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by

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RÉSUMÉ. — Le traitement semi-classique de l'équation de Schrö-
dinger à une dimension est rendu libre de toute approximation. En effet,
pour un potentiel analytique, la méthode BKW en paramètres complexes
se formalise de manière à inclure par resommation exacte toutes les contri-
butions au développement semi-classique : les termes asymptotiques ainsi
que les exponentiellement petits dans la constante de Planck. Pour ce faire,
on se ramène par transformation de Borel au problème de la propagation
des singularités d'une fonction analytique sur une surface de Riemann,
lequel se résout par l'équation de Hamilton-Jacobi complexe avec condi-
tions aux limites dépendant des détails du problème (méthode de Balian-
Bloch). En particulier le phénomène de Stokes est induit par le rebrousse-
ment des solutions de Hamilton-Jacobi aux points tournants (un cas spécial
de la théorie de Picard-Lefschetz); il en résulte des formules exactes de
raccordement pour les solutions semi-classiques autour des points tour-
nants, d'où découlent toutes les applications. Le problème du spectre de
l'oscillateur quartique vient alors illustrer toutes les étapes de la méthode
ainsi que la typologie des résultats à espérer. Ceux-ci ne sont pas assez
explicites pour vraiment résoudre l'équation de Schrödinger, mais four-
nissent la structure analytique globale de certaines fonctions spectrales
sous une forme récursive du type résurgent d'Ecalle, dans une géométrie
qui est, dans un problème spectral, contrôlée par l'ensemble de toutes
les orbites classiques fermées, réelles ou complexes. Cette structure induit

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des équations fonctionnelles pour les déterminants de Fredholm, qui
généralisent la formule des compléments pour la fonction Gamma d'Euler
et qui évoquent certains résultats de Sibuya-Cameron. Pour les poten-
tiels homogènes (dont le pur quartique), le produit final actuel est une
infinité de règles de somme numériquement vérifiables portant sur les
valeurs propres. Divers développements ultérieurs de la méthode sont
discutés comme potentiellement concevables.

ABSTRACT. — The semi-classical treatment of the one-dimensional
Schrödinger equation is made free from all approximation. For an analytic
potential indeed, the WKB method in complex parameters can be forma-
lized so as to include by exact resummation all contributions to the semi-
classical expansion: the asymptotic as well as the exponentially small
terms in Planck's constant. This is achieved by reduction, via a Borel
transformation, to the propagation problem for the singularities of an
analytic function on a Riemann surface, which is solved by the complex
Hamilton-Jacobi equation with boundary conditions depending on
the details of the problem (Balian-Bloch method). In particular the Stokes
phenomenon is induced by the cuspidal singularity of the Hamilton-Jacobi
solution at turning points (a special case of Picard-Lefschetz theory);
exact matching formulas for semi-classical solutions throughout the com-
plex plane are thus obtained for all subsequent applications. The problem
of the quartic oscillator spectrum their comes to illustrate all the steps of
the method as well as the general features of the results to be expected.
Those are not explicit enough to actually solve the Schrödinger equa-
tion, but consist in the global analytic structures of certain spectral functions
in a recursive resurgent form of the Ecalle type, and whose geometry is,
in a spectral problem, governed by the set of all the closed classical orbits
(real or complex). Those structures induce functional equations for the
Fredholm determinants, that generalize the reflection formula for the Euler
Gamma function and also remind of certain Sibuya-Cameron results.
For the pure quartic and other homogeneous potentials, the present
ultimate output is an infinite set of eigenvalue sum rules, numerically
verifiable. Various further developments of the method are discussed as
conceivable.

As indicated by its title, this work serves a dual purpose: on one hand,
extend and strengthen recent non perturbative results about a special
one-dimensional Schrödinger operator, the homogeneous quartic oscil-
lator [1]-[3]: $-d^2/dq^2 + q^4$; and on the other hand, clarify the theory

of semiclassical expansions (WKB or Liouville-Green method [4]-[7]) in one complex coordinate. The two ideas develop in parallel because our results on the quartic oscillator are of a semiclassical nature, and in turn the quartic oscillator is among the simplest nontrivial models on which to probe semiclassical methods that ultimately apply in full generality (in the same way as the harmonic + quartic oscillator was used to study perturbation theory).

The interest of the WKB method for applications is that is goes far beyond perturbation theory. Although it generates asymptotic expansions too (in powers of Planck's constant $\hbar$), it does so in a more powerful way because it is a nonlinear « dynamical approximation » which fully accounts for some global qualitative features of the interaction (like its degree if it is a polynomial): this combination of generality and efficiency of the semi-classical approach is responsible for its many successes and justifies all efforts to improve it.

This work specifically describes the complex one-dimensional WKB method in the Balian-Bloch representation of quantum mechanics [11]. We shall demonstrate that this choice, apart from being natural and convenient, provides above all an exact (neither asymptotic nor approximate) interpretation of the semi-classical treatment, as an algorithm aimed at describing the analytic structure of the Borel transforms with respect to $\hbar^{-1}$ of various dynamically relevant functions. The WKB procedure establishes that these Borel transforms are multiply-valued (ramified) analytic functions, that their branch points are distributed according to the complex periods of the classical motion, and that each discontinuity is a nonlinear function of the Borel transform itself suitably translated (« analytic bootstrap »). Details will be mostly shown upon the quartic oscillator, but all the rules of the game will apply in principle to any polynomial, and perhaps analytic, potential (the general Sturm-Liouville problem). Throughout this work, emphasis will be put on the operational description of the method at the expense of axiomatic rigor: computer algebra was more relevant indeed than classical analysis in our search for explicit results. Thus, some mathematical hypotheses, all validated by their numerical implications, will be left unproved (we would call this approach *physical mathematics*, i. e. application of physical ideas to mathematical problems).

Our general plan is the following. Preliminary sections 1 and 2 focus upon the stationary phase expansion as a model case to illustrate the resummation of divergent series and the problem of subdominant terms therein ([2], § 1). A Borel transformation relates the subject to the convolution properties of ramified analytic functions, discussed in section 3. The paper really begins with section 4 (formulation of the Schrödinger eigenvalue problem and review of the standard WKB treatment) and develops fully in sections 5-6, where WKB theory is formalized exactly with the help

of the Balian-Bloch method and of Dingle's [4] ideas, especially to the effect of clarifying the general connection problem: sections 4-6 might thus be of separate interest to practitioners of the WKB method. The central sections 7 and 8 illustrate the theory by concrete applications, actually the models from which the general procedure was inferred: especially the quartic oscillator (section 8). Section 9 sketches an alternate, more abstract description of the WKB method in the language of mono-dromy theory and Riemann-Hilbert problems: this shows how the WKB procedure works for arbitrary potentials to generate the same type of « analytic bootstrap » features as observed in the quartic case, and remar-kably coinciding with some « resurgent » structures of ramified functions observed and formalized by Ecalle [32] in seemingly unrelated mathema-tical contexts. The final section 10 was added in an afterthought once we found that the preceding analyticity results had some quite concrete consequences: functional equations for the Fredholm determinants [34] that imply, for homogeneous potentials q^{2M}, arithmetical identities on the spectrum that generalize ancient formulas for the Riemann Zeta function in the harmonic oscillator case; those functional equations have some relation to earlier work by Sibuya-Cameron [40]. Finally four appendices present intermediate results of separate interest, Appendix C being partly joint work with D. Chudnovsky and G. Chudnovsky (unpublished).

Our theory is also presented in a summarized form [41] readable as an introduction; the results of section 10 were separately announced some time ago [34] to compensate for our slowness in preparing this final draft.

We finally stress what we believe to be the salient features of our results for applications. The practical user might be deterred by the complexity of the formalism; this to some extent was forced upon us once we chose to clarify familiar existing WKB theory in its established language rather than bluntly state the final method, which is simple enough by itself but not motivating. We therefore emphasize our belief that, however abstract and marginal our improvements upon WKB theory may appear, they are liable to concern even casual applications of the method. Because on the one hand they make it clear that the usually neglected divergence of the full semi-classical series is the obstacle to the unequivocal and consistent understanding of subdominant phenomena like tunneling; and on the other hand they show that the WKB framework provides by itself full control over that problem via Borel transformation (whereas this consti-tues a major difficulty in linear perturbation theory). All that should still increase our confidence in the known as well as in the yet hidden resources of the semi-classical approach.

Note : we have only listed a few references that we encountered as rele-vant during our work. More complete and impartial bibliographies can be gathered from [4]-[6], [11] [13] [20]) (*for instance*).

1. A TALE OF SUBDOMINANCE

A very simple and common case of subdominance is found in the asymptotic expansion of an integral of the form:

$$F(x) = \int_\Gamma e^{-xs(u)}du \qquad (1.1)$$

when the parameter $x \to +\infty$; Γ is a complex path without endpoints and s is an analytic function such that the integral converges. Under the further simplifying assumptions $\left(\text{here } ' \text{ denotes } \dfrac{d}{du}\right)$:

— the zeros $u_j \in \mathbb{C}$ of s' (critical or saddle points) are isolated and non-degenerate $(s''(u_j) \neq 0 \; \forall j)$,

— the critical values $s_j = s(u_j)$ are all distinct, and one of them, say s_0, satisfies:

$$\text{Re } s_0 < \text{Re } s_j \qquad \forall j \neq 0, \qquad (1.2)$$

then the *saddle-point method* yields for F the expansion:

$$F(x) \sim e^{-s_0 x}\left(\sum_{k=0}^{\infty} F^{(0)}_{k+1/2}x^{-k-1/2}\right) = e^{-s_0 x}F^{(0)}\{x\}. \qquad (1.3)$$

$F^{(0)}\{x\}$ denotes the formal series $\displaystyle\sum_{0}^{\infty} F^{(0)}_{k+1/2}x^{-k-1/2}$, whose coefficients $F^{(0)}_{k+1/2}$ are well-known combinations of the derivatives $s^{(n)}(u_0)$, $2 \leq n \leq 2(k+1)$ ([4], chaps. 5.3 and 6; see also eqs. (2.5-8) below).

If instead of (1.2) we suppose that several s_j, say for $0 \leq j \leq J < \infty$, have the same minimal real part

$$\text{Re } s_0 = \ldots = \text{Re } s_J < \text{Re } s_j \qquad (\forall j > J), \qquad (1.4)$$

then each critical point $u_j (0 \leq j \leq J)$ acts independently of the others, resulting in an expansion :

$$F(x) \sim \sum_{j=0}^{J} e^{-s_j x}F^{(j)}\{x\}, \qquad F^{(j)}\{x\} = \sum_{k=0}^{\infty} F^{(j)}_{k+1/2}x^{-k-1/2} \qquad (1.5)$$

This sort of expression is completely and uniquely defined in the asymptotic sense: each term with given j can be fully separated from the others as $x \to \infty$ because of its different oscillatory factor $e^{-ix\text{Im}s_j}$ (for instance, in the limiting case of (1.1) when $\Gamma = \mathbb{R}$ and the function s is allowed

to become purely imaginary on Γ, an expansion of the form (1.5) is known to result from the stationary phase method).

We now want to ascribe a role in the saddle-point method to the other critical points $u_j (j > J)$. Let us start from the simplest situation $J = 1$ in (1.4); we may put the origin of the s-plane at $s(u_0)$, so that $s_0 = 0$ and $\operatorname{Re} s_1 = 0$:

$$F(x) \sim F^{(0)}\{x\} + e^{-s_1 x} F^{(1)}\{x\} \qquad (1.6)$$

Letting s depend also on some external parameter(s) θ, we now displace s_1 continuously into the half-plane $\operatorname{Re} s_1 > 0$ while keeping $s_0 = 0$. As soon as $\operatorname{Re} s_1 > 0$, it is asymptotically consistent to drop from the expansion (1.6) the term $e^{-s_1 x} F^{(1)}\{x\}$, now exponentially small (*subdominant*) relatively to all terms of the (*dominant*) series $F^{(0)}\{x\}$. This suppression however creates a discontinuity with respect to the external parameter(s), which is undesirable not only aesthetically but also practically, in view of numerical applications. In the classical example of the Bessel function $J_\nu(z)$ [1], the large z expansions consist of two competing terms that are alternatively dominant and subdominant in various sectors of the complex z plane (here $x = |z|$, $\theta = \operatorname{Arg} z$); if we move off the real axis where both terms have equal strength, we feel that by continuity the subdominant term should still be kept in order to reach optimal numerical accuracy from the expansion [1] [4] [5]. But the information possibly contained in the subdominant term can only be exploited if we can assess the precise meaning of this term, i. e. isolate it *unambiguously* from the background created by the dominant series.

We are now at the heart of the problem: it is impossible to disentangle the two dependent expansion scales x^{-1} and $e^{-s_1 x}$ in (1.6) on purely asymptotic grounds. The only way would be to substract from the function $F(x)$ « the sum » $F^{(0)}(x)$ of the dominant part, leaving the subdominant term as the asymptotic expansion of the remainder:

$$F(x) - F^{(0)}(x) \sim e^{-s_1 x} F^{(1)}\{x\}. \qquad (1.7)$$

But it is in the nature of the series $F^{(0)}\{x\}$, contributed by the critical point u_0, to *diverge* for all x precisely because there is another critical point u_1 [2] [4]; convergence of $F^{(0)}$ requires u_0 to be the only complex critical point of s, which means $s(u) = a(u - u_0)^2 + c$, the trivial gaussian case.

Therefore, any approach to subdominance *implies* that a principle for assigning a « sum » $F^{(0)}(x)$ to a divergent series like $F^{(0)}\{x\}$ has been selected-among the many conceivable ones compatible with the asymptotic requirement that the function $F^{(0)}(x)$ should expand to $F^{(0)}\{x\}$. The subdominant series is then *defined* by eq. (1.7) (and so on if necessary) and it will depend drastically on the resummation method being used. Conversely, if our problem is to resum a divergent series $F^{(0)}\{x\}$ to a

given function F(x) (as in quantum perturbation theory), subdominant terms indicate a discrepancy or bias induced by the resummation method under consideration [1]. It is thus impossible to dissociate the two topics of subdominance and of series resummation.

We could view this fact simply as a one-way incentive to study subdominance, given the pervasiveness of resummation methods in the quantum physics literature. But in the other way around, if often happens that resummation is only invoked when a perturbative expansion has been pushed far enough to make its divergence numerically visible. Therefore there is room to argue that resummation problems have not received due attention in the study of subdominant phenomena *per se*. We do not refer here to mathematical works [4] [5] mostly concerned with special functions (these admit closed integral representations of the form (1.1), which make their resummation easier: see next section), but rather to the literature about the complex semiclassical methods in quantum mechanics, which is very clearly reviewed in [6], § 1-4 (Ref. [7] describes a more recent and complete state of the subject): items like the Stokes phenomenon, tunneling, connection formulas all involve manipulation of subdominant quantities at some or at all stages, yet the resummation problem is never raised. We said earlier that this problem has to be solved somehow in order to accommodate subdominance, hence it is solved without ever being raised, i. e. some tacit resummation is performed, which may well be inefficient or worse, inconsistent. We claim that such inconsistencies are responsible for the present obscurities and limitations of the complex WKB method, and that they can be removed (see sections 5-6).

2. THE SADDLE-POINT METHOD

We return to our previous function F(x) endowed with an exact representation of the form:

$$F(x) = \int_\Gamma e^{-xs(u)} du \qquad (1.1)$$

This situation is well understood, and a variant of *Borel resummation* is known to be the most natural approach to this simple case [4] [2] [9]. As the correct treatment is not however widely used, we repeat it here in order to emphasize the geometrical and topological problems raised by the saddle-point method.

We start by taking $s = s(u)$ as the integration variable in (1.1):

$$F(x) = \int_C e^{-xs} \rho(s) ds \qquad (2.1)$$

Here $\rho(s) = \dfrac{du}{ds}$ is a multivalued function, which becomes single-valued on the Riemann surface $\mathscr{S}$ of the inverse function $u(s)$; C is the image path $s(\Gamma) \subset \mathscr{S}$; eq. (2.1) is a *Laplace representation* of F(x). The branch points of ρ are the *critical values* s_j.

The first step in the customary treatment is to distort the path Γ in eq.(1.1) to a path of steepest descent. In the s variable, this amounts to pushing the path C on $\mathscr{S}$ as far to the right as permitted by the branch points s_j, so as to benefit maximally from the decrease of the exponential factor in (2.1); by Cauchy's theorem the integral taken along the new contour C' is the same (Fig. 1). We also call C' a (union of) *path(s) of steepest descent*. For convenience we perform an integration by parts at the same time:

$$F(x) = x \int_{C'} e^{-xs} u(s) ds \tag{2.2}$$

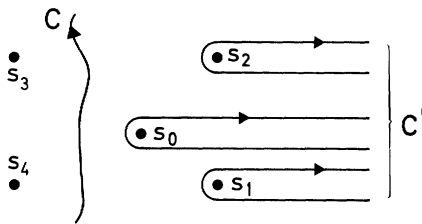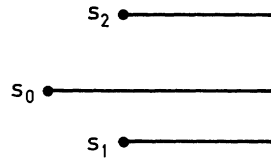
FIG. 1.   FIG. 2.

We admit that in order to focus on typical situations we have skipped some serious mathematical difficulties, caused by: branch points of ρ at finite limiting values of $s(u)$, $|u| \to \infty$; a possibly nontemperate growth of $\rho(s)$, $|s| \to \infty$; the presence of infinitely many critical values. The easiest way to evade those problems is to think of $s(u)$ as a polynomial.

If all Im s_j are *distinct*, eq. (2.2) yields an expression of F(x) as the sum of Laplace transforms (in the ordinary sense) of the *discontinuities* of $u(s)$ across horizontal cuts drawn from *certain* critical values s_j, denoted as *active* (Fig. 2):

$$F(x) = x \sum_{\{active\}} \int_{s_j}^{\infty} e^{-xs} \Delta_{s_j} u(s) ds \tag{2.3}$$

$$\Delta_{s_j} u(s) = u(s + i0) - u(s - i0)|_{\mathrm{Im}\,s = \mathrm{Im}\,s_j}$$

We call (2.3) the *standard* representation of F, and we shall now derive from it the asymptotic expansion of F(x) under the assumption that (1.2)

is satisfied by the active critical values. This makes all terms of (2.3) exponentially small except:

$$F^{(0)}(x) = x \int_{s_0}^{\infty} e^{-xs} \Delta_{s_0} u(s) ds \qquad (2.4)$$

By inverting the Taylor series of $s(u)$ at the critical point u_0, namely

$$s(u) = s_0 + \frac{s''(u_0)}{2}(u - u_0)^2 + \sum_{3}^{\infty} \frac{s^{(k)}(u_0)}{k!}(u - u_0)^k \qquad (2.5)$$

we obtain the local Puiseux expansion of $u(s)$ around s_0, which we choose to write as

$$u(s) = \sum_{n=0}^{\infty} \frac{u_{n/2}^{(0)}(s - s_0)^{n/2}}{2\Gamma(1 + n/2)} = u_0 + \left(\frac{2}{s''(u_0)}\right)^{1/2}(s - s_0)^{1/2} + \dots \qquad (2.6)$$

The coefficients in (2.6) obtained by this algebraic process are universal combinations of the derivatives of s at u_0; eq. (2.6) shows that two sheets of $\mathscr{S}$ connect around s_0. The discontinuity of u on the cut from s_0 only involves the half-integral coefficients from (2.6):

$$\Delta_{s_0} u(s_0 + t) = \sum_{k=0}^{\infty} \frac{u_{k+1/2}^{(0)} t^{k+1/2}}{\Gamma(k + 3/2)}, \qquad t = s - s_0 > 0 \qquad (2.7)$$

Substitution of (2.7) into (2.4) is allowed in the sense of asymptotic expansions for $x \to +\infty$; the result is

$$F^{(0)}(x) \sim e^{-s_0 x}\left(\sum_{k=0}^{\infty} F_{k+1/2}^{(0)} x^{-k-1/2}\right), \qquad F_{k+1/2}^{(0)} = u_{k+1/2}^{(0)} \qquad (2.8)$$

Now we recall a few definitions: given a formal series in descending powers of x with a possible exponential prefactor:

$$F\{x\} = e^{-s_0 x}\sum_{\lambda} F_\lambda x^{-\lambda} \qquad (2.9)$$

we define its *Borel transform* (cf. [4], chap. 21.4) as:

$$F_B(s) = \sum_{\lambda} \frac{F_\lambda}{\Gamma(1 + \lambda)}(s - s_0)^\lambda \qquad (2.10)$$

provided (2.10) has non-zero radius of convergence around s_0. If moreover $F_B(s_0 + t)$ is analytic on the whole positive axis and if the integral

$$F(x) = x \int_0^{\infty} F_B(s_0 + t)e^{-x(s_0 + t)} dt \qquad (2.11)$$

converges for $x > 0$ large enough, we call F(x) the *Borel sum* of the series
(2.9). It is a resummation of the (possibly divergent) series (2.9), because
(2.9) is asymptotic to F(x) for $x \rightarrow + \infty$ (by the same argument that led
us from (2.7) to (2.8)).

Hence the expansion (2.8), which *is* the desired saddle-point expansion
(1.3), has as its Borel transform the discontinuity at s_0 of $u(s)$ by eq. (2.7),
and $F^{(0)}(x)$ defined by (2.4) is the Borel sum of that expansion.

Now the same analysis applies independently to each other term in (2.3),
whose asymptotic series will however be subdominant if condition (1.2)
holds. Eq (2.3) thus defines an unambiguous decomposition of F(x) into
dominant and subdominant terms; the expansion of each term, and con-
versely its Borel resummation, happen *independently* of the other terms
(this extends the well-known statement of independence of the critical
points in the stationary phase method). We must however correct this
statement in several ways:

1) critical points do interact since the situation grows more complicated
if two or more of them are allowed to coalesce; we shall exclude that case
here.

2) The effective construction of the standard form (2.3) conveniently
splits in two parts: we start by computing the local expansion like (2.7)
for $\Delta_{s_j} u$ at each critical value s_j: this is the purely *algebraic* and *local* part
of the procedure; then the steepest descent argument applied to the most
general contour C will result in:

$$F(x) = x \sum_j m_j \int_{s_j}^{\infty} e^{-xs} \Delta_{s_j} u(s) ds \qquad (2.12)$$

with multiplicities $m_j \in \mathbb{Z}$ to be determined; this is a *topological* aspect of
the problem, involving the *global* geometry of $\mathscr{S}$ and of the initial inte-
gration path C in $\mathscr{S}$; given the function $s(u)$ that uniformizes $\mathscr{S}$ and given
the contour Γ in (1.1), this problem is in principle solved by a formal algo-
rithm, at least for polynomial s [8]. In typically simple situations we expect
$|m_j| \leq 1$, we then only have to select the active values ($m_j \neq 0$) and to
fix for these: $m_j = \pm 1$; the latter step involves a labeling of the sheets
of $\mathscr{S}$ around s_j that removes the $s''(u_0)^{-1/2}$ sign ambiguity in (2.6) and (2.7)
(index correction), and we can always fix the signs so that (2.12) becomes
the simplified formula (2.3). But we now see that (2.3) conceals the output
of a nontrivial topological argument in which all critical values cooperate
together as the branch points of $\mathscr{S}$.

Remark. — If the integral (1.1) is multidimensional ($u \in \mathbb{C}^n$, $\dim_{\mathbb{R}} \Gamma = n$), F(x)
also admits a Laplace representation (2.1) with $\rho(s) = (2\pi i)^{-1} \int_{\Gamma} \dfrac{du}{s - s(u)}$,

the branch points of $\mathcal{S}$ are again the critical values and the algebraic part of the method is known, but the topological part is still under investigation [8]. This should make us very cautions towards any argument relying on the saddle-point method in the complex domain ($\mathbb{C}^n$, $n \geq 2$). However, in the limiting case of the stationary phase method ($\Gamma = \mathbb{R}^n$, s imaginary, only dominant terms wanted), the contour deformation can be made infinitesimal and the topological step can be avoided, thereby allowing full construction of the asymptotic series.

3) A third case for interaction of distinct critical values lies in the divergence of the series in (1.3), (2.8), etc: the circle of convergence of the Puiseux series (2.6) is indeed limited by the branch point s_1 nearest to s_0 *among those* lying on one of the two sheets that meet at s_0.

This implies for $n \to \infty$, by Darboux's theorem ([4] [2]):

$$| u_{n/2}^{(0)} | \sim (2\pi)^{-1} \Gamma\left(\frac{n-1}{2}\right) | s_1 - s_0 |^{-(n-1)/2} | u_{1/2}^{(1)} | \qquad (2.13)$$

where $u_{1/2}^{(1)}$ is the leading term in the same discontinuity expansion as (2.7) but around s_1. The rate of divergence of the asymptotic series contributed by s_0 is thus controlled by the singular part of u at another branch point s_1 whose determination involves some knowledge about the global topology of $\mathcal{S}$ (the whole analysis is actually nothing but the saddle-point evaluation of another integral like (1.1)). The same treatment can then be repeated for the Puiseux series at every other s_j.

4) Another case of interaction between critical values is known as the *Stokes phenomenon*. This happens when a critical value s_2 lies on the cut drawn from another active critical value s_0. Then the Laplace integral taken along the path encircling s_0 (Fig. 3) cannot be resolved into independent contributions associated with s_0 and s_2; equivalently, the discontinuity $\Delta_{s_0} u$ has a singularity on the integration axis in (2.3), hence it is not Borel summable. Note that the problem arises only if s_2 lies at the junction of the two sheets around s_0 and not on some other sheet of $\mathcal{S}$.
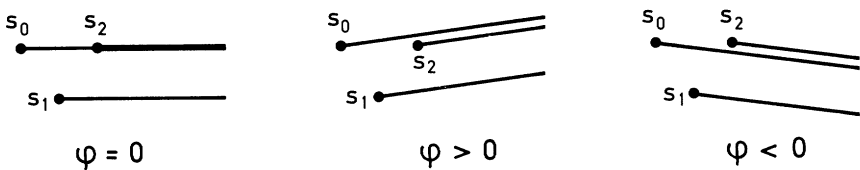

FIG. 3.

We see two ways of understanding this ambiguity dynamically:

a) either we let x become complex; as $\theta = \text{Arg } x$ varies, the cuts are turned by $-\theta$ (they are the directions of steepest descent of e^{-xs}) and the ambiguity

is lifted (it reappears at all discrete values of the form $\theta = \mathrm{Arg}\,(s_j - s_k)$).

  b) Or, if x has to be kept real, we still have no compelling reason to orient the cuts of $\mathscr{S}$ in the direction of steepest descent: cuts are not intrinsic features of $\mathscr{S}$ but only convenient visualizing devices. We are thus free to tilt them by an angle $\varphi\left(\,|\,\varphi\,| < \dfrac{\pi}{2}\right)$ (or to distort them in more general ways) so that they avoid the other s_j.

  Although the two solutions are similar and not mutually exclusive, they offer interesting complementary interpretations:

  a) produces the traditional description of the *Stokes phenomenon*: as $\theta = \mathrm{Arg}\,x$ crosses the forbidden value 0, the active critical value s_2 disappears in the second sheet and becomes inactive (or *vice-versa*); the corresponding subdominant term vanishes (or appears) *discontinuously* precisely when it becomes maximally subdominant as θ varies, i. e. when $|\,e^{-(s_0 - s_2)x}\,|$ is maximal.

  b) Amounts to a change of *resummation prescription* since the Laplace integration paths are modified, and the subdominant terms exhibit their sensitivity to such a change (as explained in the previous section). The ambiguity is lifted with x kept real, but in a prescription-dependent way. If we work with this φ-resummation (where the integration paths make an angle φ with the direction of steepest descent: Fig. 3) and if $\theta = \mathrm{Arg}\,x$ varies again, we see that the Stokes discontinuity for s_2 now occurs at $\theta = -\varphi$: its location is entirely governed by the resummation prescription. The only advantage of the steepest descent ($\varphi = 0$) choice is that for given s_j and $|\,x\,|$, the relative order of magnitude of the discontinuity viewed as a numerical nuisance is smallest (being $|\,e^{-(s_2 - s_0)\,x\,e^{-i\varphi}}\,|$ for general φ).

  In conclusion, we see that all results of this section are by-products of the analytical and topological structure of the Riemann surface $\mathscr{S}$: all relevant information is contained (coded) in that structure, whose description becomes our primary goal. Another helpful notion, which will stay valid elsewhere, is « analytic continuation of algebra »: thanks to the factorization of the problem into an algebraic and a topological part, the algebraic form of the subdominant terms is to all orders the same as if they were continued from a dominant position, once their topological weight has been properly ascribed (this can be done at the level of leading powers, for instance). In other words, everything looks (locally in the external parameters) as though distinct saddle-points did not interfere, similarly to the usual stationary phase situation, and the computation of subdominant terms is not contaminated at all by the presence of dominant quantities: this is contrary to asymptotic intuition and greatly eases the manipulation of subdominant terms. This formal analyticity in the external parameters

must however break down somewhere in order to achieve global consistency: the Stokes discontinuities can be somewhat displaced but not suppressed altogether.

A final remark will help the transition to next section: since all our troubles arose from the reduction of (2.1) to the standard form (2.3), we might think of skipping that step altogether. But the saddle-point method is an untypical, and thereby easier, asymptotic problem in one respect: in most cases as in section 5, the quantity $F(x)$ to be expanded is not known in any closed analytical form like (1.1), and neither is $u(s)$ in (2.2). Asymptotics provides us at best with a knowledge of the *singularities* of u, i. e. of u modulo analytic functions, and a standard form like (2.3) is a necessary step towards an explicit reconstruction (resummation) of F in terms of the singular part of u alone.

3. ANALYTIC LAPLACE TRANSFORMS
OF RAMIFIED FUNCTIONS

We have grouped here some mathematical tools and notations motivated by the contents of the previous section.

We let f denote an analytic ramified (= multivalued) function f, also viewed as a single-valued function defined on its Riemann surface $\mathscr{S}$. We call $\tilde{\mathscr{C}}$ the vector space of all such functions satisfying:

a) f has no natural boundary: $\mathscr{S}$ covers the whole complex plane;

b) f has (at most countably many) *isolated* branch points;

c) each branch point s_0 is of algebraic or logarithmic type
(a pole at s_0 is also allowed in order to make the space $\tilde{\mathscr{C}}$ stable under differentiations);

d) f satisfies a *temperate growth condition* when $|s| \to \infty$ in any sheet of $\mathscr{S}$ (on a path which keeps at some distance $d > 0$ from all branch points in the sheet, and on which Arg s remains bounded): a precise form of the condition could be

or
$$\left. \begin{array}{ll} |d^k f(s)/ds^k| < C e^{a|s|} & \forall k \in \mathbb{N}, \quad \forall a, \quad C > 0 \\ |d^k f(s)/ds^k| < C_k (1 + |s|)^{m_k} & \forall k \in \mathbb{N} \end{array} \right\} \quad (3.1)$$

but either weaker or stronger growth conditions are equally conceivable.

With such a function space $\tilde{\mathscr{C}}$ we hold a candidate to provide a rigorous setting for our subsequent Laplace calculus, but some properties of $\tilde{\mathscr{C}}$ (see sections 5-6) will remain *conjectural* in this work. Our primary need is that the space $\tilde{\mathscr{C}}$ should satisfy *those* properties, rather than comply rigidly with *a priori* requirements. The optimal definition of $\tilde{\mathscr{C}}$ (together

with a topology) is left here as an open question (see however the last paragraph of section 8).

We also introduce the subspaces $\widetilde{\mathscr{C}}_2$ (resp. $\widetilde{\mathscr{C}}_\infty$) of functions $f \in \widetilde{\mathscr{C}}$ with all their branch points of order 2 (resp. of logarithmic type). For instance, if $s(u)$ is a polynomial having only quadratic critical points, then the inverse function $u(s)$ used in section 2 and all its derivatives belong to the space $\widetilde{\mathscr{C}}_2$; whereas the space $\widetilde{\mathscr{C}}_\infty$ will occur naturally in WKB theory (sections 5-6).

As we must later allow countably many branch points to accumulate at infinity, the Riemann surface $\mathscr{S}$ cannot be asked to be regular there, contrary to conventional ones that cover the whole Riemann sphere $\overline{\mathbb{C}}$.

Definition of the analytic Laplace transforms of $f \in \widetilde{\mathscr{C}}$;

Whenever $s_1 \in \mathscr{S}$ is a regular point or carries an integrable singularity of f, and the half-line $\{ \operatorname{Arg}(s - s_1) = \varphi \} \subset \mathscr{S}$ meets no other singularity of f, we pose:

$$L_{s_1}^\varphi f(x) = x \int_{s_1}^{e^{i\varphi}\infty} f(s)e^{-xs}ds. \tag{3.2}$$

This is an analytic function in the half-plane $\operatorname{Re}(e^{i\varphi}x) > 0$ thanks to the growth condition (3.1). Here the angle φ is defined mod 2π if s_1 is a regular point, mod 4π if s_1 is a square root branch point, and φ is a real number if s_1 is a logarithmic branch point: in all cases the integration path, a *straight half-line*, lives on the Riemann surface $\mathscr{S}$. The normalizing factor x in eq. (3.2) is not required but it will be convenient for us.

We can immediately extend def. (3.2) to the case where f has a non-integrable singularity at s_1, since the formula:

$$\int_{s_1}^{e^{i\varphi}\infty} \frac{(s - s_1)^\alpha}{\Gamma(1 + \alpha)} e^{-xs}ds = e^{-s_1 x}x^{-\alpha-1} \qquad (\alpha \neq -1, -2, \ldots) \tag{3.3}$$

generalizes to negative integers α by interpreting $\dfrac{(s - s_1)^\alpha}{\Gamma(1 + \alpha)}$ as the distribution $\delta^{(-\alpha-1)}(e^{-i\varphi}(s - s_1))$ on the straight line $\{ s_1 + e^{i\varphi}\mathbb{R} \}$ (as in standard operational calculus [10]).

Straightforward properties:

$$x \int_{s_1}^{e^{i\varphi}\infty} f(s - s_0)e^{-xs}ds = e^{-xs_0}L_{s_1-s_0}^\varphi f(x) \tag{3.4}$$

$$x \int_{s_1}^{e^{i\varphi}\infty} f(e^{-i\varphi_0}s)e^{-xs}ds = [L_{s_1 e^{i\varphi_0}}^{\varphi+\varphi_0}f](e^{i\varphi_0}x) \tag{3.5}$$

$$\int_{s_1}^{e^{i\varphi}\infty} \frac{d}{ds}[\theta(e^{-i\varphi}(s - s_1))f(s)]e^{-xs}ds = L_{s_1}^\varphi f(x) \tag{3.6}$$

where the derivative is taken in the sense of distributions on the line $e^{i\varphi}\mathbb{R}$ (θ is the Heaviside step function).

Relation with Borel transformation (cf. eqs. (2.9-11)); for all φ, the function $L^{\varphi}_{s_1} f$ admits for $\operatorname{Re}(e^{i\varphi}x) \to +\infty$ the asymptotic expansion

$$L^{\varphi}_{s_1} f(x) \sim e^{-s_1 x}\left(\sum_\lambda F_\lambda x^{-\lambda}\right) \tag{3.7}$$

specified by the relation:

$$(\Sigma F_\lambda x^{-\lambda})_{\mathrm{B}}(t) \equiv f(s_1 + t) \tag{3.8}$$

Uniqueness: any decomposition of a function F:

$$F(x) = \sum_{j=0}^{\mathrm{K}} L^{\varphi}_{s_j} f_j \qquad (\mathrm{K} < \infty)$$

with prescribed φ is unique, provided all integration paths are disjoint and each $f_j \in \widetilde{\mathscr{C}}$ is analytic on $\{\operatorname{Arg}(s - s_j) = \varphi\}$. For instance the standard form (2.3) of the function (1.1) is unique (if a finite sum).

Proof. — We may take $\varphi = 0$. We then isolate the s_j with minimal real part as $s_0, \ldots, s_{\mathrm{J}}$ (cf. (1.4)). By eq. (3.7), F has an asymptotic expansion of the form (1.5) such that $f_j = F^{(j)}_{\mathrm{B}}$ for $0 \leq j \leq \mathrm{J}$; but an expansion like (1.5) is unique hence s_j and f_j for $0 \leq j \leq \mathrm{J}$ are uniquely determined. We then repeat the argument for the function $\left(F - \sum_0^{\mathrm{J}} L^0_{s_j} f_j\right)$ and so on until all terms are exhausted (barring technical details, this uniqueness property should extend to many infinite decompositions too).

Discontinuity formulas: we shall extensively use the variation properties of $L^{\varphi}_{s_1} f$ as φ varies in the allowed sector $\left\{\left|\varphi + \operatorname{Arg} x\right| < \dfrac{\pi}{2}\right\}$. Clearly $L^{\varphi}_{s_1} f$ is independent of φ in every sector of $\mathscr{S}$ containing no other branch point of f. But whenever the half-line $\{\operatorname{Arg}(s - s_1) = \varphi\}$ contains one branch point s_2:

$$L^{\varphi+0}_{s_1} f = L^{\varphi-0}_{s_1} f + L^{\varphi}_{s_2}(\Delta_{s_2} f) \tag{3.9}$$

where $\Delta_{s_2} f$ is the discontinuity of f across the half-line $\{\operatorname{Arg}(s - s_2) = \varphi\}$:

$$\Delta_{s_2} f(s) = f(se^{i0}) - f(se^{-i0}) \tag{3.10}$$

Apart from a finite sum of derivatives of the Dirac measure at s_2 (produced by the pole terms: $\Delta_{s_2}(s - s_2)^{-n} = \dfrac{2\pi i(-1)^n}{(n-1)!} \delta^{(n-1)}(e^{-i\varphi}(s - s_2))$, the rest of the discontinuity is analytic in s and belongs to the same space $\widetilde{\mathscr{C}}$ (or $\widetilde{\mathscr{C}}_2$, $\widetilde{\mathscr{C}}_\infty$) as f.

If we allow several branch points $s_2, s_3, s_4 \ldots$ (at increasing distances)

on the same half-line $\{\,\mathrm{Arg}\,(s - s_1) = \varphi\,\}$, formula (3.9) is ambiguous unless all integration paths are tilted one way or another to become *disjoint*. Typically:

$$L_{s_1}^{\varphi + 0} f - L_{s_1}^{\varphi - 0} f = L_{s_2}^{\varphi_2}(\Delta_{s_2} f) + L_{s_3}^{\varphi_3}(\Delta_{s_3} f) + \ldots \qquad (3.11)$$

where each φ_j is assigned the value $\varphi \pm 0$ in increasing order of the index j. The proof of formulas (3.9-11) is obvious: the difference $L_{s_1}^{\varphi + 0} f - L_{s_1}^{\varphi - 0} f$ is a Laplace integral of f on a path Γ without end points, enclosing s_2, $s_3, \ldots$; when Γ is collapsed onto a system of *disjoint* cuts from $s_2, s_3 \ldots$ to infinity, each cut $\{\,\mathrm{Arg}\,(s - s_j) = \varphi_j\,\}$, with due reference to the sheet of φ on which it is positioned by the previous assignments of φ_k $(k < j)$, contributes precisely $L_{s_j}^{\varphi_j}(\Delta_{s_j} f)$ thanks to the Cauchy theorem (Fig. 4); the contribution from any pole at s_j is correctly evaluated by (3.3). The convenient choice of all $\varphi_j = \varphi + 0$ (resp. $\varphi - 0$) will be referred to as the e^{+i0} (resp. e^{-i0}) convention.
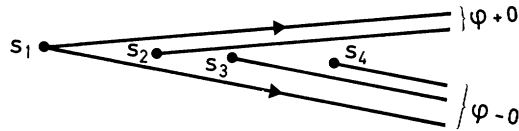

FIG. 4.

Analytic convolution: if $f_1, f_2 \in \mathscr{C}$ and are moreover holomorphic in some disk $|z| < R$, we define their analytic convolution as:

$$(f_1 \ast f_2)(z) = \frac{d}{dz} \int_0^z f_1(z - y) f_2(y) dy \qquad (3.12)$$

the initial integration path being the straight segment from 0 to z. Trivially $f_1 \ast f_2$ is holomorphic in $|z| < R$, continues analytically to a temperate function in the whole plane deprived of radial cuts issued from the singularities s_j of f_1 or f_2, and is designed to satisfy for every other direction φ:

$$L_0^\varphi(f_1 \ast f_2) = L_0^\varphi f_1 L_0^\varphi f_2 \qquad (3.13)$$

If φ is the argument of just one branch point s_1, say of f_1, we find by using the discontinuity formula (3.9) in both ways (a dot denotes the dummy variable in a function):

$$
\begin{aligned}
L_0^{\varphi + 0}(f_1 \ast f_2) - L_0^{\varphi - 0}(f_1 \ast f_2) &= L_{s_1}^\varphi(\Delta_{s_1} f_1) L_0^\varphi f_2 \\
&= e^{-x s_1} L_0^\varphi(\Delta_{s_1} f_1(s_1 + .)) L_0^\varphi f_2 \\
&= e^{-x s_1} L_0^\varphi(\Delta_{s_1} f_1(s_1 + .) \ast f_2) \\
&= L_{s_1}^\varphi[(\Delta_{s_1} f_1(s_1 + .) \ast f_2)(- s_1 + .)] \\
\Rightarrow \Delta_{s_1}(f_1 \ast f_2)(s) &= (\Delta_{s_1} f_1(s_1 + .) \ast f_2)(s - s_1)
\end{aligned}
$$

$$(3.14)$$

i. e. $f_1 \overset{*}{\ast} f_2$ has at s_1 a discontinuity equal to the discontinuity of f_1 translated to the origin, convoluted with f_2 and translated back to s_1.

If several branch points have the same argument it is convenient to work with either one of the conventions e^{+i0} or e^{-i0} throughout.

For instance assume s_1 is a branch point for f_1 and s_2 for f_2 with

$$\operatorname{Arg} s_1 = \operatorname{Arg} s_2 = \varphi,$$

and write:

$$L_0^{\varphi+0}(f_1 \overset{*}{\ast} f_2) = L_0^{\varphi+0} f_1 L_0^{\varphi+0} f_2 = (L_0^{\varphi-0} f_1 + L_{s_1}^{\varphi-0} \Delta_{s_1} f_1)(L_0^{\varphi-0} f_2 + L_{s_2}^{\varphi-0} \Delta_{s_2} f_2)$$

$$L_0^{\varphi+0}(f_1 \overset{*}{\ast} f_2) - L_0^{\varphi-0}(f_1 \overset{*}{\ast} f_2) = L_{s_1}^{\varphi-0}(\Delta_{s_1} f_1) L_0^{\varphi-0} f_2 + L_0^{\varphi-0} f_1 L_{s_2}^{\varphi-0}(\Delta_{s_2} f_2)$$

$$+ L_{s_1}^{\varphi-0}(\Delta_{s_1} f_1) L_{s_2}^{\varphi-0}(\Delta_{s_2} f_2) \quad (3.15)$$

The first term is the same as before, the second one is obtained by exchanging f_1 and f_2 and produces a discontinuity at s_2, but there is also a third term at $(s_1 + s_2)$, which becomes the convolution product of the discontinuities of the factors once everything has been translated to the origin; all three terms are visualized in the e^{-i0} convention.

Since convolution preserves the growth condition (3.1), it is possible to imagine a recursive argument based on eqs. (3.15) to prove that $(f_1 \overset{*}{\ast} f_2)$ satisfies (3.1) in all its sheets, that its set of branch points is the union of the two branch point sets and of their pointwise sum, thus $f_1, f_2 \in \widetilde{\mathscr{C}}$ would imply $f_1 \overset{*}{\ast} f_2 \in \widetilde{\mathscr{C}}$; however we have not worked that point out in detail, because we study this formalism not so much for itself as for its computational convenience in later applications. We therefore prefer to list a few more practical formulas derived from (3.13), such as:

$$L_0^{\varphi}(f \overset{*}{\ast}{}^n) = (L_0^{\varphi} f)^n \quad (3.16)$$

where $f \overset{*}{\ast}{}^n$ denotes the n-th convolution power of f. Also, if $f(0) = 0$ and if $g(Z)$ is an analytic function about $Z = 0$, then repeated application of (3.16) on the Taylor series of $g(L_0^{\varphi} f)$, wherever it converges, leads to:

$$L_0^{\varphi}(g \overset{*}{\ast}(f)) = g(L_0^{\varphi} f) \quad (3.17)$$

where the mapping $g \overset{*}{\ast}$ denotes the image of g under Borel transformation. If emphasis is shifted uppon the formal expansion $F(x)$ of which $f \in \widetilde{\mathscr{C}}$ is the Borel transform ($f = F_B$), the latter formulas read:

$$(\forall \varphi): \qquad L_0^{\varphi}((F^n)_B) = (L_0^{\varphi} F_B)^n, \quad L_0^{\varphi}[g(F)]_B = g(L_0^{\varphi} F_B); \quad (3.18)$$

they mean that φ-resummation of formal power series commutes with algebraic and functional operations, or that φ-resummation *automatically* turns formal relations (involving such operations) into *exact* identities, provided only that the Borel transforms have non-zero radii of convergence and reasonable global analytic properties (e. g. belong to the space $\widetilde{\mathscr{C}}$).

Here is now a discontinuity formula generated by (3.16) for instance, when s_1 is the unique branch point of f at given argument φ:

$$(\mathrm{L}_0^{\varphi+0} - \mathrm{L}_0^{\varphi-0})(f^{\dot{*}n}) = \sum_{k=1}^{n} \binom{k}{n}(\mathrm{L}_{s_1}^{\varphi-0}\Delta_{s_1}f)^k (\mathrm{L}_0^{\varphi-0}f)^{n-k} \qquad (3.19)$$

which means, under the e^{-i0} convention ($\binom{k}{n}$ is the binomial coefficient):

$$(\Delta_{ks_1}f^{\dot{*}n})(s) = \binom{k}{n}[(\Delta_{s_1}f(s_1 + .))^{\dot{*}k} \dot{*} f^{\dot{*}(n-k)}](s - ks_1) \qquad (1 \le k \le n) \quad (3.20)$$

Discontinuity formulas like (3.14-3.15), (3.19-3.20) and their obvious generalizations will be constantly invoked in sections 7-8 in order to locate and evaluate the analytic singularities of given functions $f(s)$, precisely through the computation of $\mathrm{L}_0^{\varphi}f$ for all φ and the subsequent detection of all (exponentially small) jumps—*Stokes discontinuities*—of this expression as φ explores the full angular range; we shall refer to this device under the eye-catching name of « radar method ».

Complementary remark: the domain of analyticity in x of a function like $(\mathrm{L}_0^{\varphi}f)(x)$ ($f \in \mathscr{C}$ and analytic at $s = 0$) is initially the half-plane $\{ \mathrm{Re}(e^{i\varphi}x) > 0 \}$; but $\mathrm{L}_0^{\varphi}f$ is invariant as φ explores the largest sector Σ containing no first sheet branch points, so the domain gets trivially enlarged to the sector:

$$\{ x \mid \mathrm{Re}(e^{i\varphi'}x) > 0 \quad \text{for some} \quad \varphi' \in \Sigma \} \qquad (3.21)$$

4. THE WKB METHOD IN ONE DIMENSION (A REVIEW)

The relevance of the Laplace representation (and of its corollary, Borel resummation) to semiclassical theory in quantum mechanics will only appear in the Balian-Bloch treatment of the Schrödinger equation in terms of classical paths [*11*] (next section). Before that we give a brief review of the standard WKB method in one dimension with a complex coordinate: this will also fix notations. We shall work at a fixed (possibly complex) energy E since we are ultimately interested in spectral properties, and we shall assume the potential V to be *analytic* throughout. The stationary Schrödinger equation reads:

$$(- \hbar^2 d^2/dq^2 + \mathrm{V}(q))\psi(q) = \mathrm{E}\psi(q) \qquad (4.1)$$

and its solutions in the complex q plane are known to have the exact form ([*12*]; Chap. VI.7)

$$\psi = u^{-1/2} \exp \frac{i}{\hbar} \int u\,dq \qquad (4.2)$$

where $u(q, \hbar^{-1})$ is a solution of

$$u^2 - p^2 = \hbar^2 (u^{-1/2})'' u^{1/2} ; \tag{4.3}$$

$\left(p(q) = \pm (E - V(q))^{1/2} \text{ is the classical momentum, and } ' = \dfrac{d}{dq} \right)$. The non-

linear equation (4.3) is only superior to (4.1) in that we can take $\hbar$ as a
small expansion parameter (corresponding to x^{-1} in the notations of sec-
tion 2). For either choice of sign of the function $p(q)$, eq. (4.3) can be solved
in formal powers of $\hbar^2$, as:

$$u(q, \hbar^{-1}) = p(q) + \sum_{n=1}^{\infty} \hbar^{2n} u_{2n}(q, p(q)) \tag{4.4}$$

(this is called the WKB expansion). The terms u_{2n} come out recursively
as polynomials of $V'(q)$, $V''(q)$ ... and of p^{-1}, odd in p. For instance:

$$u_2 = \frac{V''(q)}{8p^3} + \frac{5V'(q)^2}{32p^5} \tag{4.5}$$

The *turning points* (real or complex zeros of $E - V(q)$) are at the same time
the branch points of $p(q)$ and of the expansion (4.4), but they are singular
points for the latter. We assume all turning points to be *simple* (i. e. simple
roots of $V(q) = E$) and we shall denote them as $q'_1, q'_2, q'_3 \ldots$.

Given a solution u of (4.3), $(-u)$ is also a solution, and we deduce from
(4.2) that

$$\psi = b_+ u^{-1/2} e^{\frac{i}{\hbar} \int u dq} + b_- u^{-1/2} e^{-\frac{i}{\hbar} \int u dq} \quad (b_\pm \text{ constants})$$

represents the general solution of the Schrödinger equation (4.1). But
this exact statement becomes applicable only if used jointly with some
approximation to u based on the expansion (4.4). Typically in the litte-
rature one takes some *asymptotic* approximation $U(q, \hbar^{-1})$ to $u(q, \hbar^{-1})$
valid as $\hbar \to 0$ or $q \to \infty$ or both, and defines « quasi-solutions » or
« WKB waves »:

$$\phi_\pm(q, \hbar^{-1}) = U^{-1/2} e^{\pm \frac{i}{\hbar} \int U dq} \tag{4.6}$$

Under rather mild and explicit assumptions, $\phi_\pm$ are shown to approximate
actual solutions of (4.1) in the initial asymptotic sense, but only *locally* in q
(global agreement is impossible since any solution ψ is a single-valued
analytic function whereas $\phi_\pm$ are multivalued). More precisely we can
write the general solution as

$$\psi = b_+(q)\phi_+ + b_-(q)\phi_- \tag{4.7}$$

with $b_\pm$ constant only to the order of approximation used and only within

limited ranges of variation of q. We are then left with the problem of eva-
luating the 2×2 *connection matrix* $F(q_2, q_1)$ such that [13]

$$\begin{pmatrix} b_+(q_2) \\ b_-(q_2) \end{pmatrix} = F(q_2, q_1) \begin{pmatrix} b_+(q_1) \\ b_-(q_1) \end{pmatrix} \qquad (4.8)$$

for points q_1, q_2 in general position, where $F(q_2, q_1)$ may quite differ
from the identity matrix.

The general scheme that we have just vaguely outlined can be implemented
in a variety of ways for current practice. A battery of mathematical theo-
rems covering many cases of interest is given for instance in [5] [13]; for
a more pragmatic approach we refer to the review [6] for its comparative
study of many variants of the WKB method. In any case we stress that
nowhere does the connection problem (4.8) receive a general, complete
and consistent solution, as is well shown in [6]. That is because, as the
approximation (4.6) is continued in the complex q plane, one of the solu-
tion components ϕ_+ or ϕ_- becomes subdominant and its coefficient can
no longer be controlled in any conventional asymptotic approach.

On the positive side here are some standard theorems that justify approxi-
mations like (4.6) and solve the connection problem in a limited number
of cases. We give them under simplifying assumptions that can be weakened
but suffice for most cases of interest (for more complete statements and
for the proofs see [5], Chap. 6 and [14]). We begin by taking q and E real,
and a real (analytic) potential V satisfying for $q \rightarrow \pm \infty$:

$$| \operatorname{Im} (E - V(q))^{1/2} | \geq C > 0 \qquad (4.9)$$

$$| V^{(n)}/V | \leq C_n | q |^{-n} \qquad (4.10)$$

The first condition is satisfied in the energy range of the discrete spectrum,
which is what we have in mind; the second one holds for a polynomial
potential, for instance.

There exist then two solutions ψ_1 and ψ_2 of the Schrödinger equation (4.1)
satisfying for $q \rightarrow + \infty$:

$$\psi_{\frac{1}{2}}(q, \hbar^{-1}) = p(q)^{-1/2} \exp \left(\pm i\hbar^{-1} \int_{q_+}^q p(q')dq' \right)[1 + o(1)] \quad (4.11)$$

The origin of integration q_+ is fixed but arbitrary, apart from the requirement
that the interval $[q_+, + \infty]$ contain no turning point. Because of (4.9),
one solution ψ_1 or ψ_2 grows exponentially fast (is *dominant*) relative to
the other (*recessive*) one, and the two are linearly independent. Conse-
quently, the recessive solution is *uniquely* specified by the asymptotic condi-
tion (4.11) but the dominant one is not.

A similar statements holds independently for $q \rightarrow - \infty$, or for $q \rightarrow \infty$

along a complex path avoiding all turning points; in the latter case (4.9) is replaced by:

$$\text{« Im} \left(\int_{q_0}^{q} p\, dq \right) \text{ is monotone and tends to } \pm \infty \text{ as } q \to \infty \text{ »}$$

(for polynomial V, this is true in almost all complex directions).

We shall henceforth consider *confining* potentials, i. e.:

$$V(q) \to + \infty \qquad \text{for} \qquad q \to \pm \infty \qquad (4.12)$$

and by convention we set $\inf \{ V(q) \}_{q \in \mathbb{R}} (> - \infty)$ to the value 0. Then the Schrödinger Hamiltonian has a purely discrete and positive spectrum $E_0 < E_1 < E_2 \ldots, E_k \to + \infty$ [15]. We begin by taking the parameter $E < 0$ in eq. (4.1) and we introduce different notations from (4.11) to qualify the *recessive* solutions of the Schrödinger equation for $q \to \pm \infty$ respectively:

$$\left. \begin{aligned} \psi_{+,q_0}(q_1, \hbar^{-1}) &\sim p(q_1)^{-1/2} \exp \left(i\hbar^{-1} \int_{q_0}^{q_1} p(q)dq \right), q_1 \to - \infty \\[2mm] \psi_{-,q_0}(q_1, \hbar^{-1}) &\sim p(q_1)^{-1/2} \exp \left(-i\hbar^{-1} \int_{q_0}^{q_1} p(q)dq \right), q_1 \to +\infty \end{aligned} \right\} \quad (4.13)$$

Now $q_0 \in \mathbb{R}$ is completely arbitrary since there are no real turning points for $E < 0$; the determinations in use are: $\text{Arg } p(q) = -\pi/2$, $\text{Arg } p(q)^{-1/2} = +\pi/4$ for all real q. The Wronskian:

$$W = W(\psi_-, \psi_+) = \psi_- \psi'_+ - \psi_+ \psi'_- \qquad (4.14)$$

is independent of q_1 and q_0 and by evaluating it for $q_1 \to \pm \infty$ we get:

$$\left. \begin{aligned} \psi_{+,q_0}(q_1, \hbar^{-1}) &\sim \frac{\hbar}{2i} W p(q_1)^{-1/2} \exp \left(i\hbar^{-1} \int_{q_0}^{q_1} p(q)dq \right), q_1 \to + \infty \\[2mm] \psi_{-,q_0}(q_1, \hbar^{-1}) &\sim \frac{\hbar}{2i} W p(q_1)^{-1/2} \exp \left(-i\hbar^{-1} \int_{q_0}^{q_1} p(q)dq \right), q_1 \to + \infty \end{aligned} \right\} \quad (4.15)$$

except if $W = 0$ (i. e. ψ_+ and ψ_- proportional).

By analogy with quantum scattering theory ([16]), we call

$$a(E, \hbar^{-1}) = \frac{\hbar}{2i} W(E, \hbar^{-1}) \qquad (4.16)$$

the *Jost function* of the confining potential. It is an analytic function of $E < 0$ (and of $\hbar^{-1} > 0$ as well): it vanishes whenever one solution is recessive both at $q = \pm \infty$, hence is a square-integrable eigenfunction (the zeros must be reached by analytic continuation from $E < 0$). Rather precise asymptotic estimates for $\hbar \to 0$ can be derived about the solutions (4.13)

and the Jost function. But before that, we mention an *exact* identify satisfied by the Jost function. Assume that for some C, $\varepsilon > 0$:

$$V(q) > C \, | \, q \, |^{2 + \varepsilon} \qquad (q \rightarrow \pm \infty) \tag{4.17}$$

so that the trace of the Green's function exists (converges):

$$R(E, \hbar^{-1}) = \sum_{0}^{\infty} (E_k(\hbar) - E)^{-1} \tag{4.18}$$

Then:

$$\log a(E, \hbar^{-1}) = \int_{-\infty}^{E} [- R(E', \hbar^{-1}) + \hbar^{-1} T(E')] dE' \tag{4.19}$$

where T(E) is the (finite) transit time from $q = -\infty$ to $q = +\infty$ for the classical motion at energy $(- E)$ in the potential $(- V)$:

$$T(E) = \int_{-\infty}^{\infty} \frac{dx}{2\sqrt{V - E}} \qquad (E < 0) \tag{4.20}$$

Eq. (4.19) and related formulas are proven in Appendix A; although the proof relies on WKB estimates we stress that it involves no approximation whatsoever.

We have seen so far that the « quasi-solutions » $p^{-1/2} \exp \pm \dfrac{i}{\hbar} \int p dq$ are asymptotically correct as $q \rightarrow + \infty$ or $q \rightarrow - \infty$; here p enters as the lowest order approximation to the function u in (4.4). We now discuss the asymptotic character with respect to parameters ($\hbar$ and E) of a *recessive* solution specified by its behaviour at infinity (4.13) (recall that a dominant solution cannot be specified as such).

Taking for instance the solution ψ_{+,q_0} recessive at $-\infty$, we set:

$$\psi_{+,q_0}(q_1) = p(q_1)^{-1/2} \exp \left(\frac{i}{\hbar} \int_{q_0}^{q_1} p(q) dq \right) a(q_1, E, \hbar^{-1}) \tag{4.21}$$

We call $a(q, E, \hbar^{-1})$ the amplitude-correcting factor. It would equal 1 if WKB theory were exact; in actual fact the error term $\delta = a - 1$ admits the precise bound (constants omitted) [5] [13]:

$$| \delta(q_1, E, \hbar^{-1}) | \leq \exp \left\{ \hbar \int_{-\infty}^{q_1} | p^{-1/2}(p^{-1/2})'' | dq \right\} - 1 \tag{4.22}$$

which, under assumptions (4.9-4.10), tends to zero *either* for $q_1 \rightarrow -\infty$ *or* for $\hbar \rightarrow 0$ *or* for $E \rightarrow -\infty$, uniformly with respect to fixed parameters as long as turning points are kept safely away from the interval $(-\infty, q_1)$.

Now the replacement of p in (4.21) by a higher order approximation U

to the expansion (4.4) will lead to a new error estimate instead of the δ in Eq. (4.22):

$$| \delta'(q_1, \mathrm{E}, \hbar^{-1}) | \leq \exp \left\{ 2 \int_{-\infty}^{q_1} | (2\hbar \mathrm{U})^{-1} [p^2 - \mathrm{U}^2 + \hbar^2 \mathrm{U}^{1/2} (\mathrm{U}^{-1/2})''] | \right\} dq$$

which is of *faster* decrease in all its arguments if (4.9-4.10) are satisfied. In that sense the expansion (4.2-4.4) is a compound asymptotic formula valid for $\hbar \to 0$ or $q \to -\infty$. When (4.2) is used to generate a particular recessive solution, we must only ensure, by choosing the origin of integration for the higher order terms, that these do not alter the prescribed recessive behavior by a factor. For instance, the explicit WKB expansion to all orders of ψ_{+,q_0} (defined by (4.13)) reads:

$$\psi_{+,q_0}(q_1, \hbar^{-1}) \sim u(q_1, \hbar^{-1})^{-1/2} \exp \frac{i}{\hbar} \int_{q_0}^{q_1} p \, dq \cdot \exp \frac{i}{\hbar} \int_{-\infty}^{q_1} (u - p) dq \quad (4.23)$$

because $(u - p)$ is integrable at $q = -\infty$, and $u \sim p$ when $\hbar \to 0$ or $\mathrm{E} \to -\infty$. In other words the amplitude-correcting factor $a(q_1, \mathrm{E}, \hbar^{-1})$ defined by eq. (4.21) admits the complete compound expansion (recall that u_{2n} depend on E through $p(q)$):

$$a(q_1, \mathrm{E}, \hbar^{-1}) \sim \left\{ \left(1 + \sum_{n=1}^{\infty} p(q_1)^{-1} u_{2n}(q_1) \hbar^{2n} \right)^{-1/2} \exp \left(i \sum_{n=1}^{\infty} \hbar^{2n-1} \int_{-\infty}^{q_1} u_{2n}(q) dq \right) \right\} \quad (4.24)$$

$$\sim \sum_{m=0}^{\infty} \alpha_m(q_1) \hbar^m \quad (\alpha_0 \equiv 1) \quad (4.25)$$

Formula (4.25) stands for the brute force expansion of eq. (4.24) in formal powers of $\hbar$. The fact that ψ is built from an even function of $\hbar$ in (4.2) is completely concealed in (4.25); however this form of the WKB expansion is often preferred because the α_m are obtained by recursive integration of *linear* transport equations. We shall need both forms (4.24)-(4.25) in this work, in alternance (cf [17]).

Returning now to the Jost function, we see that its definition:

$$\psi_{+,q_0}(q_1) \sim a(\mathrm{E}, \hbar^{-1}) p(q_1)^{-1/2} \exp \frac{i}{\hbar} \int_{q_0}^{q_1} p \, dq$$

$(q_1 \to +\infty)$, together with eq. (4.21), amounts to the property:

$$\lim_{q_1 \to +\infty} a(q_1, \mathrm{E}, \hbar^{-1}) = a(\mathrm{E}, \hbar^{-1}) \quad (4.26)$$

Letting $q_1 \rightarrow +\infty$ in the expansion (4.24) and recalling that $u(q_1)/p(q_1) \sim 1$ also in that limit, we get the important expansion formula [14]:

$$a(E, \hbar^{-1}) \sim \exp\left(i \sum_{n=1}^{\infty} \hbar^{2n-1} \int_{-\infty}^{+\infty} u_{2n}(q)dq \right) \qquad (4.27)$$

valid for $\hbar \rightarrow 0^+$ or $E \rightarrow -\infty$ (or both). Note that because $E < 0$, $ip > 0$ and u_{2n} is odd in p, this expansion is real, as it should be.

We have thus been able to solve semiclassically a special connection problem: for a specified solution recessive at $q = -\infty$, to compute its dominant WKB behavior at $q = +\infty$, given by the Jost function. Of course the evaluation is only asymptotic, but it is obtained to all orders (eq. (4.27)). By contrast, the coefficient b_- of the wave ϕ_- in eq. (4.7) stays completely undefined as ϕ_- is subdominant for $q \rightarrow +\infty$.

Similarly the solution ψ_{-,q_0} recessive at $q = +\infty$ can be followed up to $q = -\infty$. In both cases the two features crucial to the computation are that the solution is followed in a direction of *increasing dominance* and that *no turning points* are present on the continuation path [5] [13].

We now assert that the foregoing WKB expansion has a relevance, but a rather weak one, to the eigenvalue problem. Substituting the expansion (4.27) into the identity (4.19) and taking for granted that we may differentiate once under $\sum$ and $\int$, we find:

$$R(E, \hbar^{-1}) \sim \hbar^{-1}T(E) - i \sum_{n=1}^{\infty} \hbar^{2n-1} \int_{-\infty}^{\infty} \frac{\partial}{\partial E} u_{2n}(q)dq \qquad (4.28)$$

$$(E \rightarrow -\infty, \hbar > 0) \qquad \text{or} \qquad (\hbar \rightarrow 0^+, E < 0).$$

In other words we get from (4.27) a quantitative idea of the resolvent trace, but only far away from the spectrum in units of typical eigenvalue spacings.

But eq. (4.27) also formally bears direct resemblance to the *Bohr-Sommerfeld* expansion for *individual eigenvalues*, which we now recall. Under the same assumptions as before about the confining potential V, we now fix $E > 0$ and moreover assume only *two* simple turning points on the real axis (a « simple well » case). Then the quantum eigenvalues $E_k(\hbar)$ are given by the implicit equation [17] [2]:

$$\oint_\gamma u(q, \hbar^{-1})dq = \sum_0^{\infty} \hbar^{2n} \oint_\gamma u_{2n}(q)dq = (2k+1)\pi\hbar \qquad (4.29)$$

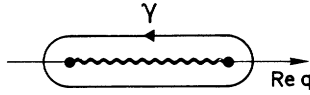where the contour γ encircles positively a cut joining the two turning points

Fig. 5.

(Fig. 5) and u_{2n} depends on E through $p(q) = (E - V(q))^{1/2}$, with the determination $p(q \pm i0) \lessgtr 0$ when $V(q) < E$. Formula (4.29) is asymptotic when $\hbar \to 0$ (E fixed) *or* $k \to \infty$ ($\hbar$ fixed). Obviously the two expansions (4.27) and (4.29) are related order by order through analytic continuation in E and deformation of integration paths in the complex q plane. The precise relationship depends on the global geometry of turning points and will be explicited better in sections 7-8.

Remark. — Eq. (4.39) naturally generates a sequence of *characteristic values* x_k for $\hbar^{-1}$ at fixed E; a functional inversion is needed to produce the eigenvalues E_k at fixed $\hbar$. We choose to discard the $E_k(\hbar)$ form here, because we find that the $x_k(E)$ are more natural and simple spectral quantities than the energy levels in *all aspects of semi-classical theory*.

5. THE WKB METHOD
IN THE BALIAN-BLOCH REPRESENTATION: LOCAL ASPECTS

We shall now present a drastically different approach to WKB theory and the connection problem, inspired in part by more recent work of Knoll-Schaeffer on the complex WKB method [7], by Dingle's interpretation of the Stokes phenomenon [4] (see sections 2, 6), and by the Balian-Bloch representation of quantum mechanics in terms of complex classical paths [11]. The idea is to derive a generalized Laplace representation for the solution $\psi(q, \hbar^{-1})$ of the Schrödinger equation (4.1) and to follow it as the coordinate q, considered as an external complex parameter, varies.

More precisely, we set $x = \hbar^{-1}$ (so as to distinguish by notation the mathematical variable x and its constant Planck's value $\hbar^{-1}$), and we postulate that the Schrödinger equation (4.1), also written:

$$\frac{\partial^2 \psi}{\partial q^2} + (E - V(q))x^2 \psi = 0 \qquad (5.1)$$

admits solutions of the form proposed by Balian and Bloch:

$$\psi(q, x) = x \int_{C(q)} e^{-xs} \tilde{\psi}(q, s) ds$$

for some infinite path $C(q) \subset \mathbb{C}$ (i. e. without endpoints). The original equation (5.1) then translates to a *homogeneous* linear partial differential equation:

$$\frac{\partial^2 \widetilde{\psi}}{\partial q^2} + (E - V(q)) \frac{\partial^2 \widetilde{\psi}}{\partial s^2} = 0 \qquad (5.3)$$

Now we want: first to apply the analysis of section 2 to Eq. (5.2) for each fixed q, i. e. express $\psi(q, x)$ as a sum of Laplace transforms of analytic *singularities* of $\widetilde{\psi}(q, .)$, and next: to follow the resulting representation as q moves in the complex plane. We therefore require $\widetilde{\psi}(q, .)$ to belong to the space $\widetilde{\mathscr{C}}$ of ramified temperate functions of s (section 3), and also to depend analytically on q since $\psi(q, x)$ must be analytic in q. We call an *admissible solution* of the Schrödinger equation (5.3) any such function $\widetilde{\psi}(q, s)$.

Although admissible solutions might *a priori* form a very small (or even empty) subset of all conceivable solutions, we believe that they actually encompass the solutions that are useful and interesting, especially for the study of the spectrum of (5.1). Our argument will develop along four stages:

1) a formal verbatim translation of WKB theory into the Balian-Bloch representation will yield the *laws of propagation* of the local singularities of an admissible solution $\widetilde{\psi}(q, .)$ as the coordinate q varies (q will act as a complex « time » variable);

2) we shall argue that the *conjecture* that $\widetilde{\psi}(q, .) \in \widetilde{\mathscr{C}}$ is plausibly consistent with this rephrasing of WKB theory.

3) In return, property $\widetilde{\psi}(q, .) \in \widetilde{\mathscr{C}}$ will allow a stronger (*global*) interpretation of WKB results than the conventional asymptotic one: it will permit to follow explicitly the evolution as q varies of the integral representation (5.2). We believe this to be the best way of posing (and solving) the general *connection problem* in WKB theory;

4) we shall see that the asymptotic conditions that specify physically interesting solutions fit very nicely into our framework, although we still have to *conjecture* (for lack of a proof) that these conditions define admissible solutions. We shall end with a concrete illustration of the complex WKB method in full use: the derivation of *Borel summability* for the Jost function expansion (4.27).

We now describe points 1) and 2) in detail and shall reserve a separate section to the global aspects 3) and 4).

1) Throughout this subsection $\widetilde{\psi}(q, s)$ is taken to be any *admissible* solution of Eq. (5.3). It is important to view $\widetilde{\psi}$ as an analytic function of the two complex variables q and s; $\widetilde{\psi}$ is then ramified along certain analytic curves in $\mathbb{C}^2$, namely the loci of the branch points $s_j(q)$ of $\widetilde{\psi}(q, .)$ as q varies. We shall call $\mathscr{S}$ the Riemann surface of $\widetilde{\psi}$ (of complex dimension 2) and $\mathscr{S}(q)$ its section above a point $q \in \mathbb{C}$. Let then $C(q)$ be a path in $\mathscr{S}(q)$ as in Fig. 6,
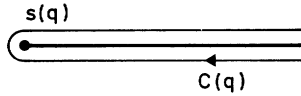
FIG. 6.

encircling negatively the branch point $s(q)$ (and none other). As in section 2, we cut the s-plane above q by the half-line $\{\, \text{Arg}\, (s - s(q)) > 0 \,\}$ and introduce the discontinuity function:

$$\Delta_{s(q)}\tilde{\psi}(q, s(q) + t) = \tilde{\psi}(q, s(q) + t + i0) - \tilde{\psi}(q, s(q) + t - i0) \quad (t > 0) \quad (5.4)$$

We now form $\psi(q, x)$ by Eq. (5.2). Due to (3.7), ψ has an asymptotic expansion for $x \to +\infty$:

$$\psi(q, x) \sim e^{-xs(q)}\Sigma\psi_\lambda(q)x^{-\lambda} \tag{5.5}$$

in $1 - 1$ correspondence with the discontinuity by Borel transformation:

$$(\Sigma\psi_\lambda(q)x^{-\lambda})_B(t) = \sum_0^\infty \psi_\lambda(q)\frac{t^\lambda}{\Gamma(1 + \lambda)} = \Delta_{s(q)}\tilde{\psi}(q, s(q) + t) \tag{5.6}$$

But ψ satisfies the Schrödinger equation (5.1), as a consequence of (5.3); then (5.5) is nothing but the WKB Ansatz for ψ. We therefore identify the unknown expression (5.5) with the known general WKB solution (4.2):

$$\psi(q, x) = c(x)u(q, x)^{-1/2} \exp\, ix \int_{q_0}^q u(q', x)dq' \tag{5.7}$$

where $u(q, x) = \sum_0^\infty u_{2n}(q)x^{-2n}$; the factor $c(x) = \sum c_\lambda x^{-\lambda}$ and the lower bound of integration q_0 provide the freedom to normalize ψ according to initial conditions.

a) To leading order we obtain that the exponent $s(q)$, as is well known, must satisfy the Hamilton-Jacobi equation [19]:

$$\left(\frac{ds}{dq}\right)^2 = V(q) - E[= (ip(q))^2]. \tag{5.8}$$

Its general solution (the classical energy-dependent action)

$$s(q) = -i \int_{q_0}^q pdq + s(q_0) \tag{5.9}$$

defines for fixed initial data a multiply-valued function of q, ramified around the turning points, whose graph in $\mathbb{C}^2$ is a connected analytic curve S: an action curve (unique up to global translation in s). Since turning points are singular for the WKB method we shall restrict the coordinate q to the

complex plane deprived from all (real or complex) turning points, denoted $\dot{\mathbb{C}}$ (from the viewpoint of the homogeneous equation (5.3), turning points are singular because the planes $\{ q = \text{constant} \}$ become characteristic above them). We also introduce a number of coverings of this punctured plane $\dot{\mathbb{C}}$:

— the universal covering $\dot{\mathbb{C}}_*$ (of which all coverings are quotient spaces);
— the double covering $\dot{\mathbb{C}}_2$ that makes $p(q)$ uniform; its two sheets join at each turning point q'_k, and p takes opposite values on the two points above given $q \in \dot{\mathbb{C}}$; hence the second family of solutions $s(q) = + i \int p dq$ is fully redundant with the first one (5.9) and need not be discussed separately;
— the quadruple covering $\dot{\mathbb{C}}_4$ (a double covering of $\dot{\mathbb{C}}_2$) that makes $p(q)^{-1/2} = (E - V(q))^{-1/4}$ uniform;
— the covering $\dot{\mathbb{C}}_{c1}$ that makes the classical action uniform: it actually covers $\dot{\mathbb{C}}_2$ since each determination of $s(q)$ induces one determination of $p(q) = i \dfrac{ds}{dq}$. To specify $\dot{\mathbb{C}}_{c1}$ we take any closed loop γ in $\dot{\mathbb{C}}_2$ and define its action period as the number

$$\omega(\gamma) = - i \int_\gamma p dq \qquad (5.10)$$

Periods form an additive group Ω. Now the loop γ lifts up to a closed curve in $\dot{\mathbb{C}}_{c1}$ if and only if $\omega(\gamma)=0$: this homologic condition completely specifies $\dot{\mathbb{C}}_{c1}$.

b) *We now match the higher order* terms of (5.5) with our explicit expression (5.7) and recover the other result known from WKB theory: that the amplitude terms in (5.5) also propagate in an explicit fashion as the coordinate q follows a path $\gamma \in \dot{\mathbb{C}}$. Namely:

$$\Sigma \psi_\mu(q) x^{-\mu} = \left\{ \left(\frac{u(q_0, x)}{u(q, x)} \right)^{1/2} \exp ix \int_{q_0}^q (u(q', x) - p(q')) dq' \right\} (\Sigma \psi_\lambda(q_0) x^{-\lambda}) \quad (5.11)$$

From now on, any expression in curly brackets is understood to be expanded by brute force in descending powers of x; in particular the exponential is to be expanded since $x[u(q, x) - p(q)] = 0(x^{-1})$, and this produces all (even and odd) integral powers of x^{-1}. We need convenient notations for the following formal expressions to be frequently used:

$$u_\gamma(x) = \exp ix \int_\gamma u(q, x) dq, \qquad a_\gamma(x) = \exp ix \int_\gamma [u(q, x) - p(q)] dq \quad (5.12)$$

for any path γ in the double covering $\dot{\mathbb{C}}_2$ (where u and p are single-valued).

We now Borel-transform Eq. (5.11) in order to get the discontinuity function (5.6); with the help of formulas (3.12-3.13) we obtain:

$$\Delta_{s(q)}\tilde{\psi}(q, s(q) + .) = \left\{ \frac{u(q_0, x)^{1/2}}{u(q, x)^{1/2}} a_\gamma(x) \right\}_{\text{B}} \dot{*} \Delta_{s(q_0)}\tilde{\psi}(q_0, s(q_0) + .) \quad (5.13)$$

where γ is a path connecting q_0 to q.

The following remarks are in order:

— the convolution kernel in (5.13) can be fully explicited to any finite order around $s = 0$ by formal iteration of Eq. (4.3) for u;

— this kernel depends on the homotopy class of the path γ linking q_0 to q, even for coinciding determinations of $s(q) = s(q_0) - i \int_\gamma p dq$. This confirms our previous idea that we should not only view the action curve S as projected on $\mathbb{C}^2$ but also as embedded in the Riemann surface $\mathscr{S}$ of $\tilde{\psi}$: homotopically distinct paths on $\dot{\mathbb{C}}$ then drive to distinct sheets of $\tilde{\psi}(q, .)$ in $\mathscr{S}(q)$.

— Formula (5.13) is rigorously true as long as analyticity is preserved, i. e. as long as the Puiseux series of $\Delta_{s(q)}\tilde{\psi}(q, s)$ around $s = s(q)$ has non-zero radius of convergence. But by linearity the singularities of the discontinuities of $\tilde{\psi}$ are (among) those of $\tilde{\psi}$ and propagate along action curves too. Only the global control of those curves in $\mathscr{S}$ will tell us when two branch points of $\tilde{\psi}$ join in $\mathscr{S}$ and destroy analyticity (the answer will be given in section 6: precisely when q is a turning point).

— The sequence of powers in the initial data $\Sigma\psi_\lambda(q_0)x^{-\lambda}$ should be preferably chosen as $\{ \lambda_n = \lambda_0 + n \}_{n\in\mathbb{N}}$ so as to be preserved by evolution under Eq. (5.11). In particular integral (resp. half-integral) exponents correspond to the choice $\tilde{\psi}(q_0, .) \in \mathscr{C}_\infty$ (resp. $\mathscr{C}_2$). In order words these two subspaces are stable by action of the kernel (5.13), at least formally (see subsection 2 below).

— Of particular interest is the relation (5.13) for $q = q_0$: it then relates the discontinuities of the same function $\tilde{\psi}(q_0, .)$ at two branch points s_1 and s_2 that are distinct in $\mathscr{S}(q_0)$ but are analytic continuations of one another along S:

$$s_2(q_0) - s_1(q_0) = - i \int_\gamma p dq = - i \int_0^1 p(\gamma(\tau))d\gamma(\tau) \quad (5.14)$$

$\{ \gamma(\tau) \}$ being a parametrization such that $\gamma(0)$ and $\gamma(1)$ lie in $\dot{\mathbb{C}}_2$ above q_0. Now we recall that $\dfrac{u(q, x)}{p(q)}$ only contains $p(q)^2$ hence is single-valued in $\dot{\mathbb{C}}$,

and so is $\left(\dfrac{u(q, x)}{p(q)}\right)^{-1/2}$ because $\dfrac{u(q, x)}{p(q)} = 1 + 0(x^{-2})$ and the $\left(-\dfrac{1}{2}\right)$ root
is evaluated by series expansion around 1. Therefore:

$$\frac{u(\gamma(0), x)^{1/2}}{u(\gamma(1), x)^{1/2}} = \frac{p(\gamma(0), x)^{1/2}}{p(\gamma(1), x)^{1/2}} = e^{-im(\gamma)\pi/2} \qquad (5.15)$$

where $m(\gamma)$ is the rotation index of γ around all turning points q'_j (a complex
generalization of the Maslov index: $m\,[\mathrm{mod}\,4]$ labels the sheets of $\mathbb{C}_4$):

$$m(\gamma) = (2\pi i)^{-1} \int_\gamma \frac{(E - V(q))'}{E - V(q)}\, dq \in \mathbb{Z} \qquad (5.16)$$

Hence (5.13) simplifies to:

$$\Delta_{s_2}\tilde{\psi}(q_0, s_2 + .) = e^{-im(\gamma)\pi/2} a_\gamma(q_0, x)_{\mathrm{B}} \mathbin{\dot{*}} \Delta_{s_1}\tilde{\psi}(q_0, s_1 + .) \quad (5.17)$$

If finally γ is a loop in the double covering $\mathbb{C}_2$ (the case of s_1 and s_2 pro-
pagating at the same speed), then the integrand $(u(q, x) - p(q))$ is single-
valued on γ, the integrals $\displaystyle\int_\gamma u dq$ and $\displaystyle\int_\gamma p dq = i\omega(\gamma)$ no longer depend on q_0,
$m(\gamma)$ is an even number, and (5.17) reduces to:

$$\Delta_{s_1+\omega(\gamma)}\tilde{\psi}(q_0, s_1+\omega(\gamma)+ .) = (-1)^{m(\gamma)/2}(a_\gamma)_{\mathrm{B}} \mathbin{\dot{*}} \Delta_{s_1}\tilde{\psi}(q_0, s_1+ .) \quad (5.18)$$

For this last case we note that the mapping $\gamma \rightarrow a_\gamma$ defines a *homomorphism*
from all cycles γ of the double covering $\mathbb{C}_2$ onto a multiplicative group of
formal series in x^{-1} with constant complex coefficients.

To summarize this subsection, we now control both the motion of the
branch points $s_j(q)$ of an admissible solution $\tilde{\psi}(q, s)$ thanks to the Hamilton-
Jacobi equation (5.8), and the transformation of the local singularity (in
the sense of Puiseux series) that accompanies the motion of each point,
thanks to Eq. (5.13). As in section 2, we have found that the same analytic
formulas (5.8)-(5.13) describe the local situation around each branch point,
be it dominant or subdominant, without interference from the other
branch points at this level; again this remarkably simple behavior follows
from the choice of Borel resummation (here carried out through the Balian-
Bloch representation) to separate the dominant and subdominant contri-
butions.

2) We have so far obtained that the branch points of $\tilde{\psi}(q, .)$ propagate
along action curves in $\mathbb{C}^2$. In order to really benefit from all the preceding
propagation formulas we must assume complete control and knowledge
over the action curves S that solve the complex Hamilton-Jacobi equation.
Given that, if at some $q_0 \in \mathbb{C}$ we know the branch points s_j of $\tilde{\psi}(q_0, .) \in \mathscr{C}$,
then we can continue $\tilde{\psi}(q, .)$ along a curve γ in $\mathbb{C}$ starting from q_0 and follow
separately the motion of each branch point $s_j(q)$ by integrating Eq. (5.9)

with initial data: $s(q_0) = s_j$ and choice of $\pm ip(q_0)$: the branch points stay isolated and no natural boundary can develop. The form of the singularities (e. g. square-root or logarithmic) is moreover preserved under (5.13). In that sense an assumption like $\tilde{\psi}(q,.) \in \tilde{\mathscr{C}}$ is consistent with WKB results. We do not know however if an estimate like (3.1) can propagate in q, even for V polynomial; such global results are notoriously rare and unpredictably difficult to prove. We are also aware of another global difficulty: the branch points of $\tilde{\psi}(q,.)$ in the s plane result from propagating each branch point s_j of $\tilde{\psi}(q_0,.)$ along all possible paths $\gamma \subset \mathbb{C}$ from q_0 to q, hence a branch point of $\tilde{\psi}(q,.)$ will occur at *each* determination of $s(q)$, and will propagate with either speed $\pm ip(q)$. The problem is that any two branch points of the same speed differ by a constant which may be any period $\omega \in \Omega$, and except if V is a polynomial of degree ≤ 4, the group of periods Ω is typically *dense* in $\mathbb{C}$, thereby contradicting our assumption *b*) about $\tilde{\mathscr{C}}$. The concrete applications of this paper will at first concern potentials of degree ≤ 4 for which Ω is a discrete lattice indeed, but their extensions to potentials of higher degrees, for which Ω is dense, will pose no problem either! (section 10). The branch points of $\tilde{\psi}(q,.)$, while dense in projection, occur in finite number on each sheet of the Riemann surface : this would only require inessential changes in the definition of the space $\tilde{\mathscr{C}}$.

All in all, up to minor adaptations of this sort or concerning the growth condition (3.1), we are confident that there exists a space $\tilde{\mathscr{C}}$ of ramified temperate functions in which the homogeneous Schrödinger equation (5.3) is well posed (we shall precisely require in point (4) of next section that a certain « scattering problem » be well posed in the space $\tilde{\mathscr{C}}$). We have not searched very hard to establish rigorous theorems about this, because we feel that whatever difficulties in constructing a proof would have no influence on the concrete results that we are primarily seeking here; this may be however an interesting mathematical problem (there has been independently developed [32] a mathematical theory of ramified functions that might serve our purposes: cf. the final remark of our section 8).

6. THE CONNECTION PROBLEM AND THE GLOBAL WKB METHOD

3) We now make an incursion into the third of the topics listed in the previous section: the transformation properties as q varies of the Balian-Bloch representation (5.2). We shall first solve that problem in the form of a *connection formula* superficially similar to those derived in the WKB literature but actually more precise. Later (§ 9) we shall sketch the relevance of our results to the global analytic structure of the Riemann surface $\mathscr{S}$.

The problem is a follows: let $\tilde{\psi}$ be ramified along a given action curve S, and suppose also given the form of ψ at some initial $q_0 \in \dot{\mathbb{C}}$:

$$\psi(q_0, x) = x \int_{C(q_0)} \tilde{\psi}(q_0, s)e^{-xs}ds \qquad (6.1)$$

If this is an admissible solution of the Schrödinger equation, then we get by analytic continuation:

$$\psi(q, x) = x \int_{C(q)} \tilde{\psi}(q, s)e^{-xs}ds \qquad (6.2)$$

where the integration path $C(q)$ is deformed continuously in $\mathscr{S}$ so as never to be crossed by any branch point of $\tilde{\psi}(q, .)$.

But the only thing about $\tilde{\psi}(q, .)$ that we can *explicitly* follow with q is its discontinuities, due to WKB theory. Therefore we must understand the evolution of the *standard form* of (6.1), i. e. where $C(q_0)$ is a steepest descent path around one branch point $s_+(q_0)$; the case of the general standard form like (2.3) follows by linear combination. We thus denote for short as:

$$\phi_j(q, x) = L^0_{s_j} \Delta_{s_j} \tilde{\psi}(q) \qquad (j = +, -, 1, 2, \text{etc...}) \qquad (6.3)$$

the Borel sum of the asymptotic contribution of the branch point $s_j(q)$. We then take as initial datum:

$$\psi(q_0, x) = x \int_{C(q_0)} \tilde{\psi}(q_0, s)e^{-xs}ds = \phi_+(q_0, s) \qquad (6.4)$$

Now we continue analytically ψ along a path $\gamma \subset \dot{\mathbb{C}}$ from q_0.

First question: how long shall we have:

$$\psi(q, x) = \phi_+(q, x) ? \qquad (6.5)$$

Answer: up to the first value $q_1 \in \gamma$ for which some other branch point $s_-(q)$, necessarily moving opposite to $s_+ \left(\dfrac{ds_-}{dq} = -\dfrac{ds_+}{dq} \right)$, crosses the cut $\{ s - s_+(q) \geq 0 \}$. Until then we can deform without obstruction $C(q_0)$ to the path of steepest descent $C(q)$ around $s_+(q)$.

Then how is (6.5) changed beyond q_1? The answer will be the *connection formula*. To derive it, we assume, again up to a slight perturbation of the parameters, that:

« s_- is the only branch point such that $s_-(q) - s_+(q_1) \geq 0$ »   (6.6)

(simple crossing; implications of multiple crossings will be discussed later).

By Cauchy's theorem, the effect of the crossing will be (Fig. 7) to add to the expression ϕ_+ in (6.5) a multiple of the similar contribution ϕ_- from the branch point s_- ($\rightarrow$ denotes analytic continuation across q_1):

$$\phi_+(q, x) \rightarrow \phi_+(q, x) + \alpha\phi_-(q, x) \qquad (6.7)$$
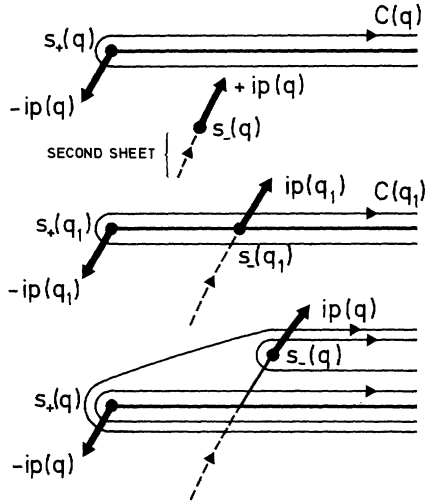
FIG. 7.

$\alpha = 0$ if and only if the crossing is apparent (occurs after projecting upon $\dot{\mathbb{C}}$ but not on the Riemann surface $\mathscr{S}$ itself); deciding whether the crossing is apparent or not is part of the problem.

The relations $\dfrac{d}{dq}(s_+ - s_-) = -2ip(q)$ and (6.6) suggest exploiting the integral curves of the direction field:

$$\mathrm{Re}\, p(q)dq = 0 \qquad (6.8)$$

(i. e. the level lines of the function $\mathrm{Im}\, s(q)$).

We begin by following the integral curve issued from q_1 up to the first point q_2 where

$$s_-(q_2) = s_+(q_2) \qquad (6.9)$$

a) q-plane   b) s-plane
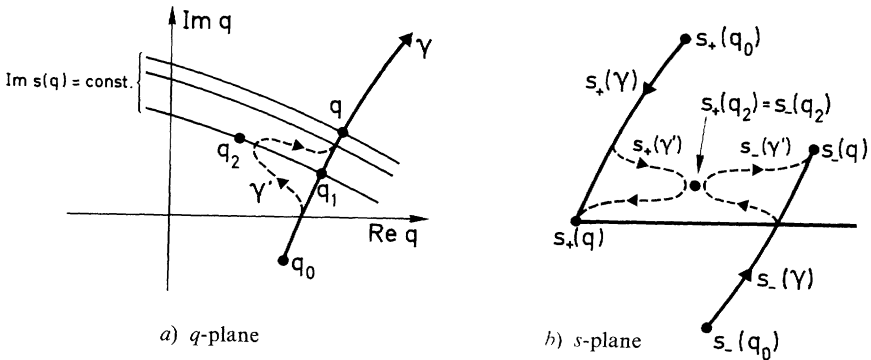
FIG. 8.

We can push accordingly the path γ along the integral curves to make $\{ s_-(q) \}$ cross the cut $\{ s - s_+(q) \geq 0 \}$ transversally and *arbitrary close to the point* $s_+(q)$ (Fig. 8); no turning point is encountered (this would contradict the simple crossing hypothesis) hence the homotopy class of γ in $\dot{\mathbb{C}}$ is unchanged. This argument shows that our connection problem is entirely governed by the local structure of the action curves above $q_2 \in \dot{\mathbb{C}}$ and that all branch points besides s_+ and s_- are decoupled spectators which can be forgotten.

In order to determine α we apply the principle of single-valuedness of the analytic function $\psi(q, x)$, in the form of uniqueness of its standard representation at points q_0 where branch points are in general position. We pick q_0 very close to q_2 and continue the solutions ϕ_+ and ϕ_- as q describes positively a small loop Γ around q_2 beginning and ending at q_0 (the relation with the usual connection method is that ϕ_+ and ϕ_- are two WKB waves with opposite frequencies). Now we must distinguish two cases:

a) $p(q_2) \neq 0$: q_2 is not a turning point and the loop Γ is homotopically trivial; the expansions (4.4) for ϕ_+ and ϕ_- are single-valued around q_2 and return to their values after continuation along Γ. In the mean while:

$$s_\pm(q) = s_\pm(q_2) \mp ip(q_2)(q - q_2) + 0(|q - q_2|^2)$$
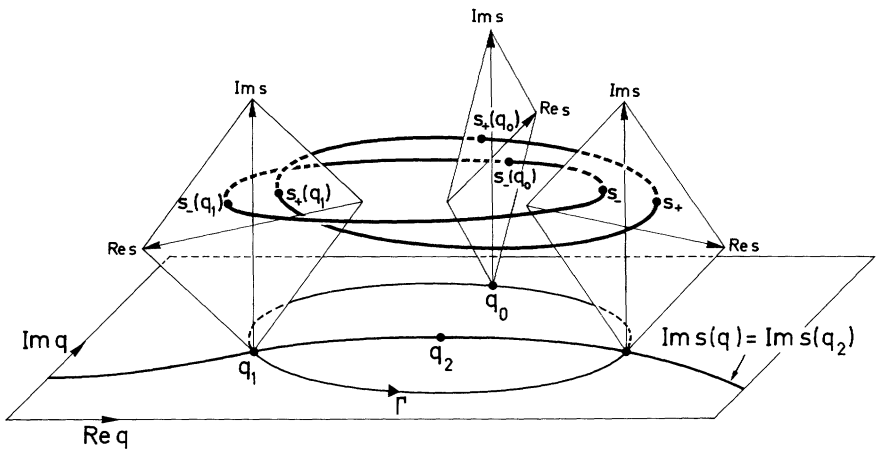
FIG. 9.

have each revolved 2π about $s_+(q_2) = s_-(q_2)$, and two crossings have occurred: one with $s_- > s_+$ and the other one with $s_+ > s_-$ (Fig. 9). Instead of continuing ϕ_+ and ϕ_- separately it is simpler to follow the coefficient matrix $\begin{pmatrix} b_+ \\ b_- \end{pmatrix}$ of the general solution $\psi(q, x)$ expanded on the

basis $(\phi_+(q, x), \phi_-(q, x))$. The effect of the crossing with $s_- > s_+$ is to multiply:

$$\begin{pmatrix} b_+ \\ b_- \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} b_+ \\ b_- \end{pmatrix} \qquad (6.10)$$

and similarly the effect of the other crossing is

$$\begin{pmatrix} b_+ \\ b_- \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_+ \\ b_- \end{pmatrix} \qquad (6.11)$$

(we recognize the F-matrix description of the connection problem as reviewed in [6]).

The single-valuedness condition after describing the loop Γ is then:

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (6.12)$$

because the standard form of ψ is unique (section 3); hence:

$$\alpha = \beta = 0 \qquad (6.13)$$

In conclusion, crossings such that $\dfrac{d}{dq}(s_+ - s_-) \neq 0$ at the coincidence point of s_+ and s_- are apparent crossings, i. e. the two branch points have no sheet of $\mathscr{S}$ in common and they ignore each other; formula (6.5) stays valid in this case on either side.

b) $p(q_2) = 0$: q_2 is a *turning point* at which the determinations s_+ and s_- coalesce with the behavior

$$s_\pm(q) - s_\pm(q_2) \sim \pm \text{ const } (q - q_2)^{3/2} \qquad (6.14)$$

As we make one turn around Γ (in the positive sense) the values s_+ and s_- exchange one another. Therefore a combination of ϕ_+ and ϕ_- can be single-valued only if their Borel transforms $\Delta_{s_+} \tilde{\psi}(q, s_+ + .)$ and $\Delta_{s_-} \tilde{\psi}(q, s_- + .)$ are themselves *exchanged by analytic continuation* around Γ. Such is the case for instance, up to normalization, with the two WKB expressions:

$$\phi_\pm = u(q, x)^{-1/2} \exp \pm ix \int_{q_0}^q u(q', x) dq' \qquad (6.15)$$

which are not single-valued around q_2 but become after one turn:

$$\phi_+ \rightarrow e^{-i\pi/2} \exp\left(ix \int_\Gamma u dq \right) \phi_- = -iu_\Gamma \phi_-$$

$$\phi_- \rightarrow e^{-i\pi/2} \exp\left(-ix \int_\Gamma u dq \right) \phi_+ = -iu_\Gamma^{-1} \phi_+ \qquad (6.16)$$

(we have Laplace transformed Eq. (5.17), and used the notation (5.12)

and the property $m(\Gamma) = +1$). In the meanwhile, s_+ and s_- have revolved 3π around $s_\pm(q_2)$ and have permuted with one another after *three* crossings, which we may assume without loss of generality to occur in the following order (Fig. 10): $s_- > s_+$, then $s_+ > s_-$ and again $s_- > s_+$. Proceeding as with Eqs. (6.10-6.11), we are to determine three multipliers α, β, γ with:

$$\begin{pmatrix} b_+ \\ b_- \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} b_+ \\ b_- \end{pmatrix} \qquad (6.17)$$

under analytic continuation around Γ. Now the continuation formulas (6.16) together with the singlevaluedness of $\psi = b_+\phi_+ + b_-\phi_-$ impose that

$$\begin{pmatrix} b_+ \\ b_- \end{pmatrix} \rightarrow \begin{pmatrix} 0 & iu_\Gamma^{-1} \\ iu_\Gamma & 0 \end{pmatrix}\begin{pmatrix} b_+ \\ b_- \end{pmatrix} \qquad (6.18)$$

and the matrix equality resulting from (6.17) and (6.18) has the unique solution:

$$\alpha = \gamma = -\beta^{-1} = iu_\Gamma \qquad (6.19)$$

Crucially, because (6.19) was derived by Laplace transformation of Eq. (5.17), the multiplier α involved in Eq. (6.7) is the *Borel sum* of the formal expansion $iu_\Gamma(x)$ (this is obvious if the whole procedure is expressed in terms of $\widetilde{\psi}(q, s)$ alone, as will be done in section 9; here we have rather kept track of the physical solution $\psi(q, x)$ throughout, so as to use familiar WKB language).

All three crossings occurs when $s_- - s_+ \in \mathbb{R}$, i. e. when q crosses an integral curve of the field (6.8) issued from the turning point q_2. There
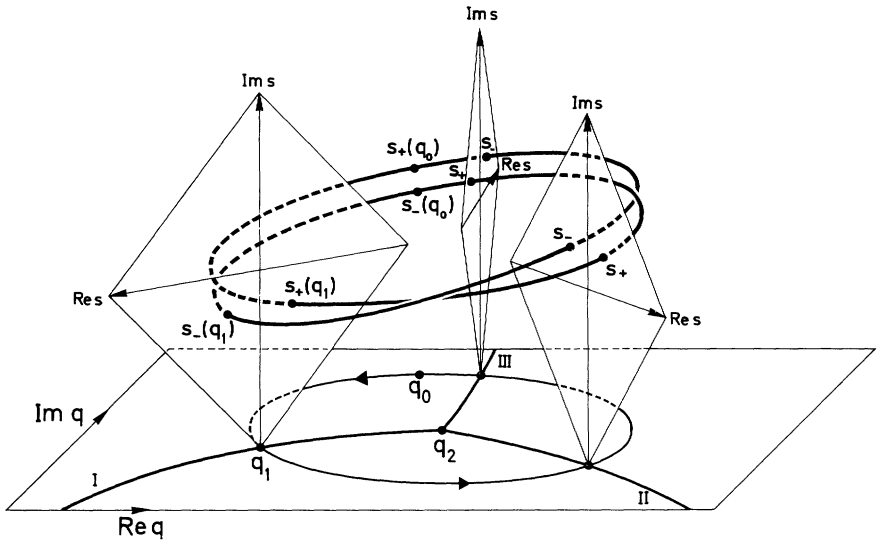
FIG. 10.

are indeed three such lines called the *Stokes lines* of the turning point q_2 (sometimes « anti-Stokes lines » [20] but we shall conform to the convention of Refs. [4] [6]); they are named I, II, III on Fig. 10. The simple crossing hypothesis (6.6) amounts to the assumption that no Stokes line ends on another turning point (i. e. they all go to infinity).

We now summarize our results for later use. We have first found that the only crossings of discontinuities that do change the decomposition of the general solution ψ on a basis of WKB series (*resummed à la Borel*) occur on the Stokes lines and concern a branch point and its *immediate* analytic continuation around the turning point (from which the Stokes line under consideration is issued). The coefficient of the dominant WKB series is unchanged, but the coefficient of the subdominant WKB series suffers a jump of α times the other coefficient, where

$$\alpha = \varepsilon i \exp \pm i \int_\Gamma u dq = \varepsilon i u_\Gamma^{\pm 1} \quad \text{(Borel-summed)}, \qquad (6.20)$$

and $+(-)$ is selected if $\phi_+(\phi_-)$ is the dominant wave on the Stokes line, and $\varepsilon = +1$ if the Stokes line is crossed counterclockwise (as seen from the turning point), else $\varepsilon = -1$; all this is a rephrasing of formula (6.19). Warning: the value (6.20) of α is specific of the relative normalization of ϕ_+ and ϕ_-, here provided by the choice of q_0 in Eq. (6.15).

All other crossings of discontinuities are apparent (fortunately, as they can be more frequent than actual crossings). These include transverse self-intersections of an action curve (case *a*), or similarly intersections of distinct action curves. Then the two branch points do not interact directly and ignore each other. It is clear in retrospect that all our restrictive assumptions about crossings need not apply to apparent crossings but only to the actual ones (case *b*).

Formulas (6.15-6.20) provide us with that we shall call a *complete connection formula* across a Stokes line belonging to a single turning point. It formally resembles the matching rules found in [6] [7] but with all corrections in powers of $\hbar = x^{-1}$ included. Indeed if we limit ourselves to the lowest order term $u = p$ of the WKB expansion, it is *then* possible (and customary) to normalize the WKB waves from the turning point q_2, i. e. set $q_0 = q_2$ in (6.15); the loop Γ shrinks to a point and (6.19) reduces to $\alpha = \beta = \gamma = i$ as found in [6]. However our formulas express more, namely the analytic continuation of the Borel-resummed WKB series: in that sense they are completely exact and reversible. Furthermore they are universal (i. e. they apply to any potential) as long as the crossing of every Stokes line involves one turning point at a time independently of all the others, and they avoid the cumbersome discussion about the choice of a « good » continuation path. Thus our results stand midway between the work of Dingle, who wrote formulas equivalent to Eqs. (6.19), ([4],

Chap. 13.3) but derived them heuristically and did not state their validity in presence of several turning points, and the work of Knoll-Schaeffer [7], who emphasized the latter feature, but only worked to leading powers in $\hbar$ hence could not see the role of Borel resummation in a consistent treatment of the Stokes phenomenon.

It seems that the obligation of Borel-resumming all WKB expansions in order to be able to apply Eqs. (6.20) is a price to pay for the advantages just mentioned. However we shall take the reverse stand that our connection formulas express information about the global analytic structure of the Borel transform, and as such they will help us understand the Borel summability properties of the WKB expansions.

4) We finally turn to the last topic of our list (previous section): fit all pieces of the complex WKB method together (from section 4 onwards) and show it at work on concrete problems.

Until now we have set up a rather abstract machinery for propagating an admissible solution $\psi(q, x)$, endowed with a representation (5.2), as the coordinate q varies, but we still have to exhibit such solutions and if possible interesting ones. Taking all propagation properties for granted, there essentially remains to argue that an initial assumption $\tilde{\psi}(q', .) \in \mathcal{C}$ for some $q' \in \mathbb{C}$ can be a reasonable input. We must be very careful here about the normalization of solutions to the initial Schrödinger equation (5.1), for the innocuous linearity property:

$$\psi(q, x) \text{ solution} \Rightarrow f(x)\psi(q, x) \text{ solution } (\forall f)$$

becomes in Fourier representation:

$$\tilde{\psi}(q, s) \text{ solution} \Rightarrow \tilde{f}(s) \overset{*}{*} \tilde{\psi}(q, s) \text{ solution}$$

But the set of singularities of a convolution product is the pointwise sum of the set of singularities of the factors (section 3): by an awkward choice of f we can generate for $\tilde{f} * \tilde{\psi}(q, .)$ very nasty singularities (e. g. accumulating points or natural boundaries) which have clearly no dynamical relevance.

Now we have seen in section 4 that the good semi-classical normalizing conditions are asymptotic ones for $|q| \to \infty$, especially in the case of recessive solutions, unambiguously specified in this way. We shall now focus on one recessive solution, for instance: $\psi(q, x) = \psi_{+, q_0}(q, x)$ such that:

$$\psi(q, x) \sim p(q)^{-1/2} e^{ix \int_{q_0}^{q} p\, dq} \qquad (q \to -\infty) \quad (4.13)\text{-}(6.21)$$

($q_0 \in \mathbb{R}$ arbitrarily chosen to the left of all turning points).

Now the right-hand side has the Fourier representation:

$$
\begin{aligned}
p(q)^{-1/2} e^{ix \int_{q_0}^{q} p\, dq} &= (2\pi i)^{-1} \int_{C(q)} \left(\frac{-p(q)^{-1/2}}{s - s(q)} \right) e^{-xs} ds \\
&= -(2\pi i)^{-1} x \int_{C(q)} p(q)^{-1/2} \log(s - s(q)) e^{-xs} ds
\end{aligned}
\qquad (6.22)
$$

where the contour $C(q)$ encircles the point $s(q) = -i \int_{q_0}^{q} p\,dq$ negatively. In this formal sense, the desired representation (5.2) *exists* for $q \to -\infty$:

$$\tilde{\psi}(q, s) \sim -(2\pi i)^{-1} p(q)^{-1/2} \log (s - s(q)) \quad (q \to -\infty) \quad (6.23)$$

Equivalently, the amplitude-correcting factor $a(q, x)$ of Eq. (4.21) has the Fourier transform:

$$\tilde{a}(q, s) = p(q)^{1/2} \tilde{\psi}(q, s + s(q)) \tag{6.24}$$

for which (6.23) amounts to:

$$\lim_{q \to -\infty} \tilde{a}(q, s) = -(2\pi i)^{-1} \log s \tag{6.25}$$

Both statements are purely formal, but the latter is easier to comprehend since a limiting function, independent of q, exists and clearly *belongs to the space* $\mathscr{C}$ (more precisely: $\mathscr{C}_\infty$).

Now full WKB theory in the Balian-Bloch representation takes the form of a scattering problem in complex coordinates. We are to solve the partial differential equation (with V polynomial, to avoid complications):

$$\frac{\partial^2 \tilde{\psi}}{\partial q^2} + (E - V(q)) \frac{\partial^2 \tilde{\psi}}{\partial s^2} = 0 \tag{5.3)-(6.26}$$

with « Cauchy data in the space $\mathscr{C}$ at $q = -\infty$ » in the sense of asymptotic conditions:

$$\left. \begin{aligned} \tilde{\psi}(q, s) \underset{q \to -\infty}{\sim} & -(2\pi i)^{-1} p(q)^{-1/2} \log (s - s(q)) \\ \frac{\partial}{\partial q} \tilde{\psi}(q, s) \underset{q \to -\infty}{\sim} & -(2\pi)^{-1} p(q)^{1/2} \frac{1}{s - s(q)} \end{aligned} \right\}$$

$$\Leftrightarrow \begin{cases} \displaystyle\lim_{q \to -\infty} \tilde{a}(q, s) = -(2\pi i)^{-1} \log s \in \mathscr{C}_\infty \\ \displaystyle\lim_{q \to -\infty} \frac{\partial}{\partial q} \tilde{a}(q, s) = 0 \in \mathscr{C}_\infty \end{cases}$$

$$\tag{6.27}$$

We *conjecture* that this problem admits a (unique) solution in the space $\mathscr{C}$ for all finite q:

$$\tilde{\psi}(q, .) \in \mathscr{C} \quad \text{for all} \quad q \in \dot{\mathbb{C}} \quad \text{(and is q-analytic)} \tag{6.28}$$

and also for infinite q in the sense that, for instance:

$$\tilde{a}(s) = \lim_{q \to +\infty} \tilde{a}(q, s) \quad \text{exists and} \quad \tilde{a}(s) \in \mathscr{C} \tag{6.29}$$

($\tilde{a}(s)$ is the Fourier transform of the Jost function $a(x)$, as seen from (4.26)). Armed with this conjecture we can now fit our WKB formulas to this particular function $\tilde{\psi}$:

a) The *branch curves* of $\tilde{\psi}(q, s)$ in $\mathbb{C}^2$ are certain solutions of the Hamilton-Jacobi equation:

$$\left(\frac{\partial s}{\partial q}\right)^2 = V(q) - E\,[= (ip(q))^2]\qquad (5.8)\text{-}(6.30)$$

Now our asymptotic conditions at $q = -\infty$ only allow us the one solution that behaves like $-i\int_{q_0}^q p\,dq$ for $q \to -\infty$ (with $\operatorname{Arg} p = -\pi/2$) i. e.:

$$s(q_0) = 0, \qquad \frac{\partial s}{\partial q}(q_0) < 0.\qquad (6.31)$$

Conditions (6.30) and (6.31) define a unique connected analytic curve $S \subset \mathbb{C}^2$, which is the graph of a multiply-valued classical action $s(q)$ (ramified at the turning points). By global analytic continuation we state that $\tilde{\psi}(q, s)$ is ramified around (and only around) that curve S. Note that half of the branches of S behave like $+i\int p\,dq$ and generate dominant solutions for $q \to -\infty$ (independent of (6.21)).

b) The discontinuity $\Delta_{s(q)}\tilde{\psi}(q, \cdot)$ propagates along S according to the general formula (5.13), but it is also subjected to the asymptotic constraint $\Delta_{s(q)}\tilde{\psi}(q, s+t) \sim p(q)^{-1/2}\theta(t)$ when $q \to -\infty$ on the branch $s(q) = -i\int_{q_0}^q p\,dq$, because of (6.23). Hence:

$$\Delta_{s(q)}\tilde{\psi}(q, s(q)+t) = \left\{ u(q, x)^{-1/2}\exp ix\int_{-\infty}^q [u(q', x)-p(q')]dq' \right\}_{\mathrm{B}}(t) \quad (6.32)$$

The discontinuity at q depends in principle on the continuation path γ from $-\infty$ to q, but this is no inconsistency since the location of the branch point $s(q)$ on the Riemann surface of $\tilde{\psi}(q, \cdot)$ will also depend on γ (even for identical values of $s(q)$). Formula (6.32) also implies that $\tilde{\psi}(q, \cdot) \in \mathscr{C}_\infty$ precisely.

c) From the asymptotic conditions (6.27) we also draw that $\Delta_{s(q)}\tilde{\psi}(q, \cdot)$ is *Borel summable* and equal to its Borel sum for q negative large enough. With the notation (6.3):

$$\psi(q, x) = \phi_+(q, x) \qquad \forall q < Q_0\qquad (6.33)$$

for some real Q_0. Indeed, in all sheets connecting around the branch point $s(q) = -i\int_{q_0}^q p\,dq$ (tending to $+\infty$ as $q \to -\infty$), conditions (6.27) force all other branch points to disappear when $q \to -\infty$ (the persistence of a branch point whose discontinuity would vanish relatively to $\Delta_{s(q)}\tilde{\psi}$ is ruled out by the propagation formula (5.13)). Therefore, when $q \to -\infty$, those branch points that move opposite to $s(q)$ must tend to $-\infty$ *uni-*

formly, and those that move parallel to $s(q)$ must not remain accessible by paths staying within a finite distance from $s(q)$. No branch point can thus cross (or lie on) the cut $(s(q), +\infty)$ when q is less than some Q_0, i. e. no obstruction to Borel resummation can occur there. Now the asymptotic limit (6.27) is obviously its Borel sum, hence the same is true for $\psi(q, x)$ when $q < Q_0$. In a nutshell, permanence of form of the Borel representation (6.5) away from a Stokes line stays true for $q \to \infty$ (even along an asymptotic direction of Stokes lines, provided we consider the recessive solution).

d) From our treatment of the connection problem we deduce that $\psi(q, x)$ stays equal to its Borel sum as long as it is continued from the former interval $(-\infty, Q_0)$ (in short: from $q = -\infty$) along a path in $\dot{\mathbb{C}}$ *avoiding all Stokes lines*. The continuation of $\psi(q, x)$ across a Stokes line is expressed through the connection formulas (6.15)-(6.20) as a new combination of two Borel sums, which can be separately continued further on. Every Borel transform $\Delta_{s(q)}\tilde{\psi}$ has non-zero radius of convergence away from turning points (that radius is the distance to the nearest branch $s'(q)$ that coalesces with $s(q)$ at some turning point: see remark at the end of this section) and is Borel summable away from Stokes lines.

In particular, in the situation where $E < \inf\limits_{q\in\mathbb{R}} V(q)$ ($= 0$ by convention), the real axis is an integral curve of (6.8) that contains no turning points, hence it cannot meet any Stokes line and $\psi(q, x)$ *stays equal to its Borel sum* $\phi_+(q, x)$ for all $q \in \mathbb{R}$. By taking the limit $q \to +\infty$ in Eq. (6.24) we also conclude that *the Jost function a(x)* equals *the Borel sum of its asymptotic expansion* (4.27) for $E < 0$ (and $x > 0$, as was assumed throughout).

e) By following $\tilde{\psi}(q, s)$ along all possible paths in $\dot{\mathbb{C}}$ starting from $-\infty$, we could in principle explore its global analytic structure thanks to the connection rules. The same is true for the transformed Jost function $\tilde{a}(s)$ provided we restrict ourselves to paths from $-\infty$ to $+\infty$ that are homotopic to the real axis in the double covering $\dot{\mathbb{C}}_2$ (because we want to land on the dominant branch of $\tilde{\psi}$ for $q \to +\infty$). But we find this procedure very cumbersome in practice because it forces us to visualize the relative motion of all branches of S in $\mathbb{C}^2$ as $q \in \dot{\mathbb{C}}$ varies arbitrarily. Being mainly interested in the Jost function (whose zeros yield the eigenvalues), we shall see that a more efficient analysis of its Borel transform $a_B(s)$ results by combining the previously derived connection formulas with the « radar method » of section 3, with x allowed to become a *complex* variable. The computation of the analytic structure of $a_B(s)$ together with the topology of its Riemann surface will thus reduce to a purely *algebraic*, combinatorial procedure, to be described on detailed examples in the next two sections (the Borel-transformed solution $\tilde{\psi}(q, s)$ itself could be studied likewise, but will no longer be considered here).

Then the only features from this section to be retained for later use are the connection formula (6.20), and the related analytic properties in s of the Borel transform $\tilde{\psi}(q, s)$:

— « analyticity (non zero radius of convergence) away from the turning points,   (6.34)

and:

— « Borel summability away from the Stokes lines »   (6.35)

Remark. — The radius of convergence in (6.34) can also be controlled by a direct analytic treatment of a Riccati equation (like Eq. (2.8) of Ref. [2]) equivalent to our Schrödinger equation. The computation is similar to one carried out in [21], section 2.

7. THE ANALYSIS OF THE JOST FUNCTION :
PRELIMINARIES

In this section we set up a list of useful formulas for the practical implementation of the theory explained in the previous chapters. We shall dwell to some extent on the case of homogeneous potentials ($V(q) = q^{2M}$, $M > 0$ an integer), borrowing the notations and some results from Ref. [3]. Finally we shall treat in detail the case of the harmonic oscillator $M = 1$ as a preparation to the next section.

We first list a selection of those previous formulas that will actually be needed. From WKB theory, we shall use:

1) the general form of a solution of the Schrödinger equation (5.1) in the complex plane ($x = \hbar^{-1}$):

$$\psi = b_+ \phi_+ + b_- \phi_-, \qquad \phi_\pm = L^0_{s_\pm(q)} \left[(u^{-1/2} e^{\pm ix \int_{q_0}^q u\,dq})_B \right] \qquad (7.1)$$

where the expansion $u = \sum_0^\infty u_{2n}(q) x^{-2n}$ is computed from the equation:

$$u^2 - p(q)^2 = \hbar^2 (u^{-1/2})'' u^{1/2}, \qquad u_0 = p \qquad (4.3)$$

and the notation $\phi_\pm$ conforms to Eq. (6.3), with $s_\pm(q) = \mp i \int_{q_0}^q p\,dq$.

2) The exact connection formula for the normalization (7.1):

$$\begin{pmatrix} b_+ \\ b_- \end{pmatrix}(q') = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} b_+ \\ b_- \end{pmatrix}(q), \qquad \alpha = iu_\Gamma \qquad (6.20)$$

when q' is reached from q by crossing *counterclockwise* a Stokes line on which ϕ_+ *dominates* (the other cases follow trivially), and the multiplier α

is *Borel-resummed*. We shall also refer to the related analyticity proper-
ties (6.34)-(6.35).

3) The two expansions, of the Jost function at forbidden energies
$(E < \inf_{\mathbb{R}} V)$, and of the Bohr-Sommerfeld eigenvalue condition in the
case of two real turning points: formulas (4.27) and (4.29), which are
deduced from (7.1) plus asymptotic constraints.
And from section 3 we shall borrow the discontinuity formulas (3.9)-
(3.20) for explicit computations.

Some contour integrals: we conveniently choose $q_0 = 0$ as the base
point for all subsequent integration paths, especially in all WKB for-
mulas like (7.1) (we assume that it is not a turning point). Let the classical
problem have 2M complex turning points, which we label $q'_k, k = 1, \ldots, 2M$
in increasing order of Arg q'_k from the value 0. In this article we raise neither
problem of 1) coinciding values of Arg q'_k (for which an ordering prescrip-
tion similar to the $e^{\pm i0}$ convention of section 3 can easily be added); 2) infini-
tely many turning points (non polynomial V *vs* its polynomial approxi-
mations).

We now place radial cuts in the complex q plane (along the half-lines
$\{ \text{Arg } (q - q'_j) = \text{Arg } q'_j \}$ and we pick in this cut plane or *first sheet* the
complex determinations of the functions $p(q)$ and $p^{-1/2}(q)$ that satisfy:

$$\text{Arg } p(0) = -\pi/2, \qquad \text{Arg } p(0)^{-1/2} = +\pi/4 \qquad (7.2)$$

(all subsequent determinations of $p(q)$ will also be specified on the *first
sheet* of $\mathbb{C}$ without mention).

Let γ_j be any loop based at $q_0 = 0$ and encircling positively once the
straight segment $[0, q'_j]$ alone. These 2M loops are important as they
generate the fundamental group $\pi_1(\hat{\mathbb{C}}, q_0)$ of the punctured plane
$\hat{\mathbb{C}} = \mathbb{C} \backslash \{ q'_1, \ldots, q'_{2M} \}$ [22]. We shall also need paths C_j encircling posi-
tively each cut. Fig. 11 depicts the cuts and contours for $V(q) = q^6$ (dashed
lines lie outside the first sheet).
Instrumental in our WKB computations are the following integrals and
the various relations connecting them as formal power series in x^{-2}:

$$\int_{\gamma_j} u(q, x)dq \qquad \text{and} \qquad \int_{C_j} [u(q, x) - p(q)]dq \qquad (7.3)$$

Before writing a few formal relations (which are actually meant to be later
resummed exactly by the trick (3.18)), we note that the integrals along C_j
converge term by term if conditions (4.9)-(4.10) hold in the relevant *com-
plex* directions.

The most general relations are those obtained by contour deformation.
We set:

$$\gamma_{jk} = \gamma_k - \gamma_j \qquad (7.4)$$
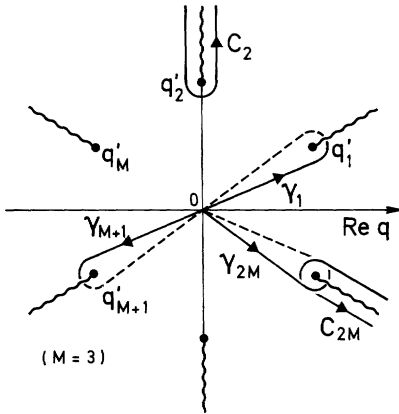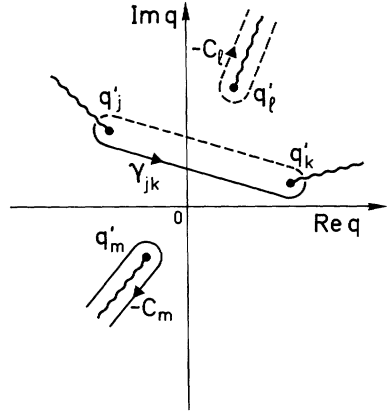
FIG. 11.   FIG. 12.

This is a loop in the double covering $\dot{\mathbb{C}}_2$ encircling positively q'_j and q'_k, such that the portion.oriented from q'_j towards q'_k lies in the first sheet of $\dot{\mathbb{C}}_2$, and the rest lies in the second sheet (Fig. 12). In the contour integral

$$\int_{\gamma_k} u\,dq - \int_{\gamma_j} u\,dq = \int_{\gamma_{jk}} p\,dq + \int_{\gamma_{jk}} (u-p)\,dq$$

we can « blow up to infinity » the contour γ_{jk} for the second term and decompose it as a sum of integrals along the C_l (Fig. 12).

The final result is:

for $j = k$:
$$0 = \sum_{l=1}^{2M} \int_{C_l} (u-p)\,dq \qquad (7.5)$$

for $j \neq k$:
$$\int_{\gamma_{jk}} u\,dq = i\omega(\gamma_{jk}) + \sum_{k<l<j} \int_{C_l} (u-p)\,dq - \sum_{j<m<k} \int_{C_m} (u-p)\,dq \qquad (7.6)$$

where $\omega(\gamma_{jk})$ is the action period (5.10) of γ_{jk}, the summation indices are restricted in the sense of *cyclic* ordering of the turning points, and the signs take into account the fact that $(u-p)$ has opposite determinations on the two sheets of $\dot{\mathbb{C}}_2$.

Other relations will express the semi-classical expansions of ultimate interest. For instance, the Jost function expansion (4.27) becomes, again by path deformation:

$$\log a(x) = ix \sum_{1}^{M} \int_{C_l} (u-p)\,dq = -ix \sum_{M+1}^{2M} \int_{C_m} (u-p)\,dq$$
$$(E < 0, \text{ Arg } p(0) = -\pi/2) \quad (7.7)$$

Similarly the Bohr-Sommerfeld condition (4.29) can be rewritten as:

$$x \int_{\gamma \varepsilon_l} u \, dq = (2k + 1)\pi \quad (E > 0; \quad \text{Arg } p(q) = 0 \quad \text{for} \quad q'_j < q < q'_i) \quad (7.8)$$

where γ_{jl} encircles the two turning points that are real $(q'_j < q'_i)$ in the positive energy case (always assuming a « simple well » case).

We remark that all quantities and relations written until now make sense for arbitrary analytic potentials (namely for polynomial ones, unless one accepts to work with infinitely many turning points). But in order to be complete we should also exploit the relations stemming from the particular symmetries of the potential. Among these:

— reality of $(V - E)$ implies, for any pair of complex conjugate turning points q'_j and q'_k (and for real x):

$$\int_{C_j} (u - p) \, dq = + \left(\int_{C_k} (u - p) \, dq \right)^*, \quad \int_{\gamma_j} u \, dq = - \left(\int_{\gamma_k} u \, dq \right)^* \quad (7.9)$$

— parity $(V(q) = V(-q))$ implies that p and u are even in q, hence

$$\left. \begin{array}{l} \displaystyle\int_{C_j} (u - p) \, dq = - \int_{C_k} (u - p) \, dq \\[3mm] \displaystyle\int_{\gamma_j} u \, dq = - \int_{\gamma_k} u \, dq = - \frac{1}{2} \int_{\gamma_{jk}} u \, dq \end{array} \right\} \quad (7.10)$$

for any pair of opposite turning points q'_j and q'_k.

— The *homogeneous case*: we now take our confining potential to have the form:

$$V(q) = q^{2M}, \quad M \text{ positive integer} \quad (7.11)$$

This restriction is by no means crucial (we shall give a few hints about the general case in § 9) but it will eliminate technicalities and highlight the abstract structure of the theory by reducing the number of free parameters to the bare minimum. When $V = q^{2M}$, the solutions u of eq. (4.3) satisfy the scaling rule:

$$u(q, x, E) = \alpha^M u(\alpha^{-1} q, \alpha^{1+M} x, \alpha^{-2M} E) \quad (\forall \alpha \in \mathbb{C}^*) \quad (7.12)$$

which makes the analytic continuations in x and in E *equivalent*; it also implies, for any path Γ and $x = |x| e^{-i\varphi}$:

$$|x| \int_{\alpha^{-1}\Gamma} u(q', |x|, \alpha^{-2M}E) \, dq' = x \int_{\Gamma} u(q, x, E) \, dq, \quad \alpha = e^{\frac{i\varphi}{M+1}} \quad (7.13)$$

In operator language: $\hat{H}(x) = -x^{-2} d^2/dq^2 + q^{2M}$ is unitary equivalent (under a coordinate dilation) to $x^{-2M/(M+1)} \hat{H}(1)$; hence any dimensionless

and coordinate free quantity like the Jost function $a(E, x)$ obeys the scaling law:

$$a(E, x) \equiv a(\lambda^{\frac{M+1}{2M}}) \qquad \text{where} \qquad \lambda = -x^{2M/(M+1)}E \qquad (7.14)$$

We can then fix E to any constant value E_0 we please and keep *complex* $x = |x| e^{-i\varphi}$ as only parameter. To study the Jost function at negative energies, $E_0 = -1$ is a convenient choice.

The classical problem (7.11) has then the 2M turning points:

$$q'_j = e^{\frac{i}{2M}(2j-1)\pi} \qquad (7.15)$$

At fixed E all power series in x^{-1} like (7.3), (7.7)-(7.8) acquire purely numerical coefficients and become easier to study. We are going to show that all such coefficients are proportional to the coefficients a_n of the *Jost function expansion* (4.27) taken in logarithmic form at $E = -1$, where $a(E, x) \equiv a(x)$ according to (7.14):

$$\log a(x) = ix \int_{-\infty}^{\infty} (u - p)dq = \sum_{n=1}^{\infty} a_n x^{1-2n} \qquad (7.16)$$

The a_n are real, M-dependent numbers, computable as $a_n = i \int_{-\infty}^{\infty} u_{2n}dq$, for instance $a_1 = M\Gamma(2-1/2M)\Gamma(1/2+1/2M)/6\sqrt{\pi}$, but no closed form is known for the general a_n. Another series of separate interest is the *Bohr-Sommerfeld expansion* (7.8) valid for positive energies, which for

$$E = |E_0| = +1$$

we denote as:

$$\sum_0^{\infty} b_n x^{1-2n} = 2\pi\left(k + \frac{1}{2}\right) \qquad (7.17)$$

Some b_n have been computed previously but at *different* normalizations [23, 1-3]; here

$$b_0 = \frac{\sqrt{\pi}\Gamma(1/2M)}{M\Gamma(3/2+1/2M)}, \qquad b_1 = -\frac{M\sqrt{\pi}\Gamma(2-1/2M)}{3\Gamma(1/2-1/2M)} \qquad (7.18)$$

Bender *et al.* [23] went as far as b_7, their notations being: $N = 2M$, and $a_n(N) = b_n(M)/2$.

We now relate explicitly the values of all contour integrals $\int_\gamma u_{2n}dq$, such as b_n, to the coefficients a_n in (7.16).

Consider first $\int_{\gamma_j} udq$, where γ_j encircles the turning point q'_j, and rotate q'_j and γ_j by $\alpha^{-1} = e^{-i(2j-1)\pi/2M}$ to bring q'_j to the positive real axis where it

can contribute to the Bohr-Sommerfeld condition. By formula (7.13):

$$x \int_{\gamma_j} u(q, x, E_0) dq = |x| \int_{\alpha^{-1}\gamma_j} u(q', |x|, e^{-i(2j-1)\pi}E_0) dq',$$

$$\text{Arg } x = -\frac{(M+1)}{2M}(2j-1)\pi \quad (7.19)$$

In the right-hand side of (7.19), u is a positive energy ($E = |E_0|$) solution, whose sign has to be specified by fixing the sign of $p(q)$ at a point, say $q=0$. We use $p(0) = [e^{-i(2j-1)\pi}E_0]^{1/2}$ with $\text{Arg } E_0^{1/2} = -\pi/2$ by Eq. (7.2), hence

$$\text{Arg } p(0) = \text{Arg } [e^{-i(2j-1)\pi}E_0]^{1/2} = -j\pi \quad (7.20)$$

Now $\alpha^{-1}\gamma_j$ is a positive contour encircling the positive turning point, therefore Eq. (7.19) with the sign correction (7.20) is half the contribution to the Bohr-Sommerfeld rule (the negative turning point contributes the same amount by parity). Finally:

$$|x| \int_{\alpha^{-1}\gamma_j} u(q', |x|, e^{-i(2j-1)\pi}E_0) = \frac{(-1)^j}{2} \sum_0^\infty b_n |x|^{1-2n}$$

$$= x \int_{\gamma_j} u(q, x, E_0)$$

for $\text{Arg } x = -\varphi_j = -\frac{(M+1)}{2M}(2j-1)\pi$, which amounts to the explicit expansion:

$$\int_{\gamma_j} u \, dq = \frac{(-1)^j}{2} \sum_{n=0}^\infty b_n e^{i\varphi_j(1-2n)} x^{-2n} = -\frac{i}{2} \sum_{n=0}^\infty (-1)^n b_n e^{i(2j-1)(1-2n)\pi/2M} x^{-2n} \quad (7.21)$$

which is our first result.

Next we return to our basic expansion (7.16). If we tilt the integration axis by $+\dfrac{\pi}{M}$, the scaling laws (7.12)-(7.13) imply:

$$\int_{-e^{i\pi/M}\infty}^{+e^{i\pi/M}\infty} (u-p)(q, x, E_0) dq = e^{-i\pi} \int_{-e^{i\pi/M}\infty}^{+e^{i\pi/M}\infty} (u-p)(q, x, e^{2i\pi}E_0) dq \quad (7.22)$$

$$= e^{i\pi/M} \int_{-\infty}^\infty (u-p)(q', xe^{\frac{i(M+1)\pi}{M}}, E_0) dq'$$

hence

$$\left[\int_{-\infty}^\infty + \int_{-e^{i\pi/M}\infty}^{+e^{i\pi/M}\infty} \right] (u-p) dq = -i \sum_{n=1}^\infty (a_n x^{-2n} + e^{i\pi/M} a_n(-xe^{i\pi/M})^{-2n}) \quad (7.23)$$

Now the second contour can be reversed and shifted to the second sheet,
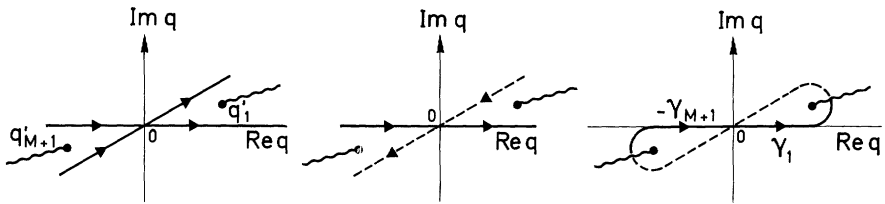
FIG. 13.

since $(u-p)$ takes its opposite determination there (fig. 13); then by contour deformation the left-hand side of (7.23) is recognized as:

$$\left[\int_{\gamma_1} - \int_{\gamma_{M+1}}\right](u-p)dq = 2\int_{\gamma_1}(u-p)dq = -i\sum_{n=1}^{\infty}(-1)^n b_n e^{i(1-2n)\pi/2M}x^{-2n} \tag{7.24}$$

according to our previous result (7.21). If we identify (7.23-7.24) term by term:

$$(n \geq 1): \quad b_n = 2(-1)^n a_n \cos(1-2n)\frac{\pi}{2M} = 2a_n \sin\left[(1-2n)\left(\frac{M+1}{2M}\right)\pi\right] \tag{7.25}$$

Formula (7.25) concretely exhibits the intimate relationship between the two expansions (7.16) and (7.17). Actually, for odd M Eq. (7.25) causes b_n to *vanish* whenever M divides $(2n-1)$; the numbers a_n are thus more fundamental (contain more information) than the numbers b_n. Finally we extend formula (7.25) to $n=0$ so as to *define* a_0 although there is no such number in the expansion (7.16). Remark: the relation (7.25) resulted implicitly from properties of the zeta function in Ref. [3] (by combination of Eqs. (27), (31), (38)). Also, $a_0 = \infty$ in the harmonic case $M = 1$.

Finally we change one sign in Eq. (7.23):

$$\left[\int_{-\infty}^{\infty} - \int_{-e^{i\pi/M}\infty}^{+e^{i\pi/M}\infty}\right](u-p)dq = -i\sum_{n=1}^{\infty}(a_n x^{-2n} - e^{i\pi/M}a_n(-xe^{i\pi/M})^{-2n})$$

and recognize the left hand side, by contour deformation (fig. 14), as:

$$\left[\int_{C_1} - \int_{C_{M+1}}\right](u-p)dq = 2\int_{C_1}(u-p)dq$$

We thus deduce the expression for this last integral and, by the scaling law, for the rotated ones:

$$\int_{C_j}(u-p)dq = \sum_{n=1}^{\infty} a_n e^{i(2j-1)(1-2n)\pi/2M}\sin\left[(2n-1)\frac{\pi}{2M}\right]x^{-2n} \tag{7.26}$$
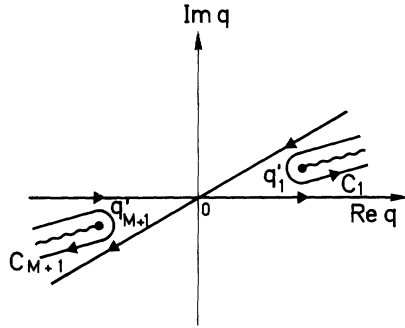
FIG. 14.

Formulas (7.21), (7.25-7.26) constitute our results, to be applied mostly in the cases $M = 1$ and 2; we found it equally easy to write them down directly for general M.

Our program will now be to recompute the Laplace representation of the Jost function $a(x)$ for every value of $\varphi = - \text{Arg } x$ in the Schrödinger equation, starting from the value $\varphi = 0$ for which we known that $a(x)$ equals its Borel sum:

$$a(x) = x \int_0^\infty e^{-xs} \left[\exp \sum_1^\infty a_n x^{1-2n} \right]_B (s) = L_0^0 a_B \qquad (7.27)$$

(a_B is also equal to $\Delta_0 \tilde{a}$ of Eq. (6.29)).

We recall that (7.27) results from the representation of a solution recessive at $- \infty$:

$$\psi_+(q) = L_{s_+(q)}^0 \left[u^{-1/2} e^{ix \int_0^q u dq} \right]_B, \qquad s_+(q) = - i \int_0^q p dq \quad (7.28)$$

which holds for *all real* q when $\varphi = 0$, because the real axis meets no Stokes lines. When we start varying φ, two important things happen:

a) The analytic continuation in x of $\psi_+(q, x)$ stops being recessive at $q = - \infty$ when $| \varphi | = \pi/2$; it does stay recessive whatever φ, when $q \to \infty$ in the continuously *rotating* complex direction defined by:

$$ix \int_0^q p dq \to - \infty \Leftrightarrow q \to - e^{\frac{i\varphi}{M+1}} \infty . \qquad (7.29)$$

For the same reason the analytic continuation of the Jost function $a(x)$ has a new definition, instead of (4.15-4.16):

$$\psi_+(q, x) \sim a(x) p^{-1/2} \exp ix \int_0^q p dq, \qquad q \to + e^{\frac{i\varphi}{M+1}} \infty \quad (7.30)$$

As φ varies, the integration paths in (7.28-7.30) should be deformed if necessary to avoid turning points and thus preserve analyticity in x.

b) The Stokes lines *move* in the q plane as their definition is changed: they are no longer tangent to the direction field (6.8), but instead they satisfy:

$$Re\left[e^{-i\varphi}\int_{q'_j}^{q} p(q')dq'\right] = 0 \qquad j = 1, 2, \ldots, 2M \qquad (7.31)$$

The reason is that the directions (6.8) corresponded to the choice of steepest descent contours to define the standard representation of (6.2). When Arg $x = -\varphi$ the steepest descent contours all rotate by $+\varphi$ as in section 2 (again we could keep Arg $x = 0$ and still rotate the contours by $+\varphi$ with the same effect, but then only for $|\varphi| < \pi/2$). Remark: each of the directions (7.29-7.30) stays asymptotic to a moving Stokes line.

The important point here is that if we solve the connection problem from $-e^{\overline{M+1}}\infty$ to $+e^{\overline{M+1}}\infty$ with Arg $x = -\varphi$ and with the definition (7.31) of Stokes lines, then we shall obtain for the Jost function its Laplace representation with all integration paths tilted by $+\varphi$:

$$a(x) = \sum_j L^{\varphi}_{s_j} f_j \qquad (7.32)$$

Our aim is to describe this collection of representations for $\varphi \in [0, 2\pi[$ and use all discontinuities in form of (7.32) for some φ to signal the singularities of the functions f_j (according to the « radar method » of section 3). The reason for which Eq. (7.27) cannot hold for arbitrary φ is a topological one. For each fixed φ, the Stokes lines separate the q plane in disjoint connected regions (call them *Stokes regions*) in which the standard representation of $\psi(q, x)$ stays constant in form. Now to derive (7.28) and its corollary Eq. (7.27), we needed one such region connecting the asymptotic directions $q \to -\infty$ and $q \to +\infty$. As φ varies, the Stokes regions are distorted but their global arrangement is preserved, except at some « critical angles » for which two Stokes lines *merge into a segment connecting two turning points*. At those angles the pattern of Stokes regions is disrupted and the representation (7.32) may change form. By definition a critical angle φ satisfies:

$$Re\, e^{-i\varphi}\int_{q'_j}^{q'_k} pdq = 0 \Leftrightarrow Re\, e^{-i\varphi}\int_{\gamma_{jk}} pdq = 0 \quad \text{for some } j, k. \quad (7.33)$$

For each angular interval delimited by two critical angles, we must then compute afresh the connection matrix F such that:

$$\psi = b_+ \phi_+ + b_- \phi_- \left(q \to -e^{\frac{i\varphi}{M+1}}\infty\right) \Rightarrow \psi = b'_+ \phi_+ + b'_- \phi_- \left(q \to +e^{\frac{i\varphi}{M+1}}\infty\right)$$

with

$$\begin{pmatrix} b'_+ \\ b'_- \end{pmatrix} = \begin{pmatrix} F^{++} & F^{+-} \\ F^{-+} & F^{--} \end{pmatrix} \begin{pmatrix} b_+ \\ b_- \end{pmatrix} \tag{7.34}$$

The computation of the Jost function then amounts to that of b'_+ under the condition $b_- = 0$: precisely, due to the normalization difference between (4.13-4.16) and (7.1):

$$a(x) = F^{++}(x) L_0^{\varphi} \left[\exp ix \int_{-e^{\frac{i\varphi}{M+1}} \infty}^{+e^{\frac{i\varphi}{M+1}} \infty} (u-p) dq \right]_B = F^{++}(x) L_0^{\varphi} [a_\Gamma]_B \tag{7.35}$$

where the integration path Γ is deformed from the real axis at $\varphi = 0$ to have endpoints $\pm e^{i\varphi/(M+1)} \infty$ and avoid all turning points as well. Our connection rules directly provide F^{++} in the form of a Borel sum $\displaystyle\sum_j L_{s,j}^{\varphi} f_j$

and finally the product rule (3.13) transforms (7.35) into the desired representation (7.32).

We shall now carry out this program explicitly in great detail for the harmonic oscillator ($M = 1$) before turning to a more interesting case in next section.

The harmonic oscillator ($M = 1$):

$$\left(-x^{-2} \frac{d^2}{dq^2} + q^2 \right) \psi = -\psi \tag{7.36}$$

The eigenvalue formula (7.17) stops at the leading term because the relation (7.25) shows that all $b_n \equiv 0$ ($n \geq 1$). This alone does not prove Borel summability of the Bohr-Sommerfeld expansion: we must bring in from other sources the information that the eigenvalue formula $b_0 x = (2k+1)\pi$ is exact. This external input does not satisfy us because we aim at deriving results about Borel summability entirely within our semiclassical (complex WKB) framework.

Let us then turn to the Jost function expansion (7·16) which is non trivial and known to be Borel summable since it was computed at negative energy. We are going to show that the connection formulas (6.20) allow to investigate the analytical structure of this Borel transform. In this case the latter can again be computed by other means, thus allowing a *check* of our method before we apply it to the quartic oscillator case.

For clarity we rewrite the various integrals just computed in this case of two turning points $q'_1 = i$, $q'_2 = -i$ (fig. 15); here $b_0 = \pi$, all other $b_n \equiv 0$:

$$\int_{\gamma_1} u dq = -\int_{\gamma_2} u dq = \pi/2 \Rightarrow \omega(\gamma_{12}) = -i \int_{\gamma_{12}} p dq = i\pi \tag{7.37}$$

$$\int_{-\infty}^{\infty} (u-p)dq = \int_{C_1} (u-p)dq = -\int_{C_2} (u-p)dq = -i \sum_{1}^{\infty} a_n x^{-2n} \quad (7.38)$$

(here the a_n are known explicitly: Eq. (7.56)).
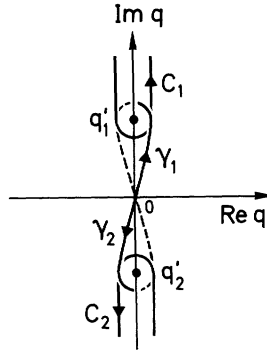
FIG. 15.

We now compute the connection matrix F from $-e^{\frac{i\varphi}{2}}\infty$ to $+e^{\frac{i\varphi}{2}}\infty$ for all angles φ. These asymptotic directions are marked by the symbols $\ominus$ and $\oplus$ on the various diagrams of Stokes regions (fig. 16 *a-e*), which also carry cuts to make the ‘WKB waves single-valued, the determinations being Arg $p(0) = -\pi/2$, Arg $p^{-1/2}(0) = +\pi/4$ as usual on the first sheet. Cuts can be arbitrary drawn from each turning point to infinity; we shall conveniently place them within Stokes regions never to be crossed. On each asymptote of a Stokes line we then indicate whichever WKB wave $\phi_{\pm}$ dominates there.

The critical angles are $\varphi = \pm \pi/2$ by Eq. (7.33), hence $a_B(s)$ can have singularities on the *imaginary axis* only.

a) $|\varphi| < \pi/2$ (fig. 16 *a-c*): we must connect the region marked A to region C across region B. According to our connection rule the F matrix from A to B is:

$$F_{BA} = \begin{pmatrix} 1 & iu_{-\gamma_1}(x) \\ 0 & 1 \end{pmatrix}, \qquad u_{-\gamma_1} = u_{\gamma_1}^{-1} = \exp -ix \int_{\gamma_1} u(q,x)dq \quad (7.39)$$

because the Stokes line between A and B is crossed anticlockwise and ϕ_- dominates. On the contrary, to go from B to C we cross clockwise a Stokes line on which ϕ_+ dominates:

$$F_{CB} = \begin{pmatrix} 1 & 0 \\ -iu_{\gamma_2}(x) & 1 \end{pmatrix}, \qquad u_{\gamma_2}(x) = \exp ix \int_{\gamma_2} udq \quad (7.40)$$

and finally, using definition (7.4):

$$F_{CA}(\varphi) = F_{CB}F_{BA} = \begin{pmatrix} 1 & iu_{-\gamma_1} \\ -iu_{\gamma_2} & 1+u_{\gamma_{12}} \end{pmatrix} \quad \text{for} \quad |\varphi| < \pi/2 \quad (7.41)$$

Reporting into (7.35) the value:

$$F^{++} = 1 \tag{7.42}$$

we recover the Borel sum formula (7.27) for $a(x)$, $|\operatorname{Arg} x| < \pi/2$. Here we could equally well have used $F_{BB} = 1$ directly since both asymptotic directions $\pm e^{i\varphi/2}\infty$ also belong to region B (the crossing of any number of Stokes lines along the asymptotic directions changes the connection matrix but not the Jost function, by the principle of exponential dominance [6]), but we also wanted to show what happens to F_{CA} itself at a critical angle, for later purposes.

 b) We now increase φ through $\pi/2$: $\pi/2$ itself is a critical angle for which the simple crossing assumption (6.6) is violated (fig. 16 d) and the connection rule (6.20) is wrong (ill-defined). For $\varphi > \pi/2$ the intermediate region B' gets *differently positioned* (fig. 16 e); the same connection rules as before apply again but now lead to *another* final result:

$$\left.\begin{array}{c} F_{B'A} = \begin{pmatrix} 1 & 0 \\ -iu_{\gamma_2} & 1 \end{pmatrix} \qquad F_{CB'} = \begin{pmatrix} 1 & iu_{-\gamma_1} \\ 0 & 1 \end{pmatrix} \\[2mm] F_{CA}(\varphi) = F_{CB'}F_{B'A} = \begin{pmatrix} 1+u_{\gamma_{12}} & iu_{-\gamma_1} \\ -iu_{\gamma_2} & 1 \end{pmatrix} \qquad \text{for} \qquad \varphi > \pi/2 \end{array}\right\} \tag{7.43}$$

(the same factors as in (7.41) but in *reversed order*). Now, by Eqs. (7.37):

$$F^{++} = 1 + u_{\gamma_{12}} = 1 + e^{-\omega x} \qquad (\omega = \omega(\gamma_{12}) = i\pi) \tag{7.44}$$

$$\Rightarrow a(x) = (1 + e^{-\omega x})L_0^\varphi a_B = L_0^\varphi a_B + L_\omega^\varphi a_B(-\omega + .) \tag{7.45}$$

But the formulas: (7.27) for $\varphi_1 < \pi/2$ and (7.45) for $\varphi_2 > \pi/2$ have a common sector of validity in the x plane $(-\varphi_2 - \pi/2 < \operatorname{Arg} x < -\varphi_1 + \pi/2)$ containing in particular the half-line $\{\operatorname{Arg} x = -\pi/2\}$ if $(\varphi_2 - \varphi_1)$ is small enough. From the resulting equality:

$$a(x) = L_0^{\frac{\pi}{2}-0}a_B = (1 + e^{-\omega x})L_0^{\frac{\pi}{2}+0}a_B \quad \left(\operatorname{Arg} x = -\frac{\pi}{2}\right) \tag{7.46}$$

we then draw a *discontinuity formula* at $\varphi = \pi/2$, of the type (3.11):

$$L_0^{\frac{\pi}{2}+0}a_B - L_0^{\frac{\pi}{2}-0}a_B = -e^{-\omega x}L_0^{\frac{\pi}{2}+0}a_B = -L_\omega^{\frac{\pi}{2}+0}a_B(-\omega+.) \tag{7.47}$$

(typical of a Stokes discontinuity is the exponentially small character of the quantity (7.47), which is $O(e^{-\omega x}) = O(e^{-\pi|x|})$ since $\operatorname{Arg} x = -\varphi = -\pi/2$).

 We have thus obtained that $a_B(s)$ has at $s=\omega$ a discontinuity which is

a) $-\pi/2 < \varphi < 0$

d) $\varphi = \pi/2$

b) $\varphi = 0$

e) $\pi/2 < \varphi < \pi$

c) $0 < \varphi < \pi/2$

FIG. 16.
—— Stokes lines.
〜〜 Cuts.
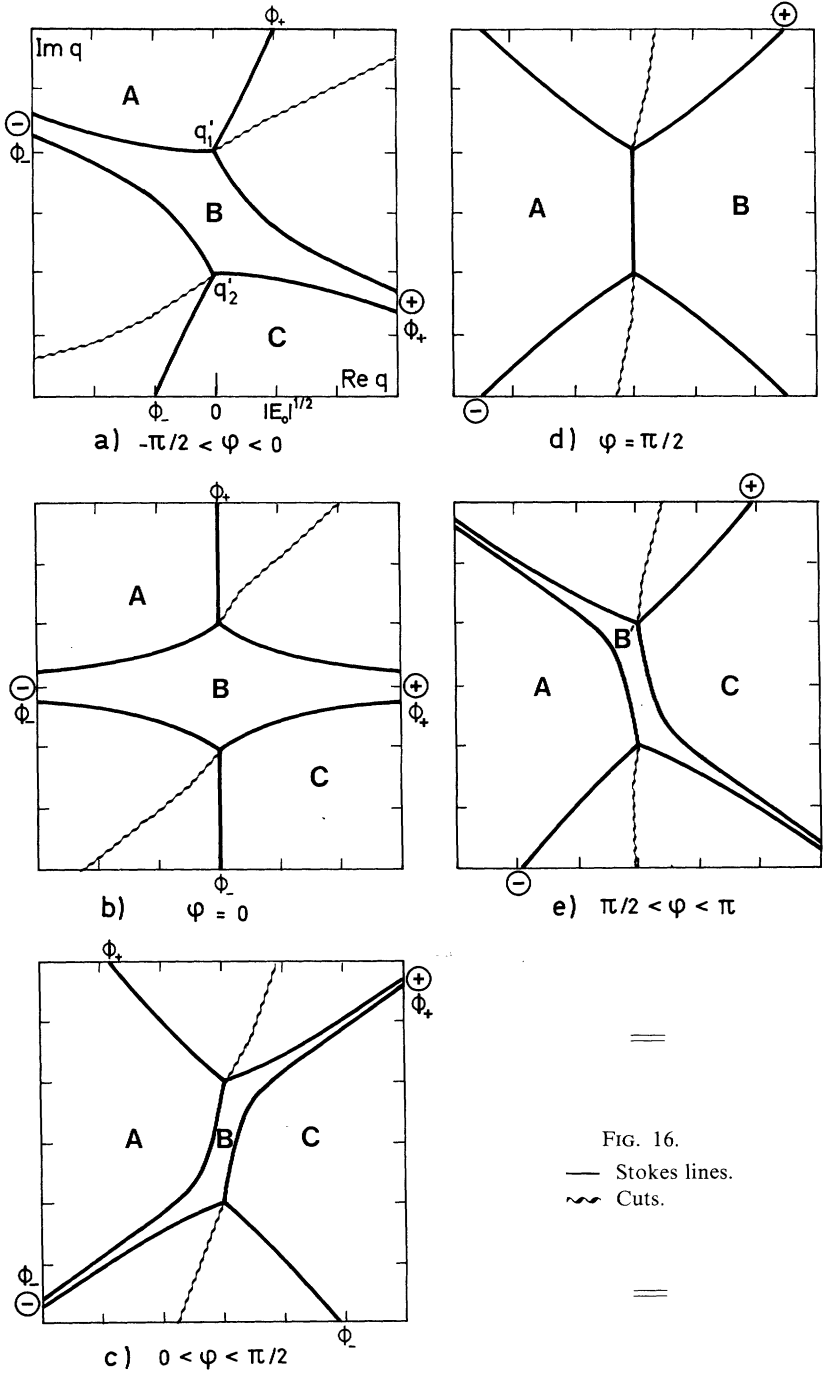
nothing else but $(-a_B)$ itself translated by ω ! Before we comment further on this result, we complete the exploration of the other values of φ: formula (7.45) holds for $\pi/2 < \varphi < 3\pi/2$, and there are similar formulas for $-3\pi/2 < \varphi < -\pi/2$:

$$a(x) = (1+e^{\omega x})L_0^{\varphi}a_B \;\Rightarrow\; L_0^{-\frac{\pi}{2}+0}a_B - L_0^{-\frac{\pi}{2}-0}a_B = +L_{-\omega}^{-\frac{\pi}{2}-0}a_B(\omega+.) \quad (7.48)$$

We have thus exhausted a full angular sector of 2π (and even more but with no extra gain) and found all the discontinuities of $a_B(s)$ in the cut plane drawn on fig. 17; the position of the cuts is *imposed* by the right-hand sides of the discontinuity formulas (7.47-7.48).
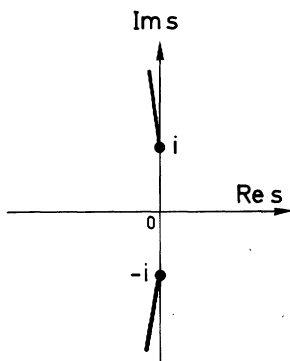
FIG. 17.

c) The following comments are worthwhile as they will suitably generalize to other potentials.

The discontinuities of a_B are very simple transforms of a_B itself: we call this feature *analytic bootstrap*. It allows us to explore all the sheets of the Riemann surface of $a_B(s)$. Indeed, the discontinuity function $a_B(-\omega+.)$ at $(+\omega)$ has itself a discontinuity $a_B(-2\omega+.)$ at 2ω, and so on. In particular the set of branch points on all sheets is $\omega\mathbb{Z}$, whose geometrical origin is clear from our derivation: it is the *group* Ω *of all action periods of* $\hat{C}_2$, defined by Eq. (5.10). Here $\Omega = \omega\mathbb{Z}$, the cyclic group of integral multiples of the primitive period $\omega(\gamma_{12}) = \omega$ computed in Eq. (7.37).

One way of exhibiting all branch points over Ω is to rotate the cuts for $a_B(s)$ differently. We can for instance tilt all cuts issued from the points $n\omega \, (n > 0)$ to the angle $\left(\dfrac{\pi}{2} - 0\right)$ instead of $\left(\dfrac{\pi}{2} + 0\right)$ as in Eq. (7.47). We could perform that operation on one cut at a time in sequential order using the bootstrap property, but this would be very inefficient, as the periodic structure of our problem allows for a much faster method. We

rewrite the equality (7.46) as:

$$L_0^{\frac{\pi}{2}+0} \, a_B = (1 + e^{-\omega x})^{-1} L_0^{\frac{\pi}{2}-0} \, a_B \tag{7.49}$$

and we expand the Taylor series in powers of $e^{-\pi|x|}$ to get:

$$L_0^{\frac{\pi}{2}+0} \, a_B - L_0^{\frac{\pi}{2}-0} \, a_B = \sum_{n=1}^{\infty} (-1)^n e^{-n\omega x} L_0^{\frac{\pi}{2}-0} \, a_B = \sum_{n=1}^{\infty} (-1)^n L_{n\omega}^{\frac{\pi}{2}-0} \, a_B(-n\omega + .) \tag{7.50}$$

which yields the desired result ! (compare with Eq. (3.11)). From the boot-strap property we knew that the discontinuity at $n\omega$ had to be proportional to a_B translated by $n\omega$, but Eq. (7.50) gives us its proper *weight* (on the particular sheet uncovered by the e^{-i0} rotation of the cuts).

We can similarly consider the simpler (odd) series:

$$(\log a)_B = \left(\sum_1^{\infty} a_n x^{1-2n} \right)_B \tag{7.51}$$

By expanding the logarithm of Eq. (7.47) we get:

$$L_0^{\frac{\pi}{2}+0} \, (\log a)_B - L_0^{\frac{\pi}{2}-0} \, (\log a)_B = -\log(1 + e^{-\omega x}) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n\omega x} \tag{7.52}$$

We have used here the second of the formulas (3.18):

$$L_0^{\varphi}[g(a)]_B = g(L_0^{\varphi} a_B) \tag{7.53}$$

for the function $g(z) = \log(1 + z)$ with $z = a(x) - 1$. Thus (7.52) can be understood as a combinatorial trick to get the global analytic structure of $(\log a)_B$ knowning that it is the « convolution logarithm » of a_B. The result (7.52) now has a very simple interpretation for the function $\dfrac{d}{ds}(\log a)_B = a_1(s)$, if we use formula (3.6) and the residue formula to get the discontinuity:

$$L_0^{\varphi}(\log a)_B = \int_0^{e^{i\varphi}\infty} a_1(s) e^{-xs} ds \tag{7.54}$$

This function $a_1(s)$ then has only *simple poles* as singularities, the residue of the pole at $+n\omega$ being $(2\pi i)^{-1}(-1)^{n+1}/n$. This result extends to negative n by parity (a_1 is an even function). Therefore $a_1(s)$ is no longer a ramified, but a single-valued meromorphic function.

d) We now compare our results with known facts about the harmonic oscillator. The following exact formula is proved in Appendix A by a generalization of identity (4.19), namely Eq. (A.16) (where we have set $E = xE_0 = -x$):

$$a(x) = \left(\frac{x}{2e} \right)^{x/2} \sqrt{2\pi} / \Gamma\left(\frac{1}{2} + \frac{x}{2} \right) \tag{7.55}$$

Note that the Stirling formula relates the a_n of Eq. (7.51) to the Bernoulli numbers:

$$a_n = (2^{2n-1} - 1)B_{2n}/[2n(2n - 1)] \qquad (7.56)$$

A Laplace representation of $a(x)$ follows by setting $(-t) = yu^2$ in the Hankel formula

$$\frac{1}{\Gamma\left(\dfrac{1}{2} + y\right)} = -(2i\pi)^{-1} \int_{C'} e^{-t}(-t)^{-\frac{1}{2} - y} dt \qquad (7.57)$$

(where C′ is a contour from $+\infty$ to $+\infty$ encircling positively the origin). We get:

$$\frac{(y/e)^y}{\Gamma\left(\dfrac{1}{2} + y\right)} = (i\pi)^{-1} y^{1/2} \int_C e^{y[u^2 - 2\log u - 1]} du \qquad (7.58)$$

for some contour C, which makes $a(x)$ amenable to the treatment described in section 2. But we prefer to focus on the Laplace representation of $\log a$, which provides a simpler check of our method. Now the useful formulas are (A.13-A.15) with $E = -x$:

$$\frac{d^2}{dx^2} \log a(x) = -\frac{1}{4} \psi'\left(\frac{1+x}{2}\right) + \frac{1}{2x} = \frac{1}{2x} - \sum_{k=0}^{\infty} [x + (2k+1)]^{-2}$$

This has the explicit Laplace representation:

$$\frac{d^2}{dx^2} \log a(x) = \int_0^\infty e^{-xs}\left[\frac{1}{2} - s\sum_{k=0}^{\infty} e^{-(2k+1)s}\right] ds = \int_0^\infty \frac{e^{-xs}}{2}\left(1 - \frac{s}{\text{sh } s}\right) ds \qquad (7.59)$$

which yields after two integrations

$$\log a(x) = \int_0^\infty \frac{e^{-xs}}{2s^2}\left(1 - \frac{s}{\text{sh } s}\right) \qquad (7.60)$$

The condition $\lim_{x \to +\infty} \log a(x) = 0$ that fixes the integration constants is ensured here by the fact that the right-hand side is integrable at $s=0$. By comparison with (7.54) we identify:

$$a_1(s) = \frac{1}{2s}\left(\frac{1}{s} - \frac{1}{\text{sh } s}\right) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{s^2 + n^2\pi^2} \qquad (7.61)$$

Thus, not only the WKB prediction (7.52-7.54) for the analytic structure (poles and residues) of $a_1(s)$ was fully *correct*, but $a_1(s)$ is also the *simplest* function compatible with it: the bare sum of its polar singularities. This raises the hope concerning the general case, that if we could impose and propagate

more stringent growth conditions than (3.1) on the functions of $\mathscr{C}$, we might come closer to an actual reconstruction of the Jost function from its singularities.

e) We can also interpret the previous result in a more physical context.

For $\varphi = \dfrac{\pi}{2} = -\operatorname{Arg} x$, the Schrödinger equation (7.36) also reads

$$- |x|^2 \psi'' - (q^2 + 1)\psi = 0 .$$

It then describes the « underdense » parabolic barrier and $1/a(x)$ is the full complex transmission factor [6]. We have derived the exact analytic structure of the Borel transforms for $a(x)$ and $\log a(x)$ respectively, starting from *first principles* of WKB theory. This makes Borel resummation numerically feasible in principle [24]. By contrast, usual evaluations of $a(x)$ resort to a comparison technique [6] based on special knowledge about the solutions of (7.36). For a realistic (not exactly parabolic) barrier, the quality of the approximation yielded by this comparison method is very hard to assess (and to improve), whereas our method is fully general and does not rely upon the idiosyncrasies of any special solvable potential. Its only present restriction is that the techniques of numerical Borel summation are not yet systematic and efficient enough to be used mechanically.

8. THE QUARTIC OSCILLATOR

This section is devoted to improving and strengthening some results of Refs. [2]-[3]. Unfortunately the normalization adopted there as well as those of earlier works [23] [1] look somewhat artificial in the more comprehensive approach taken here, and will be *discarded*. After this due apologetic warning to the reader, we now set M = 2 in all formulas of § 7 to deal with the following Schrödinger equation:

$$\left(- x^{-2} \frac{d^2}{dq^2} + q^4 \right)\psi = - \psi . \tag{8.1}$$
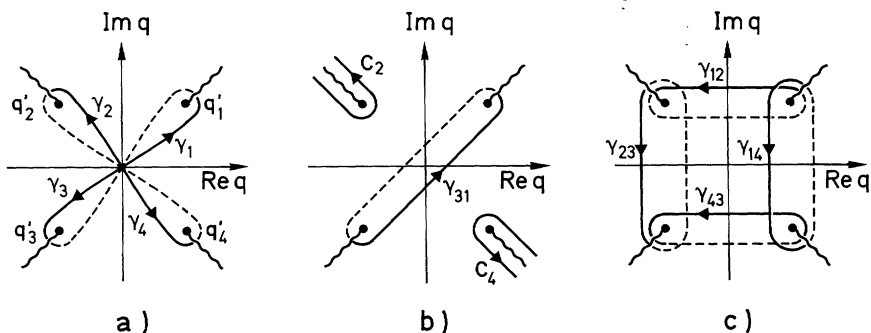
FIG. 18.

There are four turning points $q'_j = e^{i(2j-1)\pi/4}$ (Fig. 18), and:

$$\int_{\gamma_1} u dq = -\int_{\gamma_3} u dq = \frac{1}{2}\oint_{\gamma_{31}} u dq = \frac{1}{2}\sum_0^\infty b_n e^{i\left(-\frac{\pi}{4}+n\frac{\pi}{2}\right)} x^{-2n}$$

$$\int_{\gamma_2} u dq = -\int_{\gamma_4} u dq = \frac{1}{2}\oint_{\gamma_{42}} u dq = \frac{1}{2}\sum_0^\infty b_n e^{i\left(\frac{\pi}{4}-n\frac{\pi}{2}\right)} x^{-2n} \qquad (8.2)$$

At fixed $E_0 = -1$, the coefficients b_n are pure numbers:

$$b_0 = \frac{\Gamma(1/4)^2}{3}(2/\pi)^{1/2}, \ b_1 = -\Gamma(1/4)^{-2}(\pi^3/2)^{1/2},$$

$$b_2 = \frac{11\Gamma(1/4)^2}{768}(2\pi)^{-1/2}, \ \dots \qquad (8.3)$$

To fix ideas, the coefficients called « b_n » in Refs. [2]-[3] were computed at fixed $E_0 = -b_0^{4/3}$ (in the present notation) so that « b_n » $= b_0^{2n-1}b_n$ (and « b_0 » $= 1$).

The action periods form a *square lattice* Ω generated for instance by the primitive periods (Fig. 18*b*):

$$\omega = \omega(\gamma_{43}) = \omega(\gamma_{12}) = -i\left[\int_{\gamma_2} u dq - \int_{\gamma_1} u dq\right]_{x=0} = 2^{-1/2}b_0 \quad (8.4)$$

and
$$\omega' = \omega(\gamma_{23}) = \omega(\gamma_{14}) = i\omega$$

Then
$$\omega(\gamma_{13}) = \omega + \omega' = e^{i\pi/4}b_0, \ \omega(\gamma_{42}) = \omega - \omega' = e^{-i\pi/4}b_0 \qquad (8.5)$$
$$\omega(\gamma_{jk}) = -\omega(\gamma_{kj}) \qquad (\forall k, j). \qquad (8.6)$$

Applying relations (7.6)-(7.10) to γ_{13} and γ_{24} we obtain:

$$\int_{C_1}(u-p)dq = -\int_{C_3}(u-p)dq = \frac{1}{2}\oint_{\gamma_{24}}(u-p)dq = \int_{\gamma_4} u dq + \frac{e^{i\pi/4}}{2}b_0$$

$$\int_{C_2}(u-p)dq = -\int_{C_4}(u-p)dq = \frac{1}{2}\oint_{\gamma_{31}}(u-p)dq = \int_{\gamma_1} u dq - \frac{e^{-i\pi/4}}{2}b_0 \qquad (8.7)$$

Similarly, using Eqs. (7.6-7), (7.16) and (7.22) with $M = 2$:

$$\oint_{\gamma_{21}} u dq = -2^{-1/2}ib_0 + \int_{-\infty}^\infty (u-p)dq = -i\sum_0^\infty a_n x^{-2n}$$

$$\oint_{\gamma_{41}} u dq = 2^{-1/2}b_0 + \int_{-i\infty}^{+i\infty}(u-p)dq = \sum_0^\infty (-1)^n a_n x^{-2n} \qquad (8.8)$$

where the a_n are real numbers obeying Eq. (7.25) with $M = 2$:

$$a_n = \frac{-b_n}{2 \sin (2n - 1)\dfrac{3\pi}{4}} \qquad (a_0 = 2^{-1/2}b_0 \equiv \omega) \qquad (8.9)$$

(empirically the a_n alternate in sign so that $a_n = 2^{-1/2}(-1)^{n+1}\,|\,b_n\,|$ except $a_0 > 0$). Also:

$$\int_{-\infty}^{\infty} (u - p)dq = \oint_{\gamma_{21}} (u - p)dq = \oint_{\gamma_{34}} (u - p)dq$$
$$\int_{-i\infty}^{+i\infty} (u - p)dq = \int_{\gamma_{41}} (u - p)dq = \oint_{\gamma_{32}} (u - p)dq \qquad (8.10)$$

We shall mostly use the resulting relations between the exponentiated quantities u_γ and a_γ defined by Eqs. (5.12). To make notations even shorter we shall use the same symbol $a(x)$ for the Jost function and its asymptotic expansion, which otherwise should read

$$a_{\mathbb{R}}(x) = \exp \sum_1^{\infty} a_n x^{1 - 2n} \qquad (8.11)$$

and we also note

$$a'(x) = a_{i\mathbb{R}}(x) = \exp \sum_1^{\infty} a_n (ix)^{1 - 2n} \qquad (8.12)$$

Besides some obvious relations:

$$u_\gamma^{-1}(x) = u_{-\gamma}(x) = u_\gamma(- x) \quad \text{(for any path } \gamma; \text{ the same for } a_\gamma) \quad (8.13)$$
$$a'(x) = a(ix) \quad (\Leftrightarrow a_B'(s) = a_B(- is)) \qquad (8.14)$$

we basically need the following ones (to be understood in their exact resummed forms, according to Eqs. (3.13) and (3.18)):

$$\left.\begin{aligned}u_{\gamma_{21}}(x) &= u_{\gamma_{34}}(x) = e^{\omega x}a(x) \ , & a &= a_{\gamma_{21}} = a_{\gamma_{34}} \\ u_{\gamma_{41}}(x) &= u_{\gamma_{32}}(x) = e^{\omega' x}a'(x), & a' &= a_{\gamma_{41}} = a_{\gamma_{32}}\end{aligned}\right\} \qquad (8.15)$$

$$\left.\begin{aligned}u_{\gamma_{31}}(x) &= e^{(\omega + \omega')x}a_{\gamma_{31}}(x), & a_{\gamma_{31}} &= aa' \\ u_{\gamma_{24}}(x) &= e^{(\omega - \omega')x}a_{\gamma_{24}}(x), & a_{\gamma_{24}} &= a/a'\end{aligned}\right\} \qquad (8.16)$$

The main computation: we now solve the connection problem from $q = - e^{i\varphi/3}\infty$ to $q = + e^{i\varphi/3}\infty$ for all angles $\varphi = -\operatorname{Arg} x$. The critical angles in the sense of Eq. (7.33) are *all the multiples of* $\pi/4$. Figs. 19-20 show in parallel the diagrams of Stokes regions in the q plane and the resulting integration lines with their integrands in the s plane.
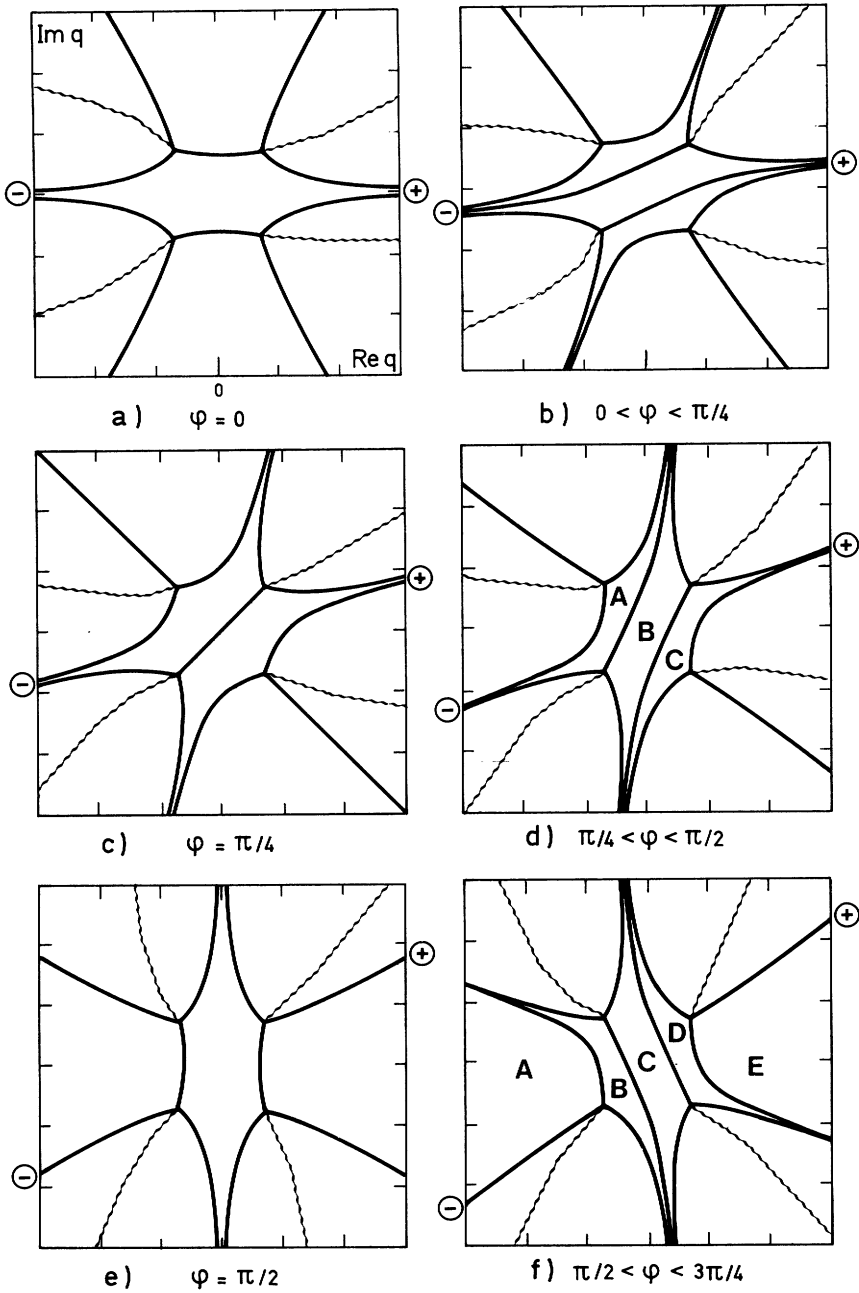
FIG. 19.

—— Stokes lines.
〜 Cuts.

g) $\varphi = 3\pi/4$

h) $3\pi/4 < \varphi < \pi$

i) $\varphi = \pi$

j) $\pi < \varphi < 5\pi/4$
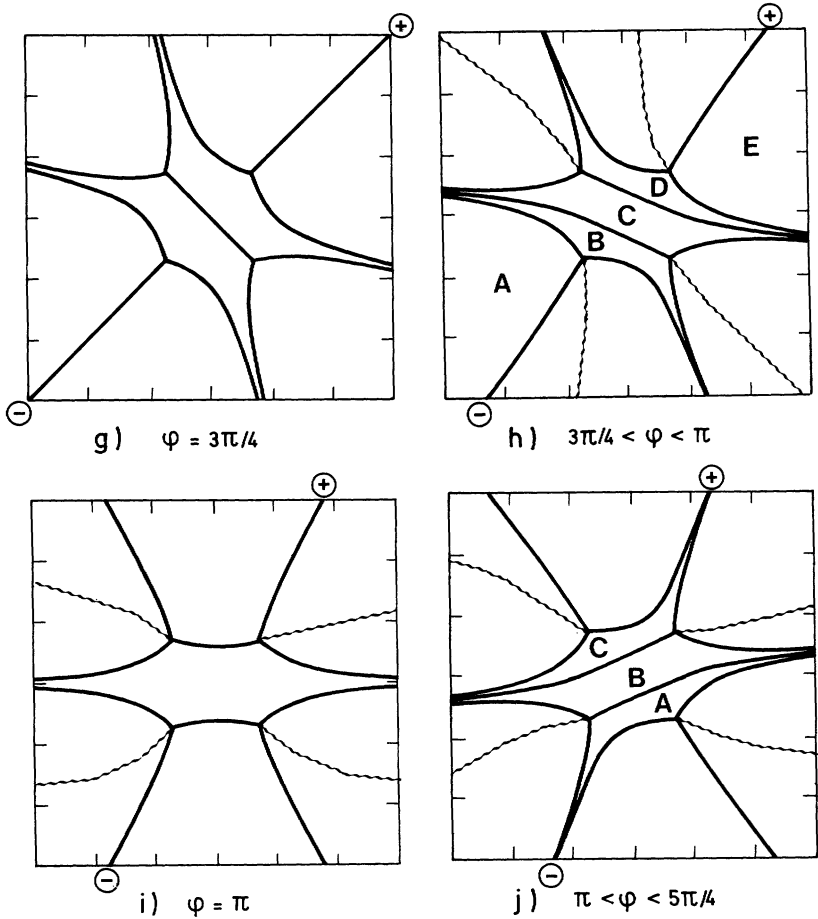
FIG. 19 *(continued)*.

a) For $|\varphi < \pi/4$ (Figs. 19a-b) the central Stokes region always connects $(- e^{i\varphi/3}\infty)$ to $(+ e^{i\varphi/3}\infty)$, hence $F^{++} = 1$ and (Fig. 20a):

$$a(x) = L_0^\varphi a_B \qquad \forall \varphi, \, |\varphi| < \pi/4. \qquad (8.17)$$

Although $\varphi = 0$ is a critical angle, no singularity of $a_B(s)$ is detected on the line $\{ \text{Arg } s = 0 \}$ (singularities do exist there but on other sheets as we shall see later: no inconsistency arises this way.

b) $\pi/4 < \varphi < \pi/2$: we compute for instance (Fig. 19d):

$$F = F_{CB}F_{BA} = \begin{pmatrix} 1 & iu_{-\gamma_1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ - iu_{\gamma_3} & 1 \end{pmatrix}$$

$$\Rightarrow F^{++} = 1 + u_{\gamma_{13}} \qquad (8.18)$$

$$a(x) = L_0^\varphi a_B + L_0^\varphi (u_{\gamma_{13}}a)_B$$
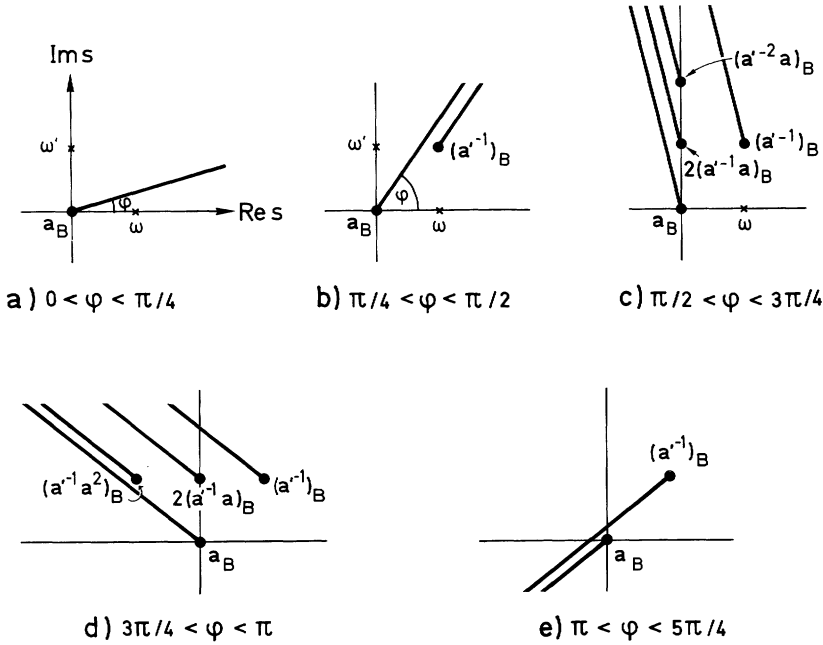
FIG. 20.

Now the relations (8.16) simplify this to:

$$a(x) = L_0^\varphi a_B + e^{-(\omega+\omega')x}L_0^\varphi(a'^{-1})_B$$
$$= L_0^\varphi a_B + L_{\omega+\omega'}^\varphi(a'^{-1})_B(-(\omega+\omega')+.) \quad (8.19)$$

which is expressed by the diagram of Fig. 20b.

We draw from (8.17) and (8.19) with Arg $x = -\pi/4$ the discontinuity formula:

$$L_0^{\pi/4+0}a_B - L_0^{\pi/4-0}a_B = -L_{\omega+\omega'}^{\pi/4+0}(a'^{-1})_B(-(\omega+\omega')+.) \quad (8.20)$$

and we conclude that $a_B(s)$ has at $s = \omega + \omega'$ a discontinuity equal to $-(a'^{-1})_B(-(\omega+\omega')+.)$, namely to the function a_B (is) *itself* translated to the branch point $(\omega+\omega')$.

c) $\pi/2 < \varphi < 3\pi/4$ (Fig. 19f):

$$F = F_{ED}F_{DC}F_{CB}F_{BA} = \begin{pmatrix} 1 & iu_{-\gamma_1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -iu_{\gamma_4} & 1 \end{pmatrix}\begin{pmatrix} 1 & iu_{-\gamma_2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -iu_{\gamma_3} & 1 \end{pmatrix}$$

$$\Rightarrow F^{++} = (1+u_{\gamma_{14}})(1+u_{\gamma_{23}})+u_{\gamma_{13}} = (1+u_{\gamma_{14}})^2 + u_{\gamma_{13}}$$

$$\Rightarrow a(x) = L_0^\varphi a_B + 2L_{\omega'}^\varphi(a'^{-1}a)_B(-\omega'+.)+L_{2\omega'}^\varphi(a'^{-2}a)_B(-2\omega'+.)$$

$$+ L_{\omega+\omega'}^\varphi(a'^{-1})_B(-(\omega+\omega')+.) \quad (8.21)$$

after use of (8.15-16). A small difficulty arises here: substraction of (8.19) and (8.21) with Arg $x = -\pi/2$ only yields a composite formula

$$\left[L_0^{\pi/2+0} - L_0^{\pi/2-0}\right]a_B + \left[L_{\omega+\omega'}^{\pi/2+0} - L_{\omega+\omega'}^{\pi/2-0}\right](a'^{-1})_B(-(\omega+\omega')+.)$$
$$= -2L_{\omega'}^{\varphi}(a'^{-1}a)_B(-\omega'+.) - L_{2\omega'}^{\varphi}(a'^{-2}a)_B(-2\omega'+.) \quad (8.22)$$

from which we want to extract $\left[L_0^{\pi/2+0} - L_0^{\pi/2-0}\right]a_B$. To sort correctly the various terms of (8.22) we note that in a discontinuity formula like (3.11), all branch points s_j in the right hand side satisfy

$$\text{Arg }(s_j - s_1) = \varphi \qquad \forall j \neq 1 \qquad (8.23)$$

Under that criterion, both terms in the right hand side of (8.22) induce discontinuities for a_B (at $s = \omega'$ and $2\omega'$) and none for the other function $(a'^{-1})_B(-(\omega+\omega')+.)$ (this is illustrated by Fig. 20c):

$$L_0^{\pi/2+0}a_B - L_0^{\pi/2-0}a_B$$
$$= -2L_{\omega'}^{\pi/2+0}(a'^{-1}a)_B(-\omega'+.) - L_{2\omega'}^{\pi/2+0}(a'^{-2}a)_B(-2\omega'+.) \quad (8.24)$$

d) $3\pi/4 < \varphi < \pi$ (Fig. 19h):

$$F = F_{ED}F_{DC}F_{CB}F_{BA} = \begin{pmatrix} 1 & iu_{-\gamma_1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & iu_{-\gamma_2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -iu_{\gamma_4} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -iu_{\gamma_3} & 1 \end{pmatrix}$$
$$\Rightarrow F^{++} = 1 + (u_{-\gamma_1} + u_{-\gamma_2})(u_{\gamma_3} + u_{\gamma_4})$$

$$a(x) = L_0^{\varphi}a_B + L_{\omega+\omega'}^{\varphi}(a'^{-1})_B(-(\omega+\omega')+.) + 2L_{\omega'}^{\varphi}(a'^{-1}a)_B(-\omega'+.)$$
$$+ L_{\omega'-\omega}^{\varphi}(a'^{-1}a^2)_B(\omega-\omega'+.) \quad (8.25)$$

Substracting from (8.21) with Arg $x = -3\pi/4$:

$$\left[L_0^{3\pi/4+0} - L_0^{3\pi/4-0}\right]a_B + \left[L_{\omega+\omega'}^{3\pi/4+0} - L_{\omega+\omega'}^{3\pi/4-0}\right](a'^{-1})_B(-(\omega+\omega')+.)$$
$$+ 2\left[L_{\omega'}^{3\pi/4+0} - L_{\omega'}^{3\pi/4-0}\right](a'^{-1}a)_B(-\omega'+.)$$
$$= L_{2\omega'}^{3\pi/4-0}(a'^{-2}a)_B(-2\omega'+.) - L_{\omega'-\omega}^{3\pi/4+0}(a'^{-1}a^2)(\omega-\omega'+.) \quad (8.26)$$

which, according to criterion (8.23), implies (Fig. 20d):

$$[L_0^{3\pi/4+0} - L_0^{3\pi/4-0}]a_B = -L_{\omega'-\omega}^{3\pi/4+0}(a'^{-1}a^2)_B(\omega-\omega'+.), \text{ etc.} \quad (8.27)$$

hence a_B has a discontinuity at $s = -\omega + \omega'$ too.

e) $\pi < \varphi < 5\pi/4$ (Fig. 19j):

$$F = F_{CB}F_{BA} = \begin{pmatrix} 1 & -iu_{-\gamma_1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ iu_{\gamma_3} & 1 \end{pmatrix}$$
$$F^{++} = 1 + u_{\gamma_{13}}, \text{ etc.}$$

but all terms in the discontinuity formula across $\varphi = \pi$ now concern the function $(a'^{-1})_B(-(\omega+\omega')+.)$, and a_B itself is analytic on $\{\text{Arg }s = \pi\}$ in the same way as it was on $\{\text{Arg }s = 0\}$.

Here the value $\varphi = 5\pi/4$ is a natural limit to the radar method, as the secondary cut that appeared at $s = \omega + \omega'$ (carrying the function

$(a'^{-1})_B(-(\omega+\omega')+.))$ ends by eating up the main integration line on which the Laplace transform of a_B itself was computed (Fig. 20e). However the whole procedure can be symetrically applied from $\varphi = 0$ to $(-5\pi/4)$. Having thus covered an angular range greater than 2π, we are sure to have obtained *all first sheet discontinuities* of $a_B(s)$. The discontinuity formulas symmetric to (8.20), (8.24) and (8.27) are explicitly:

$$L_0^{-\pi/4+0}a_B - L_0^{-\pi/4-0}a_B = L_{\omega-\omega'}^{-\pi/4-0}a_B'(-\omega+\omega'+.) \qquad (8.28)$$

$$L_0^{-\pi/2+0}a_B - L_0^{-\pi/2-0}a_B$$
$$= 2L_{-\omega'}^{-\pi/2-0}(a'a)_B(\omega'+.) + L_{-2\omega'}^{-\pi/2-0}(a'^2a)_B(2\omega'+.) \qquad (8.29)$$

$$L_0^{-3\pi/4+0}a_B - L_0^{-3\pi/4-0}a_B = L_{-\omega-\omega'}^{-3\pi/4-0}(a'a^2)_B(\omega+\omega'+.) \qquad (8.30)$$

The analytic structure of the Borel transform $a_B(s)$ is summarized on Figure 21, where the cuts have their orientations as prescribed by the discontinuity formulas, namely $e^{+i0}(e^{-i0})$ in the upper (lower) half-planes; the value of the discontinuity is also written at each branch point. *Analytic bootstrap* is again observed in a very curious form: each branch point is an action period of the form $(m\omega + m'\omega')$, and the corresponding discontinuity has the form:

$$\Delta_{m\omega+m'\omega'}a_B(s) = c_{mm'}(a^{1-m}a'^{-m'})_B(-m\omega - m'\omega' + s) \left.\begin{array}{c} \\ \\ \end{array}\right\}$$
$$= c_{mm'}\left[a_B^{*(1-m)} \dot{*} (a_B')^{*(-m')}\right](-m\omega - m'\omega' + s) \qquad (8.31)$$

where $c_{mm'}$ are certain constant multipliers given by the discontinuity
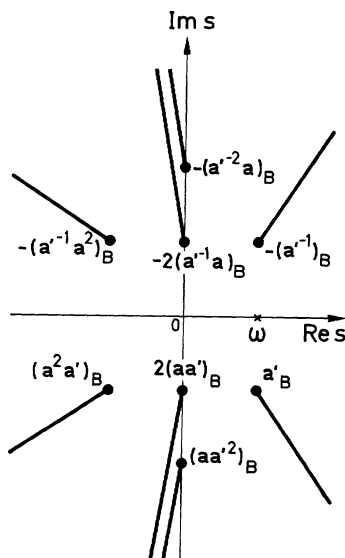
FIG. 21. — $a_B(s)$.

formulas (8.20-24-27 to 30) and f^{*n} denotes the nth convolution power of f. Clearly the structure of (8.31) can be understood from Eq. (5.18) in the limit $q_0 \to +\infty$ (when $\omega(\gamma) = m\omega + m'\omega'$, $(a_\gamma)_B = a_B^{*(-m)} \dot{*} (a'_B)^{*(-m')}$). But the new feature referred to as analytic bootstrap is that the *same* function a_B occurs on both sides of (8.31) (we recall that $a'_B(s) = a_B(-is)$).

Implications of analytic bootstrap: the fundamental consequence of (8.31) is that the structure of the discontinuity formulas will reproduce itself in *all sheets* of the Riemann surface of $a_B(s)$, thereby allowing its *global analytic structure* to be computed. Indeed this computation is equivalent, by the discontinuity formula (8.31), to the simultaneous description of the first sheet discontinuities of *all monomials*

$$\alpha_{nn'} = a_B^{*(-n)} \dot{*} (a'_B)^{\dot{*}(-n')} \qquad (n, n' \in \mathbb{Z}) \tag{8.32}$$

Let us describe in detail what happens to those monomials along the half-line $\text{Arg } s = \pi/4$, for instance. The discontinuity formula (8.20) for $a_B(s)$ is conveniently rewritten as:

$$L_0^{\pi/4 - 0}\alpha_{-1,0} = L_0^{\pi/4 + 0}[(1 + e^{-(\omega+\omega')x}\alpha_{1,1}) \dot{*} \alpha_{-1,0}] \tag{8.33}$$

where the factor $e^{-(\omega+\omega')x}$ is understood to stand at the left of the L operator when the product is expanded. The similar formula for $\alpha_{0,-1} = a'_B(s)$ must be drawn from Eq. (8.28) because of the relation (8.14):

$$L_0^{\pi/4 + 0}\alpha_{0,-1} = L_0^{\pi/4 - 0}[(1 + e^{-(\omega+\omega')x}\alpha_{1,1}) \dot{*} \alpha_{0,-1}] \tag{8.34}$$

We now apply the product formula (3.13) to Eq. (8.32) to derive a discontinuity formula similar to (3.15) and we obtain Eq. (8.36), supplemented here with the analogous formulas for the other critical angles:

$$L_0^{+0}\alpha_{nn'} \quad = L_0^{-0}[(1 + e^{-\omega x}\alpha_{1,0})^{\dot{*}(-2n')} \dot{*} \alpha_{nn'}] \tag{8.35}$$

$$L_0^{\pi/4+0}\alpha_{nn'} \quad = L_0^{\pi/4-0}[(1 + e^{-(\omega+\omega')x}\alpha_{1,1})^{\dot{*}(n-n')} \dot{*} \alpha_{nn'}] \tag{8.36}$$

$$L_0^{\pi/2+0}\alpha_{nn'} \quad = L_0^{\pi/2-0}[(1 + e^{-\omega'x}\alpha_{0,1})^{\dot{*}(2n)} \dot{*} \alpha_{nn'}] \tag{8.37}$$

$$L_0^{3\pi/4+0}\alpha_{nn'} \quad = L_0^{3\pi/4-0}[(1 + e^{-(\omega'-\omega)x}\alpha_{-1,1})^{\dot{*}(n+n')} \dot{*} \alpha_{nn'}] \tag{8.38}$$

$$L_0^{\pi+0}\alpha_{nn'} \quad = L_0^{-\pi-0}[(1 + e^{\omega x}\alpha_{-1,0})^{\dot{*}(2n')} \dot{*} \alpha_{nn'}] \tag{8.39}$$

$$L_0^{-3\pi/4+0}\alpha_{nn'} = L_0^{-3\pi/4-0}[(1 + e^{(\omega+\omega')x}\alpha_{-1,-1})^{\dot{*}(n'-n)} \dot{*} \alpha_{nn'}] \tag{8.40}$$

$$L_0^{-\pi/2+0}\alpha_{nn'} \quad = L_0^{-\pi/2-0}[(1 + e^{\omega'x}\alpha_{0,-1})^{\dot{*}(-2n)} \dot{*} \alpha_{nn'}] \tag{8.41}$$

$$L_0^{-\pi/4+0}\alpha_{nn'} \quad = L_0^{-\pi/4-0}[(1 + e^{(\omega'-\omega)x}\alpha_{1,-1})^{\dot{*}(-n-n')} \dot{*} \alpha_{nn'}] \tag{8.42}$$

We pursue the analysis of the same formula (8.36) as before. Its right hand side can be *expanded* in a convergent power series of $e^{-(\omega+\omega')x}L_0^{\pi/4-0}\alpha_{1,1}$

(an exponentially small quantity when Arg $x = -\pi/4$) and recombined using formulas (8.32) and (3.4), to yield:

$$L_0^{\pi/4+0}\alpha_{nn'} - L_0^{\pi/4-0}\alpha_{nn'}$$
$$= \sum_{m=1}^{\infty} \binom{n-n'}{m} L_{m(\omega+\omega')}^{\pi/4-0}\alpha_{n+m,n'+m}(-m(\omega+\omega')+.) \quad (8.43)$$

where $\binom{j}{k} = \dfrac{\Gamma(j+1)}{\Gamma(k+1)\Gamma(j-k+1)}$ is the binomial coefficient. Alternatively, $(1 + e^{-(\omega+\omega')x}\alpha_{1,1})^{*(n-n')}$ can be transferred to the other side, in which case we get:

$$L_0^{\pi/4+0}\alpha_{nn'} - L_0^{\pi/4-0}\alpha_{nn'}$$
$$= -\sum_{m=1}^{\infty} \binom{n'-n}{m} L_{m(\omega+\omega')}^{\pi/4+0}\alpha_{n+m,n'+m}(-m(\omega+\omega')+.) \quad (8.44)$$

We recognize (8.43) and (8.44) as two equivalent discontinuity formulas, written respectively in the e^{-i0} and e^{+i0} conventions. Let us call *adapted* the convention that leads to a finite number of terms: Eq. (8.43) if $n \geq n'$, Eq. (8.44) if $n \leq n'$. The case $n = n'$ is *neutral* in the sense that all the coefficients in both discontinuity formulas in the $e^{\pm i0}$ conventions coincide two by two: in the present case they vanish.

This whole analysis carries over to every other critical angle *separately* and results in the following global statement of analytic bootstrap: in any sheet of its Riemann surface the function $\alpha_{nn'}$ admits at every action period $s = (m\omega + m'\omega') \in \Omega$ a discontinuity of the form:

$$\Delta_{m\omega+m'\omega'}\alpha_{nn'}(s) = C_{mm'}^{nn'}(\mu)\alpha_{n+m,n'+m'}(-m\omega - m'\omega' + s) \quad (8.45)$$

where the multiplier $C_{mm'}^{nn'}$ (an integer) depends on all indices *and* on the sheet μ under scrutiny. This statement and the value of $C_{mm'}^{nn'}(\mu)$ result from cranking the various discontinuity formulas in an order assigned by the definition of μ (see § 9 for a more precise algorithm). With analytic bootstrap in mind, the ramification of $a_B(s)$ is completely described by the table of values $C_{mm'}^{nn'}(\mu)$. The basic values are those directly read off the discontinuity formulas in a given convention, e^{-i0} to fix ideas (Eqs. (8.35-42)); other values follow by iterated applications of the same discontinuity formulas. For instance Eq. (8.43) amounts to the numerical formula $C_{mm}^{nn'} = \binom{n-n'}{m}$. The complete list of basic values in the e^{-i0} convention is:

$$(\varphi = 0) \qquad : C_{m0}^{nn'} = \begin{pmatrix} -2n' \\ m \end{pmatrix} \quad (\varphi = \pi/4) \qquad : C_{mm}^{nn'} = \begin{pmatrix} n - n' \\ m \end{pmatrix}$$

$$(\varphi = \pi/2) \quad : C_{0m}^{nn'} = \begin{pmatrix} 2n \\ m \end{pmatrix} \qquad (\varphi = 3\pi/4) \qquad : C_{-m,m}^{nn'} = \begin{pmatrix} n + n' \\ m \end{pmatrix}$$

$$(\varphi = \pi) \qquad : C_{-m,0}^{nn'} = \begin{pmatrix} 2n' \\ m \end{pmatrix} \qquad (\varphi = -3\pi/4) : C_{-m,-m}^{nn'} = \begin{pmatrix} n' - n \\ m \end{pmatrix}$$

$$(\varphi = -\pi/2) : C_{0,-m}^{nn'} = \begin{pmatrix} -2n \\ m \end{pmatrix} \quad (\varphi = -\pi/4) \quad : C_{m,-m}^{nn'} = \begin{pmatrix} -n - n' \\ m \end{pmatrix}$$

$$(8.46)$$

where n, $n' \in \mathbb{Z}$ and $m \in \mathbb{N} \setminus \{ 0 \}$. All other $C_{mm'}^{nn'}$ *vanish identically* in the first sheet. Hence a very small subset of the lattice Ω carries actual discontinuities of $\alpha_{nn'}$ in the first sheet. This subset is a *star* with eight rays directed along the critical angles; of two opposite rays one contains an infinite number of branch points. Only in the adapted convention defined above are all rays finite (Fig. 22 illustrates the case of α_{21} in the adapted convention; only the weights $C_{mm'}^{nn'}$ are indicated at the branch points). Each discontinuity $\Delta_{m\omega + m'\omega'}\alpha_{nn'}$ will in turn have its own star of branch points but its center is *translated* to $s = m\omega + m'\omega'$ by formula (8.45), hence by successive iterations of (8.45) we expect to find branch points of $a_B(s)$ in the Riemann
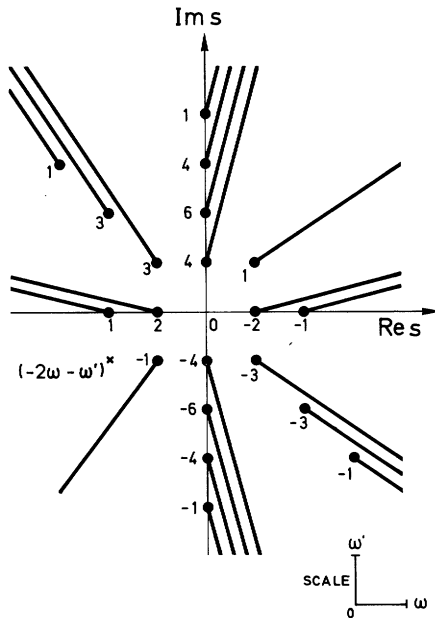
FIG. 22. — $\alpha_{21}(s)$.

surface above *every lattice point* (with one exception to be given shortly).

For special values of n and n', one line of branch points (the union of two opposite rays) is moreover missing from the first sheet, as seen by direct inspection of Eqs. (8.46):

$$\left.\begin{array}{l} \alpha_{n0} \quad \text{are analytic on the real axis } \mathbb{R} \text{ (e. g. } a_B(s) \text{ itself)} \\[4pt] \alpha_{0n'} \quad \text{are analytic on the imaginary axis } i\mathbb{R} \text{ (e. g. } a'_B(s)) \\[4pt] \alpha_{nn} \quad \text{are analytic on the axis } e^{i\pi/4}\mathbb{R} \\[4pt] \alpha_{n,-n} \text{ are analytic on the } e^{-i\pi/4}\mathbb{R} \end{array}\right\} \quad (\forall n \in \mathbb{Z}) \quad (8.47)$$

It follows that every $\alpha_{nn'}$ is *analytic* at the point $s = -(n\omega + n'\omega')$ in the first sheet, but this property now propagates to *all sheets* due to the translational structure of the discontinuity formula. In particular $s = +\omega$ is a *regular point* of $a_B(s) = \alpha_{-1,0}(s)$ in all sheets (whereas $s = 0$ should be a regular point only in the first sheet). It can be shown that $a_B(\omega) = 3/2$ and $d^k a_B/ds^k(\omega) = 0$ for $k = 4, 8, \ldots$; this remark [41] has led us to express $a_B(s)$ as the solution of an interpolation problem.

The *logarithm of the Jost function and other spectral functions*: as the particular function $\alpha_{00} = 1$ does not participate in the bootstrap game it is tempting to study in its place the odd expansion $\log a(x)$ from which all $\alpha_{nn'}$ can be recovered by rotation and exponentation. This expansion (7.16) has also a simpler relation to the eigenvalues themselves (through Eqs. (7.17) and (7.25)) and to a modified partition function $Z(s)$ introduced in Ref. [2] which we shall also describe shortly.

To obtain discontinuity formulas for:

$$(\log a)_B(s) = \sum_{n=1}^{\infty} \frac{a_n s^{2n-1}}{(2n-1)!} = \log_{\dot{*}} \alpha_{-1,0} \qquad (8.48)$$

($\log_{\dot{*}} = $ convolution logarithm), we use the same trick as for Eq. (7.52) in the harmonic case: we expand the logarithms of the discontinuity formulas (8.35-8.42) for $\alpha_{-1,0}$. For instance, Eq. (8.36) yields:

$$\begin{aligned} L_0^{\pi/4+0} & (\log a)_B \\ &= L_0^{\pi/4-0} (\log a)_B + L_0^{\pi/4-0} \log_{\dot{*}} [(1 + e^{-(\omega+\omega')x}\alpha_{1,1})^{\dot{*}(-1)}] \\ &= L_0^{\pi/4-0} (\log a)_B + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} L_{m(\omega+\omega')}^{\pi/4-0}\alpha_{mm}(-m(\omega+\omega')+ .) \end{aligned} \qquad (8.49)$$

and the *same* coefficients $(-1)^m/m$ would appear in the *other* convention e^{+i0}: this discontinuity formula is neutral with respect to the choice of $e^{\pm i0}$ (that is because all the discontinuity functions $\alpha_{mm}(-m(\omega + \omega')+ .)$ in the right-hand side are themselves analytic on the critical axis $\{ \text{Arg } s = \pi/4 \}$, by property (8.47)). The same thing happens at other critical angles.

We list the basic multipliers $C_{mm'}^{00}$ appearing in the first sheet discontinuity formulas:

$$\Delta_{m\omega+m'\omega'} (\log a)_B = C_{mm'}^{00} \alpha_{mm'}(-m\omega - m'\omega' + .) \qquad (8.50)$$

$(\varphi = 0)$	$: C_{m0}^{00} = 0$	$(\varphi = \pi)$	$: C_{-m,0}^{00} = 0$
$(\varphi = \pi/4)$	$: C_{mm}^{00} = (-1)^m/m$	$(\varphi = -3\pi/4)$	$: C_{-m,-m}^{00} = (-1)^{m+1}/m$
$(\varphi = \pi/2)$	$: C_{0m}^{00} = 2(-1)^m/m$	$(\varphi = -\pi/2)$	$: C_{0,-m}^{00} = 2(-1)^{m+1}/m$
$(\varphi = 3\pi/4)$	$: C_{-m,m}^{00} = (-1)^m/m$	$(\varphi = -\pi/4)$	$: C_{m,-m}^{00} = (-1)^{m+1}/m$

$$(8.51)$$

where $m \in \mathbb{N} \setminus \{0\}$; all other $C_{mm'}^{00}$ vanish. Like a_B itself, $(\log a)_B$ is thus analytic *on the real axis* in the first sheet. Exploration of the other sheets involves the previous discontinuity formulas (8.45-46) too; the point $s = 0$ comes out as *regular* in all sheets (it is regular in the first sheet and for all the functions occurring in (8.50)). Fig. 23 schematically depicts the structure of $(\log a)_B$; the cuts are shown superposed to express the neutrality of their orientations which, together with the antisymmetry of the function under $s \rightarrow -s$, is the main simplification that we gain by considering $(\log a)_B$ instead of a_B (contrary to the harmonic case, we cannot relate $(\log a)_B$ to a meromorphic function).

The transition to the Borel transform of the other odd series of interest, the Bohr-Sommerfeld expansion (7.17), is trivial due to the relation (8.9). We can deduce from relations (8.2) and (8.16):

$$\tilde{b}(s) \equiv \sum_1^\infty \frac{b_n s^{2n-1}}{(2n-1)!} = i^{-1} \log_* \alpha_{-1,-1}(e^{-i\pi/4}s) \qquad (8.52)$$
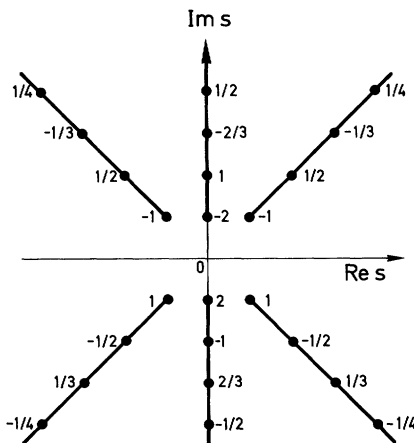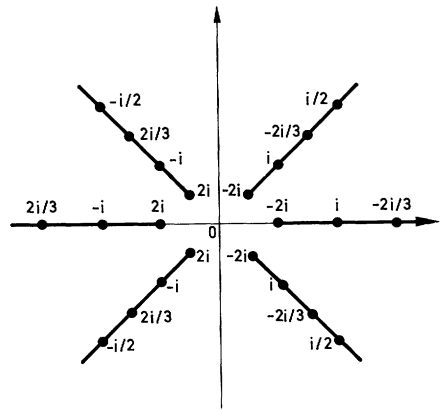
FIG. 23. — $(\log a)_B(s)$.   FIG. 24. — $\tilde{b}(s)$.

The computation is the same as the previous one up to a rotation by $(-\pi/4)$. From the first sheet discontinuity formula:

$$\Delta_{m\omega+m'\omega'}\log_{\pm}\alpha_{-1,-1} = \tilde{C}_{mm'}\alpha_{mm'}(-m\omega-m'\omega'+.)\qquad (8.53)$$

with the only non zero values:

$$\tilde{C}_{m0} = \tilde{C}_{m,-m} = \tilde{C}_{0,-m} = 2(-1)^{m+1}/m \qquad (m\in\mathbb{Z}\setminus\{0\}\qquad (8.54)$$

we deduce the discontinuity formula for $\tilde{b}(s)$ (Fig. 24):

$$\Delta_{e^{i\pi/4}(m\omega+m'\omega')}\tilde{b} = i^{-1}\tilde{C}_{mm'}\alpha_{mm'}(-m\omega-m'\omega'+e^{-i\pi/4}.)\qquad (8.55)$$

Remarks. — *a*) We recover the four nearest discontinuities at distance $|s| = \omega$ as they were computed (by a less reliable argument) in Ref. [2] where ω had the value $2^{-1/2}$.

b) $\tilde{b}(s)$ is analytic on $i\mathbb{R}$ in the first sheet and at $s = 0$ in all sheets.

c) $\tilde{b}(s)$ has singularities on the real axis hence the Bohr-Sommerfeld expansion is *not Borel summable*; this is probably a very general result (see § 9).

We now turn to a *modified partition function* studied in Refs. [2]-[3]. To motivate its definition we invoke the scaling law (7.14) satisfied by the full Jost function for $M = 2$:

$$a(\alpha E, x) = a(E, \alpha^{3/4}x)\qquad (\forall\alpha\in\mathbb{C}^*)\qquad (8.56)$$

Therefore the characteristic values x_k, defined as the positive zeros of $a(E = +1, x)$, equal $\lambda_k^{3/4}$ where λ_k are the eigenvalues of the $x = 1$ problem, in such a way that:

$$b_0 x_k \sim 2\pi(k+1/2)\quad\text{for}\quad k\to\infty\qquad (8.57)$$

by the Bohr-Sommerfeld rule. We then define the function:

$$Z(s) = \sum_{k=0}^{\infty} e^{-sx_k}\qquad (8.58)$$

already appearing in [2] [3] (respectively as $\rho(s) = iZ(-b_0 s)/b_0 s$, and $\Theta_{3/4}(s) = Z(b_0 s)$). The function $Z(s)$ is trivially holomorphic in $\{\operatorname{Re} s > 0\}$ and its singularities on the imaginary axis are the $\{ib_0 n\}_{n\in\mathbb{Z}}$ because of (8.57). We are interested in the global analytic structure of $Z(s)$ continued across the imaginary axis (by contrast the same problem would not make sense for the true partition function $\sum_k e^{-t\lambda_k}$, which has the imaginary axis as natural boundary according to results found in Ref. [25]).

We start with the obvious representation:

$$s^{-1}Z(s) = \int_0^\infty e^{-sx}\left[\sum_0^\infty \theta(x-x_k)\right]ds\qquad (8.59)$$

where θ is the Heaviside step function. By analytic continuation of the Fredholm determinant from $E < 0$ (Appendix A, Eqs. (A.2-5)):

$$-2\pi i \sum_0^\infty \theta(x - x_k) = \left[\log \Delta(x, E)\right]_{|E_0| - i0}^{|E_0| + i0}$$

$$= \left[-x \int_0^E T(E_1)dE_1 + \log a(x, E)\right]_{|E_0| - i0}^{|E_0| + i0}$$

$$= \left[a_0 y + \log a(y)\right]_{e^{i(+3\pi/4 - 0)x}}^{e^{i(-3\pi/4 + 0)x}} \qquad (x > 0)$$

$$\Rightarrow -2\pi i \sum_0^\infty \theta(x - x_k) + ib_0 x$$

$$= \log a(e^{i(-3\pi/4 + 0)}x) - \log a(e^{i(3\pi/4 - 0)}x) \quad (x > 0) \quad (8.60)$$

To get the third line we have set $E_0 = -1$ and exploited the scaling law (8.56) where $a(x) = a(E = -1, x)$; for Eq. (8.60) we have used $b_0 = a_0\sqrt{2}$. Now for $\mathrm{Arg}\ x = -\varphi = -\dfrac{3\pi}{4} + 0$ we take the standard form (8.21) for $a(x)$ which, with the notations (8.32-8.34) extended in the obvious way to half-integer exponents, factorizes as:

$$a(x) = L_0^\varphi \left[(1 + ie^{-\frac{(\omega + \omega')}{2}x}\alpha_{\frac{1}{2},\frac{1}{2}} + e^{-\omega'x}\alpha_{0,1})\right.$$

$$\left. \dot{*}\left(1 - ie^{-\frac{(\omega + \omega')}{2}x}\alpha_{\frac{1}{2},\frac{1}{2}} + e^{-\omega'x}\alpha_{0,1}\right) * \alpha_{-1,0}\right]$$

$$\Rightarrow \log a(x) = L_0^\varphi (\log a)_B$$

$$+ L_0^\varphi \left\{2\ \mathrm{Re}\ \log_{\dot{*}}\left(1 + ie^{-\frac{(\omega + \omega')}{2}x}\alpha_{\frac{1}{2},\frac{1}{2}} + e^{-\omega'x}\alpha_{0,1}\right)\right\} \quad (8.61)$$

The Taylor expansion of the logarithm is legitimate as $\mathrm{Re}\ [(\omega' \pm \omega)x] > 0$ for $\mathrm{Arg}\ x = -\dfrac{3\pi}{4} + 0$:

$$\log_{\dot{*}}\left(1 + ie^{-\frac{(\omega + \omega')}{2}x}\alpha_{\frac{1}{2},\frac{1}{2}} + e^{-\omega'x}\alpha_{0,1}\right)$$

$$= \sum_{M=1}^\infty \frac{(-1)^{M+1}}{M} \sum_{N=0}^M \binom{M}{N} i^N e^{-\left[N\frac{(\omega + \omega')}{2} + (M - N)\omega'\right]x}\alpha_{\frac{N}{2}, M - \frac{N}{2}}$$

$$\Rightarrow \log a(x) = L_0^{3\pi/4 - 0} (\log a)_B$$

$$+ \sum_{m'=1}^{\ } \sum_{m=0}^{m'} \frac{2(-1)^{m'+1}}{m + m'}\binom{m + m'}{2m} L_{m\omega + m'\omega'}^{3\pi/4 - 0}\alpha_{mm'}(-m\omega - m'\omega' + .)$$

$$\left(\mathrm{Arg}\ x = -\frac{3\pi}{4} + 0\right) \quad (8.62)$$
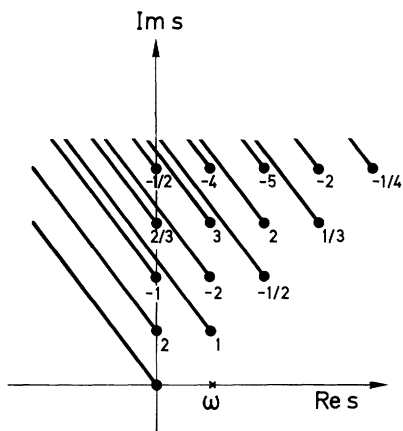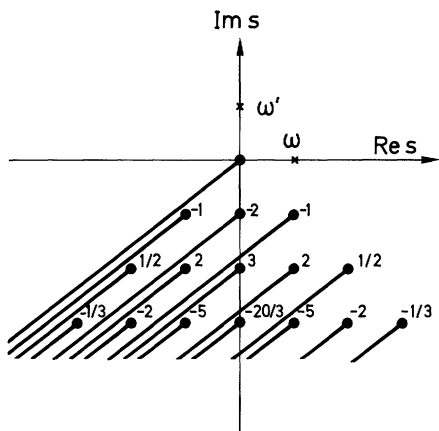
FIG. 25.

FIG. 26.

(Fig. 25). While keeping $\text{Arg } x = -\dfrac{3\pi}{4} + 0$ we can also set $\varphi = \dfrac{3\pi}{4} + 0$

and obtain from the standard form (8.25), factorized as:

$$a(x) = L_0^\varphi \Big[\Big(1 + i \big(e^{-\frac{(\omega+\omega')}{2}x} \alpha_{\frac{1}{2},\frac{1}{2}} + e^{-\frac{(\omega'-\omega)}{2}x} \alpha_{-\frac{1}{2},\frac{1}{2}} \big) \Big)$$

$$\dot* \Big(1 - i \big(e^{-\frac{(\omega+\omega')}{2}x} \alpha_{\frac{1}{2},\frac{1}{2}} + e^{-\frac{(\omega'-\omega)}{2}x} \alpha_{-\frac{1}{2},\frac{1}{2}} \big) \Big) \dot* \alpha_{-1,0} \Big]$$

the result:

$$\log a(x) = L_0^{3\pi/4+0} (\log a)_B$$

$$+ \sum_{m'=1}^{\infty} \cdot \sum_{m=-m'}^{+m'} \frac{2(-1)^{m'+1}}{2m'} \binom{2m'}{m+m'} L_{m\omega+m'\omega'}^{3\pi/4+0} \alpha_{mm'}(-m\omega - m'\omega' + .)$$

which produces by complex conjugation (Fig. 26):

$$\log a(x) = L_0^{-3\pi/4-0} (\log a)_B$$

$$+ \sum_{m'=1}^{\infty} \sum_{m=-m'}^{m'} \frac{2(-1)^{m'}}{2m'} \binom{2m'}{m+m'} L_{m\omega-m'\omega'}^{-3\pi/4-0} \alpha_{m,-m'}(-m\omega + m'\omega' + .)$$

$$\left(\text{Arg } x = \frac{3\pi}{4} - 0 \right) \quad (8.63)$$

We now change to variables $e^{i\pi/4}s$ in (8.62) and $e^{-i\pi/4}s$ in (8.63), and combine the two equations to express the right hand side of (8.60) as a sum of integrals all taken in the $e^{i(\pi-0)}$ direction (in other words we superpose

Figs. 25-26 rotated by $+\pi/4(-\pi/4)$ respectively). A little combinatorics produces the result (Fig. 27):

$$ib_0 x - 2\pi i \sum_0^\infty \theta(x - x_k) = L_0^{\pi - 0} [(\log a)_B(e^{-i\pi/4}.) - (\log a)_B(e^{i\pi/4}.)](-x)$$

$$+ \sum_{m = -\infty}^{+\infty} \sum_{m' = m}^{+\infty} C_{mm'}^- \left[L_{e^{i\pi/4}(m\omega + m'\omega')}^{\pi - 0} \alpha_{mm'}(-m\omega - m'\omega' + e^{-i\pi/4}.) \right](-x) \quad (8.64)$$

$$C_{mm'}^- = \frac{2(-1)^{m'+1}\Gamma(m + m')}{\Gamma(1 + m' - m)\Gamma(1 + 2m)} = \frac{2(-1)^{m+1}\Gamma(-2m)}{\Gamma(1 + m' - m)\Gamma(1 - m - m')}$$
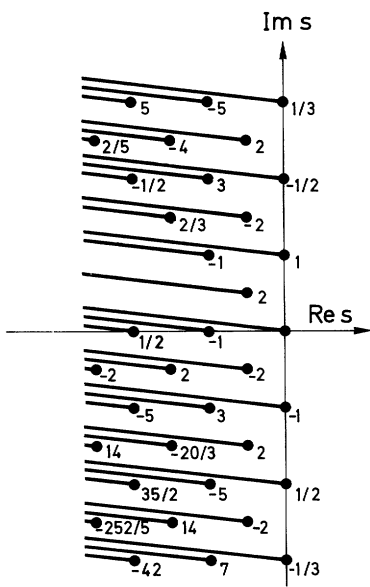
FIG. 27. — $\phi(s)$.

provided we substitute in (8.63) $\alpha_{m,-m'}(s)$ by $\alpha_{-m',-m}(-is)$ (a consequence of Eqs. (8.14)-(8.32)) ; we may recall b_0 from Eq. (8.3).

On the other hand we can take the formula inverting (8.59):

$$2\pi i \sum_0^\infty \theta(x - x_k) = \int_{c-i\infty}^{c+i\infty} s^{-1} Z(s) e^{xs} ds \quad (x > 0, c > 0) \quad (8.65)$$

which we prefer to rewrite as:

$$
\left.
\begin{aligned}
& - ib_0 x + 2\pi i \sum_0^\infty \theta(x - x_k) = x \int_{c-i\infty}^{c+i\infty} \phi(s) e^{xs} ds \\[2mm]
& \phi(s) = \frac{-b_0}{2\pi s} + \sum_{k=0}^{\prime} \int_s^{\prime} e^{-tx_k} t^{-1} dt \left(\Rightarrow Z(s) - \frac{b_0}{2\pi s} = -s \frac{d}{ds} \phi(s) \right)
\end{aligned}
\right\}
\tag{8.66}
$$

because the contribution of the leading singularity of $Z(s)$ at $s = 0$ is made explicit. Now we distort the integration path in the direction of steepest descent tilted by e^{-i0} (assuming that $\phi \in \widetilde{\mathscr{C}}$):

$$
ib_0 x - 2\pi i \sum_0^\infty \theta(x - x_k) = - \sum_{s_j} [L_{s_j}^{\pi-0} \Delta_{s_j} \phi](-x)
\tag{8.67}
$$

where the sum runs over all visible branch points in the half-plane Re $s \le 0$ (first sheet). By identification with (8.64) we get the discontinuities of ϕ:

$$
\Delta_0 \phi(s) = - \log a_B(e^{-i\pi/4}s) + \log a_B(e^{i\pi/4}s) = - i \tilde{b}(s)
\tag{8.68}
$$

(using (8.9)) ; and for $m \in \mathbb{Z}$, $m' \ge m$ and $(m, m') \ne (0, 0)$:

$$
\Delta_{e^{i\pi/4}(m\omega + m'\omega')} \phi(s) = - C_{mm'}^- \alpha_{mm'}(- m\omega - m'\omega' + e^{-i\pi/4}s)
\tag{8.69}
$$

This formula together with (8.45-8.46-8.55) allows to continue $\phi(s)$ to all sheets. For completeness we give the formula symmetrical to (8.69) when all cuts are in the direction $e^{i(\pi+0)}$:

$$
\begin{aligned}
& \Delta_{e^{i\pi/4}(m\omega + m'\omega')} \phi(s) \\
& \qquad = - C_{mm'}^+ \alpha_{mm'}(- m\omega - m'\omega' + e^{-i\pi/4}s), \quad C_{mm'}^+ = C_{-m',-m}^-
\end{aligned}
\tag{8.70}
$$

The discontinuity formulas for $Z(s)$ directly follow from (8.66):

$$
\left.
\begin{aligned}
& \Delta_0 [Z(s) - b_0(2\pi s)^{-1}] = is \frac{d}{ds} \tilde{b}(s) \\[2mm]
& \Delta_{e^{i\pi/4}(m\omega + m'\omega')} Z(s) = C_{mm'}^- s \frac{d}{ds} \alpha_{mm'}(- m\omega - m'\omega' + e^{-i\pi/4}s) \\
& \qquad\qquad (m \in \mathbb{Z}, \ m' \ge m, \ (m, m') \ne (0, 0))
\end{aligned}
\right\}
\tag{8.71}
$$

and

$$
\Delta_{m\omega + m'\omega'} s \frac{d}{ds} \alpha_{mm'}(s) = C_{mm'}^{nn'} s \frac{d}{ds} \alpha_{n+m,n'+m'}(- m\omega - m'\omega' + s)
\tag{8.72}
$$

They are clearly more complicated in form than those for $\phi(s)$ (that is why we introduced the latter function). In particular, consistency of the discontinuity formulas requires that $\frac{d}{ds} \alpha_{mm'}$ be computed *in the sense of*

distributions (section 3), and the leading term 1 in the expansion of $\alpha_{mm'}$ (a step function term in Eq. (8.69)) yields a δ discontinuity in (8.71), hence $Z(s)$ and all the functions $s\dfrac{d}{ds}\alpha_{mm'}$ appearing in (8.71-8.72) have *simple poles superimposed* upon each logarithmic branch point. The simple poles on the imaginary axis are the remnants of those of the true partition function $\Theta(s) = (2 \operatorname{sh} \pi b_0^{-1} s)^{-1}$ for a harmonic oscillator; indeed $Z(s)$ relative to the quartic oscillator is a kind of perturbation of $\Theta(s)$ thanks to formulas (8.57-8.58). The bizarre decompositions created in Eq. (8.60) and (8.66) also look more natural if they are referred to their harmonic oscillator counterparts (7.59-7.61).

Our result (8.71-8.72) extends that of Ref. [2] in the following sense. In Ref. [2] we predicted the lattice of branch points $e^{i\pi/4}\Omega$ and its relation to classical dynamics (classical trajectories of a quartic potential are elliptic functions and the points of Ω form the lattice of their action periods) and we only computed the first sheet discontinuities of $\rho(s) = iZ(-b_0 s)/b_0 s$ at the branch points with Re $s = 0$ or $\dfrac{1}{2}$ (corresponding respectively to the subsets of indices $\{\, m' = m \,\}$ and $\{\, m' = m + 1 \,\}$ in (8.71)). We had not at our disposal a precise analysis of the Stokes phenomenon that has allowed us in this work to obtain a global picture of the same function (and of a whole family of related functions $\alpha_{mm'}$ to be considered in parallel). The Poisson summation formula used in Ref. [2] can also be adapted to produce our formulas 8.69-8.72 in a more direct but less rigorous way than was done here.

We have performed numerical tests to check at least partially the validity of our results. Those tests are more accurate extensions of those carried out in Ref. [2] and concern the asymptotic behavior for $n \to \infty$ of the sequence $\{\, b_n \,\}$ or $\{\, a_n \,\}$ (with relation (8.9) in mind). Our basic tool is Darboux's theorem ([4], Chap. 7): if a function $f(s)$ is analytic around $s = 0$:

$$f(s) = \sum_{0}^{\infty} f_n s^n / n! \qquad (8.73)$$

then each branch cut $s_0 \neq 0$ carrying a discontinuity:

$$\Delta_{s_0} f(s_0 + t) = \sum_{\lambda} f_\lambda^{(s_0)} t^\lambda / \Gamma(1 + \lambda) \qquad (8.74)$$

contributes to the large order behavior of $\{\, f_n \,\}$ in (8.73) by the expansion:

$$(2\pi i)^{-1} s_0^{-n} \sum_{\lambda} f_\lambda^{(s_0)} (-s_0)^\lambda e^{-i\pi\lambda} \Gamma(n - \lambda) \qquad (8.75)$$

(this results by applying to the Cauchy formula

$$f_n = (2\pi i)^{-1} n! \oint u^{-n-1} f(u) du \qquad (8.76)$$

the saddle-point method of section 2 with n as large parameter and $\log u$ as phase function). The contributions from the nearest branch points are dominant, but subdominant terms from more distant singularities can be exhibited too, again as in section 2.

We now take $f(s) = (\log a)_{\mathrm{B}}(s)$ and apply formulas (8.50-8.51). The nearest singularities are $s_0 = \omega'$ and $s_0^* = -\omega'$; the contribution like (8.75) from s_0 itself is precisely:

$$(2\pi i)^{-1} \omega'^{-n} C_{01}^{00} \sum_{l=0}^{\infty} (\alpha_{0,1})_l \omega'^l \Gamma(n - l) \qquad (8.77)$$

where the expansion coefficients $\alpha_l = (\alpha_{0,1})_l \omega'^l$ admit the generating function $\alpha_{0,1}(\omega's)$:

$$\sum_{l=0}^{\infty} (\alpha_{0,1})_l \omega'^l \frac{s^l}{l!} = \alpha_{0,1}(\omega's) = a_{\mathrm{B}}^{\prime*(-1)}(\omega's) = a_{\mathrm{B}}(-\omega s) \qquad (8.78)$$

$$\Leftrightarrow \sum_{l=0}^{\infty} \alpha_l x^{-l} \equiv \exp \sum_{j=1}^{\infty} a_j \left(-\frac{x}{\omega}\right)^{1-2j} = \exp \sum_{j=1}^{\infty} (-\omega^{2j-1} a_j x^{1-2j}) \qquad (8.79)$$

The contribution of s_0^* is the complex conjugate of (8.77) and precisely cancels the even n terms as required by the parity of the expansion (8.48): $f_{2n} = 0$, $f_{2n-1} = a_n$. We finally obtain, using $C_{01}^{00} = -2$:

$$a_n \sim (-1)^{n+1} \omega^{1-2n} \frac{2}{\pi} \sum_{l=0}^{\infty} \alpha_l \Gamma(2n - 1 - l) \qquad (8.80)$$

Formulas (8.79-8.80) are respectively equivalent to (4.14-4.15) of Ref. [2]. Analytic bootstrap shows itself in Eq. (8.79): the sequence $\{a_n\}$ is such that the early terms $a_1, a_2 \ldots$ generate the large order behavior of the late terms ($a_n, n \to \infty$). Our test of the theory is then the following: we compute a large number (60) of values a_n, estimate the first few coefficients α_l so as to fit (8.80) and finally compare these « experimental » α_l to the theoretical values predicted by Eq. (8.79). We refer to Appendix B for details and only mention here that agreement is excellent. Our test also confirms the *absence of singularities* at $\pm \omega$ (in line with Borel summability of $(\log a)_{\mathrm{B}}$) as those would have produced detectable noise in our procedure.

Another (finer) test consists in detecting the leading subdominant contribution to a_n for $n \to \infty$, coming from the *next nearest* branch points

of $(\log a)_B$, namely the four points $s_1 = \omega + \omega'$, s_1^*, $- s_1$, $- s_1^*$. The contribution like (8.75) from s_1 is:

$$(2\pi i)^{-1}(\omega + \omega')^{-n}C_{11}^{00} \sum_{l=0}^{\infty} (\alpha_{1,1})_l(\omega + \omega')^l \Gamma(n - l) \qquad (8.81)$$

where the expansion coefficients $\gamma_l = (\alpha_{1,1})_l(\omega + \omega')^l$ admit the generating function $\alpha_{1,1}[(\omega + \omega')s]$, or equivalently (cf. (8.52)):

$$\sum_{l=0}^{\infty} \gamma_l x^{-l} \equiv \exp \sum_{j=1}^{\infty} (-1)^{j+1}(\omega\sqrt{2})^{2j-1}b_j x^{1-2j} \qquad (8.82)$$

The total subdominant series from all four branch points turns out as:

$$\delta a_n \sim \cos\left(\frac{3\pi}{4} - \frac{n\pi}{2}\right)(\omega\sqrt{2})^{1-2n}\frac{2}{\pi}\sum_{l=0}^{\infty} \gamma_l \Gamma(2n - 1 - l) \qquad (8.83)$$

(we have used $C_{11}^{00} = -1$). The equivalent asymptotic formula for b_n is:

$$\left.\begin{array}{l} b_n \sim \cos\left(\dfrac{n\pi}{2} - \dfrac{3\pi}{4}\right)\omega^{1-2n}\dfrac{4}{\pi}\sum_{0}^{\infty} \alpha_l \Gamma(2n - 1 - l) \\[4mm] \qquad - (\omega\sqrt{2})^{1-2n}\dfrac{2}{\pi}\sum_{0}^{\infty} \gamma_l \Gamma(2n - 1 - l) \end{array}\right\} \qquad (8.84)$$

where the first line corresponds to (8.80) and the second line to (8.83). The second line is especially interesting because it is the *numerical signature* of the nonsummability à la Borel of the Bohr-Sommerfeld expansion (7.17). We therefore emphasize the fact that we have indeed detected the contribution (8.83) in spite of its subdominance, and that it has « experimentally » the *right order of magnitude* (Appendix B).

Of course we are still very far from a test of the validity of our global discontinuity formulas ! What is missing here is a way of systematically exploiting the remarkable analytic structure of the functions $\alpha_{mm'}$ in order to reconstruct them numerically. When we compute the semiclassical expansion coefficients a_n one by one, we not only build the Taylor series of the Borel transform $(\log a)_B$ around $s = 0$ but we obtain *in parallel* the Taylor series of *all* its discontinuities in all sheets thanks to analytic bootstrap: we are thus approximating the ramified function $(\log a)_B$ in some global sense that we do not understand very well. In section 10 we shall give a more convincing global check of our results by deducing from them an *exact functional equation* for the Jost function, that will imply

numerical relations on the spectrum; but this will use only *part* of the content of our discontinuity formulas.

Discontinuity formulas for other spectral functions are easy to obtain once the principle of the method is understood. We shall not dwell upon this but only list a few possibilities: the functions Θ_{disc} and Θ_{reg}(introduced in Ref. [3], Eq. (78)) with application to an improved asymptotic formula for the zeta function of the quartic oscillator; the alternating function

$$Z^P(s) = \sum_0^\infty (-1)^k e^{-sx_k}$$ whose discontinuities involve the monomials $\alpha_{mm'}$

with *half-integral* indices (see also [3] and section 10), etc...

Final remark: a local form of analytic bootstrap already appears in Dingle's treatment of WKB theory ([4], Chaps. 13.7 and 14.3). We also believe that the subservience to a classical process (here to periodic orbits) of the singularities of $a_B(s)$ bears some relation to analytic S-matrix theory [33]: indeed, the Borel-transformed Jost function $a_B(s)$ is a one-dimensional S-matrix, but expressed in a sort of « proper time » variable. We suggest and hope that our dynamically simplified but very explicit results would offer new guidelines in general S-matrix theory, provided our vague analogy could be utilized even partially. For instance the problem of S-*matrix bootstrap* (to what extent is the S-matrix self-consistently determined by its global analytic structure) might benefit by an attack on our particular function $a_B(s)$ whose analytic structure was just described in full detail.

It is even more surprising that the same notion of analytic bootstrap emerges in many other fields of mathematics, as recognized by Ecalle [32] who calls the phenomenon *resurgence*. His work classifies many types of convolution algebras of ramified functions, of which our algebra $\{\alpha_{mm'}\}$ is one special case (cf. Chaps. 1-3 and 9 in [32]), and offers in Chap. 13 clues to the reconstruction of the algebra from its resurgence equations (Chap. 9), that are somewhat similar to our Eqs. (8.45) and (8.50) (but the problem of growth at infinity is not explicitly settled either). The emphasis of Ref. [32] lies at present on general classification problems whereas we have isolated specific concrete examples, but it is in future developments of [32] that we expect answers to the open questions left in our work (cf. also connections with microfunction theory [42]).

9. GENERALIZATION : CLASSICAL MECHANICS AND THE RIEMANN-HILBERT PROBLEM

A superficial reading of section 8 would suggest that our computations heavily depended on the special symmetries of the homogeneous quartic potential. In order to dispel that feeling and to shift our work from a technical

to a more general perspective, we now briefly recast our procedure of sections 5 to 8 in a more abstract setting known as the *monodromy* formalism, assuming the reader to be familiar with its basic notions [26]. It should then become clear that all our results are of a *purely semiclassical* nature and derive from *Hamilton-Jacobi theory* in one *complex* variable.

DEFINITIONS. — Let $f(s)$ be a multivalued function of one variable with the set of branch points $S = \{ s_1, s_2, \dots \}$. The global analytic structure of $f(s)$ can be described by the following objects:

1) the fundamental group $\pi_1(\mathbb{C} \backslash S, s_0) = \pi_1$ at some base point $s_0 \notin S$;

2) the vector space V spanned by all determinations of $f(s)$ above a neighborhood of s_0;

3) for each loop $\gamma \subset \mathbb{C} \backslash S$ based at s_0, the linear transformation M_γ induced on V by analytic continuation of the determinations of f along γ; M_γ only depends on the homotopy class $[\gamma] \in \pi_1$ and defines a group representation of π_1 in V; the *monodromy group* of f.

It suffices to compute the matrices M_{γ_j} for a set of loops $\{ \gamma_j \}$ generating the homotopy π_1 to control simultaneously the topology of the Riemann surface and the analytic structure of f, for we can then follow any determination of f along any circuit by matrix multiplications alone.

The relation with discontinuity formulas is the following: for any system of disjoint cuts (avoiding s_0) drawn from every branch point so that $f(s)$ becomes singlevalued we define a *dual* generating set of π_1 as follows: γ_j is a positive loop based at s_0 encircling the branch point s_j once and intersecting no cut besides the one issued from s_j (Fig. 28). Then π_1 is the free group [22] generated by the $[\gamma_j]$, V is spanned by one determination of f and all the discontinuity functions $\Delta_{s_j} f$, and

$$M_{\gamma_j} f(s) = f(s) - \Delta_{s_j} f(s) \qquad (9.1)$$
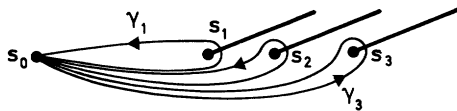
FIG. 28.

The same notions 1) to 3) carry on to functions of several complex variables for which the singular set S is an analytic manifold of complex codimension 1. We now translate into this language our previous discussions concerning in turn the Fourier transform $\tilde{\psi}(q, s)$ of the wave function recessive at $q = -\infty$, and the Borel transformed Jost function $a_B(s)$.

The solution $\tilde{\psi}(q, s)$ (section 6) is ramified along an analytic action curve $S \subset \mathbb{C}^2$. The description of its monodromy is *formally* similar to

that of a Feynman integral ramified along a Landau variety $\mathscr{L}$ of complex codimension 1 [26] ; only formally because in the Feyman case: the complex variables are projective, $\mathscr{L}$ is algebraic, the fundamental group is of finite presentation and the monodromy representation is finite dimensional. We shall nevertheless take over the algebraic ingredients well developed in the homological study of Feynman integrals, with the only justification that if we exclude a neighborhood of infinity (which is very pathological in our case) the two problems acquire similar topological features.

1) The fundamental group $\pi_1 = \pi_1(\mathbb{C}^2 \setminus S, (q_0, s_0))$ at some base point $(q_0, s_0) \notin S$ can be generated by loops γ_j in a complex plane L intersecting S *generically* (Picard-Severi theorem). It is very convenient for us to take L of the « vertical » form $L_{q_0}: \{ q = q_0 \}$: such a plane should be generic if no two determinations $s_j(q)$ of the action coincide at q_0 (in particular q_0 cannot be a turning point). To every determination $s_j(q_0)$ is associated one *generating loop* γ_j of π_1 encircling positively this and only this determination $s_j(q_0)$ in the plane L_{q_0}. It is also convenient to specify the set $\{ \gamma_j \}$ as dual to a system of horizontal cuts in L_{q_0} if no two Im $s_j(q_0)$ coincide (otherwise: to a system of cuts tilted by $e^{\pm i0}$): this makes the link with our reduction to standard form and Borel resummation (sections 5-7).

A difference with the case of one variable is that these generators γ_j are no longer independent but satisfy a set of *relations* for every *singular point* of S, according to the Van Kampen scheme: let L_{q_2} be an isolated nongeneric plane, such that two determinations s_+ and s_- coincide at $q = q_2$, and let Γ be a small loop in the complex q plane encircling the only nongeneric point q_2, with $q_0 \in \Gamma$. Each generating loop γ_j in L_{q_0} can be followed by continuous deformation (« ambient isotopy ») as the plane L_{q_0} is continuously deformed along the one parameter family of vertical planes $\{ L_q \mid q \in \Gamma \}$; γ_j ends up on a new generating loop $h(\Gamma)\gamma_j \subset L_{q_0}$, by which is defined an action h of the fundamental group of the q plane minus non-generic points upon the group $\pi_1(\mathbb{C}^2 \setminus S)$. Clearly $h(\Gamma)\gamma_j$ is homotopic to γ_j in π_1, hence we associate to the nongeneric point q_2 the following relations in π_1 :

$$h(\Gamma)\gamma_j = \gamma_j \qquad (\forall j) \qquad\qquad (9.2)$$

Now when S is an action curve associated with a Hamilton-Jacobi equation (6.30) all whose turning points are simple, there can be just two types of nongeneric points:

a) if $p(q_2) \neq 0$, S has (a) *transverse intersection(s)* above q_2.

b) if $p(q_2) = 0$, S has (a) *branch point(s)* of order 2, i. e. (a) *cusp(s)*.

If for all $q \in \Gamma$ (a loop encircling q_2) the generators $\gamma_j[q]$ (above q) are dual to a system of *horizontal cuts*, then they form a one-parameter family

of continuous deformations above Γ except at those points $q_1 \in \Gamma$ for which determinations s_j and s_k satisfy:

$$s_k(q_1) - s_j(q_1) > 0 \tag{9.3}$$

in which case the deformations of γ_j and γ_k beyond q_1 read

$$\gamma_j[q < q_1] \to \gamma_j[q > q_1] \quad \text{but} \quad \gamma_k[q < q_1] \to \gamma_j\gamma_k\gamma_j^{-1}[q > q_1] \tag{9.4}$$
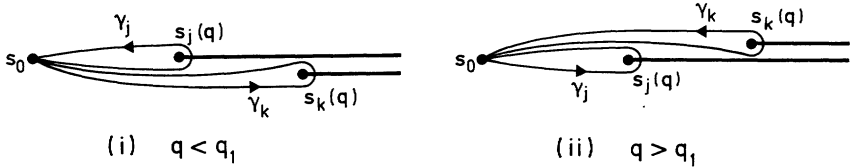

FIG. 29.

(Fig. 29), the ordering relation being induced in a neighborhood of q_1 by the positive orientation of Γ. By combining the effects (9.4) of successive crossings of the type (9.3) along Γ we can explicit the action $h(\Gamma)$ and all the relations (9.2).

a) If $p(q_2) \neq 0$ and s_+, s_- denote a pair of determinations that intersect each other at q_2, then on Γ there occur two crossings, say the first one with $s_- > s_+$ and the second one with $s_+ > s_-$. The continuous deformations of the corresponding generators γ_+ and γ_- along Γ read:

$$\gamma_+ \to \gamma_+ \to \left(\gamma_+\gamma_-\gamma_+^{-1}\right)\gamma_+\left(\gamma_+\gamma_-\gamma_+^{-1}\right)^{-1}$$

$$\gamma_- \to \gamma_+\gamma_-\gamma_+^{-1} \to \gamma_+\gamma_-\gamma_+^{-1} \tag{9.5}$$

and both relations (9.2) for $\gamma = \gamma_+$ or γ_- amount to

$$\gamma_+\gamma_- = \gamma_-\gamma_+ \tag{9.6}$$

b) If now $p(q_2) = 0$ and s_+, s_- form a cusp q_2, the same argument yields instead of (9.6) the relation:

$$\gamma_+\gamma_-\gamma_+ = \gamma_-\gamma_+\gamma_- . \tag{9.7}$$

In both cases the meaningful relations involve pairs of coalescing determinations (for any other branch s_j only the trivial relation $\gamma_j = \gamma_j$ results).

We expect that (9.6) and (9.7) generate exhaustively the relations satisfied by the generators of π_1. Then π_1 is *the group* with: one generator per branch of the action curve S, and: one relation (9.6) per self-intersection of S, one relation (9.7) per cusp of S. The abstract presentation of π_1 is thus completely governed by classical dynamics, i. e. by the geometry of the complex Hamilton-Jacobi equation (6.30).

Remark. — It is no accident that:

. (9.6) is the condition allowing the self-intersection of the two branches s_+ and s_- considered as defect lines in a solid [22].

. (9.7) is the defining relation for the trefoil knot group (the knot in question appears in Fig. 10) [27].

2) The space V of determinations of $\tilde{\psi}(q, s)$ in a neighborhood of (q_0, s_0) is generated by one particular determination $\tilde{\psi}_0(q, s)$ and by all possible discontinuities of $\tilde{\psi}(q, s)$:

$$\Delta_{s_k}(q)\tilde{\psi}(q, s) = \left\{ u(q, x)^{-1/2} \exp ix \int_k [u(q', x) - p(q')dq' \right\}_B (s - s_k(q)) \quad (9.8)$$

where k runs over all homotopy classes of paths in $\dot{\mathbb{C}}$ (the complex q plane minus all turning points) joining $q = -\infty$ to q, and $s_k(q)$ is the corresponding determination of $s(q)$: this is simply a transcription of Eq. (6.32). The space V is of countably infinite dimension (!).

3) We now consider the action of a monodromy matrix M_{γ_k} on the generating set of V defined in 2). By definition (Eq. (9.1)):

$$M_{\gamma_k}\tilde{\psi}_0 = \tilde{\psi}_0 - \Delta_{s_j(q)}\tilde{\psi} \quad (9.9)$$

and the real problem is to compute $M_{\gamma_k}(\Delta_{s_j}\tilde{\psi})$. We first assume that the determinations s_j and s_k have opposite speeds and that the values Im s_j and Im s_k are immediately adjacent. We then displace the base point q_0 on a path along which no two branches of s are related by (9.3), up to a position very close to a point q_2 at which $s_j(q_2) = s_k(q_2)$. We can then analyze the situation on Fig. 29: the discontinuity of $\Delta_{s_j}\tilde{\psi}$ is the difference of the discontinuities $\Delta_{s_k}\tilde{\psi}$ evaluated in the two configurations (*i*) and (*ii*). We now borrow the following information from the structure of WKB expressions:

. the quantities s_k and $\int_k (u - p)dq$ only depend upon the homology class of the path in $\dot{\mathbb{C}}$, which is the same in both configurations of Fig. 29, therefore the discontinuity at s_k of $\Delta_{s_j}\tilde{\psi}$ has the form:

$$\Delta_{s_k}\Delta_{s_j}\tilde{\psi} = \Delta_{s_j}\tilde{\psi} - M_{\gamma_k}\Delta_{s_j}\tilde{\psi} = -\beta\Delta_{s_k}\tilde{\psi} \quad (9.10)$$

with a multiplier β to be determined ; similarly:

$$\Delta_{s_j}\Delta_{s_k}\tilde{\psi} = \Delta_{s_k}\tilde{\psi} - M_{\gamma_j}\Delta_{s_k}\tilde{\psi} = -\alpha\Delta_{s_j}\tilde{\psi} \quad (9.11)$$

. each discontinuity $\Delta_{s_k}\tilde{\psi}$ is analytic at s_k itself:

$$\Delta_{s_k}\Delta_{s_k}\tilde{\psi} = 0 \qquad (\forall k) \quad (9.12)$$

Eqs. (9.10-9.12) imply that M_{γ_j} and M_{γ_k} leave the subspace $V_{j_k} \subset V$ spanned by $\Delta_{s_j}\tilde{\psi}$ and $\Delta_{s_k}\tilde{\psi}$ invariant, and have on this subspace the matrix form:

$$M_{\gamma_j} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \quad M_{\gamma_k} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \tag{9.13}$$

We now express compatibility of the monodromy representation with the Van Kampen relations at the nongeneric point q_2 where s_j and s_k meet. If $p_j(q_2) = - p_k(q_2) \neq 0$ relation (9.6) implies

$$M_{\gamma_j} M_{\gamma_k} = M_{\gamma_k} M_{\gamma_j} \Rightarrow M_{\gamma_j} = M_{\gamma_k} = \mathbb{1} \tag{9.14}$$

after use of (9.13); whereas if $p_j(q_2) = 0$ relation (9.7) implies

$$M_{\gamma_j} M_{\gamma_k} M_{\gamma_j} = M_{\gamma_k} M_{\gamma_j} M_{\gamma_k} \Rightarrow \alpha\beta = - 1 \tag{9.15}$$

Here we cannot determine separately α and β without information on the relative normalization of $\Delta_{s_j}\tilde{\psi}$ and $\Delta_{s_k}\tilde{\psi}$. But the reader has recognized by now that we are just resuming here our discussion of § 6. The contour change (9.4) is the homotopic counterpart of the Stokes phenomenon, (9.13) are the corresponding connection matrices, and α, β are the Stokes multipliers. Now the single-valuedness argument invoked in § 6 does fix the values of α and β in a way compatible with (9.14) or (9.15), whichever the case.

For each j let $V_j^+(V_j^-)$ be the subspace spanned by the discontinuities $\Delta_{s_k}\tilde{\psi}$ such that $\dfrac{ds_k}{dq} = \pm \dfrac{ds_j}{dq}$. Our previous argument actually shows that $M_{\gamma_j} | V_j^-$ is the identity matrix save for off-diagonal Stokes multipliers connecting only $\Delta_{s_j}\tilde{\psi}$ to its immediate analytic continuations around the turning points. That is because all crossings between s_j and other determinations s_k give rise to the relation (9.14) and can be ignored.

On the other hand the remaining part of the monodromy: $M_{\gamma_j} | V_j^+$ is prescribed only through our asymptotic specification of $\tilde{\psi}(q, s)$ for $q \to \infty$.

By requiring $\tilde{\psi}(q, s)$ to behave like $-(2\pi i)^{-1} p(q)^{-1/2} \log (s - s(q))$ for $q \to - \infty$ (Eq. (6.27)) we essentially discard the branch points going to infinity opposite to $s(q)$ and require the trace of the monodromy representation on the branch points moving parallel to $s(q)$ to reduce to the monodromy of $\log (s - s(q))$. Similarly the monodromy of the Laplace-transformed Jost function $\tilde{a}(s)$ (Eq. (6.29)) is the trace when $q \to + \infty$ of the full monodromy of $\tilde{\psi}(q, s)$, restricted to those branch points that go to infinity parallel to the adequate determination of $s(q)$ when $q \to + \infty$.

This discussion was very formal and incomplete. For instance we have not proved that (9.6)-(9.7) do exhaust all relations of π_1 and that all determinations (9.8) corresponding to homologically distinct paths are independent. Failure of either property to hold would not however inva-

lidate our results but only correspond to unexpected trivial components in our monodromy representation. Incomplete as they stand, our results then fulfill our present goal, which was to show that laborious connection formulas across conventionally defined Stokes lines (section 6) actually reflect intrinsic features of complex Hamilton-Jacobi theory: the embedding of the action curve S in $\mathbb{C}^2$ and the ensuing restrictions upon the monodromy of $\tilde{\psi}(q, s)$.

The Borel-transformed Jost function: a similar process of generalization can be directly applied to the discussion of sections 7-8 about the function $a_B(s) = \Delta_0 \tilde{a}(s)$.

The basic ingredient behind the analytic structure of $a_B(s)$ is the set of completely general relations (7.5-7.8) between the integrals (7.3), which yields multiplicative relations between the expansions u_γ and a_γ and the exponentials of action periods (cf. our notations (5.10), (5.12)). Those relations are again written at fixed E, but the difference with the homogeneous case is that the expansion terms become nontrivial functions of E (e. g. complete elliptic integrals for a general quartic potential). The analytic continuation in E that links the various integrals (7.3) becomes independent from analytic continuation in x and no simple relations like (7.12) to (7.26) connect those integrals to one another. Fortunately, although we used those relations in section 8, it was only to shorten the argument and not in a crucial way. The following alternate procedure is quite general and shows the origin of analytic bootstrap. We take as basic quantities all the expansions at fixed E:

$$a_{C_j}(x) = \exp ix \int_{C_j} (u - p)dq \qquad j = 1, \ldots, 2M \qquad (9.16)$$

where the integration paths are those of Eq. (7.3) and 2M is the number of turning points. Relation (7.5) amounts to:

$$\prod_{j=1}^{2M} a_{C_j}(x) = 1 \qquad (9.17)$$

Each quantity $a_{C_n}(x)$ can be *separately* computed as an element of a certain connection matrix. The *Anti-Stokes* (or *principal*) lines belonging to q'_n are the three curves $\left\{ \text{Im} \int_{q'}^{q} pdq = 0 \right\}$. With two adjacent such lines containing no other turning point between them, draw Fig. 30. If ϕ_+ dominates along the intermediate Stokes line, and ψ_+ is the exact wave function normalized by $\psi_+ \sim \phi_+$ along line I, then $a_{C_n}(x)$ (resummed) is the limit of ψ_+/ϕ_+ along line II (if ϕ_- dominates, exchange the roles of I and II). The analytic structure of the Borel transform $(a_{C_n})_B$ and the correct resumma-

Fig. 30.

— Stokes line
- - - Principal lines
〜〜 Cut
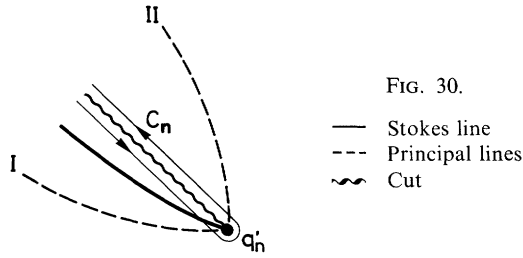
tion paths are then derived as for the Jost function, by solving the connection problem from direction I to II for all values of $\varphi = -\operatorname{Arg} x$. This connection problem is in a sense the most *basic* one: its critical angles φ (that exhibit Stokes discontinuities, i. e. branch points of $(a_{C_n})_B$) are just those where q'_n *itself* and some other turning point see their systems of Stokes lines cross, and this occurs in the *same topological pattern* as with the harmonic oscillator at $\varphi = \pi/2$ (Eqs. (7.41)-(7.43)), hence the connection matrix will be likewise changed by a *commutator term* containing only the corresponding loop integral(s) $u_{\gamma_{jk}}(x)$ (we exclude the exceptional case where a connected set of Stokes lines joins more than two turning points at a time). We now invoke (7.6), or:

$$u_{\gamma_{jk}}(x) = e^{-\omega(\gamma_{jk})x} \prod_{k < l < j} a_{C_l} \prod_{j < m < k} a_{C_m}^{-1} \qquad (9.18)$$

which implies that $(a_{C_n})_B$ has branch points at *action periods* $\omega \in \Omega$ at which the discontinuities are *convolution monomials in the* $(a_{C_l})_B$ and $(a_{C_l}^{-1})_B$ themselves, suitably translated. Therefore the collection of all discontinuity formulas will ultimately *close upon itself*: this is the general cause behind analytic bootstrap, the precise realization of which is governed by the global turning point structure of the problem, i. e. purely by *classical mechanics*.

The formulation of those facts into the monodromy language immediately suggests itself (the resulting structure is of the type studied in [32]):

1) the branch point set S is the action period group Ω:

$$\pi_1 = \pi_1(\mathbb{C}\backslash\Omega, s_0) \qquad (9.19)$$

(actually one point should be removable from Ω: see below).

2) the vector space V is spanned by all the monomials

$$(a_{C_1}^{n_1} \ldots a_{C_{2M}}^{n_{2M}})_B \qquad n_1, \ldots, n_{2M} \in \mathbb{Z} \qquad (9.20)$$

(that differ from 1: see again below); each monomial must be however *translated* in the s plane by the action period appearing in its discontinuity formulas derived from (9.18).

3) the monodromy matrix elements are computable in terms of the *multipliers* in the various discontinuity formulas.

This program is very cumbersome to carry out in the general case if only because the branch point set in projection, namely the period group Ω, is dense in $\mathbb{C}$. A few salient features can however be expected to hold in general.

— The theory will describe the analytic structure of Borel transforms with respect to $x = \hbar^{-1}$ at *fixed classical parameters* (energy included). Other types of semiclassical expansions (e. g. at fixed quantum numbers) could behave quite differently.

— The Jost function at $E < 0$ (when $V \geq 0$) will be Borel summable for $x > 0$, because no real turning points are encountered in the connection process.

— The Bohr-Sommerfeld expansion for eigenvalues ($E > 0$) will not be Borel summable; indeed, branch points on the real positive axis of its Borel transform have the form $(\omega + \omega^*)$ $(\omega, \omega^* \in \Omega)$ and should occur (in countable number) for any potential.

We now briefly return to the simpler quartic potential, starting from our results with the homogeneous case (section 8). From the discontinuity formulas (8.45) we infer that the space of determinations V to consider is spanned by the basis:

$$\{ \varphi_{nn'}(s) = \alpha_{nn'}(s - n\omega - n'\omega') \}, \quad (n, n') \in \mathbb{Z}^2 \setminus \{ 0, 0 \} \qquad (9.21)$$

(because of their exponential form the $\varphi_{nn'}$ are linearly independent) and the branch points form the lattice Ω deprived of the origin: we recall from Eq. (8.47) that all $\varphi_{nn'}$ are analytic at $s = 0$ in all sheets and that the would-be discontinuity function $\varphi_{00} = 1$ at that point is excluded from the bootstrap game (this feature should be quite general).

We then take $s_0 = 0$ as base point and pick as generators for $\pi_1 = \pi_1(\mathbb{C} \setminus (\Omega \setminus \{ 0 \}), s_0 = 0)$ a set of loops dual to radial cuts drawn with the e^{i0} convention (for instance) ; we call $\gamma_{mm'}$ the positive loop encircling the lattice point $(m\omega + m'\omega')$. The monodromy transformations have the form:

$$\mathrm{M}_{\gamma_{mm'}}\varphi_{nn'} = \varphi_{nn'} - \Delta_{m\omega + m'\omega'}\varphi_{nn'} = \varphi_{nn'} - c_{mm'}^{nn'}\varphi_{mm'} \qquad (9.22)$$

as a consequence of the structure of the discontinuity formulas (8.45). Unfortunately the multipliers in (9.22) differ from those in (8.45) because in the latter formulas cuts were radially oriented away from the zero argument of the function $\alpha_{nn'}$ (up to rotations by $e^{\pm i0}$) whereas in (9.22) they point away from the zero argument of the *translated* function $\varphi_{nn'}$ (Fig. 31 illustrates the case of φ_{21} and should be compared with Fig. 22). We now show that (9.22) can be explicitly derived from the known discontinuity relations (8.45-46) in a *finite* number of steps.
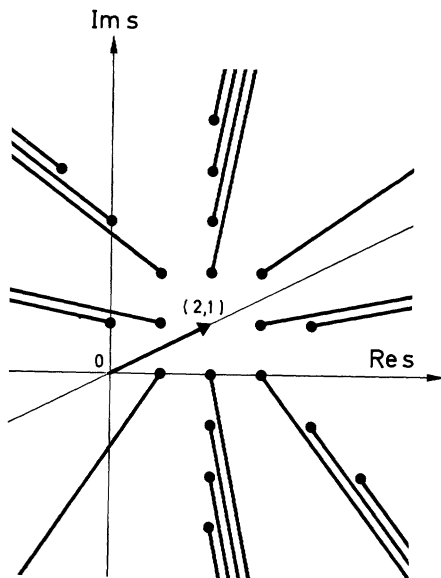
FIG. 31. — $\varphi_{21}(s)$.

We actually start from the discontinuity formulas in the *adapted convention* where only a finite number of cuts appears in the first sheet, i. e. at each critical angle independently we choose to tilt all cuts by e^{+i0} or e^{-i0} to achieve that result. For given φ in formula (8.46) (written in the e^{-i0} convention) the multipliers have the form $\begin{pmatrix} k \\ m \end{pmatrix}$: if $k > 0$ then their sequence terminates as m increases and the e^{-i0} convention is adapted; if $k < 0$ we must shift to the e^{+i0} convention where the new sequence of multipliers is $-\begin{pmatrix} -k \\ m \end{pmatrix}$; if $k = 0$ the convention is indifferent (neutral situation). A simple geometrical criterion is the following: at each branch point in the first sheet $(m\omega + m'\omega')$ of $\alpha_{nn'}$ the $e^{-i0}(e^{+i0})$ convention is adapted if the branch point lies in the half-plane $\{ nm' - mn' > 0 \, (< 0) \}$; for any branch points on the boundary line $\{ nm' - mn' = 0 \}$ the convention is indifferent.

We now show how to rotate the cuts of $\varphi_{nn'}$ issued from branch points in the half-plane H^+: $\{ nm' - mn' \geq 0 \}$ for which the e^{-i0} convention is adapted (the other half-plane is treated similarly). For $s \in H_+$ let $\chi(s)$ be the angle under which the vector $(0, n\omega + n'\omega')$ is seen: this angle is positive, and if s is a branch point our task is to turn its cut by $- \chi(s)$
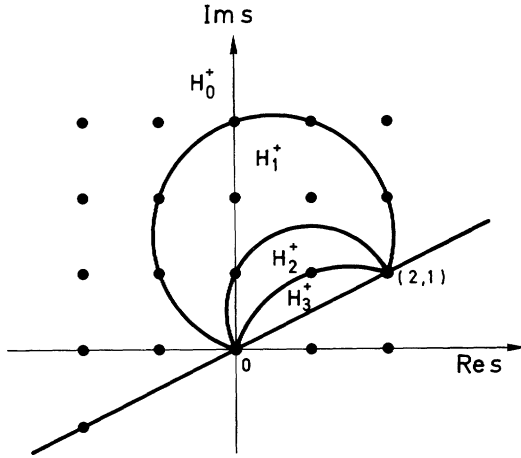
FIG. 32.

from its original position. We separate H^+ in four regions H_j^+ $(j = 0, 1, 2, 3)$ bounded by circular arcs (Fig. 32):

$$H_j^+ = \left\{ s \in \mathbb{C} \,\middle|\, \frac{j\pi}{4} < \chi(s) < (j + 1)\frac{\pi}{4} \right\} \qquad (9.23)$$

During the rotation that we inflict to each cut to orient it properly we encounter exactly j critical angles, where H_j^+ is the region containing the origin of the cut. When $j = 0$ (the case of *all but a finite number* of branch points) the cut can be oriented without changing the value of any other discontinuity. When $j \geq 1$ each encounter of a critical angle uncovers a *finite* number of new discontinuities in the e^{-i0} convention, themselves evaluated by (8.45-46). Each discontinuity present in the final picture gives rise to a non zero monodromy matrix element by virtue of formula (9.1); we have not attempted to express the result in closed form.

Remark. — The monodromy representation just described is only a direct summand of the full monodromy described by Eqs. (9.20). Indeed, with the basic definitions and notations (8.32) and (9.21) extended to rational n, n' from (8.7) and (8.16):

$$(a_{C_1})_B = (a_{C_3}^{-1})_B = (a^{1/2}a'^{-1/2})_B = \alpha_{-\frac{1}{2},\frac{1}{2}}$$
$$(a_{C_2})_B = (a_{C_4}^{-1})_B = (a^{1/2}a'^{1/2})_B = \alpha_{-\frac{1}{2},-\frac{1}{2}} \qquad (9.24)$$

The first sheet analytic structures of $\alpha_{-1/2,1/2}$ and $\alpha_{-1/2,-1/2}$ are shown on Figs. 33 a) and b) respectively, in the adapted convention ; they are indeed geometrically simpler than that $a_B = \alpha_{-1,0}$ (Fig. 21) which is now
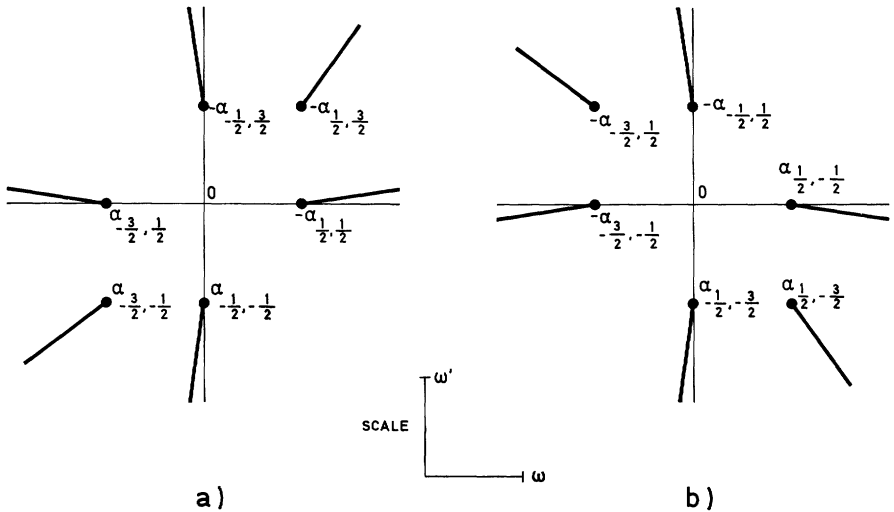
FIG. 33.

a derived result via the convolution formula $a_{\mathrm{B}} = \alpha_{-1/2, 1/2} \ast \alpha_{-1/2, -1/2}$.

A crucial property of the monodromy should now be clear: it is solely determined by the complex geometry of the turning points. If we now consider a non-homogeneous quartic potential, where now the energy, coupling constant... are external parameters, the geometry of turning points is unchanged under local continuous deformations in the parameters; such deformations are *isomonodromic*, which feature brings the semiclassical method in unexpected relation to the *Riemann-Hilbert problem* (see for instance [28] [39]; of course the deformations should avoid exceptional points where for instance two turning points coalesce).

The classical Riemann-Hilbert problem is to find all analytic functions with a given monodromy, provided the number of branch points in the Riemann sphere and the dimension of the representation are finite. Our problem is on one hand much worse because branch points accumulate at infinity and dim $V = \infty$; on the other hand classical dynamics imposes very rigid constraints by making the branch points form a subgroup of the translation group and the corresponding discontinuities form a convolutive representation of that subgroup. In any case this Riemann-Hilbert problem should not have a unique solution since many distinct quantal problems can share the same classical structure.

Another challenge lies in extending these ideas to multidimensional Schrödinger equations. This should be partly feasible as suggested by a theorem of Riemannian geometry [29]: $\mathrm{Tr} \exp \left(it \sqrt{-\Delta} \right)$ admits as C^∞ singular support (on the real axis) the periods of closed geodesics,

for a compact manifold of *arbitrary dimension*. Any global analytic statement of the same vein would suggest by analogy that our modified partition function $\Sigma \exp(-tx_k)(E))$ for the Schrödinger operator in any dimension has its analytic singularities at action periods; this is also in line with the ideas of Balian and Bloch [11]. On the other hand the WKB method is in essence one-dimensional and loses efficiency in higher dimensions, all the more that the classical system is far from integrability (the action periods might become everywhere dense, for instance). For completely integrable systems all hopes are permitted and a complex WKB method in 3 dimensions is proposed for instance in [30], but the abstract machinery of Refs. [26] will certainly be needed to a much greater extent than in one dimension to clarify the situation.

To summarize, we have shown in this section that the description of the analytic structure of various Borel transforms like $\tilde{\psi}(q, s)$ and $a_B(s)$ results solely from complex semi-classical rules (valid for all analytic, one-dimensional potentials) and not from any specificity of the few potentials considered in the previous sections. We may conversely state that the ultimate output of the semi-classical treatment is, at the present stage, the global monodromy structure of these Borel transforms. The semi-classical method is thus an *exact analytic tool* [4], becoming approximate only if it is selectively but needlessly mutilated to generate the *local* Taylor series of the Borel transforms. Its exact content is then irretrievably lost, but one is left with highly *efficient and explicit* approximation schemes to various physical quantities (cf. the Bohr-Sommerfeld expansion (4.29)). By contrast, we do not know whether and how the global and exact part of our WKB results, too abstract to be directly useful, could be nearly as much explicited in full generality. In special cases though, we have just obtained encouraging partial answers to this question [34]: these will occupy the next—which is also the last—section of this work.

10. FREDHOLM DETERMINANTS AND THEIR FUNCTIONAL EQUATIONS

Sections 7-9 have established the capacity of the WKB method to yield exact yet implicit results in the form of analytic discontinuity formulas for certain Borel transforms of the problem. Starting with the quartic oscillator, we shall now explicit part of those results, first as exact *functional equations* obeyed by objects of *direct physical relevance*, namely the Jost function itself and the Fredholm determinant, and next as an infinite set of *algebraic identities on the eigenvalue moments*, i. e. on the values of the zeta function (to follow this section better we recommend prior reading of Ref. [3]). The numerical check of those identities will give compelling evidence for the *global* validity of our interpretation of WKB theory.

We shall then illustrate the flexibility and generality of the method upon a complementary connection problem that not only produces one more functional equation in the quartic case, but also immediately extends to *higher degree potentials*. Finally we shall borrow ideas from section 9 to conclude that similar functional equations should arise for all *polynomial interactions* in one degree of freedom, as could be inferred from earlier work by Sibuya-Cameron [40].

Spectral functions and scaling revisited: we have already defined the Fredholm determinant $\Delta(E)$ and resolvent trace $R(E)$ in Appendix A:

$$\Delta(E, x) = \sum_{k=0}^{\infty} (1 - E/E_k(x)) = \exp - \int_0^E R(\varepsilon, x)d\varepsilon \qquad (10.1)$$

where $\{E_k(x)\}$ are the eigenvalues of the operator

$$\hat{H}(x) = - x^{-2}d^2/dq^2 + V(q) \quad (x \equiv \hbar^{-1}) \qquad (10.2)$$

In the homogeneous case $V(q) = q^{2M}$ ($M \geq 2$ to ensure convergence of (10.1)), the scaling rules (7.12-7.14) imply that the dimensionless quantity $\Delta(E, x)$, like the Jost function $a(E. x)$, is only a function of the special combination:

$$\lambda = - x^{\frac{2M}{M+1}}E \qquad (10.3)$$

Hence all formulas of sections 7-8, initially meant for *variable* x at fixed $E_0 = - 1$, are equally well read as formulas at fixed $x_0 = 1$ for *variable* $E \equiv - \lambda$, provided we set:

$$x \equiv (- E)^{\frac{M+1}{2M}} \qquad (10.4)$$

Either interpretation will be needed at some later stage: this distinction between the good variable for Borel transformation (x), and the variable of analyticity (E), seems unavoidable and important. Now we choose to fix $x_0 = 1$ and vary the spectral parameter E of $\hat{H}(1)$: under the identification (10.4), the Jost function expansion (7.16) is reinterpreted as:

$$\left.\begin{array}{c} \log a[x \equiv (- E)^{-i_0}] \sim \sum_{n=1}^{\infty} a_n(- E)^{-i_n} \quad (E \to - \infty) \\[2em] i_n \equiv \dfrac{M + 1}{2M}(2n - 1) \end{array}\right\} \qquad (10.5)$$

where

(we recall that the a_n depend upon M, too). From there we can get to the

expansion coefficients for the resolvent trace R(E), by combining the identity (A.3) at $x = 1$ with the formal differentiation of Eq. (10.5):

$$R(E) = T(E) - \frac{d}{dE} \log a \, [x = (- E)^{-i_0}] \qquad (10.6)$$

$$\Rightarrow R(E) \sim - \sum_{n=0}^{\infty} a_n i_n (- E)^{-i_n - 1} \quad (E \to - \infty) \qquad (10.7)$$

Indeed the classical transit time T(E) (Appendix A) evaluates for $E < 0$ to:

$$T(E) = \frac{1}{2} \int_{-\infty}^{+\infty} (q^{2M} - E)^{-1/2} dq = - a_0 i_0 (- E)^{-i_0 - 1}$$
$$a_0 = \frac{\Gamma(1/2M)\Gamma(1/2 - 1/2M)}{(M + 1)\sqrt{\pi}} \qquad (10.8)$$

This *agrees* with our earlier prescription (7.25) for a_0 in terms of b_0 given by formula (7.18).

We likewise arrive at the *Fredholm determinant expansion* through the identity (A.5):

$$\log \Delta(E) \sim - \int_0^E T(\varepsilon)d\varepsilon + \int_{-\infty}^0 [R(\varepsilon) - T(\varepsilon)]d\varepsilon + \sum_{n=1}^{\infty} a_n(- E)^{-i_n}$$

$$\sim a_0(- E)^{-i_0} + \int_{-\infty}^0 [R(\varepsilon) - T(\varepsilon)]d\varepsilon + \sum_{n=1}^{\infty} a_n(- E)^{-i_n} \quad (10.9)$$

The second term in (10.9), a pure number, clearly plays a special role that justifies the following digression.

The zeta function of the operator $\hat{H}(x = 1)$ is defined by the convergent series:

$$\zeta(\sigma) = \sum_0^{\infty} E_k^{-\sigma} \quad (\text{Re } \sigma > - i_0), \qquad (10.10)$$

where the E_k are the eigenvalues of $\hat{H}(1)$. Its importance for confining potentials was recognized in Ref. [35] where with the help of the WKB expansion, $\zeta(\sigma)$ was analytically continued to a *meromorphic* function in the whole complex plane, with poles placed along the arithmetic progression $(- i_n)$. This continuation occurs via analytic regularization: for $- i_0 < \text{Re } \sigma < 1$ we know (e. g. through term-by-term comparison) the relation of $\zeta(\sigma)$ to a *Mellin transform* of the resolvent trace R(E):

$$\zeta(\sigma) = \frac{\sin \pi\sigma}{\pi} \int_{-\infty}^0 (- E)^{-\sigma} R(E)dE; \qquad (10.11)$$

but the integral (10.11) is made to diverge at Re $\sigma = -i_0$ by the leading term $T(E)$ in the expansion (10.7) of $R(E)$ at $E = -\infty$.

By a straightforward argument ([10], Chap. I.3), this term produces a pole for $\zeta(\sigma)$ at $(-i_0)$, beyond which Eq. (10.11) must be replaced by:

$$\zeta(\sigma) = \frac{\sin \pi\sigma}{\pi} \int_{-\infty}^{0} (-E)^{-\sigma} [R(E) - T(E)] dE, \quad (-i_1 < \text{Re } \sigma < -i_0), \quad (10.12)$$

and so on. But this one substraction suffices to compute the derivative $\zeta'(0)$, as the point $\sigma = 0$ lies between the first two poles $(-i_0)$ and $(-i_1)$. By differentiating (10.12) (also note that (10.12) implies $\zeta(0) = 0$ [35] [3]):

$$\zeta'(0) = \int_{-\gamma}^{0} [R(E) - T(E)] dE \qquad (10.13)$$

we precisely recover the strange constant term in the expansion (10.9). We are now motivated to shift to a more natural normalization of the determinant:

$$D(E) = \exp(-\zeta'(0)) \Delta(E) \qquad (10.14)$$

Then the identity (A.5) and the expansion (10.9) respectively read:

$$D(E) \equiv [e^{a_0 x} a(x)]_{x = (-E)^{-i_0}} \qquad (10.15)$$

$$\log D(E) \sim \sum_{0}^{\infty} a_n (-E)^{-i_n} \quad (E \to -\infty) \qquad (10.16)$$

Another connection needed later between $D(E)$ and $\zeta(\sigma)$ lies in the following Taylor series around $E = 0$:

$$\log \Delta(E) = -\sum_{n=1}^{\infty} \zeta(n) E^n / n$$

$$\log D(E) = -\zeta'(0) - \sum_{n=1}^{\infty} \zeta(n) E^n / n \qquad (10.17)$$

They simply result by reordering the Taylor series for $\sum_{0}^{\infty} \log(1 - E/E_k)$, and they converge for $|E| < E_0$ (the ground state).

More properties of $\zeta(\sigma)$ appear in Ref. [3], also later in this section and in Appendix C.

Functional equation, quartic case ($M = 2$): we now return to the discussion of the quartic oscillator in section 8, where the fixed E_0—variable x convention was established from Eq. (8.1) onwards. Our purpose is to show that

the full collection of discontinuity formulas considered there, namely (8.20), (8.24), (8.27-30), carries a consistency condition in the form of a *functional equation for the Jost function* $a(x)$. Each discontinuity formula expresses the jump of the Laplace transform $L_0^\varphi a_B$ across a critical value of the angle φ; consistency requires that the sum of all jumps over an angular circuit of 2π be equal to zero.

We first recast the symmetry relations (8.13-14) in a more precise form that will result in an actually shorter computation. Being the *exponential of an odd series* in x, the Jost function (of *any* potential) satisfies the *formal* relation (8.13), i. e.

$$a(x) = a^{-1}(-x) \qquad (10.18)$$

Returning to the pattern of derivation of formulas (3.18), we Borel-transform Eq. (10.18) to the identity:

$$a_B(s) = a_B^{*(-1)}(-s) \qquad (10.19)$$

which is now analytic instead of formal, as it consists of power series convergent in a disk. By applying the Laplace transforms of section 3 to both sides of (10.19) we end up with:

$$(L_0^\varphi a_B)(x) = (L_0^{\varphi+\pi} a_B)^{-1}(-x) \qquad (10.20)$$

whenever the left-hand side is defined, namely: for any φ in a sector Σ without first sheet branch points of $a_B(s)$, and for all x in the associated sector (3.21). Eq. (10.20) defines an *exactly resummed* form of (10.18) in every such sector Σ; it holds for arbitrary potentials exactly as Eq. (10.18) did.

The other formal relation of interest for the specific potential q^4, Eq. (8.14): $a'(x) = a(ix)$, likewise admits the resummed form:

$$(L_0^\varphi a_B')(x) = (L_0^{\varphi-\pi/2} a_B)(ix) \qquad (10.21)$$

We now rewrite the discontinuity formulas themselves, referring to Fig. 21 for the analytic structure $a_B(s)$. The symmetry relation (10.20) allows us to restrict to the angular range $0 \leq \varphi \leq \pi$ instead of $[0, 2\pi]$. The only jumps of $L_0^\varphi a_B$ then occur at $\varphi = \pi/4$, $\pi/2$ and $3\pi/4$; we select a representative angle in each remaining angular sector, for instance: $\varphi_0 = 0, \pi/4 < \varphi_1 < \pi/2 < \varphi_2 < 3\pi/4 < \varphi_3 < \pi$, and we conveniently pose (cf. Eq. (3.4)):

$$u_\varphi(x) = e^{\omega x}(L_0^\varphi a_B)(x) = L_{-\omega}^\varphi [a_B(\omega + .)](x)$$
$$u_\varphi'(x) = e^{\omega' x}(L_0^\varphi a_B')(x) = L_{-\omega'}^\varphi [a_B'(\omega' + .)](x) \qquad (10.22)$$

(We recall from Eqs. (8.4)-(8.9) that $\omega = -i\omega' = a_0$). Relations (10.20-21) then translate to:

$$u_\varphi(x) = u_{\varphi+\pi}^{-1}(-x), \; u_\varphi'(x) = u_{\varphi-\pi/2}(ix) \qquad (10.23)$$

and the discontinuity formulas also take a simpler form. The first one, Eq. (8.20) at $\varphi = \pi/4$, reads:

$$u_{\varphi_1}(x) - u_0(x) = - u'^{-1}_{\varphi_1}(x) \quad (\text{Re } (e^{i\pi/4}x) > 0) \tag{10.24}$$

and is instantly reduced with the help of (10.23) to a form containing only functions u_φ with $0 \le \varphi < \pi$:

$$u_{\varphi_1}(x) - u_0(x) = - u_0^{-1}(ix) \quad (- 3\pi/4 < \text{Arg } x < \pi/4) \tag{10.25}$$

In detail: $u'^{-1}_{\varphi_1}(x) = u^{-1}_{\varphi_1 - \pi/2}(ix) = u_0^{-1}(ix)$, since $(\varphi_1 - \pi/2)$ lies in the same angular sector as $\varphi = 0$; the sector of validity in (10.25) is the intersection of the sectors (3.21) computed on either side.

The next discontinuity formulas: (8.24) at $\varphi = \pi/2$ and (8.27) at $\varphi = 3\pi/4$, likewise become:

$$\text{or} \quad \begin{aligned} u_{\varphi_2} - u_{\varphi_1} &= - 2u'^{-1}_{\varphi_2}u_{\varphi_2} - u'^{-2}_{\varphi_2}u_{\varphi_2} \\ u_{\varphi_1}(x) &= u_{\varphi_2}(x)[1 + u_0^{-1}(ix)]^2 \end{aligned} \quad (- \pi < \text{Arg } x < 0) \tag{10.26}$$

and

$$\text{or} \quad \begin{aligned} u_{\varphi_3} - u_{\varphi_2} &= - u'^{-1}_{\varphi_3}u_{\varphi_3}^2 \\ u_{\varphi_3}(x) - u_{\varphi_2}(x) &= - u_{\varphi_1}^{-1}(ix)u_{\varphi_3}^2(x) \end{aligned} \quad (- 5\pi/4 < \text{Arg } x < - \pi/4) \tag{10.27}$$

It is superflous to proceed further because φ_3 lies in the same angular sector as π, which allows us to recall the identity (10.20):

$$u_{\varphi_3}(x) = u_0^{-1}(- x) \quad (|\text{Arg } (- x)| < 3\pi/4) \tag{10.28}$$

Eqs. (10.25-28) now form a system of four functional relations on the four functions u_0, u_{φ_1}, u_{φ_2}, u_{φ_3}. But u_0 is up to a factor $e^{\omega x}$ the *ordinary Borel sum* of $a_B(s)$, which is the *Jost function itself* (as proved in section 7):

$$u_0(x) = e^{\omega x}a(x) \quad (|\text{Arg } x| < 3\pi/4) \tag{10.29}$$

So, if we eliminate the other three functions from the system (10.25-28), we obtain a consistency condition on u_0 alone, i. e. a *functional equation to be obeyed by the Jost function*:

$$u_0(x)u_0(ix)u_0(- ix) = u_0(x) + u_0(ix) + u_0(- ix) + 2 \quad (|\text{Arg } x| < \pi/4) \tag{10.30}$$

The fact that u_0 is not an entire function of its argument severely restricts the usefulness of such a result. Fortunately we can overcome this limitation by returning now to the *energy* E as basic variable, according to the substitution rule (10.4) for M = 2. Indeed the determinant D(E) *is* an entire function of its argument, whose relation (10.15) to the Jost function, in the light of Eq. (10.29), amounts to:

$$u_0(x) \equiv D(- x^{4/3}) \quad (|\text{Arg } x| < 3\pi/4) \tag{10.31}$$

since $a_0 = \omega$ and $-i_0 = 3/4$. This identity finally transforms (10.30) into a *functional equation for the determinant*:

$$D(E)D(jE)D(j^2E) = D(E) + D(jE) + D(j^2E) + 2 \quad (j \equiv e^{2\pi i/3}) \quad (10.32)$$

which is not only more symmetrical, but also valid everywhere in E since D is an *entire* function.

In particular we are now allowed an *independent check of the whole theory* by reexpanding around $E = 0$ the relation (10.32) that we just obtained by this resummation process around $E = \infty$. The relevant Taylor series (10.17) around $E = 0$ involves the *values of the zeta function at integers*, which are otherwise numerically accessible (Table 5, and Ref. [3]). Due to its ternary symmetry, Eq. (10.32) will impose one constraint upon every third Taylor coefficient, resulting in a countable set of *arithmetical identities*; each of these can then be verified numerically. The first ones read:

— at order zero: $X = e^{-\zeta'(0)}$ solves $X^3 = 3X + 2$, and since $X = 2$ is the only positive root this means:

$$\zeta'(0) = -\log 2 \tag{10.33}$$

This result was found before [3] but by a more fortuitous argument, repeated in Appendix C, Eqs. (C.16-18).

We may incidentally eliminate $e^{-\zeta'(0)} = 2$ from (10.32) to obtain:

$$4\Delta(E)\Delta(jE)\Delta(j^2E) = \Delta(E) + \Delta(jE) + \Delta(j^2E) + 1 \tag{10.34}$$

— at order 3:

$$\zeta(3) = \zeta(1)^3/6 - \zeta(1)\zeta(2)/2 \tag{10.35}$$

In view of the fact that $\zeta(1)$ and $\zeta(2)$ admit closed analytical expressions for general M (Appendix C), so does now $\zeta(3)$ for $M = 2$;

— at order 6:

$$\zeta(6) = \frac{41}{3\,240}\,\zeta(1)^6 - \frac{11}{216}\,\zeta(1)^4\zeta(2) + \frac{1}{72}\,\zeta(1)^2\zeta(2)^2$$
$$+ \frac{1}{24}\,\zeta(2)^3 + \frac{1}{4}\,\zeta(1)^2\zeta(4) - \frac{1}{4}\,\zeta(2)\zeta(4) - \frac{2}{5}\,\zeta(1)\zeta(5) \tag{10.36}$$

but $\zeta(4)$ and $\zeta(5)$ are only known in integral form (Eq. (C.19)), no better than $\zeta(6)$. More generally $\zeta(3n)$ gets equated to a polynomial in the $\zeta(m)$ $(1 \leq m < 3n)$ homogeneous of degree $(-3n)$ in the eigenvalues E_k. We have validated all such relations to about 10 significant digits up to $\zeta(15)$ (which comprises 70 terms).

As our present result will soon appear as a special instance of a very general exact formula, we postpone any further comments until then.

Functional equation, harmonic case ($M = 1$): for the sake of comparison

we shall briefly repeat the same argument upon the harmonic oscillator, but we stress that this is a highly idiosyncratic example not just because some of its aspects are delusively simpler (as expected), but also because it is *irregular* in one respect: the resolvent trace and the determinant diverge if (and only if) M = 1, therefore the crucial identity (10.15) fails ($a_0 = \infty$ indeed), and its substitute has a special structure, with one more substraction term imposed by Eqs. (A.13)-(A.18):

$$[e^{-\zeta'(0)}\overline{\Delta}(E)] = \overline{D}(E) = (- E/e)^{E/2}e^{\zeta_0 E}a(x)_{x=-E} \qquad (10.37)$$

(ζ_0 is the finite part of $\zeta(\sigma)$ at $\sigma = 1$, namely $(\gamma + \log 2)/2$ if Eq. (10.41) below is used, but its explicit value is irrelevant here).

At the level of the Jost function itself there is no change at all and we can stick to the same procedure as before to get the functional equation for $a(x)$. We note that $\varphi = \pm \pi/2$ are now the only excluded angles; hence the single discontinuity formula (7.46), or:

$$a(x) = L_0^0 a_B(x) = (1 + e^{-\omega x})L_0 a_B(x) \qquad (- \pi < \text{Arg } x < 0) \quad (10.38)$$

where $\omega = i\pi$, together with the general symmetry relation (10.20), suffice to produce a functional equation:

$$a(ix)a(- ix) = (1 + e^{-\pi x}) \qquad (|\text{Arg } x| < \pi/2) \qquad (10.39)$$

In view of the explicit formula (7.55) for $a(x)$, we recognize here the well-known *reflection formula for the Euler Gamma function*. That this formula results—in a contrived way—from the Gamma function's involvement in the spectral theory of a Schrödinger operator investigated semi-classically, would be an anecdotic remark if it were not at the heart of the generalization process leading to the functional relation (10.32) and similar forthcoming ones.

In terms of the regularized Fredholm determinant, (10.39) becomes:

$$\overline{D}(E)\overline{D}(- E) = 2 \cos (\pi E/2) \qquad (10.40)$$

As the zeta function of the harmonic oscillator is proportional to the Riemann zeta function ζ_R:

$$\zeta(\sigma) = (1 - 2^{-\sigma})\zeta_R(\sigma), \qquad (10.41)$$

the expansion of this functional equation in powers of E will restore the well-known values for $\zeta_R(2n)$ [36].

We can clarify here a common misunderstanding about the privileged status of the harmonic oscillator with respect to the WKB method. It is usually stated that the harmonic oscillator is one of a few exceptional systems to which the semi-classical method applies exactly, whereas our description of sections 5-6 has made the WKB method *always exact*. The real difference lies in the *level of explicitation* of the output. In the quartic case (which is a model for the general case, as suggested in section 9 and

later in this section) the discontinuity formulas (8.20-30) are non linear and implicit (recursive), as well as the resulting functional equation (10.32) and identities·(10.35-36). In the harmonic case the discontinuity formulas (7.47-48) happen to be explicit and linear(-isable) and their corollaries as well; even the exact individual eigenvalues show up as zeros in the functional equation (10.40), whereas they are hopelessy scrambled up in Eq. (10.32) for the quartic case. Also note that the latter equation cannot determine the overall *scale* of the spectrum; it is only in the harmonic case that scale invariance is broken through the additional regularization.

The inhomogeneous quartic case: this discussion about functional equations provides another opportunity to stress that the WKB method is completely general, that its outcome is primarily sensitive to the degree of the potential (the number of turning points) and therefore that our earlier restriction to a homogeneous case was only a matter of convenience. The technical adaptations to more general potentials were outlined in section 9 and rely on the basic idea that the inclusion of terms of lower degree only effects an isomonodromic deformation upon the Borel transforms. Concretely, the whole of section 8 can be rephrased for the following inhomogeneous quartic oscillator:

$$- x^{-2}d^2/dq^2 + \kappa q^2 + q^4 \qquad (\kappa \in \mathbb{R}) \tag{10.42}$$

The only important modification concerns Eq. (8.14) and relations of a similar nature like $\omega' = i\omega$, as they become:

$$a'(x, E, \kappa) = a(ix, E, -\kappa) \tag{10.43}$$

$$\omega'(E, \kappa) = i\omega(E, -\kappa) \tag{10.44}$$

The lattice Ω thus gets *deformed* as κ, and also E, vary (it stays rectangular for $0 < \kappa^2 < -4E$). As for the scaling property (10.3) that played a crucial role in this section, it can be extended likewise but will now affect κ too (Symanzik scaling):

$$a(x; E, \kappa) \equiv a(1; x^{4/3}E, x^{2/3}\kappa) \qquad \text{(and likewise for } \Delta). \tag{10.45}$$

The functional equation for the determinant will consequently involve *both* variables of analyticity E and κ (at $x = 1$):

$$D(E,\kappa)D(jE, j^2\kappa)D(j^2E, j\kappa)= D(E,\kappa) + D(jE, j^2\kappa) + D(j^2E, j\kappa) + 2 \tag{10.46}$$

But our analytic description, the existence of a functional equation and even the *basic structure* of the latter all persist in letting $\kappa \neq 0$, and should be considered as *purely semi-classical results* to which homogeneity was irrelevant. Unfortunately Eq. (10.46) seems much harder to exploit for $\kappa \neq 0$, as it now couples the spectrum of the operator (10.42) to the spectra of two other, not even hermitian, operators; this reduces our hope to

produce for $\kappa \neq 0$ any *numerical* results nearly as striking as (10.33-10.36); we have accordingly made no serious check of (10.46).

Functional equation for the alternating determinant: we showed in Ref. [3] that for an *even* potential any manipulation upon the usual spectral functions $\Delta(E)$, $R(E)$, $\zeta(\sigma)$ could be paralleled by one upon associated *alternating* spectral functions so as to instantly double the output of information. The latter functions are defined as:

$$\Delta^P(E) = \prod_{k=0}^{\infty} (1 - E/E_k)^{(-1)^k}$$

$$R^P(\lambda) = \sum_{k=0}^{\infty} (-1)^k (E_k - E)^{-1} \qquad (10.47)$$

$$\zeta^P(\sigma) = \sum_{k=0}^{\infty} (-1)^k E_k^{-\sigma}$$

and bear the same relations to one another as the ordinary Δ, R, ζ. We shall separately discuss their properties because: 1) even potentials are frequently encountered, 2) $\Delta^P(E)$ obeys a functional equation that is more basic than (i. e. implies) the one obeyed by $\Delta(E)$, and 3) this equation for Δ^P is also much easier to extend from the case q^4 to the case q^{2M} than it would have been for the equation obeyed by $\Delta(E)$ alone.

To get a functional equation for Δ^P by WKB theory we need to relate it, in the same way as Eq. (A.5) did for Δ, to a quantity that can be followed by integration along complex coordinate paths. The amplitude-correcting factor $a(q, E, x)$ of section 4 is such an object: we may recall that the Jost function was precisely studied in section 7 through its expression

$$a(E, x) = \lim_{q \to +\infty} a(q, E, x).$$

The relation analogous to (A.5) satisfied by $D^P(E, x) = e^{-(\zeta^P)'(0)} \Delta^P(E, x)$ is;

$$D^P(E, x) \equiv x^{-1} \frac{\partial}{\partial q} (\log \psi_+(q, x)) |_{q=0} \qquad (10.48)$$

where moreover:

$$x^{-1} \frac{\partial}{\partial q} [\log \psi_+(q, x)]_{q=0} = \frac{(-E)^{1/2} a(E, x)}{a(q = 0, E, x)^2} \sim iu(q = 0, E, x) \qquad (10.49)$$

Here $E < 0$, $x > 0$, and the solution

$$\psi_+ = u^{-1/2} \exp ix \int_0^q u \, dq \qquad (10.50)$$

is recessive at $q = -\infty$. The identities (10.48-49) are derived in Appendix D for an *arbitrary* confining, even potential. To exploit the first equality (10.49) we must know the Jost function and in addition solve a *second connection problem*, from $q = -\infty$ to $q = 0$, to get the denominator; whereas the second equality (10.49) will give control over the algebraic structure of the expansion, most easily accessible through Eqs. (4.3-4) for u (the second equality (10.49) is purely formal).

For clarity we explain how the general principles of WKB theory govern this new computation in the case $V(q) = q^4$. All the material for the connection problem has already been set up in section 8 (Fig. 19): instead of following the recessive solution from $q = -\infty$ up to $q = +\infty$, we now only have to end up at the central Stokes region containing $q = 0$. As in section 8 we fix $E_0 = -1$ and solve the connection problem for all $\varphi = -\mathrm{Arg}\,x$ to get the analytic structure of the Borel transform $a_B(q = 0, s)$ of $a(q = 0, E = -1, x)$. The critical angles are again the multiples of $\pi/4$, but again for $\varphi = 0$ we can link $q = -\infty$ to $q = 0$ without crossing any Stokes lines, i. e. for $x > 0$, $a(q = 0, x)$ *is Borel summable and equal to its Borel sum* like the Jost function $a(x)$; and again this situation persists for $|\mathrm{Arg}\,x| < \pi/4$.

Let us now compute the discontinuity formula at $\varphi = \pi/4$ following the pattern of Eqs. (8.17-8.20). We have just established that

$$a(0, x) = L_0^\varphi[a_B(0, s)] \qquad \forall \varphi, |\varphi| < \pi/4 \qquad (10.51)$$

For $\pi/4 < \varphi < \pi/2$ the connection matrix is (Fig. 19d):

$$F_{BA} = \begin{pmatrix} 1 & 0 \\ -iu_{\gamma_3} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -iu_{\gamma_1}^{-1} & 1 \end{pmatrix} \qquad (10.52)$$

which by definition means for q in the Stokes region B:

$$\left.\begin{array}{l} \psi_+(q, x) = L_0^\varphi\left[\left(b_+u^{-1/2}e^{ix\int_0^q u} + b_-u^{-1/2}e^{-ix\int_0^q u}\right)_B\right] \\[2mm] \begin{pmatrix} b_+ \\ b_- \end{pmatrix} = F_{BA}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -iu_{\gamma_1}^{-1} \end{pmatrix} \end{array}\right\} \qquad (10.53)$$

with

hence for $q = 0$

$$\psi_+(0, x) = L_0^\varphi\left[(1 - iu_{\gamma_1}^{-1})u^{-1/2}\right]_B$$

but this is $p(0)^{-1/2}a(0, x)$ by the definition (4.21). Therefore:

$$a(0, x) = L_0^\varphi\left[(1 - iu_{\gamma_1}^{-1})a(0, x)\right]_B \qquad (\pi/4 < \varphi < \pi/2) \qquad (10.54)$$

Substracting from (10.51) and using

$$u_{\gamma_1}^{-1} = \left(u_{\gamma_{31}}\right)^{-1/2} = e^{-(\omega + \omega')x/2}[a(x)a'(x)]^{-1/2} \qquad (10.55)$$

(derived from (8.2) and (8.16)), we arrive at the discontinuity formula that parallels (8.20):

$$L_0^{\pi/4-0} a_B(0, s) = L_0^{\pi/4+0} [(1 - ie^{-(\omega+\omega')x/2}(aa')^{-1/2}(x))a(0, x)]_B \left.\vphantom{\begin{matrix} a \\ a \end{matrix}}\right\}$$
$$= [1 - i(u_{\pi/4+0} u'_{\pi/4+0})^{-1/2}]L_0^{\pi/4+0} [a_B(0, s)] \left.\vphantom{\begin{matrix} a \\ a \end{matrix}}\right\} \qquad (10.56)$$

The second line makes use of the notation (10.22), in terms of which the analogous formulas for the Jost function (Eq. (8.20), or equivalently (10.24)) reads:

$$L_0^{\pi/4-0} a_B = [1 + (u_{\pi/4+0} u'_{\pi/4+0})^{-1}]L_0^{\pi/4+0} a_B \qquad (10.57)$$

The only quantity of actual interest to us is the combination appearing in Eq. (10.49):

$$a^P(x) \equiv a(x)a(0, x)^{-2} \qquad (10.58)$$

and so denoted to stress its parallelism with the Jost function. We then divide the discontinuity formula (10.57) for $a(x)$ by the square of (10.56), to find:

$$L_0^{\pi/4-0} a_B^P = \frac{1 + i(u_{\pi/4+0} u'_{\pi/4+0})^{-1/2}}{1 - i(u_{\pi/4+0} u'_{\pi/4+0})^{-1/2}} L_0^{\pi/4+0} a_B^P \qquad (10.59)$$

Now the function $a_B^P(s)$ has such a high degree of *symmetry* that in contrast with $a_B(s)$ we can stop the computation here (we only mention that the full set of branch points of $a_B(0, s)$ would now be the union of the lattice Ω of action periods and its dual $\hat{\Omega}$, the translate of Ω by $(\omega + \omega')/2$).

This symmetry of $a_B^P(s)$ results from the algebraic structure of the expansion $u(q = 0, x) = \sum_0^\infty x^{-2n} u_{2n}(q = 0)$ occuring in the right-hand side of (10.49). As shown in [2] by a recursive argument and a little dimensional analysis, all u_{2n} have the structure:

$$u_{2n}(q) = \sum_{l=0}^{[3n/4]} u_{n,l} q^{3n-4l} p^{-3n+2l+1} \quad ([t] = \text{integer part of } t) \quad (10.60)$$

$$\Rightarrow u_{2n}(0) = 0 \text{ except for } n \text{ even} \qquad (10.61)$$

Hence the analytic function $a_B^P(s)$ is only a function of s^4:

$$a_B^P(is) = a_B^P(s) \qquad (10.62)$$

Then the analyticity of $a_B^P(s)$ in $|\text{Arg } s| < \pi/4$, and the knowledge of its discontinuities on $\{\text{Arg } s = \pi/4\}$ through Eq. (10.59), completely fix its first sheet analytic structure (Fig. 34).

We now show that the set of discontinuity formulas (generated from (10.59) through iterated rotations by $\pi/2$) is consistent only if $a^P(x) = (L_0^0 a_B^P)(x)$
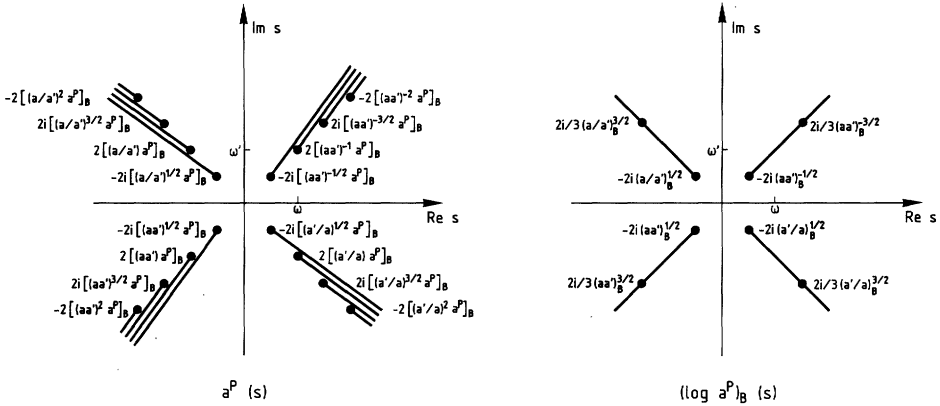
Left diagram (a^P (s)), axis Im s / Re s, with ω' and ω:

$-2[(a/a')^2\, a^P]_B$
$2i[(a/a')^{3/2}\, a^P]_B$
$2[(a/a')\, a^P]_B$
$-2i[(a/a')^{1/2}\, a^P]_B$

$-2[(aa')^{-2}\, a^P]_B$
$2i[(aa')^{-3/2}\, a^P]_B$
$2[(aa')^{-1}\, a^P]_B$
$-2i[(aa')^{-1/2}\, a^P]_B$

$-2i[(aa')^{1/2}\, a^P]_B$
$2[(aa')\, a^P]_B$
$2i[(aa')^{3/2}\, a^P]_B$
$-2[(aa')^2\, a^P]_B$

$-2i[(a'/a)^{1/2}\, a^P]_B$
$2[(a'/a)\, a^P]_B$
$2i[(a'/a)^{3/2}\, a^P]_B$
$-2[(a'/a)^2\, a^P]_B$

$a^P\ (s)$

Right diagram ($(\log a^P)_B$ (s)), axis Im s / Re s, with ω' and ω:

$2i/3(a/a')_B^{3/2}$
$-2i(a/a')_B^{1/2}$

$2i/3(aa')_B^{-3/2}$
$-2i(aa')_B^{-1/2}$

$2i/3(aa')_B^{3/2}$
$-2i(aa')_B^{1/2}$

$2i/3(a'/a)_B^{3/2}$
$-2i(a'/a)_B^{1/2}$

$(\log a^P)_B\ (s)$

FIG. 34.

satisfies a *functional relation*. We begin by resumming (10.62) to the identity:

$$(L_0^\varphi a_B^P)(x) = (L_0^{\varphi - \pi/2} a_B^P)(ix) \qquad \forall \varphi \neq \frac{\pi}{4} + k\frac{\pi}{2} \qquad (10.63)$$

Now the structure of the analyticity sectors allows us to substitute in Eq. (10.59):

$$L_0^{\pi/4 - 0} a_B^P = L_0^0 a_B^P, \qquad L_0^{\pi/4 + 0} a_B^P = L_0^{\pi/2} a_B^P,$$

$$u_{\pi/4 + 0}(x) = u_0(x) - u_0^{-1}(ix), \qquad u'_{\pi/4 + 0}(x) = u_{-\pi/4 + 0}(ix) = u_0(ix) \qquad (10.64)$$

$\left(\text{we have used Eqs. (10.23-10.25) with } \varphi = \varphi_1 = \dfrac{\pi}{4} + 0\right)$. Then Eqs. (10.59) and (10.63) together imply:

$$\frac{a^P(ix)}{a^P(x)} = \frac{1 - i[u_0(x)u_0(ix) - 1]^{-1/2}}{1 + i[u_0(x)u_0(ix) - 1]^{-1/2}} \qquad (-\pi < \text{Arg } x < \pi/2) \quad (10.65)$$

where only the Jost function appears on the right hand side $(u_0(x) = e^{\omega x} a(x))$.

Finally we replace the functions a and a^P by the determinants D and Δ^P through Eqs. (10.31) and (10.48-10.49), and simultaneously change from x to the analyticity variable E, to end up with the functional equation:

$$\frac{\Delta^P(e^{i\pi/3}E)}{\Delta^P(e^{-i\pi/3}E)} = e^{i\pi/3}\frac{1 - i[D(e^{i\pi/3}E)D(e^{-i\pi/3}E) - 1]^{-1/2}}{1 + i[D(e^{i\pi/3}E)D(e^{-i\pi/3}E) - 1]^{-1/2}} \qquad (10.66)$$

(the factor $e^{i\pi/3}$ is contributed by the $(-E)^{1/2}$ term in (10.49)).

A remarkable property of our computation is that it instantly carries over to the general homogeneous case $V(q) = q^{2M}$ $(M \neq 1)$. Indeed the same algebraic analysis that led us to (10.62) now shows that $a_B^P(s)$ is a function of s^{2M} alone. It therefore suffices to compute its *first* discontinuity

encountered as φ increases from 0: at $\varphi = \pi/2M$. At *this* discontinuity the change in the pattern of those Stokes regions that are involved in the connection process is *topologically the same* as for the quartic case (Fig. 35,
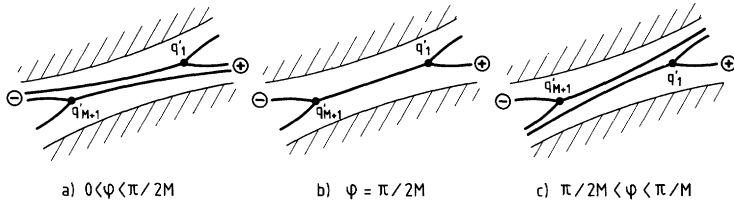

a) $0 \langle \varphi \langle \pi/2M$ b) $\varphi = \pi/2M$ c) $\pi/2M \langle \varphi \langle \pi/M$

FIG. 35.

to be compared with Fig. 19*b-d*), resulting in the *same discontinuity formulas* (10.56-10.57) at $\varphi = \pi/2M$, and so on. This argument does not work at higher critical angles where many more Stokes regions are involved, but we would only have to go to those angles if we needed the full analytic structure of $a_B(s)$ itself, which will not be the case here. The final functional equation is identical to (10.66) with $\pi/3$ replaced by $\dfrac{\pi}{M+1} \equiv 2\mu\pi$:

$$\frac{\Delta^P(e^{2i\mu\pi}E)}{\Delta^P(e^{-2i\mu\pi}E)} = e^{2i\mu\pi} \frac{1 - i[D(e^{2i\mu\pi}E)D(e^{-2i\mu\pi}E) - 1]^{-1/2}}{1 + i[D(e^{2i\mu\pi}E)D(e^{-2i\mu\pi}E) - 1]^{-1/2}} \quad (10.67)$$

Although this equation can be made rational, its most pleasant form by far is trigonometric. We pose:

$$\phi(E) \equiv \text{Arc sin } [D(e^{2i\mu\pi}E)D(e^{-2i\mu\pi}E)]^{-1/2} \quad (10.68)$$

to get:

$$\frac{\Delta^P(e^{2i\mu\pi}E)}{\Delta^P(e^{-2i\mu\pi}E)} = e^{2i\mu\pi - 2i\phi(E)} \quad \left(\mu \equiv \frac{1}{2M+2}\right) \quad (10.69)$$

On the form (10.69) it is obvious that the following cyclic consistency condition must be satisfied:

$$\sum_{k=0}^{M} \phi(e^{4i\mu\pi k}E) = \pi/2 \quad (10.70)$$

According to (10.68), this is a functional equation for $D(E)$ alone, that exhibits $(M+1)$-ary symmetry. The polynomial form of (10.70), which exists, reduces indeed to Eq. (10.32) for $M = 2$, but already for $M = 3$ it has degree 16, with 75 terms ! A direct attempt to derive it without the use of the auxiliary function $\Delta^P(E)$ would probably have been intractable

for $M > 2$. As for $M = 1$, the exceptional substraction term present in the identity (10.37) modifies the definition of $\phi(E)$ and nothing else:

$$\phi(E) \equiv \text{Arc sin} \{ e^{-\pi E/4} [D(iE)D(-iE)]^{-1/2} \}. \qquad (10.71)$$

Then the functional equation (10.69) amounts to:

$$\frac{\Delta^P(E)}{\Delta^P(-E)} = \cot \frac{\pi}{4}(1 + E) \qquad (10.72)$$

which otherwise results again from the reflection formula, given that:

$$\Delta^P(E) = \frac{\Gamma(1/4)\,\Gamma((3 - E)/4)}{\Gamma(3/4)\,\Gamma((1 - E)/4)} \qquad (M = 1). \qquad (10.73)$$

Arithmetical identities between values of the zeta functions; exactly as for the special equation (10.32), we expand the functional system (10.68-10.69) in powers of E around $E = 0$ to get a countable set of numerical identities that, in addition to their possible intrinsic value, provide very sensitive tests for the validity of our various Borel summation arguments.

Indeed if we plug the Taylor series (10.17) and its companion for Δ^P into (10.68-69), we get after a little algebra the identity:

$$\zeta'(0) + \sum_{n=1}^{\infty} \frac{\cos 2n\mu\pi}{n} \zeta(n)E^n \equiv \log \sin \left(\mu\pi + \sum_{n=1}^{\infty} \frac{\sin 2n\mu\pi}{n} \zeta^P(n)E^n \right) \quad (10.74)$$

(exception: for $M = 1$ the indeterminate $n = 1$ term in the left-hand side gets replaced by $\pi/4$). This implies:

— at zeroth order,

$$\zeta'(0) = \log \sin \mu\pi, \qquad (10.75)$$

an already known result ([3] and Eq. (C.18)), which moreover eliminates any objection to the multiple-valuedness of the functions involved in formulas (10.67-10.68);

— at any higher order n, an expression for the combination:

$$\cot \mu\pi \sin 2n\mu\pi \qquad \zeta^P(n) - \cos 2n\mu\pi \qquad \zeta(n) \qquad (10.76)$$

as a polynomial in the $\zeta^P(k)$ alone, or (at one's choice) in the $\zeta(k)$ alone, for $k < n$, those expressions being homogeneous of degree $(-n)$ in the eigenvalues. Whenever $n = p(M + 1)$, the member (10.76) involves only $\zeta(n)$, whereas for M odd and $n = \left(p + \frac{1}{2}\right)(M + 1)$, it involves only $\zeta^P(n)$. The first identities read:

$$(n = 1) : \cot \pi\mu \sin 2\pi\mu \qquad \zeta^P(1) = \cos 2\pi\mu \qquad \zeta(1) \qquad (M \neq 1) \quad (10.77)$$
$$\zeta^P(1) = \pi/4 \qquad\qquad\qquad (M = 1) ; \quad (10.78)$$

the latter is of course a classic, and the former was proved in [3] as explained in Appendix C, Eq. (C.23). By contrast, all subsequent relations seem new:

$$(n = 2): \cot \mu\pi \sin 4\mu\pi \quad \zeta^P(2) - \cos 4\mu\pi \quad \zeta(2) = [2 \cos \mu\pi \quad \zeta^P(1)]^2 \quad (10.79)$$

($\zeta^P(2)$ and $\zeta(2)$ were already known in closed form (Appendix C), yet this relation was unsuspected).

$$(n = 3): \cot \mu\pi \sin 6\mu\pi \quad \zeta^P(3) - \cos 6\mu\pi \quad \zeta(3)$$
$$= 4 \cos 2\mu\pi \, [3 \cos 2\mu\pi \quad \zeta^P(1)\zeta^P(2) - 2 \cos^2 \mu\pi \quad \zeta^P(1)^3] \quad (10.80)$$
etc...

For the *harmonic oscillator* (M = 1) this sequence of identities will only reproduce the well-known values involving Bernoulli and Euler numbers [36]:

$$\zeta(2n) = \sum_{k=0}^{\infty} (2k + 1)^{-2n} = \frac{(2^{2n} - 1)\pi^{2n}}{2(2n)!} | B_{2n} |,$$

$$\zeta^P(2n + 1) = \sum_{k=0}^{\infty} (-1)^k (2k + 1)^{-(2n+1)} = \frac{(\pi/2)^{2n+1}}{2(2n)!} | E_{2n} | \quad (10.81)$$

For the *quartic oscillator* (M = 2) the first identities are very simple too:

$$\zeta(1) = 3\zeta^P(1)$$
$$3\zeta^P(2) + \zeta(2) = 6\zeta^P(1)^2 \quad\quad\quad (10.82)$$
$$2\zeta(3) = 9(\zeta^P(1)\zeta^P(2) - \zeta^P(1)^3) ;$$

The elimination of ζ^P restores Eq. (10.35).
For the *sextic oscillator* (M = 3):

$$\zeta(1) = (\sqrt{2} + 1)\zeta^P(1), \quad \zeta^P(2) = \sqrt{2}\zeta^P(1)^2, \ldots \quad (10.83)$$

Direct numerical evidence has confirmed the many particular instances that we have checked, with significant accuracy (9 or more digits): cf. Table 5.

Functional equation for general potentials: having indirectly shown through Eqs. (10.67-10.70) that for a homogeneous potential of any degree q^{2M} the Fredholm determinant satisfies a functional equation that admits a polynomial form, we can argue by the deformation argument of section 9 that a functional equation *of the same structure* will persist if terms of lower degree are added (and similarly for the relation between Δ and Δ^P too, if the potential remains even). We have made no attemps at explicitation, since the polynomial structure for q^6 is already very complicated, but the transition from (10.32) to (10.46) for the degree 4 is certainly suggestive of the general case. At this point we refer to earlier work of Sibuya-Cameron on polynomial potentials [40] (of which we were regret-

tably unaware when we announced our results [34]), because [40] established *directly* functional equations of a very similar structure, but for a *Stokes multiplier*; the functional connection of this quantity, akin to one of our expressions (9.16), to the determinant and thereby to the spectrum, was however not recognized or at least explicited in [40]; this connection is nevertheless intimate. As for analytic potentials with infinitely many turning points, the ideas of section 9 should be locally valid inasmuch as each discontinuity formula will involve interaction of a *finite* number of turning points at a time, but at the global level it seems difficult to explicit the functional equation for D(E) except perhaps case by case: it is moreover unlikely that such an equation would take a polynomial form.

General discussion: although the exact spectral results of this section might prove interesting for their own sake (perhaps in arithmetic) we shall only consider here their relevance to our central theme: WKB theory.

Even though our last results could conceivably have been derived otherwise (cf. [40]), they still confirm the capacity of the WKB method to probe the structure of the general one-dimensional Schrödinger equation in a completely exact way: as for the potentials q^{2M}, the method has even produced exact identities for the spectrum in a numerical and verifiable form. These results rely heavily on the *global accuracy* of our analytic description of various Borel transforms, and thus *validate* the latter (up to and including our unproved belief that those transforms belong to a space $\mathscr{C}$ of temperate functions to make Borel resummation possible). We stress that formulas (10.75-10.83) are arithmetical relations concealing no possible expansion parameter, and this excludes the situation where all of them would hold only to some deceptively good approximation.

Had we been conditioned to expect WKB results for the spectrum of $V(q) = q^4$ for instance, we would have preconceived them as standard semi-classical eigenvalue formulas like Bohr-Sommerfeld, but somehow corrected to become exact.

Something very different has emerged: *functional equations for determinants* and *eigenvalue sum rules*, both unexpected and quite unpredictable by extrapolation from ordinary WKB theory. They constitute properties of the fully resummed semi-classical expansion, and yet they are expressible without requiring us to achieve Borel summation explicitly. Our present incapacity to perform the latter step accounts for the regrettable gap between the old asymptotic WKB results and the new exact ones.

At this point, our best assessment of WKB theory is that it is liable, relying on first principles, to describe exactly the *global analytic structure of the Borel transforms* of various spectral functions in general one-dimensional problems, and that consistency of the output imposes *functional equations upon the Fredholm determinants*. As explained at the end of section 9, those are too abstract and implicit results in general to be nearly

as useful as the conventional asymptotic ones. Hence it is important to ask whether the pure semi-classical approach has reached here some natural limit, or what else it is still reasonable to expect from it. The surprising character of our latest results makes any extrapolation difficult, and we can only offer some clues.

Even within our model problem of understanding the spectrum of the potential q^{2M}, there is room for improvement. On the one hand, this problem offers a few more exact results [3] that our present WKB analysis has not reproduced: there is curiously the value (D.18) of $(\zeta^P)'(0)$ as opposed to the recoverable $\zeta'(0)$ (Eq. (10.75)), and the separate values of $\zeta(1)$, $\zeta(2)$, $\zeta^P(2)$ (Appendix C). On the other hand we know that the functional equations only express part of the analytic information of section 8, if only because they mysteriously lose all reference to the *scale* of the spectrum (regarding the previous statement, $\zeta'(0)$ is scale invariant but not $(\zeta^P)'(0)$ [3]). The recovery of the remaining content is probably much harder (excluding the harmonic case) and amounts to a direct attack on the discontinuity formulas like (8.20-8.30) with the purpose of making the Borel summation process more explicit. Possible avenues of exploration are the techniques of Ref. [32], the Riemann-Hilbert problem [28], uniformization theory and non-linear integral equations, but none of them promises obvious or immediate returns.

Concerning now the extension of the method to other problems, we have been rather positive throughout this section for the general *one-dimensional* Schrödinger equation. We may also add that any WKB result bearing on the spectrum is likely to have a counterpart of the same vein relative to the *eigenfunctions* which we have left aside here ; that transposition should also be easy.

A much deeper challenge lies in understanding how much of this exact approach persists in the case of several degrees of freedom. General wisdom from conventional WKB theory suggests that such an extension should range from: fully possible but technically costly (for classically integrable systems), to: virtually impossible, for classically ergodic systems (see also relevant remarks at the end of section 9).

In way of conclusion, let us risk the wager that the old correspondence principle, for all the fascination it has exerted on generations of physicists by coupling the two unlike worlds of quantum and classical mechanics, still has in stock many secrets for us to discover.

APPENDIX A

In scattering theory it is well-known that the Jost function can be identified with the Fredholm determinant of an operator describing the interaction [16]. We now prove a similar result for confining potentials in one dimension, letting semi-classical solutions replace free solutions as reference waves (the special case of homogeneous potentials is already treated in [3]).

Under the assumption (4.17): $V(q) > C|q|^{2+\varepsilon}$, the Green's function kernel $R(E; q, q')$ (for the operator $\hat{H} = -\hbar^2 d^2/dq^2 + V(q)$ has a trace:

$$R(E) = \int_{-\infty}^{\infty} R(E; q, q)dq = \sum_{k=0}^{\infty} (E_k - E)^{-1} \qquad (A.1)$$

and the following Fredholm determinant exists (converges):

$$\Delta(E) = \sum_{k=0}^{\infty} (1 - E/E_k) = \exp - \int_0^E R(E')dE' \qquad (A.2)$$

all that because the Bohr-Sommerfeld rule causes E_k to increase faster than $k^{1+\varepsilon'}$ for some $\varepsilon' > 0$. We shall then prove the identities (for $E < 0$):

$$(\log a(E, \hbar^{-1}))^{\cdot} = -R(E, \hbar^{-1}) + \hbar^{-1}T(E) \qquad (A.3)$$

$$a(E, \hbar^{-1}) = \exp \int_{-\infty}^E (-R(E_1) + \hbar^{-1}T(E_1))dE_1 \qquad (A.4)$$

$$a(E, \hbar^{-1}) = \Delta(E) \exp \left[\hbar^{-1} \int_0^E T(E_1)dE_1 + \int_{-\infty}^0 (-R(E_1) + \hbar^{-1}T(E_1))dE_1 \right] \quad (A.5)$$

Our notations will be $\cdot = \dfrac{\partial}{\partial E}$; $' = \dfrac{\partial}{\partial q}$; and

$$T(E) = \int_{-\infty}^{\infty} \frac{dq}{2\sqrt{V(q) - E}} \Rightarrow \int_0^E T(E_1)dE_1 = \int_{-\infty}^{\infty} \left[\sqrt{V(q)} - \sqrt{V(q) - E} \right]dq \quad (\dot{A}.6)$$

Proof. — It is well known (or immediate to check) that the Green's function kernel admits the representation:

$$R(E; q, q') = \hbar^{-2}W(\psi_-, \psi_+)^{-1}\psi_+(q_<)\psi_-(q_>) \qquad (A.7)$$

$$\Rightarrow R(E) = \hbar^{-2}W^{-1} \int_{-\infty}^{\infty} \psi_+(q)\psi_-(q)dq \qquad (A.8)$$

when inserted into (A.1); $\psi_\pm$ are any two solutions recessive at $q = \mp \infty$, W is their Wronskian $(\psi_-\psi'_+ - \psi_+\psi'_-)$, and $q_< = \min(q, q'), q_> = \max(q, q')$. Now we can find explicit primitives for the product $\psi_+\psi_-$ by combining the equations $\psi_+[\hat{H} - E)\psi_-] = 0$ and $\psi_-[\hat{H} - E)\psi_+]^{\cdot} = 0$, and separately the same equations with ψ_+ and ψ_- permuted:

$$\hbar^{-2}\psi_+\psi_- = f' = \dot{g}, \quad f = \psi'_-\dot{\psi}_+ - \psi_-\dot{\psi}'_+, \quad g = \psi'_+\dot{\psi}_- - \psi_+\dot{\psi}'_-$$

and due to $(g - f) = W$, Eq. (A.8) becomes:

$$R(E) = W^{-1}[g(E, +\infty) - f(E, -\infty) - \dot{W}]. \qquad (A.9)$$

Now the right-hand side only involves the asymptotic behavior of $\psi_\pm$ for $q \to \pm \infty$. We can thus use WKB results for the solutions $\psi_{\pm,q_0}$ with $E < 0$ given by Eq. (4.13). For instance we get

$$\left.\begin{array}{l} \psi'_{-,q_0}(q_1) \sim - i\hbar^{-1}p(q_1)^{1/2} \exp - i\hbar^{-1} \int_{q_0}^{q_1} p\,dq \\[4mm] \dot{\psi}_{-,q_0}(q_1) \sim - i\hbar^{-1}p(q_1)^{-1/2} \left(\int_{q_0}^{q_1} \frac{dq}{2p}\right) \exp - i\hbar^{-1} \int_{q_0}^{q_1} p\,dq \end{array}\right\} \quad (q_1 \to +\infty) \quad \text{(A.10)}$$

by differentiating (4.13) and keeping only nonvanishing contributions (this can be justified [5]). From (A.10) and similar relations derived from (4.15) we obtain:

$$g(E, +\infty) = \hbar^{-2}a(E) \int_{q_0}^\infty dq/p(q), \qquad f(E, -\infty) = -\hbar^{-2}a(E) \int_{-x}^{q_0} dq/p(q)$$

and finally, recalling that $W(E) = 2i\hbar^{-1}a(E)$ and that $p(q) = -i\sqrt{V(q) - E}$, we deduce directly (A.3) from (A.9). (A.4-5) follow by integration, taking into account the other information from WKB theory (Eq. (4.27)) that fixes the integration constant:

$$\lim_{E \to -\infty} a(E) = 1 \qquad \text{(A.11)}$$

If we now allow potentials harmonic at ∞: $V(q) \sim q^2$, then E_k is proportional to k and $R(E)$, $\Delta(E)$ diverge (as well as the classical quantity $T(E)$). However the following *definitions*:

$$\left.\begin{array}{l} \dot{R}(E) = \sum_0^\infty (E_k - E)^2 \\[6mm] \dot{T}(E) = \int_{-\infty}^\infty \frac{1}{2}[V - E]^{-1/2}{}^{\cdot}\, dq \end{array}\right\} \qquad \text{(A.12)}$$

do converge, and lead to the relations:

$$(\log a(E))^{\cdot\cdot} = -\dot{R}(E) + \hbar^{-1}\dot{T}(E) \qquad \text{(A.13)}$$

$$a(E) = \exp \int_{-\infty}^{E_1} dE_1 \int_{-\infty}^{E_1} dE_2(-\dot{R}(E_2) + \hbar^{-1}\dot{T}(E_2)) \qquad \text{(A.14)}$$

that are substitutes to (A.3) and (A.4). Here (A.13) is integrated twice to produce (A.14), and both constants of integration are still determined by (A.11).

Example. — For the harmonic oscillator $-\dfrac{d^2}{dq^2} + q^2$ ($E_k = (2k + 1)$, $\hbar = 1$):

$$\left.\begin{array}{l} \dot{R}(E) = \frac{1}{4}\psi'\left(\frac{1 - E}{2}\right)\left(\psi'(t) = \frac{d^2}{dt^2}\log\Gamma(t)\right) \\[5mm] \dot{T}(E) = +\frac{1}{4}\int_{-\infty}^\infty (q^2 - E)^{-3/2}dq = \frac{-1}{2E} \end{array}\right\} \quad (E < 0) \qquad \text{(A.15)}$$

hence the Jost function is *exactly*, by Eq. (A.14):

$$a(E) = \left(-\frac{E}{2e}\right)^{-E/2} \sqrt{2\pi}\Big/\Gamma\left(\frac{1 - E}{2}\right) \quad (E < 0) \qquad \text{(A.16)}$$

and reasonable substitutes for R(E) and $\Delta(E)$ are the following functions:

$$\overline{R}(E) = \int_0^E \overline{R}(E')dE' = \frac{1}{2}\left[\psi\left(\frac{1}{2}\right) - \psi\left(\frac{1-E}{2}\right)\right] \tag{A.17}$$

$$\overline{\Delta}(E) = \exp{-\int_0^E \overline{R}(E')dE'} = \sqrt{\pi}e^{(\log 2 + \gamma/2)E}/\Gamma\left(\frac{1-E}{2}\right) \tag{A.18}$$

where $\psi(t) = \dfrac{d}{dt}\log\Gamma(t)$, $\psi\left(\dfrac{1}{2}\right) = -\gamma - 2\log 2$, and γ is Euler's constant [36]. When $\hbar = x^{-1} \neq 1$, it suffices to replace E by xE in formulas (A.16-18).

APPENDIX B

We have collected here some numerical evidence that confirms within the attainable accuracy our analytical results about the quartic oscillator (section 8) : in actual fact the numerical studies preceded and motivated the analytic developments.

The first visible effect has been an *exponentially small shift* (up or down parity-wise) of the eigenvalues from their WKB asymptotic approximations, as in tunneling. This discrepancy can be traced [1]-[2] to the branch points of $Z(s)$ on the line $\text{Re } s = -\omega\sqrt{2}$, but it is hardly visible in the dominant background contributed by the branch points lying on the imaginary axis: no sensitive test of the theory can be based on this effect.

Better evidence is provided by the *large order behavior* of the expansion coefficients a_n or b_n, for which previously subdominant branch points become dominant. By the algorithm explained in Ref. [2] we have computed a_n up to $n = 60$ with an estimated accuracy of 34 digits (not guaranteed for n close to 60). The first eleven b_n (from which the a_n follow by relation (8.9)) also appear in arithmetic form in [3]. Here we list (Table 1) the numerical values:

$$A_n = \omega^{2n-1} \frac{\pi}{2} a_n/\Gamma(2n-1) \tag{B.1}$$

Now we can test the validity first of our asymptotic expansion (8.80), namely:

$$A_n \sim (-1)^{n+1} \left(\alpha_0 + \frac{\alpha_1}{2n-2} + \frac{\alpha_2}{(2n-2)(2n-3)} + \dots \right) \quad (n \to +\infty) \tag{B.2}$$

with the α_k *explicitly* generated by formula (8.79):

$$\alpha_0 = 1, \, \alpha_1 = -\omega a_1 = -\pi/6, \, \dots \tag{B.3}$$

The computed A_n have the sign of $(-1)^{n+1}$ indeed, so that we may focus on the absolute values of both sides of (B.2). The leading term α_0 is then numerically extracted from table 1 by the following procedure: we form the *Neville table* [31]:

$$\begin{aligned} s_{n,1}^{(0)} &= |A_n| \\ s_{n,p}^{(0)} &= \left[(2n-p)s_{n,p-1}^{(0)} - (2n-3p+2)s_{n-1,p-1}^{(0)} \right]/(2p-2) \end{aligned} \tag{B.4}$$

It can be seen by recursion that:

$$s_{n,p}^{(0)} = \alpha_0 + O(n^{-p}) \tag{B.5}$$

With our a priori knowledge of α_0 we can then extract α_1 likewise from the sequence:

$$s_{n,1}^{(1)} = (2n-2)(s_{n,1}^{(0)} - \alpha_0) \text{ etc } \dots \tag{B.6}$$

and similarly obtain in succession for every k a double-entry table $s_{n,p}^{(k)}$ such that

$$s_{n,p}^{(k)} = \alpha_k + O(n^{-p}) \tag{B.7}$$

Hence by increasing p we *accelerate* the convergence of $s_{n,p}^{(k)}$ to α_k for $n \to \infty$; but numerical *noise* increases with p (and with k too). As a trade-off, we fit α_k by the numerical value found in that column of the table which exhibits the most regular trend. No error analysis has been attempted. To fix ideas we show in table 2 part of the Neville tables built from table 1, for $k = 0$ and 7. Table 3 compares the theoretical values of α_k with our numerical fits from the Neville tables, as far as noise permits: it provides evidence for the reality of analytic bootstrap and for the validity of our discontinuity formulas, since the expansion (B.2) is sensitive to the discontinuities on the circle $|s| = \omega$ in Eq. (8.50), and any attempt to fit a wrong asymptotic expansion (B.2) would have produced completely erratic or diverging Neville Tables.

A more delicate test is that of the *subdominant contributions* to A_n coming from the disconti-nuities on the larger circle $|s| = \omega\sqrt{2}$ in Eq. (8.50). From Eqs. (8.82-8.83), these contribu-tions read:

$$\delta A_n \sim 2^{1.2} \cos\left(\frac{3\pi}{4} - \frac{n\pi}{2}\right)\left(\gamma_0 + \frac{\gamma_1}{2n-2} + \frac{\gamma_2}{(2n-2)(2n-3)} + \ldots\right) \qquad (B.8)$$

with $\gamma_0 = 1$, $\gamma_1 = -\pi/3 \ldots$

To exhibit (B.8) concretely, we have substracted from the computed A_n (table 1) many (around 20) terms of its dominant (divergent) expansion (B.2). This *ad hoc* procedure defines quite stable numerical discrepancies δA_n for n large enough, except that noise takes over if we increase n too much because $\delta A_n/A_n \rightarrow 0$. We now fit these numbers against our theoretical prediction (B.8) in the same way as before. Again the signs agree, and table 4 shows slices of the Neville tables for γ_0 to γ_3 built from the sequence $|\delta A_n|$, with the interesting region of intermediate n marked by us. This new evidence, less convincing by itself because of the higher noise level, combines however very favorably with the previous one.

Finally the arithmetical identities studied for their own sake in section 10 provide many more opportunities to test numerically the *global* validity of section 8.

APPENDIX C

(includes joint results with D. Chudnovsky and G. Chudnovsky, 1979)

Our purpose here is to derive more or less explicit formulas for the values at positive integers n of the *zeta functions* for the homogeneous Schrödinger operator

$$\hat{H} = - d^2/dq^2 + q^{2M}$$

(we set $\hbar = 1$ throughout this Appendix). The zeta functions $\zeta(\sigma)$ and $\zeta^P(\sigma)$ were defined at Eqs. (10.10)-(10.47) respectively.

' Our computations rely on a number of classical formulas involving the Bessel functions K_μ and I_μ [37] and the generalized hypergeometric functions $_pF_q(^{a_1\ldots a_p}_{b_1\ldots b_q}; z)$ [38]. A list of such formulas follows.

The *Weber-Schafheitlin* integrals (special case):

$$\mathscr{I}_1 = \int_0^\infty z^{-\lambda}K_\mu(z)^2 dz = \frac{\sqrt{\pi}}{4}\Gamma\left(\frac{1-\lambda+2\mu}{2}\right)\Gamma\left(\frac{1-\lambda}{2}\right)\Gamma\left(\frac{1-\lambda-2\mu}{2}\right)\Gamma\left(1-\frac{\lambda}{2}\right)^{-1} \quad \text{(C.1)}$$

$$\mathscr{I}_2 = \int_0^\infty z^{-\lambda}K_\mu(z)I_\mu(z) dz = \frac{1}{4\sqrt{\pi}}\Gamma\left(\frac{1-\lambda+2\mu}{2}\right)\Gamma\left(\frac{1-\lambda}{2}\right)\Gamma\left(\frac{1+\lambda+2\mu}{2}\right)^{-1}\Gamma\left(\frac{\lambda}{2}\right) \quad \text{(C.2)}$$

$$\mathscr{I}_2/\mathscr{I}_1 = \cos\pi\left(\frac{\lambda}{2}+\mu\right)(\pi\sin\lambda\pi/2)^{-1} \quad \text{(C.3)}$$

Continuation of $K_\mu(z)$ to all sheets

$$K_\mu(ze^{im\pi}) = e^{-im\mu\pi}K_\mu(z) - \frac{i\pi\sin m\mu\pi}{\sin\mu\pi}I_\mu(z) \quad (m \in \mathbb{Z}); \quad \text{(C.4)}$$

in particular, when $m = M + 1 = (2\mu)^{-1}$:

$$K_\mu(ze^{i(M+1)\pi}) = -i(K_\mu(z) + \pi(\sin\mu\pi)^{-1}I_\mu(z)) \quad \text{(C.5)}$$

Connection between K_μ and I_μ:

$$K_\mu(z) = \pi(2\sin\mu\pi)^{-1}(I_{-\mu}(z) - I_\mu(z)) \quad \text{(C.6)}$$

and by combination with (C.5)

$$K_\mu(ze^{i(M+1)\pi}) = i\pi(2\sin\mu\pi)^{-1}(I_{-\mu}(z) + I_\mu(z)) \quad \text{(C.7)}$$

Asymptotic behavior for $z \to +\infty$:

$$K_\mu(z) \sim (\pi/2z)^{1/2}e^{-z}, \quad I_\mu(z) \sim (2\pi z)^{-1/2}e^z \quad \text{(C.8)}$$

Definition of generalized hypergeometric series:

$$_pF_q\left(\begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_q \end{matrix}; z\right) = \Gamma\left(\begin{matrix} b_1 \ldots b_q \\ a_1 \ldots a_p \end{matrix}\right)\sum_{n=0}^\infty \Gamma\left(\begin{matrix} a_1+n \ldots a_p+n \\ b_1+n \ldots b_q+n \end{matrix}\right)\frac{z^n}{n!} \quad \text{(C.9)}$$

with the notation

$$\Gamma\left(\begin{matrix} b_1 \ldots b_q \\ a_1 \ldots a_p \end{matrix}\right) = \prod_1^q \Gamma(b_k)\bigg/\prod_1^p \Gamma(a_j) \quad \text{(C.10)}$$

The series is called: *Saalschutzian* if $\sum_1^q b_k - \sum_1^p a_j = 1$; *well-poised* if $p = q + 1$ and if the parameters a_j and b_k can be ordered so that $a_2 + b_1 = a_3 + b_2 = \ldots = a_{q+1} + b_q = a_1 + 1$.

Connection with products of Bessel functions:

$$I_\mu(z)^2 = \frac{(z/2)^{2\mu}}{\Gamma(\mu + 1)^2} {}_1F_2\left(\begin{array}{c} \mu + 1/2 \\ 1 + \mu \quad 1 + 2\mu \end{array}; z^2\right) \tag{C.11}$$

$$I_\mu(z)I_{-\mu}(z) = \frac{\sin \mu\pi}{\mu\pi} {}_1F_2\left(\begin{array}{c} 1/2 \\ 1 + \mu \quad 1 - \mu \end{array}; z^2\right) \tag{C.12}$$

When $z = 1$, the *Whipple formula* expresses a special well-poised ${}_7F_6$ series as a combination of two Saalschutzian ${}_4F_3$ series (as usual, we omit writing $z = 1$);

$$
{}_7F_6\left[\begin{array}{ccccccc} a, 1+\dfrac{1}{2}a, & c, & d, & e, & f, & g, \\[2mm] \dfrac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{array}\right]
$$

$$
= \frac{\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-g)\Gamma(1+a-e-f-g)}{\Gamma(1+a)\Gamma(1+a-f-g)\Gamma(1+a-g-e)\Gamma(1+a-e-f)}
$$

$$
\times {}_4F_3\left[\begin{array}{c} 1+a-c-d,\ e,\ f,\ g\,; \\ 1+a-c,\ 1+a-d,\ e+f+g-a \end{array}\right]
$$

$$
+ \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-g)}{\Gamma(1+a)\Gamma(1+a-c-d)\Gamma(e)\Gamma(f)\Gamma(g)}
$$

$$
\times \frac{\Gamma(e+f+g-1-a)\Gamma(2+2a-c-d-e-f-g)}{\Gamma(2+2a-c-e-f-g)\Gamma(2+2a-d-e-f-g)}
$$

$$
\times {}_4F_3\left[\begin{array}{c} 2+2a-c-d-e-f-g,\ 1+a-f-g,\ 1+a-g-e,\ 1+a-e-f\,; \\ \hline 2+a-e-f-g,\ 2+2a-c-e-f-g,\ 2+2a-d-e-f-g \end{array}\right]. \tag{C.13}
$$

if $(2+2a-c-d-e-f-g) > 0$.

Our basic starting formula is the expression (A.7) for the resolvent kernel, rewritten here for $\hbar = 1$;

$$(\hat{H} - E)^{-1}(q, q') = R(E; q, q') = W(E)^{-1}\psi_+(q_<)\psi_-(q_>) \tag{C.14}$$

where $W(E) = \psi_-\psi'_+ - \psi_+\psi'_-$ is the Wronskian. Our main point is now that for $E = 0$, the solutions $\psi_\pm$ (of $\hat{H}\psi = 0$) are expressible in terms of Bessel functions using Lommel's transformation ([37], §7.2.8.; see also [3]). If we set

$$\mu = (2M + 2)^{-1}, z = q^{M+1}/(M + 1) \qquad (q > 0), \tag{C.15}$$

the solutions normalized according to (4.13) with $q_0 = 0$ read:

$$
\left.\begin{array}{l}
\psi_-(q) = \psi_+(-q) = (2i/\pi)^{1/2}(2\mu)^{\frac{1}{2}-\mu}z^\mu K_\mu(z) \\[2mm]
\psi_+(q) = \psi_-(-q) = (2i/\pi)^{1/2}(2\mu)^{\frac{1}{2}-\mu}z^\mu(K_\mu(z) + (\sin \mu\pi)^{-1}I_\mu(z))
\end{array}\right\} \ (q > 0) \tag{C.16}
$$

(the latter equality derives from (C.5) and the normalization from (C.8)). Also:

$$W(= W(E = 0)) = 2i(\sin \mu\pi)^{-1} \tag{C.17}$$

We note in passing that an explicit formula immediately follows [3]:

$$e^{-\zeta'(0)} = D(0) = a(0) = W/2i \Rightarrow \zeta'(0) = \log \sin \mu\pi \tag{C.18}$$

according to Eqs. (10.14-15) for $E = 0$, and (4.16). Another formula of similar nature is derived in Appendix D, Eq. (D.18).

Eq. (C.14) is now completely explicit for $E = 0$, and by iterating it n times we obtain

an *integral expression* for the kernel of $\hat{H}^{-n}$, on which the trace operation is readily performed. The result is;

$$\zeta(n) = \int_{\mathbb{R}^n} R(0 \; ; q_1, q_2)R(0 \; ; q_2, q_3) \ldots R(0 \; ; q_n, q_1)dq_1 \ldots dq_n \qquad (C.19)$$

We would similarly find by inserting a parity operator [3]:

$$\zeta^P(n) = \int_{\mathbb{R}^n} R(0 \; ; q_1, q_2)R(0 \; ; q_2, q_3) \ldots R(0 \; ; q_n, - q_1)dq_1 \ldots dq_n \qquad (C.20)$$

The operator $\hat{H}^{-n}$ is of trace-class and all the integrals considered are uniformly convergent, except when $M = n = 1$ (in which case $E_n = 2n + 1$, $\zeta(1) = \infty$, $\zeta^P(1) = \pi/4$).

The problem of interest is now to reduce (C.19-20) as much as possible to simpler arithmetic forms, in order of increasing n. We shall briefly treat the known case $n = 1$ [3] and focus on the case $n = 2$; no significant reduction has been achieved yet for higher values of n by the method to be described here. Our basic tool will be the Weber-Schafheitlin formulas (C.1-2).

For $n = 1$ the result is immediate:

$$\zeta^P(1) = \int_{-\infty}^{\infty} R(0 \; ; q, - q)dq = 2 \int_0^{\infty} R(0 \; ; q, - q) \text{ (by obvious symmetry).}$$

$$= 2W^{-1} \int_0^{\infty} \psi_-(q)^2 dq$$

$$= 2W^{-1}c \int_0^{\infty} z^{4\mu-1}K_\mu(z)^2 dz \quad (c = 2i\pi^{-1}(2\mu)^{2-4\mu}) \qquad (C.21)$$

by Eq. (C.16); formulas (C.1) and (C.17) then yield;

$$\zeta^P(1) = (2\sqrt{\pi})^{-1} \sin \mu\pi(2\mu)^{2-4\mu}\Gamma(\mu)\Gamma(2\mu)\Gamma(3\mu)\Gamma\left(\frac{1}{2} + 2\mu\right)^{-1} \qquad (C.22)$$

Similarly:

$$\zeta(1) = \int_{-\infty}^{\infty} R(0 \; ; q, q)dq = 2 \int_0^{\infty} R(0 \; ; q, q)dq = 2W^{-1} \int \psi_-(q)\psi_-(- q)dq$$

$$= 2W^{-1}c \int_0^{\infty} z^{4\mu-1}K_\mu(z)\left[K_\mu(z) + \frac{\pi}{\sin \mu\pi} I_\mu(z)\right]dz \text{ by Eqs. (C.16).}$$

We now use (C.1-3) to obtain the simple relation:

$$\zeta(1) = (1 + (\cos 2\mu\pi)^{-1})\zeta^P(1) = \frac{\text{tg } 2\mu\pi}{\text{tg } \mu\pi} \zeta^P(1) \qquad (C.23)$$

For $M = 1$ we recover that $\zeta(1) = \infty$ and $\zeta^P(1) = \pi/4$; for $M = 2$ we get the amusing result that the sum of the inverse eigenvalues with *even* parity is twice the sum of the inverse eigenvalues with *odd* parity ($\zeta(1) = 3\zeta^P(1)$).

For general n we propose the following strategy to reduce the n-uple integrals (C.19-20): first to reduce the integration domain using all symmetries of the integrand, then to constrain all integrations to finite ranges (and perform these term by term on the series expansions of the integrands) except for the outermost integration, to be evaluated by the Weber-Schafheitlin formulas.

We have carried out this program for $n = 2$ jointly with D. Chudnovsky and G. Chudnovsky. Starting with $\zeta(2) = \int\int_{-\infty}^{x} R(0 \; ; q_1, q_2)R(0 \; ; q_2, q_1)dq_1 dq_2$ and using

$$R(0, q_1, q_2) = R(0 \; ; q_2, q_1) = R(0 \; ; - q_1, - q_2)$$

we reduce it to:

$$\zeta(2) = 4W^{-2} \int\!\!\int_{0<|q_1|<q_2} dq_1 dq_2 \psi_+(q_1)^2 \psi_-(q_2)^2 \tag{C.24}$$

We now substitute

$$\psi_+(q_1)^2 = \psi_-(|q_1|)^2 \quad \text{when } q_1 < 0$$

$$= \psi_-(q_1)^2 + (2\mu)^{2\mu-1} \frac{\pi^2}{\sin^2 \mu\pi} z_1^{2\mu} I_{-\mu}(z_1) I_\mu(z_1) \quad \text{when } q_1 > 0$$

using (C.6-7). Eq. (C.24) then becomes, with c as in (C.21):

$$\zeta(2) = 4W^{-2}c^2 \int\!\!\int_{0<z_1<z_2} dz_1 dz_2 z_2^{4\mu-1} z_1^{4\mu-1} K_\mu(z_2)^2 [2K_\mu(z_1)^2$$
$$+ \pi^2 (\sin \mu\pi)^{-2} I_{-\mu}(z_1) I_\mu(z_1)] \quad \text{(C.25)}$$

Integration by parts shows the first term of (C.25) to be just the square of (C.21), namely $\zeta^P(1)^2$. There remains to compute:

$$\mathscr{I} = \int_0^\infty dz_2 z_2^{4\mu-1} K_\mu(z_2)^2 \int_0^{z_2} dz_1 z_1^{4\mu-1} I_{-\mu}(z_1) I_\mu(z_1) \tag{C.26}$$

Here we use the series expansion (C.12), which we may integrate term by term in the variable z_1:

$$\mathscr{I} = \frac{\sin \mu\pi}{\mu\pi} \int_0^\infty dz_2 z_2^{4\mu-1} K_\mu(z_2)^2 \left[\frac{\Gamma(1+\mu)\Gamma(1-\mu)}{\Gamma(1/2)} \sum_{n=0}^\infty \frac{\Gamma(1/2+n)}{\Gamma(1+\mu+n)\Gamma(1-\mu+n)n!} \frac{z_2^{4\mu+2n}}{(4\mu+2n)} \right]$$

and the last integral is done term by term using (C.1):

$$\mathscr{I} = \frac{\sin \mu\pi}{4\mu\sqrt{\pi}} \frac{\Gamma(1+\mu)\Gamma(1-\mu)}{\Gamma(1/2)} \sum_{n=0}^\infty \frac{\Gamma(1/2+n)\Gamma(n+3\mu)\Gamma(n+4\mu)\Gamma(n+5\mu)}{\Gamma(1+\mu+n)\Gamma(1-\mu+n)\Gamma(1/2+4\mu+n)n!(4\mu+2n)}$$

This is now recognized as:

$$\mathscr{I} = \frac{\sin \mu\pi}{8\mu\sqrt{\pi}} \Gamma\!\left(\begin{matrix} 2\mu & 3\mu & 4\mu & 5\mu \\ \frac{1}{2}+4\mu & 1+2\mu \end{matrix} \right) \mathscr{F} \left.\begin{matrix}\\\\\\\end{matrix}\right\}$$
$$\mathscr{F} = {}_5F_4\!\left(\begin{matrix} 4\mu & 2\mu & 3\mu & 5\mu & 1/2 \\ 1+2\mu, & 1+\mu, & 1-\mu, & 1/2+4\mu \end{matrix}; 1 \right) \tag{C.27}$$

according to the definition (C.9) and the notation (C.10). Finally:

$$\zeta(2) = \zeta^P(1)^2 + \frac{W^{-2}\pi^{3/2}}{4\mu^2 \sin \mu\pi} \Gamma\!\left(\begin{matrix} 3\mu & 4\mu & 5\mu \\ 1/2+\mu \end{matrix} \right) \mathscr{F} \left.\begin{matrix}\\\\\\\end{matrix}\right\}$$
$$= \zeta^P(1)^2 + \frac{\sin \mu\pi}{4\mu^2\sqrt{\pi}} (2\mu)^{4-8\mu} \Gamma\!\left(\begin{matrix} 3\mu & 4\mu & 5\mu \\ 1/2+4\mu \end{matrix} \right) \mathscr{F} \tag{C.28}$$

We now turn to $\zeta^P(2) = \int\!\!\int_{-\infty}^\infty R(0; q_1, q_2) R(0; q_2, -q_1) dq_1 dq_2$. By symmetry we find:

$$\zeta^P(2) = 8W^{-2} \int\!\!\int_{0<q_1<q_2} dq_1 dq_2 \psi_-(q_2)^2 \psi_-(q_1)\psi_+(q_1)$$
$$= 8W^{-2}c^2 \int\!\!\int_{0<z_1<z_2} dz_1 dz_2 z_2^{4\mu-1} z_1^{4\mu-1} K_\mu(z_2)^2 K_\mu(z_1)[K_\mu(z_1) + \pi(\sin \mu\pi)^{-1} I_\mu(z_1)] \quad \text{(C.29)}$$

from (C.16); we now rewrite this, using (C.6) for one of the terms $K_\mu(z_1)$, as:

$$\zeta^P(2) = 8W^{-2}c^2 \iint_{0 < z_1 < z_2} dz_1 dz_2 z_2^{4\mu-1} z_1^{4\mu-1} K_\mu(z_2)^2 \left[K_\mu(z_1)^2 + \frac{\pi^2}{2 \sin^2 \mu\pi} I_\mu(z_1)(I_{-\mu}(z_1) - I_\mu(z_1)) \right] \quad (C.30)$$

Except for the last term, this is just Eq. (C.25)! So that:

$$\left. \begin{array}{c} \zeta^P(2) = \zeta(2) - 4W^{-2}c^2\pi^2(\sin \mu\pi)^{-2}\mathscr{J}' \\[2mm] \mathscr{J}' = \displaystyle\int_0^\infty dz_2 z_2^{4\mu-1} K_\mu(z_2)^2 \int_0^{z_2} dz_1 z_1^{4\mu-1} I_\mu(z_1)^2 \end{array} \right\} \quad (C.31)$$

We compute $\mathscr{J}'$ in the same way as $\mathscr{J}$, using Eq. (C.11) instead of (C.12):

$$\left. \begin{array}{c} \mathscr{J}' = \dfrac{\sqrt{\pi}2^{-2\mu}}{24\mu} \Gamma\begin{pmatrix} 4\mu & 5\mu & 6\mu \\[1mm] 1+\mu & 1+\mu & \dfrac{1}{2}+5\mu \end{pmatrix} \mathscr{F}' = \dfrac{1}{24\mu}\Gamma\begin{pmatrix} 4\mu & 5\mu & 6\mu & \dfrac{1}{2}+\mu \\[1mm] 1+\mu & 1+2\mu & \dfrac{1}{2}+5\mu \end{pmatrix}\mathscr{F}' \\[8mm] \mathscr{F}' = {}_5F_4\begin{pmatrix} 6\mu & 3\mu & 4\mu & 5\mu & \dfrac{1}{2}+\mu \\[2mm] & 1+3\mu, & 1+2\mu, & 1+\mu, & \dfrac{1}{2}+5\mu \end{pmatrix} ; 1 \end{array} \right\} \quad (C.32)$$

Finally:

$$\left. \begin{array}{c} \zeta^P(2) = \zeta(2) - \dfrac{W^{-2}c^2\pi^2}{6\mu \sin^2 \mu\pi}\Gamma\begin{pmatrix} 4\mu & 5\mu & 6\mu & \dfrac{1}{2}+\mu \\[1mm] 1+\mu & 1+2\mu & \dfrac{1}{2}+5\mu \end{pmatrix}\mathscr{F}' \\[8mm] = \zeta(2) - \dfrac{(2\mu)^{4-8\mu}}{6\mu}\Gamma\begin{pmatrix} 4\mu & 5\mu & 6\mu & \dfrac{1}{2}+\mu \\[1mm] 1+\mu & 1+2\mu & \dfrac{1}{2}+5\mu \end{pmatrix}\mathscr{F}' \end{array} \right\} \quad (C.33)$$

To sum up, we have expressed $\zeta(2)$ and $\zeta^P(2)$ in terms of two well-poised $_5F_4$ series with argument $z = 1$ (plus gamma factors and elementary functions). Two remarks:

— the series $\mathscr{F}$ and $\mathscr{F}'$ converge at the same rate as the series $\displaystyle\sum_n n^{4-8\mu}$, which also

happens to be the same rate at which the *original* series $\displaystyle\sum_{k=0}^\infty (\pm 1)^k E_k^{-2}$ converged (by Bohr-Sommerfeld).

— for $M = 1$ ($\mu = 1/4$): cancellations between parameters allow the further reductions:

$$\mathscr{F} = {}_3F_2\begin{pmatrix} 1 & 1/2 & 1/2 \\ & 3/2 & 3/2 \end{pmatrix}; 1\end{pmatrix}; \quad \mathscr{F}' = {}_3F_2\begin{pmatrix} 1 & 3/4 & 3/4 \\ & 7/4 & 7/4 \end{pmatrix}$$

Now $\mathscr{F} = \pi^2/8$ by Dixon's theorem [38], while $\mathscr{F}'$ has no such closed form. If we recall that

$E_n = 2n+1$ when $M = 1$, then (C.28) only restores the known result: $\displaystyle\sum_0^\infty (2n+1)^{-2} = \pi^2/8$,

whereas (C.33) becomes tautological when $\mathscr{F}'$ is replaced by its defining series: we have thus learnt nothing new about the number $\sum\limits_{0}^{\infty}(-1)^n(2n+1)^{-2}$ (Catalan's constant) [36].

For general M, formulas (C.28) and (C.33) can be further reduced in a different way. If we set $a/2 = c$ or g in (C.13), we get two distinct expressions for the most general well-poised $_5F_4$ as a combination of two Saalschutzian $_4F_3$. After all the allowed permutations of the parameters, we end up with *seven* such expressions for $\mathscr{F}$, and similarly for $\mathscr{F}'$. If we discard those that become indeterminate (of the form $\infty - \infty$) for relevant values of μ corresponding to M = 1, 2 or 3, and if we favor those where one $_4F_3$ series simplifies to an $_3F_2$, then we are led to prefer slighlty the following pair of reduced expressions, that moreover share the same $_4F_3$ term:

$$\mathscr{F} = \Gamma\begin{pmatrix} \frac{1}{2} + 4\mu & 1 + 2\mu & 1 + 2\mu & \frac{1}{2} \\ 1 + 4\mu & \frac{1}{2} + 2\mu & \frac{1}{2} + 2\mu \end{pmatrix} {}_3F_2\begin{pmatrix} 1 - 4\mu, 2\mu, 2\mu \\ 1 + \mu, \ 1 - \mu \end{pmatrix} ;1$$

$$- \frac{\mu^2(1 - 8\mu)\, \mathrm{tg}\, 4\mu\pi}{(-\mu^2 + 1/4)\, \mathrm{tg}\, \mu\pi} \, {}_4F_3\begin{pmatrix} 1, \dfrac{3}{2} - 4\mu, \dfrac{1}{2} + 2\mu, \dfrac{1}{2} + 2\mu \\ \dfrac{3}{2}, \quad \dfrac{3}{2} + \mu, \dfrac{3}{2} - \mu \end{pmatrix} ; 1$$

$$\mathscr{F}' = \Gamma\begin{pmatrix} \frac{1}{2} + 5\mu & 1 + 3\mu & 1 + 3\mu & \frac{1}{2} - \mu \\ 1 + 6\mu & \frac{1}{2} + 2\mu & \frac{1}{2} + 2\mu \end{pmatrix} {}_3F_2\begin{pmatrix} 1 - 3\mu, & 3\mu, & 3\mu \\ & 1 + 2\mu, \ 1 + \mu \end{pmatrix} ; 1$$

(C.34)

$$+ \frac{9\mu^2}{(\mu - 1/2)} \Gamma\begin{pmatrix} 1 + 2\mu & 1 + \mu & \frac{1}{2} + 5\mu & \frac{3}{2} - 4\mu \\ 1 + 6\mu & 1 - 3\mu & \frac{3}{2} + \mu & \frac{3}{2} \end{pmatrix} {}_4F_3\begin{pmatrix} 1, \dfrac{3}{2} - 4\mu, \dfrac{1}{2} + 2\mu, \dfrac{1}{2} + 2\mu \\ \dfrac{3}{2}, \quad \dfrac{3}{2} + \mu, \dfrac{3}{2} - \mu \end{pmatrix} ; 1$$

Remark. — The series in (C.34) converge more slowly than the initial $_5F_4$ series: like n^{-2} for all M.

To summarize, we have obtained for general M the closed expressions:

— (C.22-23) for $\zeta^P(1)$ and $\zeta(1)$ in terms of gamma factors;

— (C.28) and (C.33) for $\zeta(2)$ and $\zeta^P(2)$ in terms of $_5F_4$ series, reducible to $_4F_3$ by (C.34).

Our strategy in principle applies to $\zeta^{(P)}(n)$ for higher values of n, but already for $n = 3$ we have not achieved a reduction beyond the level of double hypergeometric (Appell) series, that are moreover of intractable appearance.

We wish to thank S. Graffi and J. Raynal for useful suggestions.

APPENDIX D

We are going to reestablish here the identities (10.48-49) proved in [3] that play for the alternating spectral functions the same basic role as relations (A.5) or (10.15) for the ordinary spectral functions. Therefore the structure and notations of this appendix will closely follow those of Appendix A (e. g. : $\cdot = \partial/\partial E$; $' = \partial/\partial q$).

With the prerequisite that the potential $V(q)$ be an even function and the convention that inf $V = 0$, we can relax the restriction (4.17) of Appendix A to the much weaker condition of confinement: $V(q) \to +\infty$ for $|q| \to \infty$; this suffices to endow

$$\hat{H} = -\hbar^2 d^2/dq^2 + V(q)$$

with a discrete spectrum $\{E_k\} \nearrow +\infty$ [15], which in turn ensures convergence of the functions:

$$R^P(E) = \sum_{k=0}^{\infty} (-1)^k (E_k - E)^{-1} = \int_{-\infty}^{\infty} R(E; q, -q) dq \qquad (D.1)$$

and

$$\Delta^P(E) = \sum_{k=0}^{\infty} (1 - E/E_k)^{(-1)^k} = \exp - \int_0^E R^P(E') dE' \qquad (D.2)$$

Special treatment of the harmonic case is thus not needed here.

We now insert the value (A.7) of the Green's function kernel $R(E; q, q')$ into the right-hand side of (D.1) to get, for $E < 0$:

$$R^P(E) = 2\hbar^{-2} W^{-1} \int_{-\infty}^0 \psi_+(q)^2 dq \qquad (D.3)$$

provided the two solutions $\psi_\pm(q)$ have been chosen to satisfy:

$$\psi_+(q) = \psi_-(-q). \qquad (D.4)$$

The integral in (D.3) can be performed analytically, since by combining the equations $\psi_+[(\hat{H}-E)\psi_+] = 0$ and $\psi_+[(\hat{H}-E)\psi_+]\cdot = 0$ we may write $\hbar^{-2}\psi_+(q)^2 = [\dot{\psi}_+\psi'_+ - \psi_+\dot{\psi}'_+]'$; and the Wronskian W can be computed at $q = 0$ using (D.4):

$$W(E) = 2\psi_+(0)\psi'_+(0). \qquad (D.5)$$

With all that information, Eq. (D.3) becomes:

$$R^P(E) = -[\log \psi'_+/\psi_+]\dot{}_{q=0} \qquad (D.6)$$

which is the analog of Eq. (A.3) for $R(E)$; one integration upon (D.6) will likewise produce an analog to Eqs. (A.4-5):

$$\Delta^P(E) = C \frac{\psi'_+}{\psi_+} (q = 0) \qquad (D.7)$$

but we must fix the yet unknown integration constant C. We can find C by applying WKB theory to the right-hand side in the limit $E \to -\infty$. Indeed the differentiation of the expression (10.50) for ψ_+ leads to:

$$\frac{\psi'_+}{\psi_+}(q = 0) = \left(i\hbar^{-1}u - \frac{u'}{2u}\right)_{q=0} = i\hbar^{-1}u(q = 0) \qquad (D.8)$$

($u'(0) = 0$ because u is, like p, an even function of q). The WKB estimates upon Eqs. (4.4-5) imply at $q = 0$ ($p(0) = E^{1/2} = -i|E|^{1/2}$):

$$\frac{\psi'_+}{\psi_+}(q = 0) = i\hbar^{-1}E^{1/2} + 0(|E|^{-3/2}) \qquad (E \to -\infty)$$
$$= \hbar^{-1}(-E)^{1/2} + 0((-E)^{-3/2}) \qquad (D.9)$$

Thus, $\log \dfrac{\psi'_+}{\psi_+} + \log \hbar - \dfrac{1}{2}\log(-E) \to 0$ for $E \to -\infty$; whereas $\log \Delta^P(E) = 0$ for $E = 0$.

These two boundary conditions jointly determine C in (D.7) as:

$$\log(C/\hbar) = \int_{-\infty}^{-1}\left[R^P(E) + \frac{1}{2E}\right]dE + \int_{-1}^{0} R^P(E)dE \qquad (D.10)$$

In analogy with the derivation of Eq. (10.13) we now show that the right-hand side of (D.10) is $(\zeta^P)'(0)$. We start from the same Mellin representation as (10.11) for ζ^P:

$$\zeta^P(\sigma) = \frac{\sin \pi\sigma}{\pi} \int_{-\infty}^{0} (-E)^{-\sigma}R^P(E)dE \qquad (0 < \operatorname{Re}\sigma < 1) \qquad (D.11)$$

with the convergence strip fixed by the asymptotic formula arising from (D.6)-(D.9):

$$R^P(E) = -\frac{1}{2E} + 0(|E|^{-3}) \qquad (E \to -\infty) \qquad (D.12)$$

An integration by parts then extends the analyticity strip of the integral in (D.11) (we recall that $\cdot = \partial/\partial E$):

$$\zeta^P(\sigma) = \frac{\sin \pi\sigma}{\pi\sigma} \int_{-\infty}^{0} (-E)^{-\sigma}(ER^P(E))^{\cdot}dE \qquad (-2 < \operatorname{Re}\sigma < 1) \qquad (D.13)$$

We now expand (D.13) to first order around $\sigma = 0$ to find, using again (D.12):

$$\zeta^P(0) = \int_{-\infty}^{0} (ER^P(E))^{\cdot}dE = [ER^P(E)]_{-\infty}^{0} = 1/2 \qquad [3] \qquad (D.14)$$

and

$$(\zeta^P)'(0) = -\int_{-\infty}^{0} \log(-E)(ER^P(E))^{\cdot}dE$$

Again integrating by parts but now separately on each interval $[-\infty, 1]$ and $[-1, 0]$, we get:

$$(\zeta^P)'(0) = \int_{-\infty}^{-1}\left(R^P(E) + \frac{1}{2E}\right)dE + \int_{-1}^{0} R^P(E)dE \qquad (D.15)$$

which is indeed the right-hand side of (D.10). Finally Eq. (D.7) takes a form reminiscent of (10.15):

$$D^P(E) = \hbar\frac{\psi'_+}{\psi_+}(q = 0), \quad D^P(E) \equiv e^{-(\zeta^P)'(0)}\Delta^P(E) \qquad (D.16)$$

A third expression for the logarithmic derivative at $q = 0$ results from (D.5):

$$\frac{\psi'_+}{\psi_+}(q = 0) = \frac{W(E)}{2\psi_+(0)^2}$$

Under the normalization (4.13) with $q_0 = 0$ for $\psi_\pm$, we readily identify:

$$\frac{\psi'_+}{\psi_+}(q = 0) = \frac{ip(0)a(E, \hbar^{-1})}{\hbar a(q = 0, E, \hbar^{-1})^2} = \frac{(-E)^{1/2}a(E, \hbar^{-1})}{\hbar a(q = 0, E, \hbar^{-1})^2} \qquad (D.17)$$

according to formulas (4.16) for the numerator and (4.21) for the denominator. The final results of this appendix, namely Eqs. (D.8) and (D.16-17), are however completely insensitive to the normalization chosen for ψ_+.

Remark. — For the potential q^{2M} (and $\hbar = 1$), if we insert the known solution (C.16) at $E = 0$ into (D.16), we obtain the explicit formula:

$$(\zeta^P)'(0) = - \log D^P(0) = - \log \left[\frac{\psi'_+}{\psi_+} (q = 0, \ E = 0) \right]$$

$$= \log \left[\pi^{-1} \sin (\mu\pi) \Gamma^2(\mu\pi) \mu^{2M_\mu} \right] \qquad (\mu \equiv (2M + 2)^{-1})$$

(D.18)

that stands parallel to Eq. (10.75) for $\zeta'(0)$.

TABLE 1.

n	A_n	n	A_n
1	0.82246703342411321823620758332230124	31	0.99131581400968260564830185007011358
2	-0.63019715992232262931327528500942284	32	-0.99159448798393060832472872366354413
3	0.78276326102224685330501201067800982	33	0.99185584075600584268614567891226835
4	-0.97029844025951261241363572266602123	34	-0.99210143841682964570582798346278077
5	0.96514474249426887826170739606590820	35	0.99233266354760063330058363930166567
6	-0.93573459292019242154482620650180262	36	-0.99255074123948112609107557982738444
7	0.95068452113174036551752470019520080	37	0.99275676133641458790986191538208120
8	-0.96726756406673517616328968013479970	38	-0.99295169660928929717222934794672190
9	0.96987649129040209035433981729365870	39	0.99313641794550062517403173412109490
10	-0.97058151100471770950150702634293510	40	-0.99331170733799825410331685018387780
11	0.97382229084824260412407812961413900	41	0.99347826897025538761917583791642750
12	-0.97680809011300906955816399060760960	42	-0.99363673865704449672295773221172150
13	0.97860895569718312171284166961163130	43	0.99378769193538455217897327199201390
14	-0.98006124577722960928195314290038160	44	-0.99393165104009053462909372171418410
15	0.98148980904329437732852396616986610	45	0.99406909093489061937217125060012020
16	-0.98274968521391606935874569957487050	46	-0.99420044454128011710072600669735530
17	0.98380832281566760392190032760023110	47	0.99432610728799444126450158500362020
18	-0.98473948405977139355539532105889190	48	-0.99444646441082802505318960666784219100
19	0.98558015791770563746666184170694100	49	0.99456177779011993884265156902814150
20	-0.98633446497854679277098362610507290	50	-0.99467242228447014077400834126281290
21	0.98701116414715132474466643282973110	51	0.99477865513900045032124336628419865
22	-0.98762376908191655576430336827517640	52	-0.99488073499902217837129349821575900
23	0.98818194373414170209186584314803290	53	0.99497890068269579399833525222040880
24	-0.98869206845349987774993675514633100	54	-0.99507337304487214360383324779308070
25	0.98915983912355639115099340893829450	55	0.99516435663454610073093354191234930
26	-0.98959045901887688787564257884417110	56	-0.99525204117217161254926125130225470
27	0.98998824364941211811863104045515840	57	0.99533660286924523242203663948011800
28	-0.99035677130941699044480255259524050	58	-0.99541820560946123269490592098008650
29	0.99069913751277826268914133405518060	59	0.99549700200808663087297108157835180
30	-0.99101804018481323089466876706265900	60	-0.99555731343639551368199525397438512

TABLE 2.

p / n	1	2	3	4
15	0.9814898090	1.0000611315	6.7504126376	************
20	0.9863344650	0.9999119921	1.0002451153	0.9930625796
25	0.9891598391	0.9999185645	0.9999589013	0.9994874366
30	0.9910180402	0.9999473150	1.0000041575	1.0000318005
35	0.9923326635	0.9999630929	1.0000029293	1.0000015890
40	0.9933117073	0.9999727043	1.0000018471	0.9999999547
45	0.9940690909	0.9999790064	1.0000012706	1.0000000185
50	0.9946724223	0.9999833580	1.0000009113	1.0000000131
55	0.9951643566	0.9999864869	1.0000006754	1.0000000081
60	0.9955731344	0.9999888110	1.0000005143	1.0000000053

$$\alpha_0 = 1.$$

p / n	1	2	3	4
15	********	**********	*********	**********
20	59419.04	**********	*********	**********
25	11243.40	-84203.75	*********	**********
30	-957.10	-87244.21	*********	**********
35	292.43	2812.30	271828.44	**********
40	446.64	1553.94	13071.96	**********
45	437.12	423.01	-3046.56	-69072.48
50	437.26	438.44	343.19	6051.19
55	438.13	445.51	468.23	1056.26
60	438.77	444.99	443.53	398.59

$$\alpha_7 = 442.946584$$

TABLE 3.

k	α_k (theory)	α_k (fit)
0	1	1.000 000 005
1	$-0.523\ 598\ 776$	$-0.523\ 598\ 77$
2	0.137 077 839	0.137 077
3	0.778 467 349	0.778 48
4	$-0.416\ 999\ 718$	$-0.416\ 6$
5	$-11.850\ 079$	-11.85
6	6.564 854	6.6
7	442.947	44(0)
8	-242.138	$-2(50)$
9	$-24\ 707.8$	$-24(400)$

TABLE 4.

n \ p	1	2	3	4
15.	0.96316	1.00050	6.75337	29.26461
20	0.97277	0.99969	0.99983	1.00255
25	0.97839	0.99981	0.99996	0.99975
30	0.98209	0.99986	0.99996	0.99997
35	0.98471	0.99990	1.00002	1.00063
40	0.98666	0.99992	1.00007	1.00046
45	0.98817	0.99994	0.99985	0.99533
50	0.98937	0.99991	0.99896	0.98958
55	0.99035	0.99993	1.00259	1.10966
60	0.99117	1.00103	1.03589	1.52532

$$\gamma_0 = 1$$

n \ p	1	2	3	4
15	0.42252	−6.03068	−37.69172	**********
20	0.45827	0.51273	0.39787	2.36492
25	0.47717	0.54468	0.53167	0.14579
30	0.48915	0.54029	0.47088	0.34298
35	0.49778	0.54613	0.65708	2.93380
40	0.50470	0.56907	0.99041	3.53818
45	0.51041	0.55826	−0.40805	−30.10943
50	0.50905	0.19148	−8.83963	−99.15862
55	0.49397	0.18019	26.59080	**********
60	0.66096	14.81115	474.77121	**********

$$\gamma_2 = 0.54831136$$

n \ p	1	2	3	4
15	−1.03155	−25.92949	********	*********
20	−1.03481	−1.04560	−1.05090	−0.97695
25	−1.03705	−1.04566	−1.04751	−1.05735
30	−1.03862	−1.04625	−1.04856	−1.04986
35	−1.03977	−1.04644	−1.04538	−1.00794
40	−1.04064	−1.04633	−1.04126	−1.00973
45	−1.04133	−1.04663	−1.05913	−1.42798
50	−1.04195	−1.05058	−1.14644	−2.07355
55	−1.04258	−1.05021	−0.78571	10.07081
60	−1.04155	−0.92307	3.09659	60.35643

$$\gamma_1 = -1.04719755$$

n \ p	1	2	3	4
15	−3.27057	−51.92539	*********	**********
20	−3.24137	−4.41546	−8.92856	40.11999
25	−3.27272	−3.42893	−4.09474	−18.98819
30	−3.31283	−3.73805	−7.60952	−17.26029
35	−3.33529	−3.47299	3.05288	139.02600
40	−3.31431	−1.79909	29.14754	229.46889
45	−3.25982	−2.43430	−78.94305	**********
50	−3.76870	−36.95388	*********	**********
55	−5.75975	−43.67634	*********	**********
60	13.06709	*********	*********	**********

$$\gamma_3 = -3.40096455$$

TABLE 5.

	M = 2			M = 3	
n	$\zeta(n)$	$\zeta^P(n)$	n	$\zeta(n)$	$\zeta^P(n)$
1	2.289 908 804 320	0.763 302 934 770	1	1.721 346 195	0.713 004 939
2	0.996 320 827 679	0.833 155 797 907	2	0.838 749 154	0.718 952 295
3	0.860 517 138 943	0.822 472 813 464	3	0.680 579 268	0.655 376 819
4	0.796 211 192 704	0.786 494 804 331	4	0.582 204 245	0.579 516 165
5	0.747 295 110 967	0.744 760 013 169	5	0.509 234 298	0.507 930 487
6	0.703 855 987 715	0.703 190 573 374	6	0.444 390 836	0.444 090 783

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