ANNALES DE L'I. H. P., SECTION A

D. R. YAFAEV

Scattering theory for time-dependent zero-range potentials

Annales de l'I. H. P., section A, tome 40, nº 4 (1984), p. 343-359

<http://www.numdam.org/item?id=AIHPA_1984__40_4_343_0>

Ann. Inst. Henri Poincaré,
Vol. 40, n° 4, 1984, p. 343-359. *Physique théorique*

Scattering theory for time-dependent zero-range potentials

by

D. R. YAFAEV

Leningrad Branch of Mathematical Institute (LOMI),
Fontanka 27, Leningrad, 191011 USSR and (September-November 1982),
Institut Mittag-Leffler, S-18262 Djursholm, Sweden

ABSTRACT. — Scattering of a quantum particle by a time-dependent zero-range potential is considered. The purpose of the paper is to give a complete description of the asymptotic evolution of such a system for large times.

RÉSUMÉ. — On étudie la diffusion d'une particule quantique par une interaction ponctuelle dépendant du temps. Le but de l'article consiste en une description complète de l'évolution asymptotique d'un tel système pour des grands temps.

INTRODUCTION

For a quantum system, described by a constant Hamiltonian H, the set of all initial states may be split up into two classes. The first of them is the subspace $\mathcal{R}$ of states, generating the asymptotically « free » behavior of the system for large times. One of the main results of mathematical scattering theory consists in the proof (see e. g. [*1*]) of the coincidence of $\mathcal{R}$ with the absolutely continuous subspace of H. This result is called asymptotic completeness. Since the singular continuous spectrum of H is empty, it ensures that the orthogonal complement to $\mathcal{R}$ (the second class) is spanned by the eigenvectors of H. Thus, for initial data from the second class the evolution of the system depends on time only in a trivial way.

For time-dependent Hamiltonians H(t) it is usually also possible to construct initial states, that give rise to an asymptotically free evolution. Moreover, if the perturbation decays sufficiently quickly in time, then the corresponding subspace $\mathcal{R}$ coincides [2] with the whole Hilbert space $\mathcal{H}$. However, as shown in the author's papers [3] [4], even for interactions, decaying in time, there may exists states, which are asymptotically similar to bound states in case H(t) = H. More precisely, assume that the Hamiltonian H(t) has an eigenvalue $\lambda(t)$, that tends sufficiently slowly and smoothly to the bottom of the continuous spectrum, as $t \to \infty$. Then under certain assumptions on the behavior of the associated eigenvector $\psi(t)$ the time-dependent Schrödinger equation has a solution $u(t)$ with the asymptotics

$$u(t) \sim \psi(t) \exp\left(- i \int_0^t \lambda(s)ds \right), \qquad t \to \infty .$$

The corresponding initial state f_s is called pseudostationary. Its similarity to an eigenvector of H in case H(t) = H is emphasized by the orthogonality of f_s to $\mathcal{R}$. As for H(t) = H, in the time-dependent case the problem of the asymptotic completeness consists in the description of $\mathcal{R}$. The most straightforward generalization of the stationary formulation may be obtained if we replace the eigenvectors by pseudostationary states. In fact it is physically relevant to expect that under suitable assumptions the subspace $\mathcal{H} \ominus \mathcal{R}$ is spanned by pseudostationary elements.

In the present paper we consider the Schrödinger equation with a time-dependent zero-range potential or, in other words, with a point interaction of time-dependent strength. This model is especially well suited [5] for the description of states with a small coupling energy. We treated this model already in [6], where conditions for the equality $\mathcal{R} = \mathcal{H}$ were found. The existence of a pseudostationary state was also proven in [6] under proper assumptions. However, in this case the most difficult problem, i. e. the problem of the asymptotic completeness, was left open. We fill this gap here. Combined with the results of [6], this gives a complete description of the subspace $\mathcal{R}$ for zero-range potentials that depend on time in different ways. Thus, in such a model the classification of all quantum states with respect to their asymptotic behavior is obtained.

Note that the case of the periodic dependence on time was treated in [7], where the asymptotic completeness in the sense of G. Schmidt [8] was established. Namely, in [7] it was proven that for several (possibly « interacting ») zerorange potentials $\mathcal{R}$ coincides with the absolutely continuous subspace of the so-called monodromy operator of the problem. This is similar to Yajima's result [9] for the usual potential scattering, but its proof is somewhat more complicated technically.

1. MAIN RESULTS

Let us describe our model. The Schrödinger operator with a three-dimensional zero-range potential (or with a point interaction) is defined [*10*] in the Hilbert space $L_2(\mathbb{R}^3)$ as one of the self-adjoint extensions of the operator $H_{00} = -\Delta$ with the domain $\mathscr{D}(H_{00}) = C_0^\infty(\mathbb{R}^3\setminus\{0\})$. The simplest way to construct all these extensions is to decompose previously $L_2(\mathbb{R}^3)$ into the orthogonal sum of subspaces $\mathscr{H}^{(l)}$, corresponding to different values of the orbital quantum number l, $l = 0, 1, 2, \ldots$ (in spherical functions). The subspaces $\mathscr{H}^{(l)}$ are invariant with respect to H_{00} and its restrictions $H_{00}^{(l)}$ to $\mathscr{H}^{(l)}$ are defined by

$$H_{00}^{(l)}\varphi = -x^{-2}\frac{d}{dx}\left(x^2\frac{d}{dx}\varphi\right) + l(l+1)x^{-2}\varphi, \qquad x = |\vec{x}|, \qquad \vec{x}\in\mathbb{R}^3.$$

After the substitution $\varphi = x^{-1}u$, generating the unitary transformation of $\mathscr{H}^{(l)}$ onto $L_2(\mathbb{R}_+)$, the operator $H_{00}^{(l)}$ is reduced to $-d^2/dx^2 + l(l+1)x^{-2}$ with the domain $C_0^\infty(\mathbb{R}_+)$ in $L_2(\mathbb{R}_+)$. The operators $H_{00}^{(l)}$ are essentially self-adjoint for $l \geq 1$, and the deficiency indices of $H_{00}^{(0)}$ equal (1, 1). As is well-known, all self-adjoint extensions H^ω, $\omega\in\mathbb{R}$, of $H_{00}^{(0)}$ are determined by a boundary condition at $x = 0$. Namely, $H^\omega = -d^2/dx^2$ with the domain $\mathscr{D}(H^\omega)$, which consists of functions from the Sobolev space $\mathscr{W}_2^2(\mathbb{R}_+)$ obeying $u'(0) = \omega u(0)$. The operator $-d^2/dx^2$ with the boundary condition $u(0) = 0$ (i. e. $\omega = \infty$) will be denoted by H_0. Let us consider now the self-adjoint operator in $L_2(\mathbb{R}^3)$, which equals the closure of $H_{00}^{(l)}$ on the subspaces $\mathscr{H}^{(l)}$, $l \geq 1$, and coincides (after the substitution $\varphi = x^{-1}u$) with the operator H^ω (or H_0). This operator will be also denoted by H^ω (or H_0). Clearly, the operators H^ω, $\omega\in\mathbb{R}$, and H_0 exhaust all self-adjoint extensions of H_{00}.

In the original notation the operator H^ω acts as $-\Delta$, and its domain $\mathscr{D}(H^\omega)$ in $L_2(\mathbb{R}^3)$ may be described in the following way. A function $\varphi\in\mathscr{D}(H^\omega)$, if φ belongs to the Sobolev class $\mathscr{W}_2^2$ outside some neighbourhood of the point $x = 0$; in the neighbourhood of the origin φ should satisfy

$$\varphi(\vec{x}) = a_1|\vec{x}|^{-1} + a_2 + \varphi_0(\vec{x}),$$

where $\varphi_0\in\mathscr{W}_2^2$, $\varphi_0(0) = 0$ and $a_2 = \omega a_1$. The operator $H_0 = -\Delta$ with the domain $\mathscr{D}(H_0) = \mathscr{W}_2^2(\mathbb{R}^3)$ corresponds to the free particle. Since the operators H^ω are different from H_0 only on the subspace $\mathscr{H}^{(0)}$, it is equivalent to study H^ω in $L_2(\mathbb{R}^3)$ or in $L_2(\mathbb{R}_+)$. The Hamiltonian H^ω has a negative eigenvalue for $\omega < 0$, and $H^\omega \geq 0$ for $\omega \geq 0$. However, for any $\omega\in\mathbb{R}$ the quadratic form of H^ω is smaller than that of H_0, i. e. the zero-range potential is always negative. Thus, for $\omega \geq 0$ a « depth » of a zero-range potential well is not sufficient to bind a particle; for $\omega < 0$ this well contains precisely one bound state. For $\omega = 0$ the strength of a zero-range potential is critical, and the operator H^0 has a zero-energy resonance at the bottom of the continuous spectrum.

In the present paper we study the quantum system with a time-dependent zero-range potential. Namely, the Hamiltonian is given by the relation $H(t) = H^{\omega(t)}$. The evolution of the system is described by the function $u(t)$ which satisfies the Schrödinger equation and the initial condition

$$\left.\begin{array}{l} i\dfrac{\partial u}{\partial t} = H(t)u \\ u(t_0) = f \end{array}\right\} \tag{1.1}$$

For any locally bounded function $\omega(t)$ the problem (1.1) has (see section 2) a unique solution, and the propagator $U(t)$, defined by $U(t)f = u(t)$, turns out to be unitary. We study the asymptotics of $u(t)$ as $t \to \infty$. In scattering theory it is natural to compare the evolution of our system with that of a free system, corresponding to the Hamiltonian H_0. This is convenient to perform in terms of a so-called wave operator

$$\tilde{W} = s-\lim_{t\to\infty} U(t)^*U_0(t), \qquad U_0(t) = \exp(-iH_0(t-t_0)). \tag{1.2}$$

If the limit (1.2) exists, then the operator $\tilde{W}$ is necessarily isometric, and the initial data f from the range $R(\tilde{W})$ of the operator $\tilde{W}$ give rise to the solutions of (1.1) with the free asymptotics as $t \to \infty$, i. e.

$$\| U(t)f - U_0(t)f_0 \| \to 0, \qquad f_0 = \tilde{W}^*f.$$

So the problem of scattering theory is to obtain an effective description of the subspace $\mathscr{R} = \mathscr{R}(\tilde{W})$. The operators $U(t)$ and $U_0(t)$ coincide on the subspaces $\mathscr{H}^{(l)}$, $l \geq 1$, and the wave operator (1.2) is identical on these subspaces. Therefore it is sufficient to consider $H(t)$ and H_0 in the space $L_2(\mathbb{R}_+)$. Moreover, since the limit

$$W_0 = s-\lim_{t\to\infty} U^0(t)^*U_0(t)$$

exists and is unitary, the considerations of the operators $\tilde{W}$ and

$$W = s-\lim_{t\to\infty} U(t)^*U^0(t) \tag{1.3}$$

are equivalent. More precisely, the existence of one of the limits (1.2) or (1.3) ensures the existence of the other; thereby $\tilde{W} = WW_0$ and, in particular, $R(\tilde{W}) = R(W)$. From a technical point of view it is more convenient to study the operator W.

Thus, the operator $H(t)$ acts as $-d^2/dx^2$ in the space $\mathscr{H} = L_2(\mathbb{R}_+)$;

functions from its domain obey the boundary condition $u'(0) = \omega(t)u(0)$. The operator H^0 corresponds to $u'(0) = 0$. With detailed notation the problem (1.1) reads

$$\left.\begin{aligned} i\frac{\partial u}{\partial t} &= -\frac{\partial^2 u}{\partial x^2} \\ u'(0, t) &= \omega(t)u(0, t) \\ u(x, t_0) &= f(x) \end{aligned}\right\} \tag{1.4}$$

For the study of the asymptotics of $u(t)$, $t \to \infty$, the choice of t_0 in (1.4) is inessential, and one can take arbitrary large t_0. So all conditions on $\omega(t)$, arising below, contain only restrictions on its behavior at infinity. If $\omega < 0$, the operator H^ω has a negative eigenvalue $\lambda^\omega = -\omega^2$ with a normalized eigenfunction $\psi^\omega = (-2\omega)^{1/2}\exp(\omega x)$. Set $\lambda(t) = \lambda^{\omega(t)}$, $\psi(t) = \psi^{\omega(t)}$. The following assertion was proven in [6].

THEOREM 1. — a) *For any function* $\omega(t)$ *the limit* (1.3) *exists.* b) *The operator* W *is unitary if* $\omega(t) = 0(t^{-\gamma})$, $\gamma > 1/2$. c) *The operator* W *is unitary if* $\omega(t) \geq 0$. d) *Let* $\omega(t) < 0$,

$$\left.\begin{aligned} &\omega'(t)\omega^{-3}(t) \to 0, \qquad |\omega(t)|\,t \to \infty, \qquad t \to \infty, \\ &\int_{t_0}^{\infty} (\omega'^2\omega^{-4} + |\omega''|\,|\omega|^{-3})dt < \infty . \end{aligned}\right\} \tag{1.5}$$

Then the limit

$$f_s = \lim_{t\to\infty} U^*(t)\psi(t)\exp\left[-i\int_{t_0}^{t}\lambda(s)ds\right] \tag{1.6}$$

exists, $\| f_s \| = 1$ *and* f_s *is orthogonal to* R(W).

Let us discuss the results of Theorem 1. The part *a*) means that for any zero-range potential there exists a large set of initial data (namely, R(W)), giving rise to the asymptotically free evolution. The parts *b*) and *c*) imply that all states are asymptotically free in two physically different cases. In the part *b*) it is assumed that the interaction vanishes with time sufficiently quickly. This is similar to the J. Howland's result [2] for the usual potential scattering (i. e. for the perturbation of H_0 by the operator of a multiplication). In the part *c*) Hamiltonians H(t) do not have discrete spectrum. In potential scattering it is known [11] that the wave operator is unitary for the repulsive potential. In the latter case the quadratic form of H(t) is greater than that of H_0. On the contrary, the zero-range perturbation corresponds always to the attraction of particles. Thus, not only for the time-dependent potential hump but also for the well with the « undercritical » depth all states are asymptotically free. The part *c*) is applicable, in particular, to the periodic function $\omega(t)$. Compared to [6], we god rid here of the assumption on the polynomial boundedness of $\omega(t)$. Let us proceed to the part d). In the power scale conditions (1.5) imply that $\omega(t) \sim \omega_0 t^{-\gamma}$, $\omega_0 < 0$, $\gamma < 1/2$, $t \to \infty$. So the bound state $\psi(t)$ of the Hamiltonian $H(t)$ with the energy $\lambda(t)$ vanishing sufficiently slowly (or growing to $-\infty$) with time generates the « quasibound » state of the system. The corresponding initial state f_s is called pseudostationary. Thus, parts b), c) and d) exhaust all possible versions of a power behavior of $\omega(t)$. For the sake of completeness the proof of Theorem 1 (with exception of part a)) is given in section 3.

In the case $\omega(t) = \omega < 0$ the element f_s coincides with the eigenvector of the Hamiltonian $H(t) = H$, and the subspace $\mathcal{H} \ominus R(W)$ is spanned by f_s. The purpose of the present paper is to prove the analogous result for the time-dependent case. Namely, we will show that for $\omega(t) \sim \omega_0 t^{-\gamma}$, $\omega_0 < 0$, $\gamma < 1/2$, the codimension of $R(W)$ equals 1 and consequently $\mathcal{H} \ominus R(W)$ consists only of the pseudostationary element. This solves the problem of the asymptotic completeness in the scattering of a quantum particle by the time-dependent zero-range potential. Let us now give the precise formulation of our main result.

THEOREM 2. — *Let*

$$\omega(t) = \omega_0 t^{-\gamma} + 0(t^{-\gamma-\varepsilon}), \qquad \gamma < 1/2, \qquad \omega_0 < 0, \qquad \varepsilon > 0, \quad (1.7)$$

and let the relation (1.7) *be three times differentiable. Assume further that*

$$|\omega^{(k)}(t)| \leq C t^{-\gamma-k}, \qquad 0 \leq k \leq I, \qquad (1.8)$$

where I *satisfies* $I \geq 2\nu + 7$ *with*

$$\nu = \nu(\gamma) = \max\left\{\left(\frac{5\gamma}{2} - 1\right)(1 - 2\gamma)^{-1}, 1\right\}. \qquad (1.9)$$

Then the subspace $\mathcal{H} \ominus R(W)$ *has dimension* 1.

Theorem 2 contains assumptions on the behavior of a rather a large number of derivatives of $\omega(t)$. Most probably, this number is rather exaggerated and we did not try hard to lower it.

Let us now summarize shortly the content of our paper. In section 2 we introduce and study the Volterra integral equation, which is equivalent to the problem (1.4). The results given there are necessary for the proof of Theorems 1 and 2 in section 3. Our proof of Theorem 2 requires an investigation of the asymptotics of the solution of the Volterra integral equation with a kernel, that slowly decreases or even grows at infinity. This is done in the author's paper [12]. Necessary extracts from [12] are contained in section 2.

2. THE INTEGRAL EQUATION

The proofs of Theorems 1 *b*), *c*) and 2 are based on the reduction of the problem (1.4) to the Volterra integral equation for the function $v(t)=u_x(0,t)$. In our case the domains of the operators $H(t)$ depend on t. Thus, even the existence of the solution of the problem (1.1) does not follow from well-known theorems [*13*] [*14*] of abstract character.

Let us introduce some notations. Let $\hat{f}$ be the Fourier cosinus transform of a function $f \in \mathscr{H}$, i. e.

$$\hat{f}(p) = (2/\pi)^{1/2} \int_0^\infty \cos px f(x) dx .$$

Operators T, $T : \mathscr{H} \rightarrow L_2(t_0, \infty) \equiv \mathscr{M}$, and L, $L : \mathscr{M} \rightarrow \mathscr{M}$, are defined by

$$(Tf)(t) = (U^0(t)f)(0) = (2/\pi)^{1/2} \int_0^\infty e^{-ip^2(t-t_0)} \hat{f}(p) dp , \tag{2.1}$$

$$(Lv)(t) = \pi^{-1/2} e^{\pi i/4} \int_{t_0}^t (t-\tau)^{-1/2} v(\tau) d\tau ;$$

Z_t is multiplication by the characteristic function of an interval (t_0, t) in $\mathscr{M}$. Note the identity

$$TT^* = i(L^* - L) . \tag{2.2}$$

Taking complex conjugation of (2.2) we get that $T_1 T_1^* = L + L^*$, where T_1 is an operator with the kernel $(2/\pi)^{1/2} \exp [ip^2(t - t_0)]$ (cf. with (2.1)). This ensures accretiveness of L :

$$\operatorname{Re} L \geq 0 . \tag{2.3}$$

By C we denote generic constants.

The following assertion is proven in [*6*] (see also [*15*] where several zero-range potentials in $\mathbb{R}^3$ are considered).

THEOREM 3. — *Let* $\omega \in C^2_{loc}(t_0, \infty)$, $\hat{f} \in C_0^\infty(\mathbb{R}_+)$, $f(0) = 0$, *and let* $v(t)$ *be the solution of the equation*

$$v(t) + \omega(t)\pi^{-1/2} e^{\pi i/4} \int_{t_0}^t (t-\tau)^{-1/2} v(\tau) d\tau = \omega(t)(Tf)(t) . \tag{2.4}$$

Then the function

$$u(t) = U^0(t)(f - iT^* Z_t v) \tag{2.5}$$

belongs to $\mathscr{D}(H(t))$ *for* $t > t_0$, *is strongly differentiable in* $\mathscr{H}$ *with respect to* t *and satisfies* (1.1).

The equation (1.1) and self-adjointness of $H(t)$ ensure that $\| u(t) \| = \| f \|$. This, in particular, implies uniqueness of the solution of the problem (1.1).

Comparing (2.4) and (2.5) we find the following expression for the propagator $U(t)$, $U(t)f = u(t)$:

$$U(t) = U^0(t)[I - iT^*Z_t(I + \omega L)^{-1}\omega T]. \tag{2.6}$$

The Volterra integral equation (2.4) has a unique locally bounded solution for arbitrary locally bounded functions $\omega(t)$ and $(Tf)(t)$. However, in the general case the function (2.5) satisfies (1.4) only in a rather a weak sense. Without going further into technical details, we simply take the equality (2.6) as definition of the propagator $U(t)$ for an arbitrary locally bounded $\omega(t)$. The identity (2.2) permits us to verify that for any $\omega \in L_{\infty,loc}(t_0, \infty)$ the operator (2.6) is unitary in $\mathcal{H}$. For simplicity we assume later $t_0 > 0$.

The representation (2.6) for $U(t)$ is rather convenient for scattering theory. It allows us to check that $U(t)f$ has a free asymptotics as $t \to \infty$ in terms of the behavior of the solution $v(t)$ of the equation (2.4). Since for every $\varepsilon > 0$ the operator $t^{-1/4-\varepsilon}Lt^{-1/4-\varepsilon}$ is bounded in $\mathcal{M}$, by (2.2) the operator $T^*t^{-1/4-\varepsilon}$ is bounded from $\mathcal{M}$ to $\mathcal{H}$. Consequently, taking into account (2.5) we have

LEMMA 1. — *Assume that for some $f \in \mathcal{H}$ the solution $v(t)$ of (2.4) satisfies*

$$v(t) = O(t^{-3/4-\varepsilon}), \qquad \varepsilon > 0, \qquad t \to \infty. \tag{2.7}$$

Then there exists

$$s-\lim_{t\to\infty} U^0(t)^*U(t)f. \tag{2.8}$$

Note that, if the limit (1.3) exists, then the existence of the limit (2.8) is equivalent to the inclusion $f \in R(W)$.

The kernel of the integral operator in (2.4) has a singularity at the diagonal $t = \tau$. For the proofs of Theorems 1 *b*) and 2 it is convenient to replace (2.4) by an equation with a smooth kernel. For this purpose we apply the operator $I - \omega L$ to both sides of (2.4). Then we receive the equation

$$v(t) - \int_{t_0}^{t} G(t, \tau)v(\tau)d\tau = v_0(t), \tag{2.9}$$

where

$$G(t, \tau) = i\pi^{-1}\omega(t)\int_{\tau}^{t} \omega(s)(t-s)^{-1/2}(s-\tau)^{-1/2}ds \tag{2.10}$$

and

$$v_0 = (I - \omega L)\omega Tf. \tag{2.11}$$

Now we summarize some necessary information on the asymptotic behavior of solutions of Volterra integral equations. We do not assume here that a kernel G of (2.9) is defined by (2.10). Suppose that $G(t, \tau)$, $t \geq \tau$, satisfies

$$|G(t, \tau)| \leq C(t/\tau)^{\beta}\tau^{\alpha-1} \tag{2.12}$$

The numbers α and β are called respectively an order and a type of a kernel G.

For kernels of a negative order the behavior of $v(t)$ as $t \to \infty$ can be easily controlled by estimation of the absolute values of the terms of a series of iterations

$$v = (\mathrm{I} - \mathrm{G})^{-1} v_0 = \sum_{n=0}^{\infty} \mathrm{G}^n v_0 \tag{2.13}$$

(G is a Volterra operator with a kernel $\mathrm{G}(t, \tau)$). The following assertion is well-known.

LEMMA 2. — *Let the bound* (2.12) *be fulfilled for* $\alpha < 0$ *and let* $v_0(t) = 0(t^\beta)$. *Then* $v(t) = 0(t^\beta)$.

For kernels of a positive order the series (2.13) is again convergent for any fixed t but in this case it gives only an exponential bound for $v(t)$. This is not sufficient for our purposes. For the proof of Theorem 2 we need the asymptotics of $v(t)$ for kernels of a positive order. Such results are obtained in [*12*]. Here we give some extracts from this paper.

Assume that

$$\mathrm{G}(t, \tau) = g(t/\tau)\tau^{\alpha-1} + \tilde{\mathrm{G}}(t, \tau), \qquad \alpha > 0, \tag{2.14}$$

where $g(z)$ and its derivatives obey the condition

$$|g^{(j)}(z)| \leq \mathrm{C} z^{\beta - j}, \quad z \geq 1, \quad j = 0, 1, 2, \ldots, g(1) \neq 0, \tag{2.15}$$

and $\tilde{\mathrm{G}}$ satisfies

$$|\tilde{\mathrm{G}}(t, \tau)| \leq \mathrm{C}(t/\tau)^\beta \tau^{\alpha - 1 - \varepsilon}, \qquad \varepsilon > 0. \tag{2.16}$$

Assume further that

$$\operatorname{Re} \mathrm{G}(t, t) = 0. \tag{2.17}$$

Let us introduce the Mellin transform

$$(\mathrm{M}g)(p) = \int_1^\infty z^{-p-1} g(z) dz \tag{2.18}$$

of the function $g(z)$. Then $(\mathrm{M}g)(p)$ is analytic in the halfplane $\operatorname{Re} p > \beta$, and $(\mathrm{M}g)(p) = g(1)p^{-1} + 0(|p|^{-2})$, as $p \to \infty$, $\operatorname{Re} p \geq \beta + \varepsilon, \varepsilon > 0$. Denote by p_0 the maximum of the real parts of its zeros. If $(\mathrm{M}g)(p)$ does not have any zeros for $\operatorname{Re} p > \beta$, then we set $p_0 = \beta$. Now we define

$$\delta = \max\{p_0, \lambda\}, \qquad \lambda = \operatorname{Re} g'(1) g(1)^{-1}. \tag{2.19}$$

We say that a bound of the type (2.12) is J times differentiable if

$$\left| \frac{\partial^j \mathrm{G}(t, \tau)}{\partial t^{j_1} \partial \tau^{j_2}} \right| \leq \mathrm{C}(t/\tau)^\beta \tau^{\alpha-1} t^{-j_1} \tau^{-j_2}$$

for all j_1, j_2, $0 \leq j_1 + j_2 = j \leq \mathrm{J}$. The following theorem is proven in [*12*].

THEOREM 4. — *Let conditions* (2.14)-(2.17) *be fulfilled and* $\alpha > p_0 - \lambda$.

Assume that (2.16) *is three times differentiable and that the bound* (2.12) *for* $G(t, \tau)$ *may be differentiated* J *times, where* $J \geq 2\nu + 7$ *and*

$$\nu = \max \{ (\lambda - \beta)\alpha^{-1} - 1, 1 \} . \tag{2.20}$$

Suppose further that v_0 *is locally bounded, twice differentiable for sufficiently large* t *and*

$$v_0^{(j)}(t) = 0(t^{\delta - j}), \qquad j = 0, 1, 2, \qquad t \to \infty . \tag{2.21}$$

Then the solution $v(t)$ *of the equation* (2.9) *has the asymptotics*

$$v(t) = la(t) + 0(t^{\delta - \alpha + \varepsilon}), \tag{2.22}$$

where

$$a(t) = \exp \left\{ \int_{t_0}^{t} [G_1(s, s)G(s, s)^{-1} + G(s, s)]ds \right\},$$

$G_1(t, \tau) = \dfrac{\partial G(t, \tau)}{\partial t}$, $l = l(v_0)$ *is a constant and* ε *is an arbitrary positive number.*

The verification of the assumptions of Theorem 2 for the kernel (2.10) relies on the following technical

LEMMA 3. — *Let*

$$F(z, \tau) = \int_1^z (z-\sigma)^{-1/2}(\sigma-1)^{-1/2} f(\tau\sigma)d\sigma, \qquad z \geq 1, \tau \geq \tau_0 > 0 . \tag{2.23}$$

Suppose that

$$| f^{(l)}(s) | \leq Cs^{-\gamma - l}, \qquad \gamma < 1/2, \qquad 0 \leq l \leq K . \tag{2.24}$$

Then $F(z, \tau)$ *satisfies*

$$| F(z, \tau) | \leq Cz^{-\gamma}\tau^{-\gamma}, \tag{2.25}$$

and (2.25) *is* K *times differentiable.*

The proof of Lemma 3 is given in the Appendix.

To check the condition (2.21) of Theorem 4 for the function (2.11) we need the elementary.

LEMMA 4. — *Set*

$$w(t) = \int_{t_0}^{t} (t - \tau)^{-1/2} y(\tau)d\tau . \tag{2.26}$$

If $y(t) = 0(t^{-1-\varepsilon})$, $\varepsilon > 0$, *then* $w(t) = 0(t^{-1/2})$. *If, moreover,* $y(t)$ *is twice differentiable and* $y^{(j)}(t) = 0(t^{-1-j-\varepsilon})$, $j = 1, 2$, *then* $w(t)$ *also has two derivatives for* $t > t_0$ *and* $w^{(j)}(t) = 0(t^{-1/2-j})$, $j = 1, 2$, *for* $t \geq t_1 > t_0$.

Proof. — At first we make the change of variable $\tau = ts$ in (2.26). Then

$$w(t) = t^{1/2} \int_{t_0/t}^{1} (1 - s)^{-1/2} y(ts)ds \tag{2.27}$$

The integral in (2.27) clearly does not exceed Ct^{-1}, which proves the first assertion. In contrast to (2.26), the expression (2.27) for $w(t)$ may be directly differentiated:

$$w'(t) = t^{-1/2}\int_{t_0/t}^{1}(1-s)^{-1/2}[2^{-1}y(ts)+tsy'(ts)]ds + t_0t^{-3/2}(1-t_0/t)^{-1/2}y(t_0). \tag{2.28}$$

The integral in (2.28) is quite similar to that in (2.27). Thus, the first summand in (2.28) is $0(t^{-3/2})$. The second summand is evidently bounded by $t^{-3/2}$ if $t \geq t_1 > t_0$. Differentiating (2.28) once more, we can estimate $w''(t)$ in precisely the same way.

3. UNITARITY AND COMPLETENESS OF THE WAVE OPERATOR

For completeness of exposition we repeat here the proof of the unitarity of the wave operator under the assumptions of Theorems 1 *b*) and 1 *c*). The proof of the existence of W (Theorem 1 *a*)), which is quite standard, is given in [*6*], and we omit it here.

Proof of Theorem 1 *b*). — By Lemma 1 it suffices to show that for any $\hat{f} \in C_0^\infty(\mathbb{R}_+)$ the solution $v(t)$ of the equation (2.9) satisfies (2.7). Integrating by parts in (2.1) we find that for $\hat{f} \in C_0^\infty(\mathbb{R}_+)$ the functions $(Tf)(t)$ and hence $\omega(t)(Tf)(t)$ decay quicker than any power of t^{-1} as $t \to \infty$. Lemma 4 now ensures that for the function (2.11) the estimate $v_0(t) = 0(t^{-1/2-\gamma})$ holds. Under the assumption $\omega(t) = 0(t^{-\gamma})$, $\gamma > 1/2$, the kernel (2.10) satisfies the bound

$$| G(t, \tau) | \leq Ct^{-\gamma}\int_\tau^t (t-s)^{-1/2}(s-\tau)^{-1/2}s^{-\gamma}ds$$
$$= Ct^{-\gamma}\tau^{-\gamma}\int_1^{t/\tau}\left(\frac{t}{\tau}-s\right)^{-1/2}(s-1)^{-1/2}s^{-\gamma}ds \leq C_1t^{-\gamma}\tau^{-\gamma}(t/\tau)^{-1/2}.$$

This proves the estimate (2.12) for $G(t, \tau)$ with $\alpha = 1-2\gamma < 0$ and $\beta = -1/2-\gamma$. So Lemma 2 implies that the solution $v(t)$ of the equation (2.9) is bounded by $Ct^{-1/2-\gamma}$. Since $\gamma > 1/2$, this proves (2.7). It follows that $R(W) = \mathscr{H}$.

Proof of Theorem 1 *c*). — Let $\omega(t) \geq 0$. In this case the proof of unitarity of W is based on the accretiveness of the operator L (see the relation (2.3)). Suppose that $R(W) \neq \mathscr{H}$, and let $f, f \neq 0$, be orthogonal to $R(W)$.

Since the limit (1.3) exists, $V(t)f = U^0(t)^*U(t)f$ converges weakly to zero and, in particular,

$$\lim_{t\to\infty}(V(t)f, f) = 0. \tag{3.1}$$

Set

$$\zeta_t = Z_t\omega^{1/2}, \quad v_t = Z_t(I+\omega^{1/2}L\omega^{1/2})^{-1}\omega^{1/2}Tf = (I+\zeta_t L\zeta_t)^{-1}\zeta_t Tf. \quad (3.2)$$

By (2.6)

$$V(t)f = f - iT^*\zeta_t v_t. \quad (3.3)$$

Let us show that $\| v_t \| \to 0$ as $t \to \infty$. The inequality (2.3) ensures that

$$\mathrm{Re}\,((I + \zeta_t L^*\zeta_t)v_t, v_t) \geq \| v_t \|^2. \quad (3.4)$$

Comparing (2.2) and (3.2) we find that

$$(I+\zeta_t L^*\zeta_t)v_t = \zeta_t T(f - iT^*\zeta_t v_t). \quad (3.5)$$

Now we compose a scalar product of both sides of (3.5) with v_t and take its real part:

$$\mathrm{Re}\,((I+\zeta_t L^*\zeta_t)v_t, v_t) = \mathrm{Re}\,(f, T^*\zeta_t v_t). \quad (3.6)$$

On the other hand, by (3.3)

$$\mathrm{Re}\,(f, T^*\zeta_t v_t) = \mathrm{Re}\,(f, i(V(t)f - f)) = \mathrm{Im}\,(f, V(t)f).$$

According to (3.1) this expression vanishes as $t \to \infty$. Therefore on account of (3.4), (3.6) $\| v_t \| = 0(1)$, $t \to \infty$. The definition (3.2) implies that $(I+\omega^{1/2}L\omega^{1/2})^{-1}\omega^{1/2}Tf = 0$ and consequently $v_t = 0$ for all $t \geq t_0$. Thus, (3.3) is reduced to the equality $V(t)f = f$, which contradicts (3.1). This concludes the proof.

Now we apply to the proof of Theorem 2. In virtue of Theorem 1 *d*) (its proof will be given at the end of the paper) under the assumptions of Theorem 2 the pseudostationary element (1.6) exists and belongs to the subspace $\mathcal{H} \ominus R(W)$. Thus it suffices to show that the dimension of $\mathcal{H} \ominus R(W)$ is not greater than 1. Besides the results of section 2 we need here an elementary

LEMMA 5. — *Let $\mathcal{H}$ be a Hilbert space and l be a linear functional defined on some linear set $\mathcal{D}$, dense in $\mathcal{H}$. Let $\mathcal{D}_0$, $\mathcal{D}_0 \subset \mathcal{D}$, consist of elements* f, *for which $l(f) = 0$. Then the dimension $\mathcal{H} \ominus \overline{\mathcal{D}_0}$ does not exceed* 1.

Actually, if l is bounded, then by the Riesz theorem $l(f) = (f, \psi_0)$ for some $\psi_0 \in \mathcal{H}$. In this case the subspace $\mathcal{H} \ominus \overline{\mathcal{D}_0}$ is spanned by ψ_0 so that its dimension equals 1. Let l be unbounded. Then there exists $\zeta_n \in \mathcal{D}$ such that $l(\zeta_n) = 1$ and $\| \zeta_n \| \to 0$. Moreover, taking some subsequence of ζ_n one can secure that for any $\eta_n > 0$ the estimate $\| \zeta_n \| < \eta_n$ holds. Let now $\psi \in \mathcal{H}$ and $f_n \in \mathcal{D}, f_n \to \psi$ as $n \to \infty$. Consider $f_n^0 = f_n + \alpha_n\zeta_n$. If $\alpha_n = -l(f_n)$, then $l(f_n^0) = 0$ so that $f_n^0 \in \mathcal{D}_0$. One can choose ζ_n in such a way that $| l(f_n) | \cdot \| \zeta_n \| \to 0$ as $n \to \infty$. It follows that $\lim f_n^0 = \lim f_n = \psi$. Thus, in this case the set $\mathcal{D}_0$ is dense in $\mathcal{H}$, which concludes the proof.

Let us check that the kernel (2.10) satisfies the assumptions of Theorem 4. We begin with a verification of the estimate (2.12). Let the condition (1.8) hold. After the change of variable $s = \tau\sigma$ (2.10) takes the form $G(t, \tau) = i\pi^{-1}\omega(t)F(t/\tau, \tau)$, where $F(z, \tau)$ is defined by (2.23) with $f(s) = \omega(s)$. By Lemma 3 the bound $\omega(t) = 0(t^{-\gamma})$ implies the estimate (2.12) with $\alpha = 1 - 2\gamma$, $\beta = -2\gamma$ for $G(t, \tau)$. Moreover, since the bound $\omega(t) = 0(t^{-\gamma})$ is differentiable J times, the same is true for the estimate (2.12). To check conditions (2.15), (2.16) we need the assumption (1.7). Set $\tilde{\omega}(t) = \omega(t) - \omega_0 t^{-\gamma}$.

$$g(z) = i\pi^{-1}\omega_0^2 z^{-\gamma}\int_1^z (z-s)^{-1/2}(s-1)^{-1/2}s^{-\gamma}ds, \tag{3.7}$$

and let $\tilde{G}(t, \tau)$ be defined by (2.14). To prove (2.15), (2.16) for $g(z)$ and $\tilde{G}(t, \tau)$ it only remains to apply Lemma 3 for $f(s) = s^{-\gamma}$ and for $f(s) = \tilde{\omega}(s)$. In our case $G(t, \tau)$ is purely imaginary for all t and τ so that (2.17) holds. By easy computations we find that for the function (3.7) $g(1) = i\omega_0^2$, $g'(1) = -3\gamma i\omega_0^2/2$, $\lambda = \operatorname{Re} g'(1)g(1)^{-1} = -3\gamma/2$. In particular, it follows that the definition (2.20) of v is the same as (1.9).

For evaluation of δ one needs to compute the Mellin transform (2.18) of (3.7):

$$\begin{aligned}(Mg)(p) &= i\pi^{-1}\omega_0^2\int_1^\infty dz\cdot z^{-p-1-\gamma}\int_1^z ds(z-s)^{-1/2}(s-1)^{-1/2}s^{-\gamma}\\ &= i\pi^{-1}\omega_0^2\int_1^\infty dss^{-\gamma}(s-1)^{-1/2}\int_s^\infty dzz^{-p-1-\gamma}(z-s)^{-1/2}\\ &= i\pi^{-1}\omega_0^2\int_1^\infty dss^{-2\gamma-p-1/2}(s-1)^{-1/2}\int_1^\infty dzz^{-p-1-\gamma}(z-1)^{-1/2}.\end{aligned} \tag{3.8}$$

In virtue of the equality

$$\int_1^\infty z^{-q}(z-1)^{-1/2}dz = \pi^{1/2}\Gamma(q-1/2\Gamma(q)^{-1}$$

each of the integrals in the right-hand side of (3.8) can be evaluated in terms of the Γ-function:

$$(Mg)(p) = i\omega_0^2\Gamma(2\gamma+p)\Gamma(p+\gamma+1/2)\Gamma(2\gamma+p+1/2)^{-1}\Gamma(p+\gamma+1)^{-1}.$$

The zeros of $(Mg)(p)$ are real, and the greatest of them equals

$$\max\{-\gamma-1, -2\gamma-1/2\}.$$

Therefore, for $\gamma < 1/2$ the function $(Mg)(p)$ does not have zeros in the half-plane $\operatorname{Re} p > \beta = -2\gamma$ so that $p_0 = \beta$. Thus, the definition (2.19) gives the equality $\delta = \delta(\gamma) = \max\{-3\gamma/2, -2\gamma\}$. The condition $\alpha > p_0 - \lambda$ of Theorem 4 is equivalent to $\gamma < 2/3$.

Let us consider the free term $v_0(t)$ in the equation (2.9); $v_0(t)$ is defined by (2.11). As in the proof of Theorem 1 *b*), we assume that $\hat{f} \in C_0^\infty(\mathbb{R}_+)$. Then $\omega(t)(Tf)(t)$ and its two derivatives decay quicker than any power of t^{-1} as $t \to \infty$. Now Lemma 4 ensures that $v_0^{(j)}(t) = 0(t^{-1/2-\gamma-j}), j=0,1,2$. Thus, the condition (2.21) also holds.

In our case $\delta - \alpha < -3/4$ and therefore the relation (2.22) of Theorem 4 reads

$$v(t) = la(t) + 0(t^{-\frac{3}{4}-\varepsilon}), \qquad \varepsilon > 0. \tag{3.9}$$

For the kernel (2.10) $G(t,t)=i\omega^2(t)$, $G_1(t,t) = \frac{3}{2} i\omega'(t)\omega(t)$ and, consequently,

$$a(t) = |\,\omega(t)/\omega(t_0)\,|^{3/2} \exp\left[i \int_{t_0}^{t} \omega^2(s)ds \right].$$

Note, however, that in our proof of Theorem 2 the concrete expression for $a(t)$ is inessential.

Let $\mathscr{D}$ be a set of elements, for which $\hat{f} \in C_0^\infty(\mathbb{R}_+)$. For $f \in \mathscr{D}$ the constant l in (3.9) depends obviously linearly on v_0 and therefore on f. Thus, $l = l(f)$ defines a linear functional on $\mathscr{D}$. As in Lemma 5, a subset $\mathscr{D}_0$ of $\mathscr{D}$ is determined by the condition $l(f) = 0$ for $f \in \mathscr{D}_0$. By (3.9) for $f \in \mathscr{D}_0$ the solution of the equation (2.9) obeys (2.7). Thus, by Lemma 1 the limit (2.8) exists so that $\overline{\mathscr{D}_0} \subset R(W)$. Since by Lemma 5 the dimension of $\mathscr{H} \ominus \overline{\mathscr{D}_0}$ does not exceed 1, the same is true for $\mathscr{H} \ominus R(W)$. Theorem 1 *d*) ensures now that the subspace $\mathscr{H} \ominus R(W)$ is precisely one-dimensional. This completes the proof of Theorem 2.

In conclusion we give

Proof of Theorem 1 *d*). — To show the existence of the limit (1.6) we replace previously the function

$$\psi(x,t)\mu(t), \quad \mu(t) = \exp\left[-i \int_{t_0}^{t} \lambda(s)ds \right],$$

in (1.6) by a better approximation to a solution of the equation (1.4). It appears that a proper approximation may be defined by the formula $w(x,t) = (1 - i\xi(t)x^2)\psi(x,t)\mu(t)$, where $\xi(t)=4^{-1}\omega'(t)\omega(t)^{-1}$. At first, we prove the existence of

$$\lim_{t\to\infty} U^*(t)w(t). \tag{3.10}$$

Since $w(t) \in \mathscr{D}(H(t))$, to that end it suffices to check that

$$\left\| \frac{\partial}{\partial t} [U^*(t)w(t)] \right\| = \left\| \left(\frac{\partial}{\partial t} - i \frac{\partial^2}{\partial x^2} \right) [(1 - i\xi(t)x^2)\psi(x,t)\mu(t)] \right\| \in L_1(t_0, \infty). \tag{3.11}$$

After differentiations we find that

$$\left(\frac{\partial}{\partial t} - i \frac{\partial^2}{\partial x^2} \right) [(1 - i\xi x^2)\psi\mu] = \mu[-i\xi' x^2 \psi + (1 - i\xi x^2)\psi_t$$
$$+ i\omega^2(1 - i\xi x^2)\psi - 2\xi\psi - 4\xi x \psi_x - i(1 - i\xi x^2)\psi_{xx}].$$

Since $\psi_{xx} = \omega^2\psi$, the third and the sixth summands in the right-hand side cancel one another. Taking into account the explicit formula

$$\psi(x, t) = (-2\omega)^{1/2} \exp(\omega x),$$

we receive $\psi_x = \omega\psi$, $\psi_t = 2\xi\psi + \omega' x\psi$. It follows that

$$\begin{aligned}\left\| \frac{\partial}{\partial t} [U^*(t)w(t)] \right\| &= \| [-i\xi' x^2 + (1 - i\xi x^2)(2\xi + \omega' x) - 2\xi - \omega' x]\psi \| \\ &= \| [(\xi' + 2\xi^2)x^2 + \xi\omega' x^3]\psi \| \\ &\leq |\xi' + 2\xi^2| \cdot \| x^2\psi \| + 4^{-1}\omega'^2 |\omega|^{-1} \| x^3\psi \| \\ &\leq C(|\omega''|\, |\omega|^{-3} + \omega'^2\omega^{-4}).\end{aligned}$$

The convergence of the integral (1.5) ensures now the inclusion (3.11). This proves the existence of the limit (3.10). By (1.5)

$$|\xi| \cdot \| x^2\psi \| = C|\omega'|\, |\omega|^{-3} = 0(1), \qquad t \to \infty,$$

so that the limits (3.10) and (1.6) exist simultaneously.

Let us show finally that the pseudostationary element f_s is orthogonal to R(W). It is sufficient to check that

$$\lim_{t\to\infty} (U^0(t)f, \psi(t)) = 0$$

for $\hat{f} \in C_0^\infty(\mathbb{R}_+)$. Let $b = \inf \operatorname{supp} \hat{f} > 0$. Integrating by parts in the relation

$$(U^0(t)f)(x) = (2\pi)^{-1/2} \int_0^\infty (e^{ipx - ip^2t} + e^{-ipx - ip^2t})\hat{f}(p)dp,$$

we find that for any $n > 0$

$$\sup_{x \leq bt} |(U^0(t)f)(x)| \leq C_n t^{-n}. \tag{3.12}$$

We need (3.12) only for $n = 1/2$. Let us now consider

$$(U^0(t)f, \psi(t)) = |2\omega|^{1/2}\left(\int_0^{bt} + \int_{bt}^\infty\right)(U^0(t)f)(x)e^{\omega x}dx. \tag{3.13}$$

By (3.12) the first summand in the right-hand side is bounded by $C(|\omega| t)^{-1/2}$. In virtue of the Schwarz inequality and the unitarity of $U^0(t)$, the second summand may be estimated by

$$\| f \| \left(2|\omega| \int_{bt}^\infty e^{2\omega x}dx\right)^{1/2} = \| f \| \left(\int_{2b|\omega|t}^\infty e^{-y}dy\right)^{1/2}.$$

According to (1.5) $|\omega| t \to \infty$ so that both sides of (3.13) tend to zero as $t \to \infty$. Thus the pseudostationary element is necessarily orthogonal to the range of the wave operator.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor L. Carleson and the staff of the Institut Mittag-Leffler for their hospitality during my stay at the Institute.

APPENDIX

Proof of Lemma 3. — Set $\Omega_0(z, s) = (z - s)^{-1/2}(s - 1)^{-1/2}$ and define functions $\Omega_n(z, s)$, $n \geq 1$, $z > s$, by the recurrence relation

$$\Omega_n(z, s) = \int_1^s \frac{\partial}{\partial z}\Omega_{n-1}(z, \sigma)d\sigma . \tag{A.1}$$

It follows that

$$\Omega_n(z, s) = [(n-1)!]^{-1}\int_1^s \frac{\partial^n \Omega_0(z, \sigma)}{\partial z^n}(s - \sigma)^{n-1}d\sigma$$
$$= a_n \int_1^s (z-\sigma)^{-n-1/2}(\sigma - 1)^{-1/2}(s-\sigma)^{n-1}d\sigma , \tag{A.2}$$

where $a_n = (-1)^n(2n-1)!![2^n(n-1)!]^{-1}$. In particular, (A.2) implies that

$$\frac{d}{dz}\int_1^z \Omega_n(z, s)ds = 0, \qquad n \geq 0 . \tag{A.3}$$

To obtain a bound for $\Omega_n(z, s)$ we rewrite (A.2) as

$$\Omega_n(z, s) = a_n(z-1)^{-1}\zeta_n\left(\frac{s-1}{z-1}\right)$$

with

$$\zeta_n(x) = \int_0^x (1-y)^{-n-1/2}(x-y)^{n-1}y^{-1/2}dy, \qquad x \in (0, 1) .$$

Since $|\zeta_n(x)| \leq Cx^{n-1/2}(1-x)^{-1/2}$, $\Omega_n(z, s)$ satisfies

$$|\Omega_n(z, s)| \leq C(z-1)^{-n}(s-1)^{n-1/2}(z-s)^{-1/2} . \tag{A.4}$$

For the proof of Lemma 3 one needs to show that under the assumption (2.24)

$$\left|\frac{\partial^k F(z, \tau)}{\partial z^n \partial \tau^m}\right| \leq Cz^{-\gamma-n}\tau^{-\gamma-m}, \qquad k = n + m, \qquad z \geq 1, \qquad \tau \geq \tau_0 . \tag{A.5}$$

To take a derivative of $F(z, \tau)$ (see (2.23)) with respect to z, we must beforehand integrate by parts in (2.23):

$$F(z, \tau) = \int_1^z f(\tau s)d_s \int_1^s \Omega_0(z, \sigma)d\sigma = f(\tau z)\int_1^z \Omega_0(z, \sigma)d\sigma - \tau\int_1^z f'(\tau s)\left(\int_1^s \Omega_0(z, \sigma)d\sigma\right)ds .$$

The last relation may be differentiated directly. On account of the definition (A.1) for Ω_1 and the identity (A.3) for Ω_0. we get

$$\frac{\partial F(z, \tau)}{\partial z} = -\tau\int_1^z f'(\tau s)\Omega_1(z, s)ds .$$

Repeating this procedure (of integrating by parts and differentiating with respect to z) n times we find that

$$\frac{\partial^n F(z, \tau)}{\partial z^n} = (-\tau)^n \int_1^z f^{(n)}(\tau s)\Omega_n(z, s)ds .$$

Taking here m derivatives with respect to τ, we finally receive

$$\frac{\partial^k F(z, \tau)}{\partial z^n \partial \tau^m} = (-\tau)^n \int_1^z f^{(k)}(\tau s)s^m\Omega_n(z, s)ds . \tag{A.6}$$

Now we can insert bounds (2.24) for $f^{(k)}$ and (A.4) for Ω_n into (A.6):

$$\left|\frac{\partial^k F(z,\tau)}{\partial z^n \partial \tau^m}\right| \le C\tau^{-\gamma-m}(z-1)^{-n}\int_1^z (z-s)^{-1/2}(s-1)^{n-1/2}s^{-\gamma-n}ds. \tag{A.7}$$

Since the integral in (A.7) is $0((z-1)^n)$ as $z \to 1$, the inequality (A.5) for $\tau \ge \tau_0$ and $z \in [1, z_0]$, where z_0 is any fixed number, is an immediate consequence of (A.7). Thus, it suffices to prove (A.5) for $\tau \ge \tau_0$ and $z \ge z_0 > 2$. By (A.7), to that end it remains to show that

$$\int_1^z (z-s)^{-1/2}(s-1)^{n-1/2}s^{-\gamma-n}ds \le Cz^{-\gamma}. \tag{A.8}$$

On account of the bound $(s-1)^n s^{-n} \le 1$, the proof of (A.8) is reduced to the case $n = 0$. Let us split the integral in (A.8) into the sum of integrals over $\left(1, \frac{z}{2}\right)$ and $\left(\frac{z}{2}, z\right)$. Thus, the left-hand side of (A.8) does not exceed

$$z^{-1/2}\int_1^{z/2} s^{-\gamma}(s-1)^{-1/2}ds + z^{-\gamma-1/2}\int_{z/2}^z (z-s)^{-1/2}ds \le Cz^{-\gamma}.$$

This concludes the proof of (A.8) and, consequently, of Lemma 3.

REFERENCES

[1] M. REED, B. SIMON, *Methods of modern mathematical physics*. III. Scattering Theory, New York, Academic Press, 1979.

[2] J. HOWLAND, *Math. Ann.*, t. **207**, 1974, p. 315-335.

[3] D. R. YAFAEV, *Soviet Math. Dokl.*, t. **19**, 1978, p. 1517-1521.

[4] D. R. YAFAEV, *Math. USSR Sbornik*, t. **39**, 1981, p. 169-188.

[5] Ju. N. DEMKOV, V. N. OSTROVSKIJ, *The method of zerorange potentials in atomic physics*. Leningrad, Leningrad State Univ., 1975 (Russian).

[6] D. R. YAFAEV, *Teor. Mat. Fis.*, t. **43**, 1980, p. 228-239 (Russian).

[7] M. R. SAJAPOVA, D. R. YAFAEV, *Topics in Math. Physics*, t. **10**, 1982, p. 252-266 (Russian).

[8] G. SCHMIDT, *Indiana Univ. Math. J.*, t. **24**, 1975, p. 925-935.

[9] K. YAJIMA, *J. Math. Soc. Japan*, t. **29**, 1977, p. 729-743.

[10] F. A. BEREZIN, L. D. FADDEEV, *Sov. Math.*, t. **2**, 1961, p. 372-374.

[11] D. R. YAFAEV, *Funct. Anal.*, t. **14**, 1980, p. 91-92 (Russian).

[12] D. R. YAFAEV, *On the asymptotics of solutions of Volterra integral equations*, Institut Mittag-Leffler, Report No. 8, 1983.

[13] T. KATO, *J. Math. Soc. Japan*, t. **5**, 1953, p. 208-234.

[14] M. REED, B. SIMON, *Methods of modern mathematical physics*. II. Fourier analysis, self-adjointness. New York, Academic Press, 1975.

[15] M. R. SAJAPOVA, D. R. YAFAEV, *The operator of evolution for time-dependent zero-range potentials*, Trudy MIAN, t. **159**, 1982, p. 167-174 (Russian).

(Manuscrit reçu le 20 juin 1983)