \| = (\Sigma \zeta_i^2)^{1/2}$ and

$$| \zeta | = (\Sigma | \zeta_i |^2)^{1/2} = (\| \mathrm{Re}\, \zeta \|^2 + \| \mathrm{Im}\, \zeta \|^2)^{1/2}$$

for $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$. Note: $\| x \| = | x |$ for $x \in \mathbb{R}^n$.

To apply our technique we assume henceforth the following

CONDITION 2.1. — The Fourier transform $\hat{V}(p)$ of $V(x)$ is the restriction to $\mathbb{R}^n$ of a function $\hat{V}(\zeta)$, analytic in the domain

$$\Lambda_{\delta,\alpha} \equiv \{ \zeta \in \mathbb{C}^n \mid \| \mathrm{Im}\, \zeta \| \leqslant \delta \| \mathrm{Re}\, \zeta \| (1 + \sigma \| \mathrm{Re}\, \zeta \|)^{-\alpha} \} \tag{2.1}$$

for some δ, $\sigma > 0$ and α and which satisfies

$$| \hat{V}(\zeta) | \leqslant C \| \mathrm{Re}\, \zeta \|^{-n+\beta} (1 + \| \mathrm{Re}\, \zeta \|)^{2-\gamma} \tag{2.2}$$

for some $\gamma > \beta > 0$.

LEMMA 2.2. — Let $V(x) = \sum_{i=1}^{m} V_i(x - R_i)$, where $R_i \in \mathbb{R}^n$ and $V_i(x)$ can be written as $V_i = U_i + W_i$ with

$$U_i(x) = \| x \|^{-2+\delta_i} g_i(x) \qquad \text{and} \qquad | D^\sigma g_i(x) | \leqslant C e^{-\varepsilon_i \| x \|}$$

for $| \sigma | \leqslant m > n$ and some δ_i, $\varepsilon_i > 0$, and $W_i(x)$ are dilation-analytic and spherically symmetric with analytic continuations obeying (as a function of the radial variable only)

$$| \hat{W}_i(z) | \leqslant C | z |^{-n+\beta} (1 + | z |)^{2-\gamma} . \tag{2.3}$$

Then V satisfies condition 2.1 with $\alpha = 1$ and some δ.

Proof. — Note first that if $V(x)$ satisfies condition 2.1, then so does $V(x - R)$ for any $R \in \mathbb{R}^n$ and with the same δ and $\alpha' = \max(\alpha, 1)$. Clearly, the class of potentials in question is also closed under the addition.

Furthermore, g_i have analytic continuations into the sets

$$\{ \zeta \in \mathbb{C}^n \mid \| \mathrm{Im}\, \zeta \| < \varepsilon_i' \} ,$$

where $\varepsilon_i' < \varepsilon_i$, which satisfy the estimates

$$| \hat{g}_i(\zeta) | \leqslant C(1 + | \zeta |)^{-m} .$$

Using elementary properties of the convolution one concludes that $\hat{U}_i$ have analytic continuations to the same sets and these continuations verify the estimate

$$|\hat{U}_i(\zeta)| \leqslant C(1 + |\zeta|)^{-n+2-\delta_i}.$$

The Fourier transform of W_i is also spherically-symmetric. Let

$$\hat{W}_i(p) = h(\| p \|).$$

By the Babbitt-Balslev theorem [BB1, th. 4.1] h is the restriction of a function $h(z)$, analytic in a sector

$$\{ z \in \mathbb{C} \mid |\arg z| < \varphi \} \qquad \text{for some } \varphi > 0.$$

Hence $\hat{W}_i(p)$ is a restriction to $\mathbb{R}^n$ of the function $h(\|\zeta\|)$ analytic in

$$\{ \zeta \in \mathbb{C}^n \mid |\arg \|\zeta\|| < \varphi \}.$$

This domain contains $\Lambda_{\delta,0}$, provided $\frac{\pi}{4} \geqslant \varphi \geqslant \frac{\delta}{1-\delta}$.

Indeed, using

$$\|\zeta\|^2 = \|\operatorname{Re}\zeta\|^2 - \|\operatorname{Im}\zeta\|^2 + 2i \operatorname{Re}\zeta \cdot \operatorname{Im}\zeta,$$

we compute for $|\arg \|\zeta\|| \leqslant \frac{\pi}{4}$,

$$|\arg \|\zeta\|| \leqslant \frac{1}{2} |\tan(\arg \|\zeta\|^2)| = \frac{|\operatorname{Re}\zeta \cdot \operatorname{Im}\zeta|}{\|\operatorname{Re}\zeta\|^2 - \|\operatorname{Im}\zeta\|^2}.$$

If $\|\operatorname{Im}\zeta\| \leqslant \delta \|\operatorname{Re}\zeta\|$, this gives

$$|\arg \|\zeta\|| \leqslant \frac{\delta}{1-\delta}.$$

Now, $h(\|\zeta\|)$ obeys estimate (2.2) on $\Lambda_{\delta,0}$, $\delta = \frac{\varphi}{1+\varphi}$, as follows from (2.3) and the inequality

$$(1 - \delta^2) \|\operatorname{Re}\zeta\|^2 \leqslant |\|\zeta\|^2| \leqslant \sqrt{1+\delta^2} \|\operatorname{Re}\zeta\|^2. \qquad \square$$

Denote by $\mathscr{A}_0$ the set of vectors u from $L^2(\mathbb{R}^n)$ whose Fourier transforms $\hat{u}$ are the restrictions of functions analytic in domains (2.1) and obeying the estimate

$$|\hat{u}(\zeta)| \leqslant C(1 + \|\operatorname{Re}\zeta\|)^{-\mu}, \qquad \mu > \frac{n}{2}.$$

The next theorem overlaps considerably with the Balslev theorem [B].

THEOREM 2.3. — Assume V satisfies condition 2.1. Then $\langle u, (H-\lambda)^{-1} v \rangle$ with $u, v \in \mathscr{A}_0$ admits a meromorphic continuation from $\mathbb{C}^+$ (resp. $\mathbb{C}^-$) to a $\overline{\mathbb{C}^-}$ – (resp. $\overline{\mathbb{C}^+}$) neighbourhood of λ. The poles of this continuation occur at and only at the eigenvalues of a certain non-self-adjoint operator constructed below (see (3.16), (3.17)). They are independent of u and v.

3. COMPLEX TRANSFORMATION METHOD

In this section we develop the complex transformation method which is our principal tool in tackling the resonance problem.

Let f be a real, bounded and boundedly differentiable vector-field on $\mathbb{R}^n$. This vector field defines uniquely the flow $\varphi_\theta : \mathbb{R}^n \to \mathbb{R}^n$,

$$\frac{d}{d\theta}\varphi_\theta(p) = f(\varphi_\theta(p)) \qquad \text{and} \qquad \varphi_0(p) = p\,. \tag{3.1}$$

Define the one-parameter family

$$U_f(\theta) : \psi(p) \to [J(\varphi_\theta)(p)]^{1/2}\psi(\varphi_\theta(p))\,, \tag{3.2}$$

where $J(\varphi)$ is the Jacobian of $\varphi : \mathbb{R}^n \to \mathbb{R}^n$. Note that since φ_θ is a flow, $J(\varphi_\theta)(p) > 0$ for all $p \in \mathbb{R}^n$ and θ. $U_f(\theta)$ is, actually, a group of unitary operators:

$$U_f(\theta)^* U_f(\theta) = \mathbb{1} \tag{3.3}$$

and

$$U_f(\theta) U_f(\delta) = U_f(\theta + \delta)\,. \tag{3.4}$$

The unitarity follows readily from the definition and the group property follows from the corresponding property of the flow:

$$\varphi_\theta(\varphi_\delta(p)) = \varphi_{\theta+\delta}(p)\,.$$

REMARK 3.1. — A straightforward computation gives the expression for the generator D_f of $U_f(\theta)$:

$$D_f = \frac{1}{2}\,[f(p)\cdot\nabla_p + \nabla_p\cdot f(p)]\,. \tag{3.5}$$

For $f(p) = p$, this, as expected, is the generator of dilations. As a generator of a one-parameter unitary group, D_f is self-adjoint. Independently, this can be verified using the commutator theorem [RSII].

Earlier and in a different context the group $U_f(\theta)$ was considered by E. Mourre [Mo] (we thank E. Mourre for pointing this out to us during our communication about the present work).

We impose the following restriction on the vector-field f:

CONDITION 3.2. — f is the restriction of a function (denoted by the same symbol f) analytic in the tube

$$\{\, \zeta \in \mathbb{C}^n \mid \|\operatorname{Im}\zeta\| < b \,\} \tag{3.7}$$

and obeying there the estimates

$$| f(\zeta) | \leqslant C \| \operatorname{Re} \zeta \| (1 + \| \operatorname{Re} \zeta \|)^{-\alpha} \qquad \text{for some } \alpha \geqslant 1 \tag{3.8}$$

and

$$||| Df(\zeta) ||| \leqslant C. \tag{3.9}$$

Here $||| \cdot |||$ denotes the (operator) norm for $n \times n$ complex matrices and $(Df)(\zeta)$ stands for the derivative of f at ζ (an $n \times n$ matrix). Let

$$M = \sup_{\|\operatorname{Im} \zeta\| \leqslant b} | f(\zeta) |.$$

LEMME 3.3. — Assume f obeys condition 3.2. Then φ_θ and $J(\varphi_\theta)$ have analytic continuations in θ into the complex strip $\{ z \in \mathbb{C} \mid | \operatorname{Im} z | < bM^{-1} \}$ and these continuations verify the estimates for $y \in \left(0, \frac{1}{3} C\right)$

$$\frac{1}{2} \| p \| \leqslant \| \operatorname{Re} \varphi_{iy}(p) \| \leqslant \frac{3}{2} \| p \|, \tag{3.10}$$

$$\frac{1}{2} \| p - q \| \leqslant \| \operatorname{Re} (\varphi_{iy}(p) - \varphi_{iy}(q) \| \leqslant \frac{3}{2} \| p - q \| \tag{3.11}$$

and

$$| J(\varphi_{iy})(p) | \leqslant e^{nCy}. \tag{3.12}$$

Moreover, Image φ_{iy} and Image φ_{iy} – Image $\varphi_{iy} \subset \Lambda_{\delta,\alpha}$ with $\delta = 3Cy$.

Proof. — The existence of the analytic continuations is guaranteed by the general O. D. E. theory (see [CL, pp. 34-36]). To prove the estimates we use a standard tool, the integral equation

$$\varphi_\theta(p) = p + \int_0^\theta f(\varphi_z(p)) dz, \tag{3.13}$$

where the integral is taken along a complex path joining 0 and θ, say, along the straight line. We have

$$\begin{aligned} | \varphi_{iy}(p) | &\leqslant \| p \| + \int_0^y | f(\varphi_{is}(p)) | ds \leqslant \| p \| + C \int_0^y | \varphi_{is}(p) | ds \\ &\leqslant \| p \| + Cy \sup_{0 \leqslant s \leqslant y} | \varphi_{is}(p) |. \end{aligned}$$

This implies for $Cy < 1$,

$$\sup_{0 \leqslant s \leqslant y} | \varphi_{is}(p) | \leqslant (1 - Cy)^{-1} \| p \|. \tag{3.14}$$

This inequality yields

$$\int_0^y | f(\varphi_{is}(p)) | ds \leqslant C \int_0^y | \varphi_{is}(p) | ds \leqslant Cy(1 - Cy)^{-1} \| p \| \leqslant \frac{1}{2} \| p \|.$$

This together with equation (3.13) gives

$$\frac{1}{2}\| p \| \leqslant \| \operatorname{Re} \varphi_{iy}(p) \| \leqslant \frac{3}{2} \| p \| .$$

(3.11) is proved in the same way as (3.10) but one uses (3.9) instead of $| f(\zeta) | \leqslant C \| \operatorname{Re} \zeta \|$.

To show (3.12), one uses the known formula [CL, p. 36]

$$J(\varphi_\theta)(p) = \exp \int_0^\theta \operatorname{tr}(Df)(\varphi_s(p))ds ,$$

where $\operatorname{tr}(Df)(p)$ is the trace of the matrix $(Df)(p) \equiv \left[\frac{\partial f_i}{\partial p_j}(p) \right]$, and remarks that $| \operatorname{Tr} A | \leqslant n ||| A |||$.

Finally we show that Image φ_{iy} and Image φ_{iy} − Image $\varphi_{iy} \subset \Lambda_{\delta,\alpha}$. Using estimate (4.10) and equation (3.13) we obtain

$$\begin{aligned} \| \operatorname{Im} \varphi_{iy}(p) \| &\leqslant C \int_0^y \| \operatorname{Re} \varphi_{is}(p) \| (1 + \| \operatorname{Re} \varphi_{is}(p) \|)^{-\alpha} ds \\ &\leqslant Cy(1 - Cy)^{-1} \| \operatorname{Re} \varphi_{iy}(p) \| \left(1 + \frac{1}{3} \| \operatorname{Re} \varphi_{iy}(p) \| \right)^{-\alpha} . \end{aligned}$$

This inequality yields

$$\| \operatorname{Im} (\varphi_{iy}(p) - \varphi_{iy}(q)) \| \leqslant 9Cy \left(1 + \frac{1}{3} \| \operatorname{Re} (\varphi_{iy}(p) - \varphi_{iy}(q)) \| \right)^{-\alpha+1} .$$

Furthermore, equation (3.13), condition (3.9) and estimate (3.11) give

$$\| \operatorname{Im} (\varphi_{iy}(p) - \varphi_{iy}(q)) \| \leqslant 3Cy \| \operatorname{Re} (\varphi_{iy}(p) - \varphi_{iy}(q) \| .$$

The last inequalities imply that Image φ_{iy} and Image φ_{iy} − Image $\varphi_{iy} \subset \Lambda_{\delta,\alpha}$ with $\delta = 3Cy$. □

Henceforth we are working in the *momentum representation.* Let T be the multiplication operator on $L^2(\mathbb{R}^n)$ by $\| p \|^2$ (the Fourier transform of the Laplacian). Due to equation (3.10), $U_f(\theta)$ maps D(T) into D(T) (= D(H)) for *real* θ. Consider the family

$$H(\theta) \equiv U_f(\theta) H U_f(\theta)^{-1} . \tag{3.15}$$

We compute

$$H(\theta) = T(\theta) + V(\theta) , \tag{3.16}$$

where

$$T(\theta) = T_{\varphi_\theta} \quad \text{and} \quad V(\theta) = V_{\varphi_\theta} . \tag{3.17}$$

Here we have used the following notations

$$(T_\varphi \psi)(p) = \| \varphi(p) \|^2 \psi(p) \tag{3.18}$$

and (for V satisfying condition 2.1 and for φ obeying Image φ − Image φ $\subset \Lambda_{\alpha,\delta}$)

$$(V_\varphi\psi)(p) = [J(\varphi)(p)]^{1/2}\int \hat{V}(\varphi(p) - \varphi(q))[J(\varphi)(q)]^{1/2}\psi(q)dq. \quad (3.19)$$

The definition of $T(\theta)$ implies

$$\sigma(T(\theta)) = \| \text{Image } \varphi \|^2. \quad (3.20)$$

LEMMA 3.4. — If the vector-field f obeys condition 3.2, then $H(\theta)$ has an analytic continuation in θ into the strip $\{z\in\mathbb{C} \mid |\mathrm{Im}\, z| < bM^{-1}\}$. This continuation is defined and closed in the same domain D(T). Moreover,

$$\sigma_{\mathrm{ess}}(H(\theta)) = \sigma(T(\theta)). \quad (3.21)$$

Proof. — The lemma follows from condition 2.1 on V, lemma 3.3 and theorem 4.1 of the next section. □

Now we derive the property of $H(\theta)$ to which the method owes its existence.

LEMMA 3.5. — Let $\lambda \neq 0$ be real and let the vector field f satisfy condition 3.2 and the conditions

$$||| Df(\zeta) ||| \leqslant C(1 + \| \mathrm{Re}\, \zeta \|)^{-\alpha} \quad (3.22)$$

and

$$p \cdot f(p) > 0 \qquad \text{on the sphere} \qquad \| p \|^2 = \lambda. \quad (3.23)$$

Then for $\mathrm{Im}\, \theta > 0$, sufficiently small, $\sigma_{\mathrm{ess}}(H(\theta))$ is disjoint from a connected open set containing

$$\{z\in\mathbb{C} \mid \mathrm{Im}\, z > C(\mathrm{Im}\, \theta)(1 + |\mathrm{Re}\, z|)^{-\alpha}\} \cup \{z\in\mathbb{C} \mid |z-\lambda| \leqslant c(\mathrm{Im}\, \theta)\} \quad (3.24)$$

for some positive C and c.

Proof. — The statement follows from equations (3.20) and (3.21) and lemma 3.6 below. □

LEMMA 3.6. — Under the conditions of lemma 3.5, $\{\| \varphi_\theta(p) \|^2 \forall p\in\mathbb{R}^n\}$ is disjoint from a connected open complex set containing (3.24).

Proof. — We use the identity

$$\| \varphi_{iy}(p) \|^2 = \| p \|^2 + \left\| \int_0^y f(\varphi_{is}(p))ds \right\|^2 + 2\int_0^y p \cdot f(\varphi_{is}(p))ds.$$

Using again equation (3.13) we estimate

$$| \varphi_{is}(p) - p | \leqslant sM(p),$$

where $M(p) = \sup_{|\zeta| \leqslant M} | f(p + \zeta) |$ (recall that $M = \sup_{\|\mathrm{Im}\, \zeta\| \leqslant b} | f(\zeta) |$). This gives

$$\int_0^y p \cdot (f(\varphi_{is}(p)) - f(p))ds \leqslant M_1(p)\int_0^y | \varphi_{is}(p) - p | \, ds \leqslant \| p \| \, M_1(p)M(p)\frac{y^2}{2},$$

where

$$M_1(p) = \sup_{|\zeta| \leqslant M} |||(Df)(p + \zeta)|||.$$

Next we have

$$\left| \int_0^y f(\varphi_s(p))ds \right| \leqslant yM(p),$$

which implies

$$\operatorname{Im} \| \varphi_{iy}(p) \|^2 \geqslant 2yp \cdot f(p) - y^2(M(p)^2 + \frac{1}{2} M(p) \| p \| M_1(p)).$$

This inequality and the conditions on $f(p)$ imply the statements of the lemma. □

4. SPECTRUM OF $T_\varphi + V_\varphi$

In this section we determine the essential spectrum of operators of the form $T_\varphi + V_\varphi$, where T_φ and V_φ are defined in (3.18) and (3.19). We restrict $\varphi : \mathbb{R}^n \to \mathbb{C}^n$ as

$$\text{Image } \varphi - \text{Image } \varphi \subset \Lambda_{\delta,\alpha} \tag{4.1}$$

and there exist constants $C_i > c_i > 0$ s. t.

$$c_1 \| p \| \leqslant \| \operatorname{Re} \varphi(p) \| \leqslant C_2 \| p \|, \tag{4.2}$$

$$c_2 \| p - q \| = \| \operatorname{Re} (\varphi(p) - \varphi(q)) \| \leqslant C_2 \| p - q \|, \tag{4.3}$$

$$\| J(\varphi)(p) \| \leqslant C_3. \tag{4.4}$$

THEOREM 4.1. — Let V obey condition 2.1 and let φ satisfy conditions (4.1)-(4.4). Then $T_\varphi + V_\varphi$ is defined as a closed operator in D(T) and

$$\sigma_{\text{ess}}(T_\varphi + V_\varphi) = \sigma(T_\varphi) \equiv \| \varphi(\mathbb{R}^n) \|^2. \tag{4.5}$$

Proof. — Observe first that, by (4.1) and (4.2), $D(T_\varphi) = D(T)$. Moreover,

$$\sigma_{\text{ess}}(T_\varphi) = \sigma(T_\varphi) \equiv \| \text{Image } \varphi \|^2. \tag{4.6}$$

By an abstract result of [Sig2, th. 5], it suffices to show that V_φ is T_φ-compact (in the restricted sense: $D(V_\varphi) \supset D(T_\varphi)$, $\rho(T_\varphi \neq \emptyset$ and $V_\varphi(T_\varphi - \lambda)^{-1}$ is compact for some, and therefore all, $\lambda \in \rho(T_\varphi)$).

LEMMA 4.2. — Let V satisfy condition 2.1 and let φ obey (4.1)-(4.4). Then V_φ defines a T_φ-compact operator.

Proof. — Conditions (2.2) and (4.2)-(4.4) imply that

$$| J_\varphi(p) |^{1/2} | \hat{V}(\varphi(p) - \varphi(q)) | | J_\varphi(q) |^{1/2} \leqslant CW(p - q), \tag{4.7}$$

where $W(p) = \| p \|^{-\nu+\beta}(1 + \| p \|)^{2-\gamma} \in (L^r + L^1)(\mathbb{R}^n)$ with $r < \dfrac{\nu}{\nu - 2}$.

Taking this into consideration and using the Young inequality (see [RSII]) we obtain

$$\| V_{\varphi}u \| \leqslant a \| Tu \| + b \| u \|$$

for some a and b. Using now (4.1) and (4.2) and definition (2.1) of $\Lambda_{\delta,\alpha}$ we derive

$$\| V_{\varphi} u \| \leqslant C(\| T_{\varphi}u \| + b \| u \|)$$

(this means that V_{φ} is T_{φ}-bounded). This implies that $V_{\varphi}(T_{\varphi} - \lambda)^{-1}$ is bounded for $\lambda \in \rho(T_{\varphi})$.

To demonstrate the compactness we note first that due to (4.2) and (4.7) the kernel of $V_{\varphi}(T_{\varphi} - \lambda)^{-1}$ is of the form

$$h(q, p)W(q - p)(1 + \| p \|)^{-2},$$

where h is a bounded function. Observe now that the operators with the kernels

$$h(p, q)W(p - q)(1 + \| p \|)^{-2}\chi(\|p \|/n)(1 - \chi(n \| p - q \|)),$$

where $\chi \in C_0^{\infty}(\mathbb{R})$ and $\chi(t) = 1$ for $| t | \leqslant 1$, are Hilbert Schmidt and converge to $V_{\varphi}(T_{\varphi} - \lambda)^{-1}$ as $n \to \infty$ in the uniform operator topology. Hence $V_{\varphi}(T_{\varphi} - \lambda)^{-1}$ is compact as a norm-limit of compact operators [RSI, thm VI.12, p. 200].

The statement of theorem 4.1 follows from lemma 4.2, the generalized Weyl theorem (see [Sig2]) and equation (4.6). □

5. PROOF OF THEOREM 2.3

We begin with a definition and a preliminary discussion of a general character. Let $\mathscr{A}$ be the set of all $U_f(\theta)$-analytic vectors, i. e.
$\mathscr{A} = \{ g \in L^2(\mathbb{R}^n) \mid U_f(\theta)g$ has an analytic continuation from $\mathbb{R}$ to an open complex set $\mathcal{O}$ having a nonempty intersection with $\mathbb{R} \}$.
Note first that the analyticity domain $\mathcal{O}$ of a vector g can be automatically extended to the strip

$$S_{\delta} = \{ z \in \mathbb{C} \mid | \operatorname{Im} z | < \delta \}, \qquad \text{where} \qquad \delta = \sup_{z \in \mathcal{O}} | \operatorname{Im} z |.$$

Indeed, if $g(\theta)$ is an analytic continuation of $U(\theta)g$ into $\mathcal{O}$, then

$$g(\theta + \delta) = U_f(\delta)g(\theta) \qquad \text{with} \qquad \delta \in \mathbb{R}$$

defines an analytic extension of $g(\theta)$ to S_{δ}.

Secondly, $\mathscr{A}$ is dense in $L^2(\mathbb{R}^n)$ (it contains, for instance,

$$\bigcup_{\text{compact } \Delta} E_{D_f}(\Delta)L^2(\mathbb{R}^n),$$

where $E_{D_f}(\Delta)$ is the spectral projection for the self-adjoint operator D_f). Moreover, if $u \in \mathscr{A}_0$ (defined in section 2) and f satisfies condition 3.2, then by lemma 3.3, $u \in \mathscr{A}$.

Proof of theorem 2.3. — Given $\lambda > 0$, pick a vector-field f, satisfying the conditions of lemma 3.5. Let $U_f(\theta)$ be the corresponding unitary group.

We apply the standard Combes argument. By virtue of unitarity of $U_f(\theta)$ for real θ, we have

$$\langle u, (H - z)^{-1} v \rangle = \langle u(\theta), (H(\theta) - z)^{-1} v(\theta) \rangle, \tag{5.1}$$

where $\theta \in \mathbb{R}$ and $z \in \mathbb{C}^+$. We have used the notation $g(\theta) = U_f(\theta)g$. If $u, v \in \mathscr{A}$, then the r. h. s. can be analytically continued in θ into a strip of $\mathbb{C}^+$ (along $\mathbb{R}$) (see lemma 3.4). Equality (5.1) does, of course, stay valid for complex θ. After that, using lemma 3.5, we extend the r. h. s. from $\mathbb{C}^+$ to the disc $|z - \lambda| < c(\text{Im } \theta)$ figuring in the statement of lemma 3.5. This furnishes the desired meromorphic continuation of the l. h. s. of (5.1). Clearly, the poles of this continuation occur at and only at the eigenvalues of $H(\zeta)$, Im $\zeta > 0$, which by the standard Combes argument (see e. g. [Hu] [RSIV]) are independent of ζ as long as they stay away from $\sigma_{\text{ess}}(H(\zeta))$. □

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REFERENCES

[AC] J. AGUILAR and J. M. COMBES, On a class of analytic perturbations of one body Schrödinger operators, *Commun. Math. Phys.*, t. **22**, 1971, p. 269-279.

[BB1] D. BABBITT and E. BALSLEV, A characterization of dilation-analytic potentials and vectors, *J. Funct. Anal.*, t. **18**, 1975, p. 1-14.

[BB2] D. BABBITT and E. BALSLEV, Local distortion technique and unitarity of the S-matrix for the 2-body problem, *J. Math. Anal. Appl.*, t. **54**, 1976, p. 316-347.

[B] E. BALSLEV, *Local spectral deformation techniques for Schrödinger operators*, Mittag-Leffler preprint, 1982.

[BC] E. BALSLEV and J. M. COMBES, Spectral properties of Schrödinger Hamiltonians with dilation analytic potentials, *Commun. Math. Phys.*, t. **22**, 1971, p. 280-294.

[CL] E. A. CODDINGTON and N. LEVINSON, *Theory of Ordinary Differential Equations*, McGraw-Hill, 1955.

[C] J. M. COMBES, *An algebraic approach to quantum scattering*, 1969, unpublished manuscript.

[Hu] W. HUNZIKER, The Schrödinger eigenvalue problem for N-particle systems, in *The Schrödinger Equation*, W. Thirring and P. Urban, eds., Springer-Verlag, 1977.

[J1] A. JENSEN, Local distortion technique, resonances and poles of the S-matrix, *J. Math. Anal. Appl.*, t. **59**, 1977, p. 505-513.

[J2] A. JENSEN, Resonances in an abstract analytic scattering theory, *Ann. Inst. H. Poincaré*, t. **33**, 1980, p. 209-223.

[Mo] E. MOURRE, Absence of singular spectrum for certain self-adjoint operators, *Commun. Math. Phys.*, t. **78**, 1981, p. 391-408.

[McCR] C. W. MCCURDY, Jr. and T. N. RESCIGNO, Extension of the method of complex basis functions to molecular resonances, *Phys. Rev. Lett.*, t. **41**, 1978, p. 1364-1368.

[N] J. NUTTALL, Analytic continuation of the off-energy-shell scattering amplitude, *J. Math. Phys.*, t. **8**, 1967, p. 873-877.

[RS] M. REED and B. SIMON, *Methods of Modern Mathematical Physics, I, II, IV*, Academic Press.

[San] Sanibel WORKSHOP on Complex Scaling, 1978, in *Int. J. Quant. Chemistry*, t. **14**, 1978.

[Sig1] I. M. SIGAL, Scattering Theory for Many-Body Quantum Mechanical Systems-Rigorous Results, *Springer*, *Lecture Notes in Mathematics* N1011, 1983.

[Sig2] I. M. SIGAL, A generalized Weyl theorem and the L^p-spectra of Schrödinger operators, *J. Oper. Theory (to appear)*.

[Sim1] B. SIMON, Quadratic form technique and the Balslev-Combes theorem, *Commun. Math. Phys.*, t. **27**, 1972, p. 1-9.

[Sim2] B. SIMON, Resonances and complex scaling: A rigorous overview, *J. Quant. Chem.*, t. **14**, 1978, p. 529-542.

[Sim3] B. SIMON, The definition of molecular resonance curves by the method of exterior complex scaling, *Phys. Lett.*, t. **71 A**, 1979, p. 211-214.

[T] L. THOMAS, On the spectral properties of some one-particle Schrödinger Hamiltonians, *Helv. Phys. Acta*, t. **45**, 1972, p. 1057-1065.

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