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## Bound states and scattering states for time periodic Hamiltonians

by

Kenji YAJIMA <sup>(1)</sup> and Hitoshi KITADA <sup>(2)</sup>

**ABSTRACT.** — Let  $U(t, s)$  be the propagator for the Schrödinger equation with a time periodic Hamiltonian  $H(t + \omega) = H(t)$  and let

$$\mathcal{H} = \mathcal{H}_p(s) \oplus \mathcal{H}_c(s)$$

be the decomposition of the state space  $\mathcal{H} = L^2(\mathbb{R}^n)$  into the point and the continuous spectral sub-spaces of the Floquet operator  $U(s + \omega, s)$ . We show that the wave packet  $u(t) = U(t, s)u(s)$  remains localized for all time  $t \in \mathbb{R}$  if and only if  $u(s) \in \mathcal{H}_p(s)$  and that  $u(t)$  decays locally as  $t \rightarrow \pm \infty$  in the time mean if and only if  $u(s) \in \mathcal{H}_c(s)$ , extending Ruelle-Amrein-Georgescu's theorem for the time independent Hamiltonians.

**RÉSUMÉ.** — Soit  $U(t, s)$  le propagateur pour l'équation de Schrödinger avec un Hamiltonien périodique en temps  $H(t + \omega) = H(t)$  et soit

$$\mathcal{H} = \mathcal{H}_p(s) \oplus \mathcal{H}_c(s)$$

la décomposition de l'espace des états  $\mathcal{H} = L^2(\mathbb{R}^n)$  en sous-espaces spectraux ponctuel et continu de l'opérateur de Floquet  $U(s + \omega, s)$ . On montre que le paquet d'onde  $u(t) = U(t, s)u(s)$  reste localisé pour tout temps  $t \in \mathbb{R}$  si et seulement si  $u(s) \in \mathcal{H}_p(s)$  et que  $u(t)$  décroît localement en moyenne dans le temps quand  $t \rightarrow \pm \infty$  si et seulement si  $u(s) \in \mathcal{H}_c(s)$ , étendant ainsi au cas périodique le théorème de Ruelle-Amrein-Georgescu pour les Hamiltoniens indépendants du temps.

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## § 1. INTRODUCTION AND THEOREMS

Several years ago Ruelle [11] and subsequently Amrein-Georgescu [1] showed that the point and the continuous spectral subspaces of a quantum mechanical Hamiltonian  $H$  can be characterized in terms of the space-time behavior of the solutions  $\exp(-itH)u_0$  of the time-dependent Schrödinger equation  $i\partial u/\partial t = Hu$ . A function  $u_0 \in L^2(\mathbb{R}^n)$  belongs to the point spectral subspace of  $H$  if and only if  $\exp(-itH)u_0$  remains localized for all time  $t \in \mathbb{R}$ : for any  $\varepsilon > 0$ , there exists  $R > 0$  such that

$\|\exp(-itH)u_0\|_{L^2(|x| \leq R)} \geq (1 - \varepsilon)\|u_0\|$  for all  $t \in \mathbb{R}$ ;  $u_0 \in L^2(\mathbb{R}^n)$  belongs to the continuous subspace of  $H$  if and only if  $\exp(-itH)u_0$  decays locally as  $t \rightarrow \pm\infty$  in the time mean: for any  $R > 0$ ,

$$T^{-1} \int_0^T \|\exp(-itH)u_0\|_{L^2(|x| \leq R)}^2 dt$$

converges to zero as  $T \rightarrow \pm\infty$ . This characterization theorem confirms the physical intuition that the states in the point and the continuous subspaces correspond respectively to the bound and the scattering states and its significance was even increased when Enss [2] used it as a corner stone in his innovative proof of the asymptotic completeness of the wave operators. The purpose of this paper is to show such a characterization of the bound and the scattering states is possible for time-periodic Hamiltonians.

We consider Schrödinger equations in the Hilbert space  $\mathcal{H} = L^2(\mathbb{R}^n)$  with time-periodic Hamiltonians  $H(t)$ :

$$(1.1) \quad i\partial u/\partial t = (-\Delta + V(t))u \equiv H(t)u, \quad H(t + \omega) = H(t), \quad \omega > 0.$$

Under a suitable condition on  $V(t)$ , Eq. (1.1) generates a unitary propagator  $U(t, s)$ . For periodic systems, the important object which plays the role of  $H$  in the time independent case is the Floquet operator  $U(s + \omega, s)$ . We shall show that the point and the continuous subspaces of  $U(s + \omega, s)$  can be characterized in terms of the space-time behavior of the solution  $U(t, s)u_0$  exactly the same way as in the time independent case. For a unitary (or selfadjoint) operator  $T$  in a Hilbert space  $\mathcal{X}$ ,  $\mathcal{X}_p(T)$  and  $\mathcal{X}_c(T)$  are respectively the point and the continuous spectral subspaces of  $T$ . We shall postpone the description of the conditions on  $V(t)$  to Section 4.

**THEOREM.** — Let  $V(t)$  satisfy either Assumption (A) or (B) of Section 4. Let  $u \in \mathcal{H}$  and  $s \in \mathbb{R}^1$ . Then:

(1) Following three statements are equivalent:

$$(1.a) \quad u \in \mathcal{X}_p(U(s + \omega, s)).$$

(1.b) For any  $\varepsilon > 0$ , there exists an  $R > 0$  such that

$$\inf_t \|U(t, s)u\|_{L^2(|x| \leq R)} \geq (1 - \varepsilon) \|u\|.$$

(1.c) For any  $\varepsilon > 0$ , there exists an  $R > 0$  such that

$$\inf_N \|U(s + \omega, s)^N u\|_{L^2(|x| \leq R)} \geq (1 - \varepsilon) \|u\|.$$

(2) Following three statements are equivalent:

(2.a) 
$$u \in \mathcal{H}_c(U(s + \omega, s)).$$

(2.b) For any  $R > 0$ ,

$$\lim_{T \rightarrow \pm\infty} \frac{1}{T} \int_0^T dt \|U(t, s)u\|_{L^2(|x| \leq R)}^2 = 0.$$

(2.c) For any  $R > 0$ ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^N \|U(s + \omega, s)^k u\|_{L^2(|x| \leq R)}^2 = 0.$$

REMARK. 1. — Since

$$U(s' + \omega, s') = U(s', s)U(s + \omega, s)U(s', s)^{-1},$$

$$\mathcal{H}_p(U(s' + \omega, s')) = U(s', s)\mathcal{H}_p(U(s + \omega, s))$$

and 
$$\mathcal{H}_c(U(s' + \omega, s')) = U(s', s)\mathcal{H}_c(U(s + \omega, s)).$$

2. We mention here that an extension of the Ruelle theorem to another class of time dependent Hamiltonians is obtained by Hagedorn [15].

We sketch here the outline of the proof. The key idea is the introduction of a big Hilbert space  $\mathcal{H} = L^2(\mathbb{T}_\omega) \otimes \mathcal{H}$ ,  $\mathbb{T}_\omega = \mathbb{R}/\omega\mathbb{Z}$ , and the « grand generator »  $K = -i\partial/\partial t + H(t)$  for the equation (1.1) (cf. Howland [3], [4], Yajima [13], [14]). We first observe in Section 2 that, in general, if  $J$  is a multiplicative operator  $(Jf)(t) = J(t)f(t)$  and  $J(K - z)^{-1}$  is a compact operator then for any  $u \in \mathcal{H}_c(U(s + \omega, s))$ ,  $\lim_{T \rightarrow \pm\infty} \frac{1}{T} \int_0^T \|J(t)U(t, s)u\|^2 dt = 0$  (Theorem 2.4).

This reduces the proof of the theorem to showing that the operator  $J = \mathbb{1} \otimes F(|x| \leq R)$ ,  $F(|x| \leq R)$  being the multiplication operator by the characteristic function of  $|x| \leq R$ , is  $-i\partial/\partial t - \Delta + V(t)$  relatively compact. We shall give in Section 3 an abstract condition for  $V(t)$  so that  $F(|x| \leq R)(K - z)^{-1}$  is compact. In Section 4 we check that the potentials we are considering indeed satisfy this condition.

The following notation will be used. For a Hilbert space  $\mathcal{X}$ ,  $\mathcal{B}(\mathcal{X})$  is the Banach algebra of all bounded operators on  $\mathcal{X}$ . For a selfadjoint operator  $T$ ,  $\{E(\cdot, T)\}$  is the spectral measure for  $T$ .  $\mathcal{D}(\mathbb{R}^n)$  is  $C_0^\infty(\mathbb{R}^n)$  endowed with the usual topology.

## §2. PERIODIC PROPAGATORS AND THEIR GRAND GENERATORS

In this section  $\mathcal{H}$  is an arbitrary separable Hilbert space.

**DEFINITION 2.1.** — A family of operators  $\{U(t, s), -\infty < t, s < \infty\}$  on  $\mathcal{H}$  is called a Strongly Continuous Periodic Unitary Propagator (SCPUP) with period  $\omega > 0$  when it satisfies the following properties:

- i) For any  $(t, s) \in \mathbb{R}^2$ ,  $U(t, s)$  is a unitary operator on  $\mathcal{H}$ .
- ii) The function  $\mathbb{R}^2 \ni (t, s) \rightarrow U(t, s)$  is strongly continuous.
- iii) For any  $-\infty < t, s, r < \infty$ ,  $U(t, r)U(r, s) = U(t, s)$ .
- iv) For any  $(t, s) \in \mathbb{R}^2$ ,  $U(t + \omega, s + \omega) = U(t, s)$ .

For a SCPUP  $\{U(t, s), -\infty < t, s < \infty\}$ , we set an auxiliary Hilbert space  $\mathcal{K} = L^2(\mathbb{T}_\omega, \mathcal{H}) = L^2(\mathbb{T}_\omega) \otimes \mathcal{H}$ , where  $\mathbb{T}_\omega = \mathbb{R}/\omega\mathbb{Z}$  is equipped with the natural Lebesgue measure. We define a one parameter family of operators  $\{\mathcal{U}(\sigma), -\infty < \sigma < \infty\}$  on  $\mathcal{K}$  by the equation

$$(2.1) \quad \mathcal{U}(\sigma)f(t) = U(t, t - \sigma)f(t - \sigma), \quad f \in \mathcal{K}.$$

$\{\mathcal{U}(\sigma)\}$  is a strongly continuous unitary group on  $\mathcal{K}$ . Hence by Stone's theorem there exists a unique selfadjoint operator  $K$  on  $\mathcal{K}$  such that

$$(2.2) \quad \mathcal{U}(\sigma) = \exp(-i\sigma K), \quad -\infty < \sigma < \infty$$

(see Howland [3] and Yajima [13], [14]).

**DEFINITION 2.2.** — The selfadjoint operator  $K$  on  $\mathcal{K}$  defined as above is called the grand generator of  $\{U(t, s)\}$ .

**REMARK 2.3.** — If  $U(t, s)$  is the SCPUP obtained by solving the equation (1.1),  $K$  is formally given as  $K = -i\partial/\partial t + H(t)$ .

The following theorem which will be used in Section 3 and 4 for proving the theorem is interesting for its own sake.

**THEOREM 2.4.** — Let  $\{U(t, s), -\infty < t, s < \infty\}$  be a SCPUP on  $\mathcal{H}$  with period  $\omega$  and  $K$  be its grand generator on  $\mathcal{K} = L^2(\mathbb{T}_\omega, \mathcal{H})$ . Let  $J(t)$  be a bounded strongly measurable  $\mathcal{B}(\mathcal{H})$ -valued function of  $t \in \mathbb{T}_\omega$  and  $J$  be the bounded operator on  $\mathcal{K}$  defined as  $(Jf)(t) = J(t)f(t)$ ,  $f \in \mathcal{K}$ . Suppose that  $JE(\Delta, K)$  is a compact operator for any compact set  $\Delta \subset \mathbb{R}^1$ . Then for any  $s \in \mathbb{R}^1$  and  $u \in \mathcal{H}_c(U(s + \omega, s))$ ,

$$(2.3) \quad \lim_{T \rightarrow \pm\infty} \frac{1}{T} \int_0^T \|J(t)U(t, s)u\|^2 dt = 0$$

and

$$(2.4) \quad \lim_{N \rightarrow \pm\infty} \frac{1}{N} \sum_{k=0}^N \|J(s)U(s + \omega, s)^k u\|^2 = 0.$$

*Proof.* — We shall prove (2.3) for the case  $T \rightarrow \infty$  only. The other case may be proved by changing a few signs in the following expressions and the proof for (2.4) is similar. By the assumption on  $J$ , it is well-known (cf. Reed-Simon [10], p. 341) that for  $f \in \mathcal{H}_c(\mathbf{K})$

$$(2.5) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|J e^{-i\sigma K} f\|_{\mathcal{H}}^2 d\sigma = 0.$$

Rewriting (2.5) by using the definition (2.1) and (2.2) of  $e^{-i\sigma K}$ , we have

$$(2.6) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left\{ \int_{\mathbb{T}_\omega} \|J(t)U(t, t - \sigma)f(t - \sigma)\|^2 dt \right\} d\sigma \\ = \lim_{T \rightarrow \infty} \int_{\mathbb{T}_\omega} \left\{ \frac{1}{T} \int_0^T \|J(t + \sigma)U(t + \sigma, t)f(t)\|_{\mathcal{H}}^2 d\sigma \right\} dt = 0.$$

Note that if  $\mathcal{U}_s$  is the unitary operator on  $\mathcal{H}$  defined by

$$(\mathcal{U}_s f)(t) = U(t, s)f(t) \quad \text{for } s \leq t < s + \omega$$

and by the periodicity elsewhere,

$$(2.7) \quad e^{-i\omega K} = \mathcal{U}_s(\mathbb{1} \otimes U(s + \omega, s))\mathcal{U}_s^*.$$

Hence if  $u \in \mathcal{H}_c(U(s + \omega, s))$ ,  $f \in \mathcal{H}$  defined by  $f(t) = U(t, s)u$  for  $s \leq t < s + \omega$  and by the periodicity elsewhere belongs to  $\mathcal{H}_c(\mathbf{K})$ . Thus by (2.6) we have for  $u \in \mathcal{H}_c(U(s + \omega, s))$

$$(2.7) \quad \lim_{T \rightarrow \infty} \int_s^{s+\omega} \left\{ \frac{1}{T} \int_0^T \|J(t + \sigma)U(t + \sigma, s)u\|^2 d\sigma \right\} dt = 0.$$

Therefore for any sequence  $T_n \rightarrow \infty$  there exists a subsequence  $T'_n \rightarrow \infty$  such that

$$(2.8) \quad \lim_{T'_n \rightarrow \infty} \frac{1}{T'_n} \int_0^{T'_n} \|J(t + \sigma)U(t + \sigma, s)u\|^2 d\sigma = 0, \text{ a. e. } t \in (s, s + \omega).$$

However (2.8) holds for all  $t \in [s, s + \omega]$  since the limit in the LHS of (2.8) is independent of  $t$ . This obviously implies (2.3) for  $T \rightarrow \infty$ .

### § 3. TIME PERIODIC HAMILTONIANS

In this and the next section  $\mathcal{H} = L^2(\mathbb{R}^n)$  and we consider SCPUPs  $\{U(t, s)\}$  which are generated by Schrödinger equations with time periodic Hamiltonians:

$$(3.1) \quad i\partial u/\partial t = (-\Delta + V(t))u \equiv H(t)u, \quad V(t + \omega) = V(t), \quad \omega > 0.$$

In this section we give a sufficient condition on the potentials  $V(t)$  and  $J(t)$  of Section 1 for  $J(\mathbf{K} - z)^{-1}$  to be compact in a rather abstract form. In

the next section, we shall show that this condition is satisfied by a wide class of potentials and  $J(t) = F(|x| \leqq R)$ .

We assume the following condition.  $H^s(\mathbb{R}^n)$  is the Sobolev space of order  $s$  and for a densely defined operator  $T$ ,  $T^*$  denotes its adjoint.  $H_0 = -\Delta$  with the domain  $D(H_0) = H^2(\mathbb{R}^n)$ .

ASSUMPTION (C). — *i*) There exist densely defined closed operators (on  $\mathcal{H}$ )  $A(t)$  and  $B(t)$  which satisfy the following conditions:

*ia*)  $A(t + \omega) = A(t)$ ,  $B(t + \omega) = B(t)$  and  $V(t) = A(t)^*B(t)$ ,  $t \in \mathbb{R}^1$ .

*ib*)  $A(t)(H_0 + 1)^{-1/2}$  and  $B(t)(H_0 + 1)^{-1/2}$  are  $\mathcal{B}(\mathcal{H})$ -valued measurable functions of  $t \in \mathbb{R}^1$ .

*ic*) 
$$c_1 = \int_0^\omega \|A(t)(H_0 + 1)^{-1/2}\| dt < \infty$$

and

$$c_2 = \int_0^\omega \|B(t)(H_0 + 1)^{-1/2}\| dt < \infty.$$

*ii*) Let  $\mathcal{A}$  and  $\mathcal{B}$  be the operators on  $\mathcal{H}$  defined by  $(\mathcal{A}f)(t) = A(t)f(t)$  and  $(\mathcal{B}f)(t) = B(t)f(t)$  and  $K_0 = (-i\partial/\partial t) \otimes \mathbb{1} + \mathbb{1} \otimes H_0$ . Then

*iiia*)  $\mathcal{A}(K_0 - z)^{-1} \in \mathcal{B}(\mathcal{H})$  and  $\mathcal{B}(K_0 - z)^{-1} \in \mathcal{B}(\mathcal{H})$  for any  $z \in \mathbb{C} \setminus \mathbb{R}$ .

*iiib*)  $\mathcal{B}(K_0 - z)^{-1}\mathcal{A}^*$  can be extended to a compact operator  $Q(z)$ .

*iiic*)  $\|Q(z)\|_{\mathcal{B}(K)} \rightarrow 0$  as  $\text{Im}z \rightarrow \pm\infty$ .

*iiid*)  $J(K_0 - z)^{-1}$  and  $J[\mathcal{A}(K_0 - \bar{z})^{-1}]^*$  are compact operators for some  $z \notin \mathbb{R}$ .

*iv*) Equation (3.1) generates the SCPUP  $\{U(t, s)\}$  which satisfies the following properties:

*iva*)  $U(t, s)H^1(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$ .

*ivb*) If  $f \in H^1(\mathbb{R}^n)$ ,  $U(t, s)f$  is  $H^{-1}(\mathbb{R}^n)$ -valued strongly differentiable and

$$\begin{aligned} (i\partial/\partial t)U(t, s)f &= H(t)U(t, s)f, \\ (i\partial/\partial s)U(t, s)f &= -U(t, s)H(s)f. \end{aligned}$$

THEOREM 3.1. — Let Assumption (C) be satisfied and let  $K$  be the grand generator of  $\{U(t, s)\}$ . Then  $J(K - z)^{-1}$  is a compact operator on  $\mathcal{H}$  for any  $z \notin \mathbb{R}$  and the equation (2.3) and (2.4) hold.

REMARK 3.2. — It might be the case that Assumption (C) (*iv*) could be derived from (C)(*i*)  $\sim$  (*ii*) (or with some additional assumptions). However we ignore this question here, since it is largely irrelevant to the main question being considered here though it is an important and non-trivial question (cf. Howland [3], [4]).

For proving the theorem, we need the following

PROPOSITION 3.3. — Let Assumption (C) be satisfied and let  $K$  be the

grand generator of  $\{ U(t, s) \}$ . Then for any  $z \notin \mathbb{R}$ ,  $1 + Q(z)$  is invertible and

$$(3.2) \quad (K - z)^{-1} = (K_0 - z)^{-1} - [\mathcal{A}(K_0 - \bar{z})^{-1}]^*(1 + Q(z))^{-1}\mathcal{B}(K_0 - z)^{-1}.$$

*Proof.* — The proof is essentially a repetition of that of Lemma 3.3 of Yajima [13]. For  $f, g \in C^\infty(\mathbb{T}_\omega, \mathcal{D}(\mathbb{R}^n))$  we have by integrating (ivb) that

$$(3.3) \quad (U(t, s)f(r), g(r))_{\mathcal{H}} = (e^{-i(t-s)H_0}f(r), g(r))_{\mathcal{H}} - i \int_s^t (B(\rho)U(\rho, s)f(r), A(\rho)e^{-i(\rho-t)H_0}g(r))_{\mathcal{H}} d\rho.$$

By the definition of grand generators (3.3) implies

$$(3.4) \quad ((e^{-i\sigma K}f)(t), g(t))_{\mathcal{H}} = ((e^{-i\sigma K_0}f)(t), g(t))_{\mathcal{H}} - i \int_0^\sigma ((\mathcal{B}e^{-i\rho K}f)(\rho + (t - \sigma)), (\mathcal{A}e^{-i(\rho - \sigma)K_0}g)(\rho + (t - \sigma)))_{\mathcal{H}} d\rho.$$

Integrating (3.4) over  $\mathbb{T}_\omega$  with respect to  $t$ , we obtain du Hamel's equation

$$(3.5) \quad (e^{-i\sigma K}f, g)_{\mathcal{H}} = (e^{-i\sigma K_0}f, g)_{\mathcal{H}} - i \int_0^\sigma (\mathcal{B}e^{-i\rho K}f, \mathcal{A}e^{-i(\rho - \sigma)K_0}g)_{\mathcal{H}} d\rho.$$

Taking into account that (iva) and the definition of SCPUP imply  $\|U(t, t - \sigma)\|_{H^1 \rightarrow H^1} \leq Ce^{D|\sigma|}$  for some  $C > 0$  and  $D > 0$ , we take the Laplace transform of both sides of (3.5) for  $|\operatorname{Im} z| > D$  to obtain

$$((K - z)^{-1}f, g) = ((K_0 - z)^{-1}f, g) - (\mathcal{B}(K - z)^{-1}f, \mathcal{A}(K_0 - \bar{z})^{-1}g).$$

Since  $\mathcal{A}(K_0 - \bar{z})^{-1}$  is bounded, this implies

$$(3.6) \quad (K - z)^{-1}f = (K_0 - z)^{-1}f - [\mathcal{A}(K_0 - \bar{z})^{-1}]^*\mathcal{B}(K - z)^{-1}f, \quad f \in C^\infty(\mathbb{T}_\omega, \mathcal{D}).$$

Since  $\mathcal{B}[\mathcal{A}(K_0 - \bar{z})^{-1}]^* = Q(z) \in \mathcal{B}(\mathcal{H})$  (cf. Kato [5], p. 263) we have from (3.6) that  $(1 + Q(z))\mathcal{B}(K - z)^{-1}f = \mathcal{B}(K_0 - z)^{-1}f$ . Since  $(1 + Q(z))^{-1}$  exists for large  $|\operatorname{Im} z|$  by (iic), this implies

$$(3.7) \quad \mathcal{B}(K - z)^{-1} = (1 + Q(z))^{-1}\mathcal{B}(K_0 - z)^{-1} \in \mathcal{B}(\mathcal{H}) \text{ for large } |\operatorname{Im} z| > 0.$$

The (first) resolvent equation and (3.7) imply  $\mathcal{B}(K - z)^{-1} \in \mathcal{B}(\mathcal{H})$  for all  $z \notin \mathbb{R}$  and the (second) resolvent equation

$$(3.8) \quad (K - z)^{-1} = (K_0 - z)^{-1} - [\mathcal{A}(K_0 - \bar{z})^{-1}]^*\mathcal{B}(K - z)^{-1}$$

hold for  $z \notin \mathbb{R}$ . Then it is easy to see (cf. Proof of Lemma 2.12 of Kuroda [7]) that for all  $z \notin \mathbb{R}$   $(1 + Q(z))^{-1}$  exists and (3.7) holds. Inserting (3.7) into (3.8), we obtain (3.2).

*Proof of Theorem 3.1.* — Multiplying  $J$  to the both sides of (3.2), we have  $J(K - z)^{-1} = J(K_0 - z)^{-1} - J[\mathcal{A}(K_0 - \bar{z})^{-1}]^*(1 + Q(z))^{-1}\mathcal{B}(K_0 - z)^{-1}$ . Thus by Assumption (C) iii),  $J(K - z)^{-1}$  is a compact operator and (2.3) and (2.4) follow from Theorem 2.4.



§ 4. CONDITIONS ON  $V(t)$   
AND THE PROOF OF THEOREM

We assume that the potential  $V(t)$  satisfies one of the following assumptions.  $\langle x \rangle = (1 + |x|^2)^{1/2}$ .

ASSUMPTION (A). — There exist operator valued functions  $V_1(t)$  and  $V_2(t)$  which satisfy the following properties:

- i)  $V_j(t)$  ( $j = 1, 2$ ) is a symmetric operator.
- ii)  $V_1(t)$  is a  $\mathcal{B}(\mathcal{H})$ -valued  $C^1$ -function of  $t \in \mathbb{T}_\omega$ .
- iii)  $V_2(t)$  is the multiplication operator by  $V_2(t, x)$ .

There exist  $1 \leq p < n/2 < q \leq \infty$  such that  $V_2(t, \cdot)$  is an  $L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ -valued  $C^1$ -function of  $t \in \mathbb{T}_\omega$ .

- iv)  $V(t) = V_1(t) + V_2(t)$ .

ASSUMPTION (B). — There exist operator valued functions  $V_1(t)$  and  $V_2(t)$  which satisfy the properties i) ii) iv) of Assumption (A) and iii') there exist constants  $0 < \delta < 1$  and  $\varepsilon > 0$  such that

$$D(t) \equiv \langle x \rangle^{\frac{1}{2} + \varepsilon} (H_0 + 1)^{-\delta/4} V_2(t) (H_0 + 1)^{-\delta/4} \langle x \rangle^{\frac{1}{2} + \varepsilon}$$

has a bounded extension and it is a  $\mathcal{B}(\mathcal{H})$ -valued  $C^1$ -function of  $t$ .

We now start proving the theorem. Along with Eq. (1.1) we consider the equation

$$(4.1) \quad i\partial u/\partial t = (-\Delta + V_2(t))u \equiv H_2(t)u.$$

By Kysinski's theorem ([9], cf. also [12]), (1.1) and (4.1) generate SCPUPs  $\{U(t, s)\}$  and  $\{U_2(t, s)\}$ , respectively, which satisfy iv) of Assumption (C). We write their grand generators as  $K$  and  $K_2$ .  $\mathcal{V}_1$  is the operator defined as  $(\mathcal{V}_1 f)(t) = V_1(t)f(t)$ ,  $f \in \mathcal{H}$ .  $\mathcal{V}_1 \in \mathcal{B}(\mathcal{H})$ .

LEMMA 4.1. — For any  $z \notin \mathbb{R}$ ,

$$(4.2) \quad (K - z)^{-1} = (K_2 - z)^{-1} (1 + \mathcal{V}_1(K_2 - z)^{-1})^{-1}.$$

*Proof.* — Since  $V_1(t) \in C^1(\mathbb{T}_\omega, \mathcal{B}(\mathcal{H}))$ , we have the du Hamel's equation

$$(4.3) \quad \begin{aligned} e^{-i\sigma K} f(t) &= U(t, t - \sigma) f(t - \sigma) = U_2(t, t - \sigma) f(t - \sigma) \\ &- i \int_0^\sigma U(t, t - \sigma + \rho) V_1(t - \sigma + \rho) U_2(t - \sigma + \rho, t - \sigma) f(t - \sigma) d\rho \\ &= e^{-i\sigma K_2} f(t) - i \int_0^\sigma e^{-i(\sigma - \rho)K} \mathcal{V}_1 e^{-i\rho K_2} f(t) d\rho. \end{aligned}$$

Taking the Laplace transform of both sides of (4.3), we have

$$(4.4) \quad (K - z)^{-1} = (K_2 - z)^{-1} - (K - z)^{-1} \mathcal{V}_1 (K_2 - z)^{-1}.$$

Since  $K$  and  $K_2$  are selfadjoint and  $\mathcal{V}_1$  is bounded, (4.2) follows from (4.4) by a standard argument.

**LEMMA 4.2.** — Let  $0 < \gamma < 1/2$  and  $\delta > 1/2$ . Let  $K_0 = (-i\partial/\partial t) \otimes \mathbb{1} + \mathbb{1} \otimes H_0$  on  $\mathcal{H} = L^2(\mathbb{T}_\omega, \mathcal{H})$ . Then  $\langle x \rangle^{-\delta} (H_0 + 1)^{\gamma/2} |(K_0 - z)^{-1}|^{1/2}$  is a compact operator in  $\mathcal{H}$  for all  $z \notin \mathbb{R}$  and

$$(4.5) \quad \lim_{|\operatorname{Im} z| \rightarrow \infty} \| \langle x \rangle^{-\delta} (H_0 + 1)^{\gamma/2} |(K_0 - z)^{-1}|^{1/2} \| = 0.$$

*Proof.* — Let us consider the operators  $T(n, z)$  ( $n=0, \pm 1, \pm 2, \dots, z \notin \mathbb{R}$ ) on  $\mathcal{H}$  defined as  $T(n, z) = |(H_0 - n - z)^{-1}|^{1/2} (H_0 + 1)^{\gamma/2} \langle x \rangle^{-\delta}$ . By Rellich's compactness theorem and a standard approximation to  $T(n, z)^*$ , it is easy to see that  $T(n, z)$  is a compact operator on  $\mathcal{H}$  for any  $z \notin \mathbb{R}$  and  $n \in \mathbb{Z}$ . We denote  $z = \lambda + i\mu$ . Since

$$\| T(n, z) \| \leq \| |(H_0 - n - z)^{-1}|^{1/2} (H_0 + 1)^{\gamma/2} \| = \| ((\xi^2 - n - \lambda)^2 + \mu^2)^{-1/4} (\xi^2 + 1)^{\gamma/2} \|_{L^\infty(\mathbb{R}^n)}$$

and

$$\sup_{n \leq N} \| ((\xi^2 - n - \lambda)^2 + \mu^2)^{-1/4} (\xi^2 + 1)^{\gamma/2} \|_{L^\infty} = \| ((\xi^2 - N - \lambda)^2 + \mu^2)^{-1/4} (\xi^2 + 1)^{\gamma/2} \|_{L^\infty} \rightarrow 0$$

as  $\mu \rightarrow \pm \infty$  (for fixed  $N$  and  $\lambda$ ) or  $N \rightarrow -\infty$  (for fixed  $\lambda$ ) it follows that for any  $\varepsilon > 0$

$$(4.6) \quad \lim_{\mu \rightarrow \pm \infty} \sup_{n \leq N} \| T(n, \lambda + i\mu) \| = 0 \quad N \in \mathbb{Z}, \quad \lambda \in \mathbb{R};$$

$$(4.7) \quad \lim_{N \rightarrow -\infty} \sup_{n \leq N} \sup_{|\mu| \geq \varepsilon} \| T(n, \lambda + i\mu) \| = 0, \quad \lambda \in \mathbb{R}.$$

When  $n$  is very large ( $n \geq \lambda + 1$ ), we take  $L$  and estimate

$$(4.8) \quad \| T(n, z)f \|^2 = \int ((\xi^2 - n - \lambda)^2 + \mu^2)^{-1/2} (\xi^2 + 1)^\gamma | \mathcal{F}(\langle x \rangle^{-\delta} f)(\xi) |^2 d\xi$$

by splitting the integral into  $D_1 = \{ \xi : L(\xi^2 + 1)^\gamma < | \xi^2 - n - \lambda | \}$  and  $D_2 = \mathbb{R}^n \setminus D_1$ . Here  $\mathcal{F}$  is the Fourier transform and we used Plancherel's formula. We write  $\hat{f}_\delta(\xi) = \mathcal{F}(\langle x \rangle^{-\delta} f)(\xi)$ . Since  $\delta > 1/2$ ,

$$(4.9) \quad \sup_{\rho > 0} \rho^{n-1} \int_{S^{n-1}} | \hat{f}_\delta(\rho\omega) |^2 d\omega \leq C_\delta \| f \|^2$$

by Sobolev's embedding theorem (cf. Kuroda [8], p. 4.26).

Clearly

$$(4.10) \quad \int_{D_1} ((\xi^2 - n - \lambda)^2 + \mu^2)^{-1/2} (\xi^2 + 1)^\gamma | \hat{f}_\delta(\xi) |^2 d\xi \leq L^{-1} \| f \|^2.$$

Writing  $M = n + \lambda$ , we see easily that then  $LM^{\gamma-1} < 1/4$  and  $M > 2$ ,  $D_2 \subset D_3 = \{M - 3LM^\gamma < \xi^2 < M + 3LM^\gamma\}$ . Thus

$$\begin{aligned}
 (4.11) \quad & \int_{D_2} ((\xi^2 - M)^2 + \mu^2)^{-1/2} (\xi^2 + 1)^\gamma \widehat{f}_\delta(\xi)^2 d\xi \\
 & \leq (2M+1)^\gamma \int_{(M-3LM^\gamma)^{1/2}}^{(M+3LM^\gamma)^{1/2}} d\rho \left\{ ((\rho^2 - M)^2 + \mu^2)^{-1/2} \rho^{n-1} \int_{S^{n-1}} |f_\delta(\rho\omega)|^2 d\omega \right\} \\
 & \leq 3C_\delta M^{\gamma-1/2} \|f\|^2 \int_{(M-3LM^\gamma)^{1/2}}^{(M+3LM^\gamma)^{1/2}} ((\rho^2 - M)^2 + \mu^2)^{-1/2} 2\rho d\rho \\
 & \leq 3C_\delta M^{\gamma-1/2} \|f\|^2 \int_{-3LM^\gamma}^{3LM^\gamma} (\rho^2 + \mu^2)^{-1/2} d\rho \\
 & \leq (6C_\delta M^{\gamma-1/2} \int_0^{M/\mu} (\rho^2 + 1)^{-1/2} d\rho) \|f\|^2.
 \end{aligned}$$

Combining (4.8), (4.10) and (4.11), we have that for any  $L > 0$

$$\|T(n, \lambda + i\mu)\| \leq 2(L^{-1} + 6C_\delta M^{\gamma-1/2} \int_0^{M/\mu} (\rho^2 + 1)^{-1/2} d\rho)^{1/2}$$

provided  $M = n + \lambda > 2$  and  $M^{\gamma-1} < (4L)^{-1}$ . Thus for any  $\varepsilon > 0$

$$(4.12) \quad \limsup_{n \rightarrow \infty} \sup_{|\mu| > \varepsilon} \|T(n, \lambda + i\mu)\| = 0, \quad \lambda \in \mathbb{R}.$$

Notice now that identifying  $l^2(\mathbb{Z}, \mathcal{H}) = \bigoplus_{-\infty}^{\infty} \mathcal{H}$ , we have by the Fourier transform with respect to  $t$ ,

$$(4.13) \quad \mathcal{F}_t \langle x \rangle^{-\delta} (H_0 + 1)^{\gamma/2} |K_0 - z|^{-1/2} \mathcal{F}_t^{-1} = \bigoplus_{n=-\infty}^{\infty} T(n, \bar{z})^*.$$

(4.6), (4.7), (4.12) and (4.13) imply that

$$\langle x \rangle^{-\delta} (H_0 + 1)^{\gamma/2} |K_0 - z|^{-1/2}$$

is compact since each  $T(n, \bar{z})^*$  is compact and that

$$\lim_{\mu \rightarrow \pm \infty} \| \langle x \rangle^{-\delta} (H_0 + 1)^{\gamma/2} |K_0 - z|^{-1/2} \| = \lim_{\mu \rightarrow \pm \infty} \sup_n \|T(n, \bar{z})\| = 0.$$

This completes the proof of the lemma.

### Completion of the proof of Theorem

Let us write  $\mathcal{H}_b(s) = \{u \in \mathcal{H} : \text{Property (1.b) is satisfied}\}$ ,

$$\mathcal{H}'_b(s) = \{u \in \mathcal{H} : \text{Property (1.c) is satisfied}\},$$

$$\mathcal{H}_{sc}(s) = \{u \in \mathcal{H} : \text{Property (2.b) is satisfied}\}$$

and  $\mathcal{H}'_{sc}(s) = \{ u \in \mathcal{H} : \text{Property (2.c) is satisfied} \}$ .

We only show that

$$\mathcal{H}_p(s) \equiv \mathcal{H}_p(U(s + \omega, s)) = \mathcal{H}_b(s)$$

and

$$\mathcal{H}_c(s) \equiv \mathcal{H}_c(U(s + \omega, s)) = \mathcal{H}_{sc}(s)$$

since the proof for  $\mathcal{H}_p(s) = \mathcal{H}'_b(s)$  and  $\mathcal{H}_c(s) = \mathcal{H}'_{sc}(s)$  is similar.

Since  $\mathcal{H}_p(s) \oplus \mathcal{H}_c(s) = \mathcal{H}$ , it clearly suffices to show that

$$(4.14) \quad \mathcal{H}_b(s) \text{ is orthogonal to } \mathcal{H}_{sc}(s);$$

$$(4.15) \quad \mathcal{H}_p(s) \subset \mathcal{H}_b(s);$$

$$(4.16) \quad \mathcal{H}_c(s) \subset \mathcal{H}_{sc}(s).$$

Let  $u \in \mathcal{H}_b(s)$  and  $v \in \mathcal{H}_{sc}(s)$ . For any  $\varepsilon > 0$ , there exists  $R > 0$  such that (1.b) is satisfied. Then

$$(4.17) \quad \begin{aligned} |(u, v)| &= |(U(t, s)u, U(t, s)v)| = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |(U(t, s)u, U(t, s)v)| dt \\ &\leq \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \{ \|U(t, s)u\|_{L^2(|x| \geq R)} \|v\| + \|u\| \|U(t, s)v\|_{L^2(|x| \leq R)} \} dt \\ &\leq \varepsilon \|u\| \|v\|. \end{aligned}$$

Since  $\varepsilon > 0$  can be chosen arbitrarily small, (4.17) implies  $(u, v) = 0$  and this implies (4.14).

Suppose  $u \in \mathcal{H}_p(s)$  now. For any  $\varepsilon > 0$ , there exists a finite set of eigenfunctions  $u_j, j = 1, \dots, N$ , of  $U(s + \omega, s)$  with eigenvalues  $e^{-i\omega\lambda_j}$  such that

$$\left\| u - \sum_{j=1}^N u_j \right\| < (\varepsilon/2) \|u\|.$$

Since  $e^{i\lambda_j t} U(t, s)u_j$  is a periodic function of  $t$ ,  $\{ U(t, s)u_j : -\infty < t < \infty \}$  forms a precompact subset of  $\mathcal{H}$  and hence so does

$$\left\{ U(t, s) \sum_{j=1}^N u_j : -\infty < t < \infty \right\}.$$

Since  $F(|x| \leq R)$  converges strongly to the identity operator in  $\mathcal{H}$ , it converges uniformly on the precompact set

$$\left\{ U(t, s) \sum_{j=1}^N u_j : -\infty < t < \infty \right\}.$$

Hence one can choose  $R > 0$  such that

$$\inf_t \left\| U(t, s) \sum_{j=1}^N u_j \right\|_{L^2(|x| \leq R)} \geq (1 - \varepsilon/2) \left\| \sum_{j=1}^N u_j \right\|.$$

Thus

$$\inf_t \| U(t, s)u \|_{L^2(|x| \leq R)} \geq \inf_t \left\| U(t, s) \sum_{j=1}^N u_j \right\| - \left\| u - \sum_{j=1}^N u_j \right\| > (1 - \varepsilon) \| u \|.$$

This implies  $u \in \mathcal{H}_b(s)$  and (4.15) is proved. Finally we prove (4.16). We set  $J = \mathbb{1} \otimes F(|x| \leq R)$ . Once we prove  $J(K_2 - z)^{-1}$  is compact, Lemma 4.1 implies  $J(K - z)^{-1}$  is compact and hence by Theorem 2.4, (4.16) follows. To show  $J(K_2 - z)^{-1}$  is compact, we apply Proposition 3.3 replacing  $K$  by  $K_2$  and taking  $J = \mathbb{1} \otimes F(|x| \leq R)$ . Thus it suffices to check that Assumption (C) is satisfied with  $V_2(t)$  in place of  $V(t)$ . We already remarked that (C) *iv*) is satisfied.

*i*) The case when  $V(t)$  satisfies Assumption (A). We set  $A(t) = |V_2(t, x)|^{1/2}$  and  $B(t) = |V_2(t, x)|^{1/2} \text{ sign } V_2(t, x)$ . Then (C) *i*) follows by Sobolev's embedding theorem, *ii*) is proved in Lemma 3.1 Yajima [13] and since  $F(|x| \leq R)$  is no worse than  $B(t)$ ,  $J[\mathcal{A}(K_0 - \bar{z})^{-1}]^*$  is compact. Compactness of  $F(|x| \leq R)(K_0 - z)^{-1}$  follows immediately from Lemma 2.8 of Yajima [14].

*ii*) The case when  $V(t)$  satisfies Assumption (B). We set

$$A(t) = \langle x \rangle^{-\frac{1}{2} - \varepsilon} (H_0 + 1)^{\delta/4}$$

$$\text{and } B(t) = D(t) \langle x \rangle^{-\frac{1}{2} - \varepsilon} (H_0 + 1)^{\delta/4}.$$

Assumption (C) *i*) is clearly satisfied. By Lemma 4.2, (C) *ii*) and (C) *iii*) hold obviously. This completes the proof of the theorem.

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