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An upper bound for the local time-decay of scattering solutions for the Schrödinger equation with Coulomb potential

by

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ABSTRACT. — We show that the large-time decay for some scattering states of the Coulomb Schrödinger operator can be estimated by $t^{-1} | \ln t |^{6n}$ using suitable « energy cut-off » norms.

RÉSUMÉ. — On montre, en utilisant des normes contenant un cut-off convenable en énergie, que la décroissance aux grands temps de certains états de diffusion de l'opérateur de Schrödinger coulombien peut être estimée par $t^{-1} | \ln t |^{6n}$.

1. INTRODUCTION

Consider the Coulomb-Schrödinger operator $H := H_0 + g | x |^{-1}$ in the Hilbert space $\mathscr{H} := L^2(\mathbb{R}^n)$, $(n \geq 3)$ where $H_0 := \overline{(-\Delta) \restriction C_0^\infty(\mathbb{R}^n)}$ and $g \in \mathbb{R}$, $g \neq 0$. Note H is a self-adjoint operator in $L^2(\mathbb{R}^n)$ with domain the Sobolev space $D(H) = \mathscr{H}_2(\mathbb{R}^n)$.

We are interested in the « time evolution »

$$(1) \qquad \varphi(t) := e^{-itH}\varphi \quad \text{for large} \quad t \in \mathbb{R}^+, \qquad (\varphi \in \mathscr{H}).$$

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This is the (Hilbert space-) solution of the Schrödinger equation

$$(2) \qquad -i\frac{d}{dt}\varphi(t) = \mathrm{H}\varphi(t), \qquad t \in \mathbb{R}^+, \qquad \varphi(0) = \varphi$$

where $\varphi \in \mathrm{D(H)}$.

Since H has no singular continuous spectrum (see [*12*, p. 186], compare also [*2*]) we have $\mathcal{H} = \mathcal{H}_{ac}(\mathrm{H}) \oplus \mathcal{H}_p(\mathrm{H})$ (see [*7*, p. 519]) and there are two types of (formal) solutions of (2): the bound states $e^{-it\mathrm{H}}\varphi$ where $\varphi \in \mathcal{H}_p(\mathrm{H})$ (i. e. φ is an eigenvector of H) and the scattering solutions where $\varphi \in \mathcal{H}_{ac}(\mathrm{H})$.

Bound states stay « localized » in a bounded region of $\mathbb{R}^n$ for all times t while scattering solutions leave any bounded region for $t \to \infty$ (see [*1*, p. 260]).

In this paper we will make the last statement more precise, i. e. we shall prove the estimate

$$(3) \qquad \| e^{-it\mathrm{H}}\varphi \|_{\mathrm{L}^2_n} \leq ct^{-1} | \ln t |^{6n} \| \varphi \|_{\mathrm{N}}, \qquad t \geq t_0$$

for φ in a dense set of $\mathcal{H}_{ac}(\mathrm{H})$ and suitable constants t_0 and c.

On the left hand side of (3) we use the « localizing » weighted norm $\| \psi \|_{\mathrm{L}^2_n} := \| (1 + x^2)^{-n}\psi \|_{\mathrm{L}^2}$ whereas on the right $\| \cdot \|_{\mathrm{N}}$ is a suitable « energy cut-off » norm which will be defined below.

Local time-decay of scattering states for Schrödinger operators with short-range potentials has been discussed by many authors; see for example [*4*] [*5*] [*6*] [*9*] [*10*]. Very recently Kitada [*8*] proved, using pseudo-differential operator techniques, an estimate similar to (3) which included long range potentials but only for states with sufficiently high energy.

We shall prove estimate (3) by relatively elementary calculations similar to those of Dollard [*3*] in his proof of existence of modified wave operators.

The essence of our proof is the device of a suitable norm which « supresses» low energies by a weight function.

2. RESULTS

Let $\mathrm{H} := \mathrm{H}_0 + g | x |^{-1}$ ($x \in \mathbb{R}^n$, $n \geq 3$) as in the introduction. It is well known that the modified wave operator

$$\Omega_{\mathrm{D}}^- := \underset{t\to\infty}{s\text{-lim}}\, e^{-it\mathrm{H}}\mathrm{U}_{\mathrm{D}}(t)$$

exists [*3*], where

$$\mathrm{U}_{\mathrm{D}}(t) := \exp\left\{ -i(\mathrm{H}_0 t + \frac{g}{2|p|}\ln(t4p^2)\right\}, \qquad t \in \mathbb{R}^+$$

Notice, here and in the following we denote $p := -i\nabla$ and define the operator $\mathrm{F}(p)\varphi := (\mathrm{F}(k)\hat{\varphi})^{\check{}}$ for any real or complex-valued function F on $\mathbb{R}^n$. ($\hat{}$ and $\check{}$ denote the Fourier transform and its inverse respectively).

Now let

$$M := \left\{ \psi \in L^2(\mathbb{R}^n) \mid \| \psi \|_M := \sup_{k \in \mathbb{R}^n} \left| (1 + k^2)^{2n} e^{\frac{1}{|k|}} \sum_{|l| \leq 6n} D_k^l \hat{\psi}(k) \right| < \infty \right\}$$

and

$$N := \Omega_D^- M$$

where we used the usual notation

$$D_k^l := \partial_{k_1}^{l_1} \partial_{k_2}^{l_2} \ldots \partial_{k_n}^{l_n}, \qquad l = \langle l_1, l_2, \ldots, l_n \rangle$$

and

$$|l| := \sum_{i=1}^{n} l_1 .$$

Then M is dense in $\mathscr{H}$ and N is dense in $\mathscr{H}_{ac}(H)$ (in the L^2-sense). We might N understand physically as scattering states with small low (and high) energy parts.

Define the norm

$$\| \varphi \|_N := \| (\Omega_D^-)^{-1} \varphi \|_M \qquad \text{for} \qquad \varphi \in N .$$

We will prove the following.

THEOREM. — There are constants c and t_0 such that

(3) $\| e^{-itH} \varphi \|_{L^2_n} \leq ct^{-1} |\ln t|^{6n} \| \varphi \|_N, \quad$ for $\quad t \geq t_0 \quad$ and $\quad \varphi \in N$

In order to prove the theorem we require some technical lemmas (compare [3] and [11, p. 171]).

Let $t_0 := e$ and denote

(4) $$\psi_c(x, t) := e^{-iA_D(p,t)} \psi(x), \qquad \psi \in M, \quad x \in \mathbb{R}^n, \quad t \geq t_0$$

where

(5) $$A_D(k, t) := \frac{g}{2|k|} \ln(t 4k^2), \qquad k \in \mathbb{R}^n$$

We first show.

LEMMA 1. — Let $\psi \in M$. Then for any multiindex $l \in N^n$ with $|l| \leq 2n$

(6) $$\sup_{y \in \mathbb{R}^n} |(1 + y^2)^{3n} D_y^l \psi_c(y, t)| \leq c |\ln t|^{6n} \| \psi \|_M$$

provided $t \geq t_0$, where c is a constant depending on g and n.

Proof. — Notice first that if $\tilde{l}_i \in N$, $i \in \{1, \ldots, n\}$ and $t \geq t_0$ then by some calculation for a suitable $c_1 > 0$

$$|\partial_{k_i}^{\tilde{l}_i} e^{-iA_D(k,t)}| \leq c_1 |\ln t|^{\tilde{l}_i} e^{\frac{1}{|k|}}, \qquad k \in \mathbb{R}^n$$

Therefore for any multiindex $\tilde{l} \in \mathbb{N}^n$ and $t \geq t_0$

(7) $$| D_k^{\tilde{l}} e^{-iA_D(k,t)} | \leq c_2 | \ln t |^{|\tilde{l}|} e^{\frac{1}{|k|}}, \qquad k \in \mathbb{R}^n, \qquad c_2 > 0 \text{ suitable}.$$

Hence for any multiindex $j \in \mathbb{N}^n$ with $|j| \leq 6n$

$$| D_k^j e^{-iA_D(k,t)} \hat{\psi}(k) | \leq c_3 \sum_{\substack{|\tilde{l}| \leq 6n \\ |l| \leq 6n}} | D_k^{\tilde{l}} e^{-iA_D(k,t)} | \quad | D_k^l \hat{\psi}(k) |$$
$$\leq c_4 | \ln t |^{6n} e^{\frac{1}{|k|}} \sum_{|l| \leq 6n} | D_k^l \hat{\psi}(k) |$$

where the constants c_3 and c_4 depend on j and g.

On the other hand we have

$$\sup_{y \in \mathbb{R}^n} | (1 + y^2)^{3n} D_y^l \psi_c(y, t) | = \sup_{y \in \mathbb{R}^n} \left| (2\pi)^{-\frac{n}{2}} \int dk e^{iky} (1 - \Delta_k)^{3n} k^l e^{-iA_D(k,t)} \hat{\psi}(k) \right|$$
$$\leq c_5 \sup_{k \in \mathbb{R}^n} \left| (1 + k^2)^{2n} \sum_{|j| \leq 6n} D_k^j (e^{-iA_D(k,t)} \hat{\psi}(k) \right|.$$

where $l \in \mathbb{N}^n$, $|l| \leq 2n$ and $c_5 > 0$ suitable. Using (8), the last term above can be dominated by

$$c_6 | \ln t |^{6n} \sup_{k \in \mathbb{R}^n} \left((1 + k^2)^{2n} e^{\frac{1}{|k|}} \sum_{|l| \leq 6n} | D_k^l \hat{\psi}(k) | \right) \leq c | \ln t |^{6n} \| \psi \|_M$$

for suitable c_6 and c.

Therefore (6) follows. ■

Now Introduce the notation

(9) $$R_\psi(x, t) := U_D(t)\psi(x) - (2it)^{-\frac{n}{2}} \eta(x, t) \hat{\psi}\left(\frac{x}{2t}\right); \quad (x \in \mathbb{R}^n, t > t_0)$$

where

(10) $$\eta(x, t) := \exp\left(i \frac{x^2}{4t}\right) \exp\left(i \frac{gt}{|x|} \ln \frac{x^2}{t}\right).$$

Then we have

LEMMA 2. — For $\psi \in M$ and a suitable c

(11) $$| R_\psi(x, t) | \leq c t^{-\left(\frac{n}{2}+1\right)} | \ln t |^{6n} \left(1 + \left(\frac{x}{t}\right)^2 \right)^{-n} \| \psi \|_M, \quad (x \in \mathbb{R}^n, \quad t \geq t_0)$$

Proof. — Let $x \in \mathbb{R}^n$, $t \geq t_0$. Since e^{-itH_0} has a representation as an integral operator, we have (using the notation (4) and (5))

$$U^D(t)\psi(x) = (4\pi i t)^{-\frac{n}{2}} e^{i\frac{x^2}{4t}} \int dy e^{-\frac{xy}{2t}} e^{i\frac{y^2}{4t}} \psi_c(y, t)$$

Now we use the identity

$$(2it)^{-\frac{n}{2}}\eta(x,t)\hat{\psi}\left(\frac{x}{2t}\right) = (4\pi it)^{-\frac{n}{2}} e^{i\frac{x^2}{4t}} \int dy e^{-\frac{xy}{2t}} \psi_c(y,t)$$

to obtain (see [3, 11])

$$\text{(12)} \qquad R_\psi(x,t) = (4\pi it)^{-\frac{n}{2}} e^{i\frac{x^2}{4t}} \int dy e^{-i\frac{xy}{2t}} (e^{i\frac{y^2}{4t}} - 1)\psi_c(y,t)$$

since $\left| e^{i\frac{y^2}{4t}} - 1 \right| \le \frac{y^2}{4t}$ it is easy to see that

$$\text{(13)} \qquad | R_\psi(x,t) | \le c_7 t^{-\left(\frac{n}{2}+1\right)} \| \psi \|_M, \qquad c_7 > 0 \text{ suitable}.$$

Furthermore by integration by parts

$$\left| \left(\frac{x}{t}\right)^{2n} R_\psi(x,t) \right| \le c_8 t^{-\frac{n}{2}} \left| \int dy e^{-i\frac{xy}{2t}} (-\Delta_y)^n \{ (e^{-\frac{y^2}{4t}} - 1)\psi_c(y,t) \} \right|$$

$$\le c_9 t^{-\frac{n}{2}} \int dy \left| \frac{e^{-i\frac{xy}{2t}}}{(1+y^2)^{3n}} \right| \sum_{|l|,|\tilde{l}| \le 2n} \left| D_y^{\tilde{l}}(e^{i\frac{y^2}{4t}} - 1) \right| \left| (1+y^2)^{3n} D_y^l \psi_c(y,t) \right|$$

$$\le c_9 t^{-\left(\frac{n}{2}+1\right)} \left(\max_{|\tilde{l}| \le 2n} \int dy \frac{D_y^{\tilde{l}}(e^{i\frac{y^2}{4t}} - 1)t}{(1+y^2)^{3n}} \right) \max_{|l| \le 2n} \sup_{y \in \mathbb{R}^n} \left| (1+y^2)^{3n} D_y^l \psi_c(y,t) \right|.$$

Since the integral in this last expression is bounded we get using Lemma 1

$$\text{(14)} \qquad \left| \left(\frac{x}{t}\right)^{2n} R_\psi(x,t) \right| \le c_{10} t^{-\left(\frac{n}{2}+1\right)} | \ln t |^{6n} \| \psi \|_M .$$

Combining (13) and (14) yields

$$\left| \left(1 + \left(\frac{x}{t}\right)^2\right)^n R_\psi(x,t) \right| \le c t^{-\left(\frac{n}{2}+1\right)} | \ln t |^{6n} \| \psi \|_M, \quad c > 0 \text{ suitable}.$$

which implies Lemma 2. ■

LEMMA 3. — For $\psi \in M$ and a suitable $c > 0$

$$\text{(15)} \qquad \| U_D(t)\psi \|_{L_n^2} \le c t^{-\frac{n}{2}} | \ln t |^{4n} \| \psi \|_M, \qquad t \ge t_0$$

Proof. — Because the Hilbert-Schmidt norm of

$\| (1 + x^2)^{-n} e^{-itH_0} (1 + x^2)^{-n} \|$ is bounded by $t^{-\frac{n}{2}} c_{11}$

for suitable c_{11}, we get

$$\begin{aligned}
&\| (1 + x^2)^{-n} \mathrm{U_D}(t)\psi \| \\
&\qquad \leq \| (1 + x^2)^{-n} e^{-it\mathrm{H}_0}(1 + x^2)^{-n} \| \; \| (1 + x^2)^n \psi_c(x, t) \| \\
&\qquad \leq c_{11} t^{-\frac{n}{2}} \left(\int dx (1 + x^2)^{-2n} \right)^{\frac{1}{2}} \sup_{x \in \mathbb{R}^n} | (1 + x^2)^{2n} \psi_c(x, t) | \\
&\qquad \leq c t^{-\frac{n}{2}} | \ln t |^{4n} \| \psi \|_{\mathrm{M}} .
\end{aligned}$$

The last inequality follows by a calculation similar to that in the proof of Lemma 1. ■

Finally we prove the Theorem.

Proof (of the Theorem). —Let $\varphi \in \mathrm{N}$, denote $\psi := (\Omega_{\mathrm{D}}^-)^{-1}\varphi$ thus $\psi \in \mathrm{M}$. Consider

$$(16) \quad \left\{ \begin{aligned}
&\| e^{-it\mathrm{H}}\varphi \|_{\mathrm{L}^2} \leqslant \| (1+x^2)^{-n}(e^{-it\mathrm{H}}\varphi - \mathrm{U_D}(t)\psi) \| + \| (1+x^2)^{-n} \mathrm{U_D}(t)\psi \| \\
&\leq \| (1+x^2)^{-n} e^{-it\mathrm{H}} \| \; \| \varphi - e^{it\mathrm{H}} \mathrm{U_D}(t)\psi \| + \| (1+x^2)^{-n} \mathrm{U_D}(t)\psi \|
\end{aligned} \right.$$

Now denote

$$h(t) := e^{it\mathrm{H}} \mathrm{U_D}(t)\psi .$$

Then $h(\infty) = \varphi$ and we have

$$h'(t) = i e^{it\mathrm{H}}(\mathrm{H} - \mathrm{H}_0 - g(2\,|\,p\,|\,t)^{-1}) \mathrm{U_D}(t)\psi$$

Therefore

$$\| h'(t) \| \leq | g | \; \| [\,| x |^{-1} - (2\,|\,p\,|\,t)^{-1}] \mathrm{U_D}(t)\psi \|$$

Denote $\psi_1 := (\,|\,k\,|^{-1}\hat{\psi})\check{}$.

Then we have

$$| x |^{-1} \hat{\psi}\left(\frac{x}{2t}\right) = (2t)^{-1} \hat{\psi}_1\left(\frac{x}{2t}\right) \qquad \text{(see [11])}.$$

Now we use the notation (9), (10) and Lemma 2 and get for $t \geq t_0$

$$\begin{aligned}
&\| h'(t) \| \leq | g | \; \| \, | x |^{-1} \mathrm{R}_\psi(x, t) \| + | g | \; \| (2t)^{-1} \mathrm{R}_{\psi_1}(x, t) \| \\
&\qquad \leq c_{12} t^{-\left(\frac{n}{2}+1\right)} | \ln t |^{6n} t^{\frac{n-2}{2}} \left(\int \frac{u^{n-1} du}{u^2(1 + u^2)^{2n}} \right)^{\frac{1}{2}} \| \psi \|_{\mathrm{M}} \\
&\qquad + c_{13} t^{-\left(\frac{n}{2}+2\right)} | \ln t |^{6n} t^{\frac{n}{2}} \left(\int \frac{u^{n-1} du}{(1+u^2)^{2n}} \right)^{\frac{1}{2}} \| \psi \|_{\mathrm{M}} .
\end{aligned}$$

Thus we get

$$(17) \qquad \| h'(t) \| \leq c_{14} t^{-2} | \ln t |^{6n} \| \psi \|_{\mathrm{M}} .$$

By (16), (17) and Lemma 3 we have

$$\begin{aligned}\| e^{-itH}\rho \|_{L^2_n} &\leq \| h(\infty) - h(t) \| + \| (1 + x^2)^{-n} U_D(t)\psi \| \\ &\leq \int_t^{\infty} \| h'(s) \| ds + \| (1 + x^2)^{-n} U_D(t)\psi \| \\ &\leq c_{15} \left\{ \int_t^{\infty} s^{-2} | \ln s |^{6n} ds + t^{-\frac{n}{2}} | \ln t |^{4n} \right\} \| \psi \|_M \\ &\leq ct^{-1} | \ln t |^{6n} \| \psi \|_M\end{aligned}$$

Since $\| \psi \|_M = \| \varphi \|_N$ we have shown (3). ■

Remark. — Note that an estimate similar to (3) holds if one replaces t by $-t$ for $t \leq -t_0$ provided one replaces Ω_0^- by Ω_D^+ in the proofs.

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