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Existence, uniqueness and iterative construction of motions of charged particles with retarded interactions (*)

by

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ABSTRACT. — The paper provides an existence and uniqueness proof and an iterative construction with an error estimate for the solution of a system of truncated equations of motion for the scattering problem of classical electrodynamics with initial conditions in the infinite past. It is assumed that the fields produced by the particles are given by the velocity-dependent terms of the retarded Liénard-Wiechert fields and that each particle moves according to the Lorentz force corresponding to the field produced by the other particle or particles. The acceleration-dependent parts of the retarded Liénard-Wiechert fields as well as the radiation reaction terms are disregarded. It is proved that one can choose, as initial values, the three components of the velocity of each particle and, in addition, three real numbers for each particle which can be looked at as initial values for the location of the particle in the infinite past.

RÉSUMÉ. — On donne une preuve d'existence et d'unicité et une construction itérative avec estimation d'erreur de la solution d'un système d'équations du mouvement tronquées pour la diffusion en électrodynamique classique avec conditions initiales à l'infini dans le passé. On suppose que les champs produits par les particules sont donnés par les termes dépendant de la vitesse des champs retardés de Liénard-Wiechert et que chaque particule se meut sous l'action de la force de Lorentz associée au champ produit par les autres particules. On omet les termes dépendant de l'accé-

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lération des champs retardés de Liénard-Wiechert, ainsi que les termes de réaction du rayonnement. On prouve qu'il est possible de choisir comme valeurs initiales, les trois composantes de la vitesse de chaque particule et en outre, pour chaque particule, trois nombres réels qu'on peut considérer comme les valeurs initiales de la position de cette particule à l'infini dans le passé.

§ 0. INTRODUCTION

In this paper we treat some mathematical aspects of the scattering problem for a system of classical point particles with retarded interactions. Since the main purpose is to handle retardations rigorously, a simplified form of electromagnetic interaction is used: each particle is taken to move according to the Lorentz force due to the velocity-dependent, r^{-2}-parts of the retarded Liénard-Wiechert fields produced by the other particles; the acceleration-dependent, r^{-1}-parts of the fields as well as the radiation reaction forces are disregarded.

The equation of motion for a particle of charge q and mass m moving in an electric field $\vec{E}$ and a magnetic induction $\vec{B}$ in an inertial frame is

$$\vec{a} = \frac{q}{m}\sqrt{1 - |\vec{v}|^2}(\vec{E} - (\vec{E}\cdot\vec{v})\vec{v} + \vec{v} \times \vec{B}) \tag{0.1}$$

where $\vec{v}$ is the velocity and $\vec{a}$ the acceleration of the particle at the time t considered (we take $c = 1$). Let $\vec{x}$ be its position at time t. If the electromagnetic field is produced by another particle of charge $\tilde{q}$ which had the position $\vec{y}$ and the velocity $\vec{w}$ at the retarded time $t - \tau$ where τ is the light-travel time of a light signal sent from the second particle and arriving at the first particle at time t then the velocity-dependent parts of $\vec{E}$ and $\vec{B}$ are given by

$$\vec{E} = \tilde{q}\,\frac{1 - |\vec{w}|^2}{[\tau - (\vec{x} - \vec{y})\cdot\vec{w}]^3}\,(\vec{x} - \vec{y} - \tau\vec{w}) \tag{0.2}$$

$$\vec{B} = \frac{1}{\tau}\cdot(\vec{x} - \vec{y}) \times \vec{E} \qquad \text{(where we have taken } 4\pi\varepsilon_0 = 1\text{)}.$$

From (0.2) follows $(\vec{x} - \vec{y} - \tau\vec{w}) \times \vec{E} = 0$. Thus

$$\vec{B} = \vec{w} \times \vec{E} \tag{0.3}$$

If there are more than two particles we have to add up all contributions from the other particles.

We shall prove (theorem 4 in § 4) that for every solution of these equations for which the distances between the particles grow at least linearly for $t \to -\infty$ and the speeds of the particles are bounded away from the speed of light there are, for each of the particles, three vector $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ such that for the position $\vec{x}(t)$ of the particle at time t we have

$$\lim_{t \to -\infty} (\vec{x}(t) - [\vec{\alpha} \cdot t + \vec{\beta} \cdot \ln(-t) + \vec{\gamma}]) = 0 \tag{0.4}$$

$$\lim_{t \to -\infty} \vec{x}'(t) = \vec{\alpha} \tag{0.5}$$

The vector $\vec{\alpha}$ can be regarded as initial value in the infinite past for the velocity of that particle. $\vec{\beta}$ is uniquely determined by the initial values for the velocities of the particles. We may regard $\vec{\gamma}$ as a substitute for the initial value of the position of the particle under consideration. Further we establish in theorem 4 that for every choice of initial values $(\vec{\alpha}, \vec{\gamma})$ such that no two velocities are equal to each other there is, for sufficiently early times, a unique solution of the equations of motion which obeys the initial conditions (0.4) and (0.5). We give an iterative construction of the solution and an error estimate.

The idea of the proof is the following: The motion of the particles can be described by a function x giving for each time t the corresponding element $x(t)$ of configuration space. Then the equations of motion can be written as

$$x''(t) = \mathrm{F}(t, x) \tag{0.6}$$

where the second argument of F varies in a suitable function space. We choose a function x_0 which obeys the initial conditions and solves (0.6) « approximately for $t \to -\infty$ », i. e., $x_0''(t) - \mathrm{F}(t, x_0)$ has a certain fall-off behaviour for $t \to -\infty$. Then we can rewrite (0.6) together with the initial conditions as an integral equation

$$x = \mathrm{T}x$$

where

$$\mathrm{T}x(t) := x_0(t) + \int_{-\infty}^{t} \int_{-\infty}^{\tau} (\mathrm{F}(\sigma, x) - x_0''(\sigma)) d\sigma d\tau .$$

We prove that T is contractive in a suitable metric function space and has a unique fixed point (theorems 1 and 2). The proof is divided into four parts: In theorem 1 we treat the general type of equation (0.6) where F is a functional which is only assumed to obey a certain Lipschitz condition. Further we assume, in theorem 1, that a function x_0 as above exists. Under these assumptions and under certain assumptions about the domain of definition of F we prove existence and uniqueness of the solution of (0.6) and convergence of the iteration process. In theorem 2 we specialize our equation (0.6) to the equation

$$x''(t) = \mathrm{G}(p[x](t)) \tag{0.7}$$

where $p[x](t)$ is a tuple of real numbers containing all informations about positions and velocities of the particles at the time t and at the respective retarded times. G is a rather general force function which is assumed to have a certain fall-off behaviour. Also in this theorem we still have to make rather restrictive assumptions about the domain of definition of G and the set of functions in which we look for solutions. We drop these assumptions in theorem 3 where we assume G to have a fall-off behaviour, called normal fall-off. The existence of x_0 is not assumed but proved in theorem 3. Theorem 4 is the direct application of theorem 3 to our k-particle-problem (1).

The problem described here has, to my knowledge not been successfully treated before. Hsing proves in [*1*] an existence and uniqueness theorem for two particles in one space-dimension with positions and velocities of the particles given at $t = 0$. In [*2*] Driver proves a similar theorem for two identical particles, i. e., of equal mass and charge, and half retarded — half advanced interactions. He assumes that the particles move symmetrically to each other in one space — dimension and he gives data $x(0)=0$ and $x'(0) = 0$. In [*3*] Flume studies a system of two identical particles moving, in more than one space dimension, according to Diracs equations of motion. He gives, in the centre of mass-frame, the velocity of one of the particles in the infinite past, the impact parameter and the time at which the straight line connecting the two particles is at right angles to the velocities of the particles in the infinite past. He proves the existence of a space-symmetric solution for sufficiently large impact parameters.

§ 1. PRELIMINARIES

We consider our problem in a fixed inertial reference frame. The world lines of the particles are given by specifying the positions of the particles as functions of the time. If there are k particles we have k functions

$$x_1, \ldots, x_k : \mathrm{I} \to \mathbb{R}^3$$

where I is a subset of the set $\mathbb{R}$ of reals. As we want to give initial data in the infinite past we take I to be an interval not bounded from below, called an *initial interval.* We denote the number of space dimensions by n and do not restrict ourselves to $n = 3$. The functions $x_1, \ldots, x_k : \mathrm{I} \to \mathbb{R}^n$ are collected into one function $x: \mathrm{I} \to (\mathbb{R}^n)^k$. The acceleration of the *i-th* particle at the time t depends on the position and velocity of the *j-th* particle at the retarded time $t-\tau_{ij}[x](t)$ where $i, j \leqq k$ and $\tau_{ij}[x](t)$ denotes the appropriate light-travel time. Thus we have

$$\tau_{ij}[x](t) = | x_i(t) - x_j(t - \tau_{ij}[x](t)) | \tag{1.1}$$

(1) The main assumptions and assertions in the different stages of the proof (theorems 1 through 4) are summarized in a table at the end of this paper.

We denote the position and velocity of the *j-th* particle at time

$$t - \tau_{ij}[x](t) \quad \text{by} \quad z_{ij}[x](t) \text{ and } v_{ij}[x](t),$$

respectively, i. e.,

$$z_{ij}[x](t) := x_j(t - \tau_{ij}[x](t)) \tag{1.2}$$

$$v_{ij}[x](t) := x_j'(t - \tau_{ij}[x](t)) \tag{1.3}$$

The prime, ′, always means differentiation of a function whose domain of definition is a subset of $\mathbb{R}$ with respect to its argument. We take the above definitions to hold also for $i = j$, i. e.,

$$\tau_{ii}[x](t) = 0, \qquad z_{ii}[x](t) = x_i(t), \qquad v_{ii}[x](t) = x_i'(t).$$

We combine the so defined numbers into $k \times k$-matrices or k^2-tuples:

$$\tau[x](t) := (\tau_{ij}[x](t))_{i,j=1,\dots,k} \in \mathbb{R}^{k^2}$$
$$z[x](t) := (z_{ij}[x](t))_{i,j} \in (\mathbb{R}^n)^{k^2}$$
$$v[x](t) := (v_{ij}[x](t))_{i,j} \in (\mathbb{R}^n)^{k^2}$$

The triple

$$p[x](t) := (t, z[x](t), v[x](t)) \tag{1.4}$$

contains all informations about the positions and velocities of the particles at the time t and at the respective retarded times, and also about the light-travel times

$$\tau_{ij}[x](t) = |z_{ii}[x](t) - z_{ij}[x](t)| \tag{1.5}$$

The equations of motion of the particles are of the form

$$x''(t) = G(p[x](t)) \tag{0.7}$$

where $G : P \to (\mathbb{R}^n)^k$ is a given function and $P \subset \mathbb{R} \times (\mathbb{R}^n)^{k^2} \times (\mathbb{R}^n)^{k^2}$.

DEFINITION 1. — (Solution, unique solution, strictly unique solution). Let C be a condition imposed on x (e. g. an initial condition for $t \to -\infty$) and let I be an initial interval. Then a *solution* of (0.7) in I with the condition C is a function $x \in C^2(I, (\mathbb{R}^n)^k)$ such that (0.7) (2) and C hold. A solution x is said to be *unique* iff it is the only solution of (0.7) in I with condition C. x is said to be *strictly unique* iff, for every initial subinterval J of I, every solution of (0.7) in J with the condition C is the restriction of x to J. □

In the rest of this section we list a few properties of the functions $p[x](t)$. Since in the physical problem the speeds of the particles must be less than the speed of light, i. e., $|x_i'(t)| < 1$ for all i, it follows that $\tau_{ij}[x](t)$ is uniquely determined by (1.1). We shall make the assumption about x

(2) (0.7) is required to hold for all $t \in I$. Especially $p[x](t) \in P$ is required for all $t \in I$.

that $\sup_{t \in I} | x_i'(t) | < 1$ for all i which assures also the existence of $\tau_{ij}[x](t)$ (see Lemma 1).

LEMMA 1. — Let I be an initial interval and $\mu < 1$ a positive constant. Let $x \in C^1(I, (\mathbb{R}^n)^k)$ be such that $| x_i'(t) | \leqq \mu$ for all $t \in I$ and $i = 1, \ldots, k$. Then, for all $t \in I$ and $i, j = 1, \ldots, k$,

a) there is one and only one non-negative real number $\tau_{ij}[x](t)$ such that (1.1) holds

b) $$\frac{| x_i(t) - x_j(t) |}{1 + \mu} \leqq \tau_{ij}[x](t) \leqq \frac{| x_i(t) - x_j(t) |}{1 - \mu}$$

c) $$| v_{ij}[x](t) | \leqq \mu$$

where $z_{ij}[x](t)$ and $v_{ij}[x](t)$ are defined by (1.2) and (1.3). If $t \in \mathring{I}$, i. e., t is an inner point of I, and $\tau_{ij}[x](t) \neq 0$ then

d) $\tau_{ij}[x]$ is continuously differentiable at t and

$$\tau_{ij}[x]'(t) = \frac{(z_{ii}[x](t) - z_{ij}[x](t)) \cdot (v_{ii}[x](t) - v_{ij}[x](t))}{\tau_{ij}[x](t) - (z_{ii}[x](t) - z_{ij}[x](t)) \cdot v_{ij}[x](t)}$$

e) $z_{ij}[x]$ is continuously differentiable at t and

$$z_{ij}[x]'(t) = v_{ij}[x](t) \cdot (1 - \tau_{ij}[x]'(t))$$

For every $s, t \in I$ with $s \leqq t$ and every $i, j = 1, \ldots, k$ the following two inequalities hold:

f) $$-\frac{2\mu}{1 - \mu}(t - s) \leqq \tau_{ij}[x](t) - \tau_{ij}[x](s) \leqq \frac{2\mu}{1 + \mu} \cdot (t - s)$$

g) $$| z_{ij}[x](t) - z_{ij}[x](s) | \leqq \mu \frac{1 + \mu}{1 - \mu} \cdot (t - s)$$ □

Proof. — Let a function $\phi_t : \mathbb{R}_0^+ \to \mathbb{R}$ of the set of nonnegative reals into the set of reals be defined by

$$\phi_t(\tau) := | x_i(t) - x_j(t - \tau) | - \tau \quad \text{for} \quad \tau \in \mathbb{R}_0^+ .$$

Then ϕ_t is strictly monotonically decreasing and, at least where its value is positive, continuously differentiable and

$$-\mu - 1 \leqq \phi_t'(\tau) \leqq \mu - 1, \qquad \phi_t(0) = | x_i(t) - x_j(t) | .$$

This implies the existence of a $\tau > 0$ such that $\phi_t(\tau) \leqq 0$ and therefore (*a*) and (*b*). (*c*) is trivial.

The assertion (*d*) is a consequence of the implicit function theorem (see, e. g. [*4*], p. 206) applied to the map

$$(t, \tau) \mapsto \phi_t(\tau) : \{ (t, \tau) \in \mathbb{R} \times \mathbb{R} \mid t \in \mathring{I} \wedge \tau > 0 \wedge x_i(t) \neq x_j(t - \tau) \} \to \mathbb{R}$$

and of equation (1.1). (*e*) follows now immediately from (1.2).

In order to prove (f) and (g) let

$$A := \{ r \in \mathbb{R} \mid s \leqq r \leqq t \wedge \tau_{ij}[x](r) = 0 \} .$$

For $s < r < t \wedge r \notin A$, (d) and (e) hold with r instead of t. So if we set $\alpha_l := (z_{ii}[x](r) - z_{ij}[x](r)) \cdot v_{il}[x](r)$ for $l = i$ or $l = j$ then

$$\tau_{ij}[x]'(r) = \frac{\alpha_i - \alpha_j}{\tau_{ij}[x](r) - \alpha_j}$$

and $$| \alpha_l | = | z_{ii}[x](r) - z_{ij}[x](r) | \cdot | v_{il}[x](r) | \leqq \tau_{ij}[x](r) \cdot \mu$$

by (1.5) and (c). From this follows $-\dfrac{2\mu}{1-\mu} \leqq \tau_{ij}[x]'(r) \leqq \dfrac{2\mu}{1+\mu}$ and, by (e), $| z_{ij}[x]'(r) | < \mu \dfrac{1+\mu}{1-\mu}$ for all r such that $s < r < t$ and $r \notin A$. If A is empty then (f) and (g) follow immediately. Otherwise let $\tilde{t} := \sup A$ and $\tilde{s} := \inf A$. Then (f) and (g) hold for t and $\tilde{t}$ instead of t and s and also for $\tilde{s}$ and s instead of t and s. As $\tau_{ij}[x]$ is continuous and $\tilde{s}$, $\tilde{t} \in \bar{A}$ we have

$$\tau_{ij}[x](\tilde{s}) = \tau_{ij}[x](\tilde{t}) = 0 \quad \text{and} \quad z_{ij}[x](\tilde{s}) = x_j(\tilde{s}), \quad z_{ij}[x](\tilde{t}) = x_j(\tilde{t})$$

which implies that (f) and (g) hold also for $\tilde{t}$ and $\tilde{s}$ instead of t and s. Thus (f) and (g) hold. □

Lemma 2. — Let I be an initial interval and $\mu < 1$ a positive constant. Let $x, y \in C^1(I, (\mathbb{R}^n)^k)$ and $i, j \leqq k$ be such that $| x_j'(t) | \leqq \mu$ for all $t \in I$ and $\sup_{t \in I} | y_j'(t) | < 1$. Let $t \in I$ and $r := t - \tau_{ij}[y](t)$, $\alpha := \sup_{r \leqq s \leqq t} | x(s) - y(s) |$, $\beta := \sup_{r \leqq s \leqq t} | x'(s) - y'(s) |$ and $v := \sup_{s_1 < s_2 \leqq t} \dfrac{| x'(s_2) - x'(s_1) |}{s_2 - s_1}$.

Then

a) $$| \tau_{ij}[x](t) - \tau_{ij}[y](t) | \leqq \frac{2\alpha}{1 - \mu}$$

b) $$| z_{ij}[x](t) - z_{ij}[y](t) | \leqq \frac{1 + \mu}{1 - \mu} \alpha$$

c) $$| v_{ij}[x](t) - v_{ij}[y](t) | \leqq \beta + \frac{2\alpha v}{1 - \mu} \quad \text{if} \quad v < \infty$$ □

Proof. — Let ϕ_t be as in the proof of lemma 1. Then

$$\begin{aligned} | \phi_t(\tau_{ij}[y](t)) | = \big| \, | x_i(t) - x_j(t - \tau_{ij}[y](t)) | - \\ - | y_i(t) - y_j(t - \tau_{ij}[y](t)) | \, \big| \leqq | x_i(t) - y_i(t) | + \\ + | x_j(t - \tau_{ij}[y](t)) - y_j(t - \tau_{ij}[y](t)) | < 2\alpha \end{aligned}$$

and $\phi_t(\tau_{ij}[x](t)) = 0$. Since $\dfrac{\phi_t(\tau) - \phi_t(\sigma)}{\tau - \sigma} \leqq \mu - 1$ for all $\sigma, \tau \leqq 0$ with $\sigma \neq \tau$ it follows that (a) holds.

$$\begin{aligned} | z_{ij}[x](t) - z_{ij}[y](t) | = | x_j(t - \tau_{ij}[x](t)) - \\ - y_j(t - \tau_{ij}[y](t)) | \leqq | x_j(t - \tau_{ij}[x](t)) - x_j(t - \tau_{ij}[y](t)) | + \\ + | x_j(t - \tau_{ij}[y](t)) - y_j(t - \tau_{ij}[y](t)) | \leqq \mu \cdot | \tau_{ij}[x](t) - \tau_{ij}[y](t) | + \alpha . \end{aligned}$$

Thus (b) follows from (a). (c) is obtained from (a) in the same way. □

Before we go into the investigation of equation (0.7) we prove an existence and uniqueness theorem for functional differential equations of the more general type of equation (0.6).

§ 2. AN EXISTENCE AND UNIQUENESS THEOREM FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

In this section we prove a theorem stating, under certain conditions, the existence and uniqueness of solutions to a functional differential equation

$$x''(t) = \mathrm{F}(t, x) \tag{0.6}$$

with initial conditions at $t = -\infty$

$$\left.\begin{aligned} \lim_{t \to -\infty} (x(t) - x_0(t)) = 0 \\ \lim_{t \to -\infty} (x'(t) - x'_0(t)) = 0 \end{aligned}\right\} \tag{2.1}$$

Here t belongs to a given initial interval I (« time interval ») on the real line and x is a function which maps I into $\mathbb{R}^{\mathrm{N}}$ and is assumed to belong to a suitable set D of functions. F is a given mapping of I × D into $\mathbb{R}^{\mathrm{N}}$. Notice that x may enter into $\mathrm{F}(t, x)$ not only through its value $x(t)$ at the time t but also, for example, through its value at a « retarded time » or through the value of its derivative at the time t or at a retarded time. x_0 is assumed to be an « approximate solution » to (0.6), i. e., $x''_0(t) - \mathrm{F}(t, x_0)$ must fall off rapidly enough for $t \to -\infty$. The theorem also gives an error estimate for an iterative construction of x.

DEFINITION 2 ($\hat{h}$). — If $\mathrm{I} \subset \mathbb{R}$ is an initial interval, r a positive integer and $h : \mathrm{I} \to \mathbb{R}^r$ a continuous function we define a function $\hat{h} : \mathrm{I} \to \mathbb{R}^r$ by setting

$$\hat{h}(t) := \lim_{s \to -\infty} \int_s^t h(\tau) d\tau \qquad \text{for all} \quad t \in \mathrm{I}$$

if it exists. □

REMARK 1. — If $h : \mathrm{I} \to \mathbb{R}^r$ is a continuous function and $f : \mathrm{I} \to \mathbb{R}^r$ is some function then

$$\hat{h} = f \Leftrightarrow f' = h \wedge \lim_{t \to -\infty} f(t) = 0$$

where the left-hand side means that $\hat{h}$ exists and $\hat{h} = f$ and the right-hand side means that f is differentiable and $f' = h$ and $\lim_{t \to -\infty} f(t) = 0$. □

REMARK 2. — If $h : \mathrm{I} \to \mathbb{R}^r$ and $f : \mathrm{I} \to \mathbb{R}$ are continuous functions such that $\hat{f}$ exists and $|h(t)| \leqq f(t)$ for all $t \in \mathrm{I}$ then $\hat{h}$ exists and $|\hat{h}(t)| \leqq \hat{f}(t)$ for all $t \in \mathrm{I}$. □

DÉFINITION 3. — $(\mathrm{X}_{v\mathrm{N}}, \|\,.\,\|_v)$ If $\mathrm{I} \subset \mathbb{R}$ is an initial interval, $v : \mathrm{I} \to \mathbb{R}^+$ a positive valued continuous function and N a positive integer then we introduce the Banach space

$$\mathrm{X}_{v\mathrm{N}} := \{ x : \mathrm{I} \to \mathbb{R}^{\mathrm{N}} \mid v \cdot x \in \mathrm{C}_b(\mathrm{I}, \mathbb{R}^{\mathrm{N}}) \}$$

with the weighted supremum norm

$$\| x \|_v := \sup_{t \in \mathrm{I}} v(t) \cdot | x(t) | \qquad \text{for} \quad x \in \mathrm{X}_{v\mathrm{N}} .$$

Here $\mathrm{C}_b(\mathrm{I}, \mathbb{R}^{\mathrm{N}})$ is the set of continuous bounded functions of I into $\mathbb{R}^{\mathrm{N}}$ and $v \cdot x$ is defined by $(v \cdot x)(t) := v(t) \cdot x(t)$ for $t \in \mathrm{I}$. □

For the rest of this section (§ 2) we make the following *assumptions and conventions.*

Let $\mathrm{I} \subset \mathbb{R}$ be an initial interval and let $v, w, f : \mathrm{I} \to \mathbb{R}^+$ be positive valued continuous functions. For any positive integer N we define a Banach space $(\mathrm{X}_{\mathrm{N}}, \|\,.\,\|)$ by setting

$$\mathrm{X}_{\mathrm{N}} := \{ x \in \mathrm{C}^1(\mathrm{I}, \mathbb{R}^{\mathrm{N}}) \mid x \in \mathrm{X}_{v\mathrm{N}} \wedge x' \in \mathrm{X}_{w\mathrm{N}} \} \tag{2.2}$$

$$\| x \| := \| x \|_v + \| x' \|_w \qquad \text{for all} \quad x \in \mathrm{X}_{\mathrm{N}} \tag{2.3}$$

We assume that the functions v, w and f are such that

$$\hat{\hat{f}} \text{ exists} \tag{2.4}$$

$$\hat{\hat{f}} \in \mathrm{X}_1 \tag{2.5}$$

$$\|\hat{\hat{f}}\| < 1 \tag{2.6}$$

We define

$$\lambda := \|\hat{\hat{f}}\| \tag{2.7}$$

Then

$$0 \leqq \lambda < 1 \tag{2.8}$$

Let N be a given positive integer. We write just X for X_{N}. Let $\mathrm{D} \subset \mathrm{C}^1(\mathrm{I}, \mathbb{R}^{\mathrm{N}})$ be such that $x - y \in \mathrm{X}$ for all $x, y \in \mathrm{D}$ and let $\mathrm{F} : \mathrm{I} \times \mathrm{D} \to \mathbb{R}^{\mathrm{N}}$ be a map continuous in the first argument and such that

$$| \mathrm{F}(t, x) - \mathrm{F}(t, y) | \leqq f(t) \cdot \| x - y \| \tag{2.9}$$

for all $t \in I$ and $x, y \in D$. Further we assume that we have an « approximate solution » x_0 : Let $x_0 \in D \cap C^2(I, \mathbb{R}^N)$. We define, for every $x \in D$, a function $h_x : I \to \mathbb{R}^N$ by setting

$$h_x(t) := F(t, x) - x_0''(t) \qquad \text{for all} \quad t \in I \tag{2.10}$$

Our assumption about x_0 is that

$$\hat{\hat{h}}_{x_0} \text{ exists} \tag{2.11}$$

and

$$\hat{\hat{h}}_{x_0} \in X \tag{2.12}$$

(In the proofs of the theorems 3 and 4 in § 4 we shall prove the existence of such an x_0 for the problem described in the introduction, § 0). We define the set $x_0 + X := \{ x_0 + x \mid x \in X \} \subset C^1(I, \mathbb{R}^N)$. For $x, y \in x_0 + X$ we set

$$d(x, y) := \| x - y \| \tag{2.13}$$

$(x_0 + X, d)$ is a complete metric space and (D, d) is a metric subspace. Before we state theorem 1 we prove a lemma.

LEMMA 3. — For every $x \in D$, $\hat{\hat{h}}_x$ exists and $\hat{\hat{h}}_x \in X$. □

Proof. — Let $x \in D$. We define a function $h : I \to \mathbb{R}^N$ by setting

$$h(t) := F(t, x) - F(t, x_0) \qquad \text{for all} \quad t \in I .$$

Assumption (2.9) implies

$$| h(t) | \leqq f(t) \cdot \| x - x_0 \| .$$

Thus from assumptions (2.4) and (2.5) and remark 2 it follows that $\hat{\hat{h}}$ exists and that $\hat{\hat{h}} \in X$. By assumptions (2.11) and (2.12) $\hat{\hat{h}}_{x_0}$ exists and $\hat{\hat{h}}_{x_0} \in X$. Since $h_x = h_{x_0} + h$ it follows that $\hat{\hat{h}}_x$ exists and $\hat{\hat{h}}_x = \hat{\hat{h}}_{x_0} + \hat{\hat{h}} \in X$. □

Lemma 3 allows us to define a mapping $T : (D, d) \to (X_0 + X, d)$ of the metric space (D, d) into the complete metric space $(x_0 + X, d)$ by setting, for all $x \in D$,

$$Tx := x_0 + \hat{\hat{h}}_x \tag{2.14}$$

LEMMA 4. — Let $x \in D$. Then

$$x = Tx \Leftrightarrow \begin{cases} x''(t) = F(t, x) \qquad \text{for all} \qquad t \in I \\ \wedge \lim\limits_{t \to -\infty} (x(t) - x_0(t)) = 0 \\ \wedge \lim\limits_{t \to -\infty} (x'(t) - x_0'(t)) = 0 \quad \square \end{cases}$$

Proof. — Let $x \in D$. Then $x = Tx$ is, by (2.14), equivalent to $\hat{\hat{h}}_x = x - x_0$

which is, by remark 1, equivalent to $\hat{h}_x = (x - x_0)' \wedge \lim_{t \to -\infty} (x - x_0)(t) = 0$ which is, again by remark 1, equivalent to

$$h_x = (x - x_0)'' \wedge \lim_{t \to -\infty} (x - x_0)'(t) = 0 \wedge \lim_{t \to -\infty} (x - x_0)(t) = 0 .$$

By (2.10) this is equivalent to the right-hand side of the $\Leftrightarrow$-assertion of lemma 4. □

Finally we define a positive real number

$$\rho := \frac{1}{1 - \lambda} \| \hat{\hat{h}}_{x_0} \| \tag{2.15}$$

and the closed ball B with centre x_0 and radius ρ in the complete metric space $(x_0 + X, d)$:

$$B := \{ x_0 + x \mid x \in X \wedge \| q \mid x \| \leqq \rho \} \tag{2.16}$$

THEOREM 1. — T is contractive with the contraction factor λ, i. e.,

$$d(Tx, Ty) \leqq \lambda \cdot d(x, y) \qquad \text{for all} \quad x, y \in D \tag{2.17}$$

T maps $B \cap D$ into B, i. e.,

$$T(B \cap D) \subset B \tag{2.18}$$

If we assume in addition that

$$T(B \cap D) \subset D \tag{2.19}$$

and that

$$D \text{ is closed in } (x_0 + X, d) \tag{2.20}$$

then T has one and only one fixed point x and we have

$$x = \lim_{r \to \infty} T^r x_0 \quad \text{in} \quad (D, d) \tag{2.21}$$

and for all nonnegative integers r

$$\| x - T^r x_0 \| \leqq \lambda^r \rho . \tag{2.22}$$

x is the only twice differentiable element of D which solves the functional differential equation

$$x''(t) = F(t, x) \qquad \text{for all} \quad t \in I \tag{0.6}$$

with the initial conditions

$$\left. \begin{aligned} \lim_{t \to -\infty} (x(t) - x_0(t)) = 0 \\ \lim_{t \to -\infty} (x'(t) - x_0'(t)) = 0 \end{aligned} \right\} \tag{2.1}$$

□

Proof. — In order to prove that T is contractive let $x, y \in D$. We define a function $h : I \to \mathbb{R}^N$ by setting

$$h(t) := F(t, x) - F(t, y) \qquad \text{for all} \quad t \in I .$$

Assumption (2.9) implies

$$| h(t) | \leqq f(t) \cdot \| x - y \| \qquad \text{for all} \quad t \in \mathrm{I}.$$

By definition (2.10), $h = h_x - h_y$. Thus, by (2.14), remark 2 and (2.7),

$$\| \mathrm{T}x - \mathrm{T}y \| = \| \hat{\hat{h}}_x - \hat{\hat{h}}_y \| = \| \hat{\hat{h}} \| \leqq \| \hat{\hat{f}} \| \cdot \| x - y \| = \lambda \cdot \| x - y \|$$

So (2.17) holds.

For $x \in \mathrm{B} \cap \mathrm{D}$ it follows from (2.17), (2.14), (2.16) and (2.15) that

$$\| \mathrm{T}x - x_0 \| \leqq \| \mathrm{T}x - \mathrm{T}x_0 \| + \| \mathrm{T}x_0 - x_0 \| \leqq \lambda \| x - x_0 \| + \| \hat{\hat{h}}_{x_0} \| \leqq$$
$$\leqq \lambda\rho + (1 - \lambda)\rho = \rho, \quad \text{i. e.} \quad \mathrm{T}x \in \mathrm{B}.$$

So (2.18) holds.

Now let us assume (2.19) and (2.20). Then $\mathrm{B} \cap \mathrm{D}$ is closed in the complete metric space $(x_0 + \mathrm{X}, d)$ because B and D are. Thus the restriction $\mathrm{T}|_{\mathrm{B} \cap \mathrm{D}}$ of T to $(\mathrm{B} \cap \mathrm{D}, d)$ is a contractive map of a complete metric space into itself and has therefore, by Banachs fixed point theorem, one and only one fixed point x and (2.21) and (2.22) hold. Since T is contractive, x is the only fixed point of T. The last assertion of theorem 1 follows now from lemma 4. □

§ 3. AN EXISTENCE AND UNIQUENESS THEOREM FOR RETARDED DIFFERENTIAL EQUATIONS

In this section we apply theorem 1 to the special case where x enters into $\mathrm{F}(t, x)$ only through its value and the value of its first derivative at the time t and at the retarded times, i. e., through $p[x](t)$. For the rest of this paper we assume that n and k are fixed positive integers.

Throughout this section (§ 3) we assume that J is a given initial interval, that $x_0 \in \mathrm{C}^2(\mathrm{J}, (\mathbb{R}^n)^k)$ is a given function and that c_1 is a given positive constant.

If I is an initial subinterval of J and $x : \mathrm{I} \to (\mathbb{R}^n)^k$ then we say that x is <u>c_1-close</u> to x_0 iff x is differentiable and

$$| x(t) - x_0(t) | < c_1 \wedge | x'(t) - x_0'(t) | < \frac{c_1}{|t|} \qquad \text{for all} \quad t \in \mathrm{I}.$$

Let $\mathrm{A} \subset \mathbb{R} \times (\mathbb{R}^n)^{k^2} \times (\mathbb{R}^n)^{k^2}$ be a closed set such that

$$\left.\begin{array}{l}\text{for all initial subintervals I of J and all functions } x : \mathrm{I} \to (\mathbb{R}^n)^k \\ \text{which are } c_1\text{-close to } x_0 \text{ it follows that } p[x](t) \in \mathrm{A} \text{ for all } t \in \mathrm{I}.\end{array}\right\} \qquad (3.1)$$

Further we assume in this section that $\mu < 1$ is a positive constant such that

$$| v_{ij} | \leqq \mu \qquad \text{for all} \quad (t, z, v) \in A, \quad i, j \leqq k \tag{3.2}$$

Let c be a positive constant and let $G : A \to (\mathbb{R}^n)^k$ be a continuous function obeying the Lipschitz condition

$$| G(t, \overset{1}{z}, \overset{1}{v}) - G(t, \overset{2}{z}, \overset{2}{v}) | \leqq \frac{c}{|t|^3} | \overset{1}{z} - \overset{2}{z} | + \\ + \frac{c}{|t|^2} | \overset{1}{v} - \overset{2}{v} | \qquad \text{for all} \quad (t, \overset{1}{z}, \overset{1}{v}), \quad (t, \overset{2}{z}, \overset{2}{v}) \in A \tag{3.3}$$

and the fall-off condition

$$| G(t, z, v) | \leqq \frac{c}{|t|} \qquad \text{for all} \quad (t, z, v) \in A \tag{3.4}$$

We assume that x_0'' has the same fall-off behaviour:

$$| x_0''(t) | \leqq \frac{c}{|t|} \qquad \text{for all } t \in J \tag{3.5}$$

We want to treat equation (0.7) by applying theorem 1. We shall see (lemma 5) that (3.2) and (3.3) guarantee that (2.9) holds where

$$F(t, x) := G(t, p[x](t))$$

and v, w and f are chosen suitably. (3.1) is used in the proof of (2.19).

For every function $x : J \to (\mathbb{R}^n)^k$ we define a function h_x by setting

$$h_x(t) := G(p[x](t)) - x_0''(t) \tag{3.6}$$

for all $t \in J$ for which it exists.

Finally we assume that x_0 « solves the retarded differential equation (0.7) approximately », i. e.,

$$\hat{\hat{h}}_{x_0} \quad \text{exists} \tag{3.7}$$

and

$$\sup_{t\in J} | \hat{\hat{h}}_{x_0}(t) | < \frac{c_1}{2} \wedge \sup_{t\in J} (| \hat{h}_{x_0}(t) | \cdot | t |) < \frac{c_1}{2} \tag{3.8}$$

THEOREM 2. — There is an initial subinterval I of J in which the retarded differential equation

$$x''(t) = G(p[x](t)) \tag{0.7}$$

with the initial conditions

$$\left.\begin{aligned} &\lim_{t\to-\infty} (x(t) - x_0(t)) = 0 \\ &\limsup_{t\to-\infty} (| x'(t) - x_0'(t) | \cdot | t |) < \infty \end{aligned}\right\} \tag{3.9}$$

has a strictly unique solution x. x is obtained from x_0 by an iteration process according to theorem 1 and the error estimate of theorem 1 is also valid here if we choose $v(t) := 1$ and $w(t) := -t$. $\square$

Before we go into the proof of theorem 2 we first prove a lemma:

LEMMA 5. — Let I be an initial subinterval of J and let $x, y \in C^1(I, (\mathbb{R}^n)^k)$ be such that $p[x](t) \in A$ and $p[y](t) \in A$ for all $t \in I$ and

$$\frac{|x'(t) - x'(s)|}{t - s} \leqq \frac{c}{|t|} \quad \text{for all} \quad t \in I \quad \text{and} \quad s < t .$$

Then, for all $t \in I$,

$$|G(p[x](t)) - G(p[y](t))| \leqq \frac{1 + \mu + 2c}{1 - \mu} ck \cdot \frac{1}{|t|^3} \cdot \sup_{r \leqq s \leqq t} |x(s) - y(s)| + ck \cdot \frac{1}{|t|^2} \cdot \sup_{r \leqq s \leqq t} |x'(s) - y'(s)|$$

where $$r := \min_{i,j \leqq k} (t - \tau_{ij}[y](t))$$ $\square$

Proof. — From (3.3) we have

$$|G(p[x](t)) - G(p[y](t))| \leqq \frac{c}{|t|^3} |z[x](t) - z[y](t)| + \frac{c}{|t|^2} |v[x](t) - v[y](t)| .$$

From (3.2) and $p[x](t) \in A$ and $p[y](t) \in A$ it follows that $|x_j'(t)| \leqq \mu$ and $|y_j'(t)| \leqq \mu$ for $t \in I$, $j \leqq k$. Thus lemma 2 implies that

$$|z[x](t) - z[y](t)| \leqq \frac{1 + \mu}{1 - \mu} \cdot k\alpha$$

and

$$|v[x](t) - v[y](t)| \leqq \left(\beta + \frac{2\alpha v}{1 - \mu}\right) \cdot k$$

where α, β and v are as defined in lemma 2. These inequalities together with the assumption about x' yield immediately the asserted inequality. $\square$

Proof. of theorem 2. — Let $\eta := \sup_{t \in J} |\hat{\hat{h}}_{x_0}(t)| + \sup_{t \in J} (|\hat{h}_{x_0}(t)| \cdot |t|)$

Then $\eta < c_1$ by (3.8).

Let I be any initial subinterval of J such that

$$\sup I < -\frac{1 + \mu + 2c}{1 - \mu} \cdot ck \cdot \left(1 - \frac{\eta}{c_1}\right)^{-1} \tag{3.10}$$

In order to apply theorem 1 we have to define the functions $v, w, f : I \to \mathbb{R}^+$:

Let

$$v(t) := 1,\ w(t) := -t,\ f(t) := \frac{1+\mu+2c}{1-\mu}\cdot ck\cdot\frac{1}{(-t)^3} \quad \text{for} \quad t\in I.$$

Then $1 - \|\hat{\hat{f}}\| > \frac{\eta}{c_1}$. Thus (2.4), (2.5) and (2.6) hold and, with

$$\lambda := \|\hat{\hat{f}}\| \quad (2.7),$$

$$1 - \lambda > \frac{1}{c_1}\cdot\|\hat{\hat{h}}_{x_0}\|.$$

Defining ρ as in (2.15) $\rho := \frac{1}{1-\lambda}\cdot\|\hat{\hat{h}}_{x_0}\|$ we have

$$\rho < c_1 \tag{3.11}$$

Let

$$D := \{x\in x_0 + X \mid \bigwedge_{t\in I} p[x](t)\in A \wedge \wedge \bigwedge_{t\in I}\bigwedge_{s<t} \frac{|x'(t)-x'(s)|}{t-s} \leqq \frac{c}{|t|}\} \tag{3.12}$$

and let $F: I\times D \to \mathbb{R}^N$ be defined by

$$F(t,x) := G(p[x](t)) \qquad \text{for all} \quad t\in I, \quad x\in D \tag{3.13}$$

Obviously, lemma 1 (f) and (3.2) imply that $F(.,x)$ is continuous. The inequality (2.9) follows from lemma 5.

Next we have to prove $x_0\in D\cap C^2(I,\mathbb{R}^N)$. We have $x_0\in C^2(I,\mathbb{R}^N)$ by assumption (for simplicity of writing we use, not quite correctly, the symbol x_0 for the restriction $x_0|_I$ of x_0 to I in the proof of theorem 2). $x_0\in D$ follows from assumptions (3.1) and (3.5).

Obviously, for all $x\in D$ and $t\in I$, the $h_x(t)$ as defined in (3.6) exists and is the same as the $h_x(t)$ as defined in (2.10), (2.11) and (2.12) follow from (3.7) and (3.8).

To prove (2.19) let $x\in B\cap D$. We have to prove $Tx\in D$. By theorem 1, (2.18) it follows that $Tx\in B$, i. e.,

$$\|Tx - x_0\| \leqq \rho$$

Thus, by (3.11), Tx is c_1-close to x_0 and, by (3.1), $p[Tx](t)\in A$ for all $t\in I$. From (2.14) and (3.6) it follows that

$$(Tx)''(t) = G(p[x](t)) \qquad \text{for all} \quad x\in D, t\in I.$$

Thus by (3.4) we have $|(Tx)''(t)| \leqq \frac{c}{|t|}$ and hence $Tx\in D$.

In order to prove that D is closed (2.20) let (x_i) be a sequence in D which converges in $(x_0 + X, d)$ to some limit x. By (3.2) and (3.12), $| \underset{l}{x}'_j(t) | \leqq \mu$ and therefore $| x'_j(t) | \leqq \mu$ for all $t \in I$ açd $j \leqq k$ and every natural number l. Thus, by lemma 2,

$$\lim_{l \to \infty} | p[\underset{l}{x}](t) - p[x](t) | = 0 .$$

By (3.12), $p[\underset{l}{x}](t) \in A$ for all l. Since we have assumed that A is closed we have $p[x](t) \in A$. Thus $x \in D$. It follows that D is closed.

By theorem 1, there is one and only one twice differentiable $x \in D$ which solves the equation (0.6) with the initial conditions (2.1). Equation (0.6) is in our case the same as equation (0.7) if $x \in D$. If some function x is a solution of (0.7) then, obviously, $p[x](t)$ must lie in the domain of definition of G, i. e., $p[x](t) \in A$ for $t \in J$ and, because of (3.4), $| x''(t) | \leqq \frac{c}{|t|}$ for all $t \in I$. Hence for solutions x of (0.7) the property $x \in D$ is equivalent to $x - x_0 \in X$. So the property $x \in D$ strengthens the initial conditions (2.1) to (3.9). So there is a unique solution of (0.7) in I with the initial conditions (3.9). The strict uniqueness follows from the fact that the inequality (3.10) and therefore the whole proof of theorem 2 including the uniqueness of the solution is valid for any initial subinterval of I as well as for I and from the fact that a restriction of a solution in I to an initial subinterval of I is again a solution. □

§ 4. THE SCATTERING PROBLEM

In this section we apply theorem 2 to the scattering problem of electrically charged particles with initial conditions in the infinite past. We assume that the motion of the particles is governed by the velocity- dependent parts of the mutual retarded Liénard-Wiechert fields. We shall prove existence and strict uniqueness of the solution and provide an iterative construction.

Let n and k be given positive integers. We make a few definitions first:

$$P := \{ (t, z, v) \in \mathbb{R} \times (\mathbb{R}^n)^{k^2} \times (\mathbb{R}^n)^{k^2} \mid z_{ii} \neq z_{ij} \quad \text{for} \quad i \neq j \wedge | v_{ij} | < 1 \quad \text{for} \quad i,j \leqq k \} \tag{4.1}$$

P is the domain of definition of the function G in equation (0.7). We shall apply theorem 2 to the restriction of G to a closed subset A of P. For a triple $p = (t, z, v) \in P$ we define the numbers

$$\delta(p) := \min_{i \neq j} | z_{ii} - z_{ij} | \tag{4.2}$$

$$\mu(p) := \max_{i,j} | v_{ij} | \tag{4.3}$$

For positive numbers $\mu < 1$ and δ we define the set

$$\mathrm{P}_{\mu\delta} := \{ p \in \mathrm{P} \mid \mu(p) \leqq \mu \wedge \delta(p) \geqq \delta \} \tag{4.4}$$

DEFINITION 4 (« normal fall-off »). — We say that a function

$$\mathrm{G} \in \mathrm{C}^1(\mathrm{P}, (\mathbb{R}^n)^k) \ (^3)$$

has *normal fall-off* iff, for all positive $\mu < 1$, there are positive constants c and δ_0 such that, for all $\delta \geqq \delta_0$ and for all $\overset{1}{p} = (t, \overset{1}{z}, \overset{1}{v}) \in \mathrm{P}_{\mu\delta}$ and $\overset{2}{p} = (t, \overset{2}{z}, \overset{2}{v}) \in \mathrm{P}_{\mu\delta}$, the following inequalities hold:

$$| \mathrm{G}(\overset{1}{p}) | \leqq \frac{c}{\delta^2} \tag{4.5}$$

$$| \mathrm{G}(\overset{1}{p}) - \mathrm{G}(\overset{2}{p}) | \leqq \frac{c}{\delta^3} | \overset{1}{z} - \overset{2}{z} | + \frac{c}{\delta^2} | \overset{1}{v} - \overset{2}{v} | \tag{4.6}$$

□

In this section we shall investigate the set of solutions of the equation (0.7) where $\mathrm{G} : \mathrm{P} \to (\mathbb{R}^n)^k$ has normal fall-off [4]. For this purpose we shall make use of theorem 2. In order to do this we have to find a function x_0 which solves (0.7) « approximately ». We can get such a function x_0 by first observing that, for any solution x of (0.7) for which

$$v_{-\infty} := \lim_{t \to -\infty} x'(t)$$

exists, a double integration of (0.7) yields

$$x(t) = v_{-\infty} \cdot (t - t_0) + \int_{t_0}^{t} \int_{-\infty}^{\tau} \mathrm{G}(p[x](\sigma)) d\sigma d\tau + x(t_0) \tag{4.7}$$

Let

$$\mathrm{V} := \{ v \in (\mathbb{R}^n)^k \mid | v_i | < 1 \quad \text{for} \quad i \leqq k \wedge v_i \neq v_j \quad \text{for} \quad i \neq j \} \tag{4.8}$$

Of course, $v_{-\infty} \in \mathrm{V}$ if the distances between the particles grow at least linearly in $-t$ for $t \to -\infty$ and their speeds are bounded away from the speed of light. A very rough approximation to x is the function $a(t) := v_{-\infty} t$, and we shall see that the right-hand side of equation (4.7) with $p[x]$ replaced by $p[a]$ serves as a good candidate for x_0 if $v_{-\infty} \in \mathrm{V}$. This motivates the following abbreviation: Let t_0 be an arbitrary but fixed negative number, e. g., $t_0 := -1$.

---

(3) It would suffice to require a Lipschitz condition for G.

(4) As we shall see (lemma 9) the function G has normal fall-off in the case of our problem described in the introduction.

For every function $G : P \to (\mathbb{R}^n)^k$ with normal fall-off and every $v_{-\infty} \in V$ and $\gamma \in (\mathbb{R}^n)^k$ let

$$x_G[v_{-\infty}, \gamma](t) := v_{-\infty} \cdot (t - t_0) + \int_{t_0}^{t} \int_{-\infty}^{\tau} G(p[a](\sigma))d\sigma d\tau + \gamma \quad (4.9)$$

where $a(t) := v_{-\infty}t$.

We have

$$\lim_{t \to -\infty} x_G[v_{-\infty}, \gamma]'(t) = v_{-\infty} \quad (4.10)$$

and $x_G[v_{-\infty}, \gamma](t_0) = \gamma$.

In the physical problem described in the introduction we look only for solutions of (0.7) for which the system of particles is in an unbound state in the infinite past. So we make the

DEFINITION 5 (« initially unbound » functions). — Let I be an initial interval. A differentiable function $x : I \to (\mathbb{R}^n)^k$ is said to be initially unbound iff

$$\limsup_{t \to -\infty} |x_i'(t)| < 1 \quad \text{for all} \quad i \leqq k \quad (4.11)$$

$$\liminf_{t \to -\infty} \frac{|x_i(t) - x_j(t)|}{|t|} > 0 \quad \text{for} \quad i \neq j \quad (4.12)$$

□

REMARK 3 ([5]). — A function $x \in C^1(I, (\mathbb{R}^n)^k)$ is initially unbound iff

$$\limsup_{t \to -\infty} \mu(p[x](t)) < 1 \quad (4.13)$$

$$\liminf_{t \to -\infty} \frac{\delta(p[x](t))}{|t|} > 0 \quad (4.14)$$

□

THEOREM 3. — Let $G : P \to (\mathbb{R}^n)^k$ have normal fall-off. Then, for every $v_{-\infty} \in V$, $\gamma \in (\mathbb{R}^n)^k$, there is an initial interval I such that there is a strictly unique solution $x = x[v_{-\infty}, \gamma]$ of the equation

$$x''(t) = G(p[x](t)) \quad (0.7)$$

in the interval I with the initial conditions

$$\lim_{t \to -\infty} (x(t) - x_G[v_{-\infty}, \gamma](t)) = 0 \quad (4.15)$$

$$\lim_{t \to -\infty} x'(t) = v_{-\infty} \quad (4.16)$$

For x it even holds that

$$\limsup_{t \to -\infty} (|x'(t) - v_{-\infty}| \cdot |t|) < \infty \quad (4.17)$$

Every initially unbound solution of (0.7) is such an $x[v_{-\infty}, \gamma]$ with some unique $v_{-\infty} \in V$ and $\gamma \in (\mathbb{R}^n)^k$.

([5]) This follows directly from Lemma 1(*b*) and (1.5).

x is obtained in the following way:

Let x_0 be the function $x_G[v_{-\infty}, \gamma]$ or any other C^2-function obeying the conditions

$$\lim_{t \to -\infty} (x_0(t) - x_G[v_{-\infty}, \gamma](t)) = 0 \tag{4.18}$$

$$\lim_{t \to -\infty} (|x_0'(t) - x_G[v_{-\infty}, \gamma]'(t)| \cdot |t|) = 0 \tag{4.19}$$

$$\lim_{t \to -\infty} (|x_0''(t)| \cdot |t|) = 0 \tag{4.20}$$

For every function $x : I \to (\mathbb{R}^n)^k$ we define (6)

$$Tx(t) := x_0(t) + \int_{-\infty}^{t} \int_{-\infty}^{\tau} [G(p[x](\sigma)) - x_0''(\sigma)] d\sigma d\tau \tag{4.21}$$

where it exists.
Then the function $x = x[v_{-\infty}, \gamma]$ is given by

$$x(t) = \lim_{r \to \infty} T^r x_0(t) \quad \text{for} \quad t \in I \tag{4.22}$$

There are positive constants $\lambda < 1$ and ρ (which may depend on $v_{-\infty}$ and γ) such that the following error estimate holds:

$$\sup_{t \in I} |x(t) - T^r x_0(t)| + \sup_{t \in I}(|x'(t) - (T^r x_0)'(t)| \cdot |t|) \leqq \lambda^r \rho \tag{4.23}$$

for all nonnegative integers r. □

As part of the proof of theorem 3 we prove the following lemma:

LEMMA 6. — Let G have normal fall-off. Then every initially unbound solution x of (0.7) obeys (4.15) and (4.17) with some $v_{-\infty} \in V$ and $\gamma \in (\mathbb{R}^n)^k$. □

For the proofs of theorem 3 and lemma 6 we adopt the following notation:

If $f : I \to \mathbb{R}^+$ is a positive valued function then every occurrence of $O(f(t))$ stands for some $g(t)$ where $g : I \to \mathbb{R}^N$ is such that

$$\limsup_{t \to -\infty} \frac{|g(t)|}{f(t)} < \infty \qquad \text{(N is any positive integer).}$$

Every occurrence of $o(f(t))$ stands for some $g(t)$ where $g : I \to \mathbb{R}^N$ is such that

$$\lim_{t \to -\infty} \frac{|g(t)|}{f(t)} = 0.$$

(6) In this section we write just $\int_{-\infty}^{t} \ldots$ for $\lim_{s \to -\infty} \int_{s}^{t} \ldots$.

REMARK 4 (7). — If G has normal fall-off and x and y are initially unbound functions then

$$G(p[x](t)) = 0(|t|^{-2})$$
$$G(p[x](t)) - G(p[y](t)) = 0(|z[x](t) - z[y](t)|\cdot|t|^{-3}) + 0(|v[x](t) - v[y](t)|\cdot|t|^{-2}) \quad \square$$

Proof. of lemma 6. — Let x be an initially unbound solution of (0.7). Then, by remark 4 and equation (0.7),

$$x''(t) = 0(|t|^{-2}).$$

By integration it follows that $v_{-\infty} := \lim_{t\to-\infty} x'(t)$ exists and that

$$x'(t) = v_{-\infty} + 0(|t|^{-1}), \quad \text{i. e.,} \quad (4.17).$$

Thus $x'(t) - a'(t) = 0(|t|^{-1})$ and, by integrating again,

$$x(t) - a(t) = 0(\ln|t|).$$

Since $\tau[a](t) = 0(|t|)$ it follows from remark 4 and Lemma 2 that

$$G(p[x](t)) - G(p[a](t)) = 0\left(\frac{\ln|t|}{|t|^3}\right).$$

Let

$$\gamma := x(t_0) - \int_{-\infty}^{t_0}\int_{-\infty}^{\tau}[G(p[x](\sigma)) - G(p[a](\sigma))]d\sigma d\tau.$$

This exists and from (4.7) and (4.9) it follows that

$$x(t) - x_G[v_{-\infty}, \gamma](t) = \int_{-\infty}^{t}\int_{-\infty}^{\tau}[G(p[x](\sigma)) - G(p[a](\sigma))]d\sigma d\tau = 0\left(\frac{\ln|t|}{|t|}\right).$$

So also (4.15) holds. $\square$

Next we prove a proposition about the behaviour of x_0 in the infinite past and the assertion that x_0 solves (0.7) « approximately »:

LEMMA 7. — If x_0 is as assumed in theorem 3 then there is a positive constant c_3 and a negative constant t_1 such that

$$|(x_0)_i(t)-(x_0)_j(s)| \geqq c_3\cdot|t|+2 \quad \text{for} \quad s < t < t_1 \quad \text{and} \quad i,j \leqq k$$

and

$$\sup_{t<t_1} \mu(p[x_0](t)) < 1$$

and there is an initial interval J such that, with $h_{x_0} : J \to (\mathbb{R}^n)^k$ as defined in (3.6),

$$\hat{\hat{h}}_{x_0} \quad \text{exists}$$

(7) This follows immediately from remark 3.

and

$$\sup_{t\in J} | \hat{\hat{h}}_{x_0}(t) | < \frac{1}{2} \wedge \sup_{t\in J} (| \hat{h}_{x_0}(t) | \cdot | t |) < \frac{1}{2}. \quad \square$$

Proof. — By remark 4 we have, with $a(t) := v_{-\infty}t$,

$$G(p[a](t)) = 0(|t|^{-2}).$$

By integrating this equation twice and by using the equations (4.9), (4.19) and (4.18) we get

$$\begin{aligned} x_0'(t) - a'(t) &= 0(|t|^{-1}) \\ x_0(t) - a(t) &= 0(\ln|t|) \end{aligned} \tag{4.24}$$

from which follows the first part of lemma 7. Because of lemma 2 it also follows that

$$\begin{aligned} z[x_0](t) - z[a](t) &= 0(\ln|t|) \\ v[x_0](t) - v[a](t) &= 0(|t|^{-1}) \end{aligned}$$

and hence, by remark 4,

$$G(p[x_0](t)) - G(p[a](t)) = 0(|t|^{-3}\ln|t|).$$

From definition (4.9) it follows that

$$G(p[a](t)) = x_G[v_{-\infty}, \gamma]''(t).$$

From the last two equations and definition (3.6) we get

$$h_{x_0}(t) = x_G[v_{-\infty}, \gamma]''(t) - x_0''(t) + 0(|t|^{-3}\ln|t|).$$

We integrate this equation twice and make use of (4.19) and (4.18) to obtain

$$\begin{aligned} \hat{h}_{x_0}(t) &= o(|t|^{-1}) \\ \hat{\hat{h}}_{x_0}(t) &= o(1) \end{aligned}$$

which proves the remaining assertions of lemma 7. $\square$

LEMMA 8. — Let I be an initial interval and let I_0 be an initial subinterval of I. Let $G \in C^1(P, (\mathbb{R}^n)^k)$ and let $x, y : I \to (\mathbb{R}^n)^k$ be two solutions of (0.7) such that $\sup_{t\in I} | x_i'(t) | < 1$ and $\sup_{t\in I} | y_i'(t) | < 1$ for $i \leqq k$ and $x|_{I_0} = y|_{I_0}$. Then $x = y$. $\square$

Proof. — Let I_1 be the largest initial subinterval of I such that $x|_{I_1} = y|_{I_1}$. Assume that $I_1 \neq \mathring{I}$. Then let $t_1 := \sup I_1$. We have $t_1 \in I$ and therefore $p[x](t_1), p[y](t_1) \in P$.

Thus $t_1 - \tau_{ij}[x](t_1) < t_1$, $t_1 - \tau_{ij}[y](t_1) < t_1$ for all $i, j \leqq k$ with $i \neq j$. Since $\tau[x]$ and $\tau[y]$ are continuous (as we know from lemma 1) there is a $t_2 > t_1$ such that $t - \tau_{ij}[x](t) < t_1 \wedge t - \tau_{ij}[y](t) < t_1$ for $t_1 \leqq t \leqq t_2$ and $i \neq j$. Let $I_2 := \{ t \mid t < t_2 \}$, $w := x|_{I_1}$. Then $t - \tau_{ij}[x](t) \in I_1$ and thus $x_j(t - \tau_{ij}[x](t)) = w_j(t - \tau_{ij}[x](t))$ for $t \in I_2$ and $i \neq j$. Thus from (1.1) we obtain $\tau_{ij}[x](t)$ as a function of $x(t)$: $\tau_{ij}[x](t) = \varphi^{-1}(0)$ where $\varphi(\tau) := \tau - |x_i(t) - w_j(t-\tau)|$ and similarly we obtain $z_{ij}[x](t)$ and $v_{ij}[x](t)$ as functions of $x(t)$. Hence it follows from (0.7) that $x|_{I_2}$ obeys an ordinary (and non-retarded) differential equation. Since we also have $w = y|_{I_1}$ it follows from the same argument that $y|_{I_2}$ obeys the same ordinary differential equation. From the uniqueness theorem for ordinary differential equations follows $x|_{I_2} = y|_{I_2}$ in contradiction to the maximality of I_1. $\square$

Proof of theorem 3. — Let $G : P \to (\mathbb{R}^n)^k$ have normal fall-off and let $v_{-\infty} \in V$, $\gamma \in (\mathbb{R}^n)^k$ (as assumed in theorem 3). We want to apply theorem 2 with the restriction of G to a suitably chosen set A instead of G. Before we can do this we have to choose positive constants c_1, μ and c, an initial interval J and a closed set A such that $\mu < 1$ and (3.1), (3.2), (3.3), (3.4), (3.5), (3.7) and (3.8) hold.

First we choose $c_1 := 1$. Let c_3 and t_1 be as in lemma 7 and let

$$c_2 := \frac{1}{2}(1 - \sup_{t<t_1} \mu(p[x_0](t))), \qquad \mu := 1 - c_2$$

By lemma 7, $c_2 > 0$ and thus $0 < \mu < 1$.

Let δ_0 be as in definition 4 (« normal fall-off »). We take as our initial interval J the intersection of the initial interval J of lemma 7 and the initial interval

$$\left\{ t < t_1 \,\middle|\, \frac{c_1}{|t|} \leqq c_2 \wedge c_3 \cdot |t| \geqq \delta_0 \right\}.$$

From lemma 7 it follows that (3.7) and (3.8) hold. We define

$$A := \Big\{ p = (t, z, v) \in P \mid t \in J \wedge \bigwedge_{i,j \leqq k} |z_{ij} - (x_0)_j(t - |z_{ii} - z_{ij}|)| \leqq c_1 \wedge \bigwedge_{i,j \leqq k} |v_{ij} - (x_0)'_j(t - |z_{ii} - z_{ij}|)| \leqq c_2 \Big\}.$$

Then A is closed. (3.1) holds because if x is c_1-close to x_0 then

$$|z_{ij}[x](t) - (x_0)_j(t - |z_{ii}[x](t) - z_{ij}[x](t)|)| = |x_j(t - \tau_{ij}[x](t)) - (x_0)_j(t - \tau_{ij}[x](t))| < c_1$$

and similarly

$$|v_{ij}[x](t) - (x_0)'_j(t - |z_{ii}[x](t) - z_{ij}[x](t)|)| < \frac{c_1}{|t - \tau_{ij}[x](t)|} \leqq \frac{c_1}{|t|} \leqq c_2$$

and thus $p[x](t) \in A$.

From the definition of c_2 it follows that

$$|(x_0)'_j(t)| \leqq 1 - 2c_2 \qquad \text{for} \qquad t < t_1 \quad \text{and} \quad j \leqq k.$$

Thus (3.2) holds. From the first part of lemma 7 follows

$$|(x_0)_i(t)-(x_0)_j(t-|z_{ii}-z_{ij}|)| \geqq c_3\cdot|t|+2=c_3\cdot|t|+2c_1$$

for any $(t, z, v) \in A$ and $i, j \leqq k$. From the definition of A we have

$$|z_{ii} - (x_0)_i(t)| \leqq c_1$$
$$|z_{ij} - (x_0)_j(t - |z_{ii} - z_{ij}|)| \leqq c_1 .$$

Hence $|z_{ii} - z_{ij}| \geqq c_3\cdot|t|$. Thus

$$\delta(p) \geqq c_3\cdot|t| \qquad \text{for all} \qquad p = (t, z, v) \in A .$$

Now let $\overset{1}{p} = (t, \overset{1}{z}, \overset{1}{v}) \in A$, $\overset{2}{p} = (t, \overset{2}{z}, \overset{2}{v}) \in A$ and $\delta := c_3\cdot|t|$. Then $\overset{1}{p}, \overset{2}{p} \in P_{\mu,\delta}$ and $\delta \geqq \delta_0$.

Since G has normal fall-off there exists a positive constant c which does not depend on $\overset{1}{p}$ and $\overset{2}{p}$ such that (4.5) and (4.6) hold. Because of $\delta(p) \geqq c_3\cdot|t|$ and because of (4.20) it follows that there also is a (different) positive constant c such that (3.3), (3.4) and (3.5) hold.

By theorem 2, applied to the restriction $G|_A$ of the function G to the set A, there is an initial subinterval I of J in which the retarded differential equation (0.7) with the initial conditions (3.9) and with the additional condition $p[x](t) \in A$ for all t in the domain of definition of x has a strictly unique solution x.

If now y is any solution of (0.7) defined in an initial subinterval of I and obeying the initial conditions (3.9) then $p[y](t) \in A$ for sufficiently small t (the proof is analogous to the proof of (3.1)). Thus it follows from the strict uniqueness of x that there is an initial subinterval of I in which x and y coincide. By lemma 8, y is a restriction of x. So we have proved that there is a strictly unique solution x of (0.7) in I with the initial conditions (3.9). From (4.18) and (4.24) it follows that (3.9) is equivalent to (4.15) $\wedge$ (4.17). Since every function x which obeys (4.16) is initially unbound it follows from lemma 6 that, for solutions x of (0.7), it holds (4.15) $\wedge$ (4.16) $\Leftrightarrow$ (4.15) $\wedge$ (4.17). So we have proved the first part of theorem 3. All the other assertions of theorem 3 follow directly from lemma 6 and theorem 2. □

In the case of our scattering problem we have $n = 3$ and we have given real numbers $m_1, \ldots, m_k > 0$ and $q_1, \ldots, q_k \in \mathbb{R}$ representing the respective masses and electric charges of the k particles. For $p = (t, z, v) \in P$ and $i \leqq k$ the i-th component $G(p)_i$ of $G(p)$ is given as a sum of $k - 1$ terms of the type of the right-hand side of equation (0.1) representing the contributions of the $k - 1$ other particles:

$$G(p)_i = \frac{q_i}{m_i}\sqrt{1 - |v_{ii}|^2} \sum_{\substack{j=1,\ldots,k \\ j \neq i}} (E_{ij}-(E_{ij}\cdot v_{ii})v_{ii}+v_{ii} \times B_{ij}) \quad (4.25)$$

where the contributions E_{ij} and B_{ij} of the velocity-dependent parts of the electric field and of the magnetic induction at the i-th particle coming from the j-th particle are given as in (0.2) and (0.3):

$$E_{ij}=q_j\frac{1-|v_{ij}|^2}{[|z_{ii}-z_{ij}|-(z_{ii}-z_{ij})\cdot v_{ij}]^3}(z_{ii}-z_{ij}-|z_{ii}-z_{ij}|v_{ij}) \tag{4.26}$$

$$B_{ij}=v_{ij}\times E_{ij} \tag{4.27}$$

DEFINITION 6 (« truncated equation of motion »). — If $G: P \to (\mathbb{R}^3)^k$ is the function defined by (4.25), (4.26) and (4.27) then we say that (0.7) is the truncated equation of motion for a system of k particles with respective masses $m_1, \ldots, m_k$ and electric charges $q_1, \ldots, q_k$. □

THEOREM 4. — Assume $n = 3$. Let $m_1, \ldots, m_k > 0$ and $q_1, \ldots, q_k \in \mathbb{R}$. Then there is a function $f: V \to (\mathbb{R}^3)^k$ such that for every $v_{-\infty} \in V$, $\gamma \in (\mathbb{R}^3)^k$ there is an initial interval I in which the truncated equation of motion for a system of k particles with respective masses $m_1, \ldots, m_k$ and electric charges $q_1, \ldots, q_k$ with the initial conditions

$$\lim_{t\to-\infty}[x(t)-(v_{-\infty}\cdot t+f(v_{-\infty})\cdot\ln(-t)+\gamma)]=0 \tag{4.28}$$

$$\lim_{t\to-\infty}x'(t)=v_{-\infty} \tag{4.29}$$

has a strictly unique solution $x = x[v_{-\infty}, \gamma]$.

For x it even holds that

$$\limsup_{t\to-\infty}|x'(t)-v_{-\infty}|\cdot|t|<\infty.$$

Every initially unbound solution x is such a solution $x[v_{-\infty}, \gamma]$ with some $v_{-\infty} \in V$ and $\gamma \in (\mathbb{R}^3)^k$. x is obtained by an iteration process, as in theorem 3, with

$$x_0(t) := v_{-\infty}\cdot t+f(v_{-\infty})\cdot\ln(-t)+\gamma$$

and the error estimate of theorem 3 holds. □

For the proof of theorem 4 we first have to prove

LEMMA 9. — The function G defined by (4.25), (4.26) and (4.27) has normal fall-off. □

Proof. — Let μ and δ be positive real numbers such that $\mu < 1$. Let $p = (t, z, v) \in P_{\mu\delta}$. We define $\tau_{ij} := |z_{ii} - z_{ij}|$ for $i \neq j$. Then $\tau_{ij} \geqq \delta$ for $i \neq j$ and $|v_{ij}| \leqq \mu$ for all $i, j \leqq k$. In order to obtain an estimate for $|G(p)|$ we first estimate the expression for E_{ij} given in (4.26):

$$\tau_{ij}-(z_{ii}-z_{ij})\cdot v_{ij}\geqq\tau_{ij}-\tau_{ij}|v_{ij}|\geqq(1-\mu)\tau_{ij},$$

$$|z_{ii}-z_{ij}-\tau_{ij}v_{ij}|\leqq\tau_{ij}+\tau_{ij}|v_{ij}|\leqq(1+\mu)\tau_{ij}.$$

Thus $E_{ij} \leqq \dfrac{\eta_j}{\tau_{ij}^2} \leqq \dfrac{\eta_j}{\delta^2}$ where $\eta_j := q_j\dfrac{1+\mu}{(1-\mu)^3}$.

Because of (4.27) we thus have

$$|\mathrm{E}_{ij} - (\mathrm{E}_{ij} \cdot v_{ij})v_{ii} + v_{ii} \times \mathrm{B}_{ij}| \leqq \frac{\eta_j(1 + 2\mu^2)}{\delta^2}$$

and hence $|\mathrm{G}(p)_i| \leqq \frac{v_i}{\delta^2}$ where $v_i := \frac{q_i}{m_i}(1 + 2\mu^2)\sum_{j=1}^{k}\eta_j$.

Now let $v := \sqrt{\sum_{i=1}^{k} v_i^2}$. Then $|\mathrm{G}(p)| \leqq \frac{v}{\delta^2}$.

Thus (4.5) holds if we choose $c \geqq v$. A similar simple calculation shows that there is a $c \geqq v$ such that (4.6) holds for all $\delta > 0$ and all $\overset{1}{p} = (t, \overset{1}{z}, \overset{1}{v}) \in \mathrm{P}_{\mu\delta}$ and $\overset{2}{p} = (t, \overset{2}{z}, \overset{2}{v}) \in \mathrm{P}_{\mu\delta}$. □

Proof of theorem 4. — We want to apply theorem 3. So we calculate the function $x_{\mathrm{G}}[v_{-\infty}, \gamma]$ defined in (4.9). In order to do this we have first to calculate $\mathrm{G}(p[a](t))$. For simplicity we first consider only two particles. We have to evaluate $\vec{a}$ from equations (0.1), (0.2) and (0.3) for $\vec{x} = \vec{v}t$ and $\vec{y} = \vec{w}\cdot(t - \tau)$ where $\tau = |\vec{x} - \vec{y}|$. Here $\vec{v}$ and $\vec{w}$ are time-independent. A simple calculation shows

$$\vec{a} = \vec{g}(q, \tilde{q}, m, \vec{v}, \vec{w})\cdot\frac{1}{t^2}$$

where

$$\vec{g}(q, \tilde{q}, m, \vec{v}, \vec{w}) = \frac{q\tilde{q}}{m}\frac{(1 - |\vec{v}|^2)^{3/2}(1 - |\vec{w}|^2)}{\sqrt{[(\vec{v}-\vec{w})\cdot\vec{w}]^2 + |\vec{v}-\vec{w}|^2(1 - |\vec{w}|^2)}^{\,3}}\cdot(\vec{v}-\vec{w}).$$

If we now sum up all these contributions from the various particles as in (4.25) we obtain

$$\mathrm{G}(p[a]) = -f(v_{-\infty})\cdot\frac{1}{t^2} \tag{4.30}$$

where

$$(f(v))_i := -\sum_{\substack{j=1,\dots,k\\ j\neq i}} \vec{g}(q_i, q_j, m_i, v_i, v_j) \quad \text{for} \quad v \in \mathrm{V}, \quad i \leqq k.$$

From (4.9) and (4.30) we get setting $t_0 := -1$,

$$x_{\mathrm{G}}[v_{-\infty}, \gamma - v_{-\infty}] = v_{-\infty}\cdot t + f(v_{-\infty})\cdot\ln(-t) + \gamma.$$

From lemma 9 we know that G has normal fall-off. Let

$$x_0 := x_{\mathrm{G}}[v_{-\infty}, \gamma - v_{-\infty}].$$

Now we apply theorem 3, with γ replaced by $\gamma - v_{-\infty}$, and obtain immediately the assertions of theorem 4. □

REMARK 5. — Using the methods developed here we can also treat

TABLE OF MAIN ASSUMPTIONS AND ASSERTIONS IN THE THEOREMS.

	Theorem 1	Theorem 2	Theorem 3	Theorem 4
Equation	Functional differential equation $x''(t)=F(t, x)$	Retarded differential equation $x''(t)=\dot{G}(p[x](t))$	Retarded differential equation $x''(t)=G(p[x](t))$	Truncated equation of motion for a system of particles
$\exists x_0$ assumed	Yes	Yes	No, proved	No, proved and x_0 explicitly calculated
Domain of definition of $F(t, .)$ or G	$D \subset x_0 + X$ Where X Banach space (D depends on x_0)	Set $A \subset P$ « containing world tubes around the particles » (A depends on x_0)	P	P
Conditions assumed for F or G	$\mid F(t, x) - F(t, y) \mid \leqq f(t) \cdot \| x - y \|$ $T(B \cap D) \subset D$ D closed	$\mid G(t, \overset{1}{z}, \overset{1}{v}) - G(t, \overset{2}{z}, \overset{2}{v}) \mid \leqq \frac{c}{\mid t \mid^3} \mid \overset{1}{z} - \overset{2}{z} \mid + \frac{c}{\mid t \mid^2} \mid \overset{1}{v} - \overset{2}{v} \mid$ $\mid G(t, z, v) \mid \leqq \frac{c}{\mid t \mid}$ A closed conditions (3.1), (3.2) for A	« Normal fall-off » : $\mid G(\overset{1}{p}) - G(\overset{2}{p}) \mid \leqq \frac{c}{\delta^3} \mid \overset{1}{z} - \overset{2}{z} \mid + \frac{c}{\delta^2} \mid \overset{1}{v} - \overset{2}{v} \mid$ $\mid G(p) \mid \leqq \frac{c}{\delta^2}$	
Initial conditions	$x(t)=x_0(t)+o(1)$ $x'(t)=x'_0(t)+o(1)$ and what is hidden in $x \in D$	$x(t)=x_0(t)+o(1)$ $x'(t)=x'_0(t)+0\left(\frac{1}{\mid t \mid}\right)$ (the latter condition is hidden in $x \in D$ in theorem 1)	$x(t)=x_G[v_{-\infty}, \gamma](t)+o(1)$ $x'(t)=v_{-\infty}+o(1)$ or $v_{-\infty}+0\left(\frac{1}{\mid t \mid}\right)$	$x(t)=v_{-\infty}t + f(v_{-\infty}) \ln(-t) + \gamma + o(1)$ $x'(t)=v_{-\infty}+o(1)$ or $v_{-\infty}+0\left(\frac{1}{\mid t \mid}\right)$
Existence of a solution proved	Yes	In an initial subinterval	In an initial interval	In an initial interval
Uniqueness proved	Yes	In an initial subinterval	In an initial interval	In an initial interval
Strict uniqueness proved	No, meaningless	In an initial subinterval	In an initial interval	In an initial interval
			Every initially unbound solution obeys a set of initial conditions with some $v_{-\infty}$ and γ	

the same scattering problem with an additional time-variable external electromagnetic field which falls off like r^{-2} uniformly with respect to time and whose gradient falls off like r^{-3}, also uniformly with respect to time. We can directly use theorem 3 if we add a $(k + 1)$-th « fictitious » particle which always stays at the origin and obeys the equation $\vec{a} = 0$ instead of (0.1) and which produces the external electromagnetic field. We must then assume that the limits of the velocities of the particles are not only different from each other but also from zero in the frame which we have chosen.

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