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Section A :
Physique théorique.

Charged particles with short range interactions

by

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ABSTRACT. — Schrödinger Hamiltonians for charged particles with an additional force of very short range are studied by scaling techniques and with a view towards low energy parameters. We present results for the zero range limit as well as regarding the approach to it. In particular we give the leading terms for the S-matrix as the range parameter becomes small. As applications we compare the scattering lengths of charged particles and of their neutral counterparts and discuss the level shifts of mesic atoms.

RÉSUMÉ. — On étudie des Hamiltoniens de Schrödinger pour des particules chargées avec une interaction supplémentaire de très courte

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portée, par des méthodes de changement d'échelle, à basse énergie. On présente des résultats sur la limite de portée nulle et l'approche de cette limite. En particulier, on donne les termes dominants de la matrice S pour les petites valeurs de la portée. A titre d'application, on compare les longueurs de diffusion pour des particules chargées et leurs analogues neutres, et on discute les déplacements de niveau des atomes mésiques.

1. INTRODUCTION

Low energy expansions—particularly in the form of the effective range approximation in scattering theory—play an important role in the description and interpretation of particle interactions [*32*].

On the level of dynamics it was recognized long ago [*11*] [*45*] that the low energy behaviour of scattering amplitudes typical for short range interactions is modelled by « δ-function potentials »: The scattering length approximation becomes exact as the range of the potential goes to zero. Furthermore the dynamics becomes simple in the sense that closed form solutions are available in this limit for off-shell T matrices, etc., a fact that made the model especially attractive for n-body calculations where these off-shell two-body quantities are used as dynamical input (for a review and further references see [*17*]). The failure of these early calculations pointed towards necessity of a precise mathematical treatment of zero range interactions [*2*] [*5*] [*7*] [*9*], for 3 particle systems [*8*] [*31*] as well as multi-center problems [*2*] [*20*]-[*22*] [*28*]. Once these idealized models are well understood it is worthwile to go one step further, i. e. to include effective range phenomena through a careful investigation of the low energy [*26*], long distance scaling limit [*3*].

For the description of the vast majority of scattering experiments it is inevitable to also take Coulomb forces into account. As is well known, special considerations are necessary in view of the long range nature of the Coulomb potential such as in [*14*] [*36*] and references therein. This is well exemplified by a comparison of the effective range expansions [*10*] [*46*] in charged and neutral particle scattering.

In the present note we investigate Schrödinger Hamiltonians with a very short range plus Coulomb potential with the aim of modelling the low energy behaviour of strongly interacting, charged particles. (Relativistic quasipotential models of charged particles with a contact interaction such as for positronium are discussed e. g. in [*38*] [*43*]).

Section 2 is devoted to a detailed study of the model in the limit of zero range. In particular the Hamiltonians are characterized as operator extensions, accretive so as to guarantee contraction semigroups. This generality is appropriate to be able to describe absorption such as in mesic atoms [15] [16]. We give explicit expressions for scattering and bound states.

There are different types of behaviour as the zero range limit is approached by scaling transformations. This depends on the low energy properties of the short range potential. We present the different cases in Section 3 and prepare the ground for Section 4. Here the corresponding types of behaviour are studied for the T-matrix and the scattering amplitude and we give the leading terms as the range parameter becomes small. We introduce the concept of a Coulomb modified scattering length in such a way that we need not invoke spherical symmetry of the short range potential and give an explicit expression which encompasses the case of neutral particles. While zero range interactions allow only for a fit of the scattering length we can now adjust the model separately to a given value of the effective range.

Applications in Section 5 fall in two categories. First we compare the effective range expansions of charged particles and of their neutral counterparts with a view towards charge symmetry questions. To do this adequately we exhibit the model dependence of the relation between charged and neutral scattering lengths which merits some attention because of the well known uncertainties [39] e. g. in the case of nucleon-nucleon scattering. As a second application we give the level shifts of mesic atoms. In the limit of small shifts we obtain a proof of the results of approximations discussed in previous work.

2. COULOMB PLUS POINT INTERACTIONS

In this section we describe a system of two charged particles under the additional influence of a zero-range interaction.

Let H^c denote the pure Coulomb Hamiltonian in $L^2(\mathbb{R}^3)$

$$H^c = -\Delta + \frac{\gamma}{|\underline{x}|}, \qquad \mathscr{D}(H^c) = \mathscr{D}(\Delta), \qquad \gamma \in \mathbb{R} \tag{2.1}$$

and suppose H^c_α to be the closed extension of $H^c|_{C_0^\infty(\mathbb{R}^3\setminus\{0\})}$ subject to the boundary condition, with $\alpha \in \mathbb{C}$

$$\left[-4\pi\alpha|\underline{x}|h(|\underline{x}|) + \frac{|\underline{x}|h(|\underline{x}|) - (|\underline{x}|h(|\underline{x}|))|_{|x|=0}(1+\gamma|\underline{x}|\ln(|\gamma|\,|\underline{x}|)}{|\underline{x}|}\right]\Bigg|_{|\underline{x}|=0} = 0, \tag{2.2}$$

in the subspace of angular momentum zero. More precisely, after decomposition into partial waves, and unitary transformation from $L^2((0, \infty); r^2dr)$ onto $L^2((0, \infty))$ the function $g(|\underline{x}|) = |\underline{x}| h(|\underline{x}|)$ lies in the set

$$\{ g \mid g' \text{ locally absolutely continuous on } (0, \infty);$$

$$\left[-4\pi\alpha g(r) + \frac{g(r) - g(0_+)(1 + \gamma r \ln(|\gamma| r))}{r} \right] \Bigg|_{r=0_+} = 0 ; g, -g'' + \frac{\gamma}{r} g \in L^2((0, \infty)) \Bigg\}.$$

For angular momentum $l \geqq 1$, H^c_α clearly coincides with H^c.

H^c_α describes two charged particles with an additional zero range interaction. As long as the particles stay away from each other, technically for states $h \in C_0^\infty(\mathbb{R}\backslash\{0\})$, H^c_α acts like the Coulomb Hamiltonian H^c. It differs from H^c precisely because of our freedom to specify what happens as the particles come into contact, technically this is stated in eq. (2.2). The resolvent equation for H^c_α reads ([47]):

$$\begin{aligned}(H^c_\alpha - k^2)^{-1} = G_{\gamma,k} - 4\pi \{ ik + \gamma [\Psi(1 + i\gamma/2k) \\ + \ln(2k/i|\gamma|) - \Psi(1) - \Psi(2)] - 4\pi\alpha \}^{-1} g_{\gamma,k}(\overline{g}_{\gamma,k}, \cdot), \\ \operatorname{Im} k > 0, \quad k \neq -i\gamma/2n, \quad n = 1, 2 \ldots \quad \text{if} \quad \gamma < 0, \end{aligned} \quad (2.3)$$

where $G_{\gamma,k}$ stands for the pure Coulomb resolvent

$$G_{\gamma,k} = (H^c - k^2)^{-1}, \quad \operatorname{Im} k > 0, \quad k \neq -i\gamma/2n, \quad n = 1, 2 \ldots \quad \text{if} \quad \gamma < 0, \quad (2.4)$$

$g_{\gamma,k}$ denotes

$$g_{\gamma,k}(\underline{x}) = G_{\gamma,k}(\underline{x}, \underline{0}) = (4\pi |\underline{x}|)^{-1} \Gamma(1 + i\gamma/2k) W_{-\frac{i\gamma}{2k};\frac{1}{2}}(-2ik|\underline{x}|), \quad (2.5)$$

and $W_{\mu;\nu}(z)$ and $\Psi(z) = \Gamma'(z)/\Gamma(z)$ are the Whittaker and psi-function respectively ($\Gamma(z)$ being the usual gamma function) [1]. In the special case $\gamma = 0$ we denote the closed extension of $-\Delta|_{C_0^\infty(\mathbb{R}^3\backslash\{0\})}$ given by (2.2) with $\gamma = 0$ by $-\Delta_\alpha$ (see [3] [5]).

Obviously H^c_α is continuous in α in norm resolvent sense. A further property of H^c_α is given by

THEOREM 2.1. — iH^c_α (resp. $-iH^c_{\overline{\alpha}}$) generates a contraction semigroup

$$e^{-itH^c_\alpha} \text{ (resp. } e^{itH^c_{\overline{\alpha}}}), \quad t \geqq 0 \text{ if } \operatorname{Im}\alpha \leqq 0 \text{ (resp. } \operatorname{Im}\alpha \geqq 0) \text{ in } L^2(\mathbb{R}^3).$$

Proof. — We follow [12] where this result has been proven in the short-range case $\gamma = 0$. Define $H^c_0 = \overline{H^c}|_{C_0^\infty(\mathbb{R}^3\backslash\{0\})}$ (where — means closure) and

$$\phi_\pm(\underline{x}) = g_{\gamma,\sqrt{\pm i}}(\underline{x}), \qquad \sqrt{\pm i} = \cos\frac{\pi}{4}(\pm 1 + i).$$

Let the complex number θ be such that

$$4\pi\alpha=(1+e^{i\theta})^{-1}\left\{\left[i\gamma(\Psi(1+i\gamma/2\sqrt{i})-\Psi(1)-\Psi(2))/2\sqrt{i}-\frac{1}{2}\right](-2i\sqrt{i})\right.$$
$$\left.+e^{i\theta}\left[i\gamma(\Psi(1+i\gamma/2\sqrt{-i})-\Psi(1)-\Psi(2))/2\sqrt{-i}-\frac{1}{2}\right](-2i\sqrt{-i})\right\}$$

then

$$\mathcal{D}(H^c_\alpha) = \{ f + d\phi_+ + de^{i\theta}\phi_- / f \in \mathcal{D}(H^c_0);\ d \in \mathbb{C} \},$$
$$H^c_\alpha(f + d\phi_+ + de^{i\theta}\phi_-) = H^c_0 f + id\phi_+ - ide^{i\theta}\phi_- .$$

By straightforward computation

$$\operatorname{Im}\left([f+d\phi_+ +de^{i\theta}\phi_-],\ H^c_\alpha[f+d\phi_+ +de^{i\theta}\phi_-]\right)=|d|^2\,\|\phi_\pm\|^2(1-e^{-2\operatorname{Im}\theta})$$

and thus

$$\operatorname{Im}(h, H^c_\alpha h) \leqq 0 \quad \text{for all} \quad h \in \mathcal{D}(H^c_\alpha) \Leftrightarrow \operatorname{Im}\theta \leqq 0 \Leftrightarrow \operatorname{Im}\alpha \leqq 0 .$$

Thus iH^c_α is accretive [35] and therefore maximal accretive iff $\operatorname{Im}\alpha \geqq 0$. □

We also note that the operators H^c_α exhaust the class of all maximal accretive extensions of $iH^c|_{C_0^\infty(\mathbb{R}^3\setminus\{0\})}$.

Next we describe the point spectrum of H^c_α in the case $\alpha \in \mathbb{R}$. Then it is easily seen from (2.3) that H^c_α is self-adjoint.

THEOREM 2.2. — Let $\alpha \in \mathbb{R}$. If $\gamma \geqq 0$ then H^c_α has precisely one negative bound state if $\alpha < \gamma(2C - 1)/4\pi$ (C is Euler's constant i. e. C = 0,5772). The eigenvalue $E < 0$ is determined by the equation

$$4\pi\alpha = \gamma F(\gamma/2\sqrt{-E}), \quad \gamma \geqq 0,$$
$$F(x) = \Psi(1 + x)-(\ln(x^2))/2-1/2x-\Psi(1)-\Psi(2). \quad (2.6)$$

If $\alpha \geqq \gamma(2C-1)/4\pi$ the point spectrum of H^c_α is empty.

If $\gamma < 0$, then for all $\alpha \in \mathbb{R}$ there are always infinitely many negative eigenvalues associated with the s-wave given by the equation

$$4\pi\alpha = \gamma F(\gamma/2\sqrt{-E}), \quad \gamma < 0 . \quad (2.7)$$

For angular momenta $l > 0$ we get the usual Coulomb levels

$$E_n = -\gamma^2/4n^2, \qquad n = 2, 3, \ldots .$$

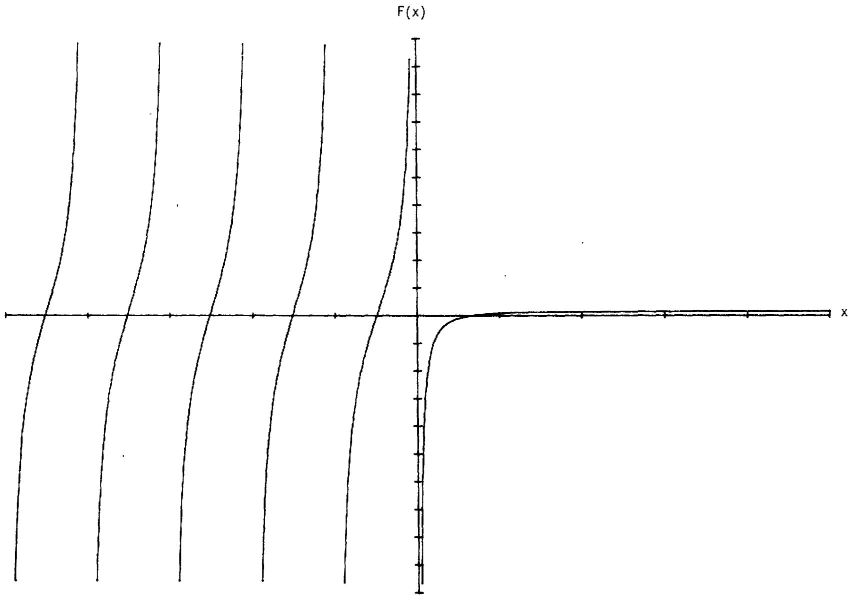
Proof. — We confine ourselves to s-waves. Following Rellich [37] who determined the spectrum for $\gamma < 0$ (2.6) and (2.7) are easily seen to be the eigenvalue equations. Let $\gamma \geqq 0$, then using ([1], p. 259).

$$F(x) = -2\int_0^\infty dt\,t(e^{2\pi t}-1)^{-1}(t^2+x^2)^{-1}-\Psi(1)-\Psi(2)$$

we obtain

$$F(0_+) = -\infty, \qquad F(+\infty) = -\Psi(1) - \Psi(2) = 2C - 1,$$

and $F'(x) > 0$ for all $x > 0$. Thus $F(x)$ for $x > 0$ is strictly increasing from $-\infty$ to $2C-1$ (see. fig. 1) and (2.6) has exactly one solution if $\alpha < \gamma(2C-1)/4\pi$. In the attractive case $\gamma < 0$, $F(x)$ for $x < 0$ is strictly increasing from $-\infty$ to $+\infty$ in each interval $(-n-1, -n)$, $n=0, 1, \ldots$ (cf. fig. 1), completing the proof. □



Now we turn to scattering theory. It is common practice to subtract the pure Coulomb part from the scattering amplitude. Indeed according to eq. (2.3) the total on-shell scattering amplitude $f_\alpha^c(k, \underline{\omega}, \underline{\omega}')$ as well as the total scattering operators S_α^c and $S_\alpha^c(k)$ in $L^2(\mathbb{R}^3)$ and $L^2(S^{(2)})$ ($S^{(2)}$ the unit sphere in $\mathbb{R}^3$) respectively split naturally into

$$f_\alpha^c(k, \underline{\omega}, \underline{\omega}') = f^c(k, \underline{\omega}, \underline{\omega}') + f_\alpha^{sc}(k, \underline{\omega}, \underline{\omega}'), \quad k > 0, \tag{2.8}$$

$$S_\alpha^c = S^c + T_\alpha^{sc}, \tag{2.9}$$

$$S_\alpha^c(k) = S^c(k) + T_\alpha^{sc}(k), \quad k > 0, \tag{2.10}$$

where f^c, S^c, $S^c(k)$ denote the corresponding Coulomb objects. Moreover we have, including the possibility of absorption (i. e. $\operatorname{Im} \alpha < 0$) in the zero range interaction

THEOREM 2.3. — Let $\operatorname{Im} \alpha \leqq 0$, then S_α^c and $S_\alpha^c(k)$, $k > 0$ are contractions in $L^2(\mathbb{R}^3)$ and $L^2(S^{(2)})$ respectively. The phase shifts $\delta_\alpha^{l,c}(k)$ associated with H_α^c fulfill

$$\begin{aligned} \delta_\alpha^{0,c}(k) &= \delta^{0,c}(k) + \delta_\alpha^{0,sc}(k), \\ \delta_\alpha^{l,c}(k) &= \delta^{l,c}(k), \quad l \geqq 1 \end{aligned} \tag{2.11}$$

where

$$e^{2i\delta^{l,c}(k)} = \frac{\Gamma(l+1+i\gamma/2k)}{\Gamma(l+1-i\gamma/2k)}, \qquad l \geqq 0,$$

$$(e^{2i\delta_\alpha^{0,sc}(k)} - 1) = -2\pi i\gamma(e^{\pi\gamma/k} - 1)^{-1}[\gamma F(i\gamma/2k) - 4\pi\alpha]^{-1}. \tag{2.12}$$

Consequently

$$f_\alpha^{sc}(k, \underline{\omega}, \underline{\omega}') = -e^{-\pi\gamma/2k}\Gamma(1+i\gamma/2k)^2[\gamma F(i\gamma/2k) - 4\pi\alpha]^{-1}, \tag{2.13}$$

and

$$\begin{aligned} (T_\alpha^{sc}(k)h)(\underline{\omega}) &= -(2\pi i)^{-1}k\int d\omega' f_\alpha^{sc}(k, \underline{\omega}, \underline{\omega}')h(\underline{\omega}') \\ &= -2ike^{-\pi\gamma/2k}\Gamma(1+i\gamma/2k)^2[\gamma F(i\gamma/2k) - 4\pi\alpha]^{-1}Y_{0,0}(\underline{\omega})(Y_{0,0}, h) \end{aligned} \tag{2.14}$$

where

$$Y_{0,0}(\underline{\omega}) = (4\pi)^{-1/2}$$

and

$$F(i\gamma/2k) = \lim_{\varepsilon \to 0_+} F((1 - i\varepsilon)i\gamma/2k).$$

Proof. — (2.13) and (2.14) follow from (2.12) which in turn is obtained by eigenfunction expansion. □

Of course if $\alpha \in \mathbb{R}$ then S_α^c, $S_\alpha^c(k)$ are unitary and

$$\sigma_{ess}(H_\alpha^c) = \sigma_{ac}(H_\alpha^c) = [0, \infty), \qquad \sigma_{sc}(H_\alpha^c) = \emptyset$$

(where σ_{ess}, σ_{ac}, σ_{sc} means the essential, absolutely continuous, singular continuous spectrum, respectively.) For the corresponding spectral and scattering properties in the case where $\gamma/|\underline{x}|$ is replaced by some short-range potential $V(\underline{x})$ and an explicit determination of the point spectrum and the scattering phase shifts in the case $V(\underline{x}) = (\beta - 1/4)/|\underline{x}|^2$, $\beta > 0$ see [*19*].

Finally we turn to the effective range expansion. For suitable spherically symmetric short-range interactions $V(r)$ in addition to the Coulomb potential γ/r the modified effective range expansion in the s-wave reads [*10*] [*25*]

$$\begin{aligned} \pi\gamma(e^{\pi\gamma/k} - 1)^{-1}[\cot(\delta^{0,sc}(k)) - i] + \gamma[\Psi(1+i\gamma/2k) + \ln(2k/i\,|\,\gamma\,|)] + ik \\ = -\frac{1}{a^{0,sc}} + (1/2)r^{0,sc}k^2 + 0(k^4) \end{aligned} \tag{2.15}$$

where $a^{0,sc}$ and $r^{0,sc}$ denote the Coulomb modified (s-wave) scattering length and effective range respectively. Under appropriate conditions on $V(r)$ (cf. [*4*]) the left hand side of (2.15) is real analytic in k^2 around $k^2 = 0$.

Inserting (2.12) into (2.15) we obtain exactly $-1/a^{0,sc}(\alpha)$ for the right hand side of (2.15)

$$\pi\gamma(e^{\pi\gamma/k}-1)^{-1}\ [\cot(\delta_\alpha^{0,sc}(k))-i]+\gamma[\Psi(1+i\gamma/2k)+\ln(2k/i\,|\,\gamma\,|)]+ik$$
$$= 4\pi\alpha+\gamma[\Psi(1)+\Psi(2)] = -\frac{1}{a^{0,sc}(\alpha)} \quad (2.16)$$

i. e. $r^{0,sc}(\alpha) = 0$ and all terms of higher order in k^2 also vanish identically ! This fact obviously justifies the name « zero-range interaction ».

3. CONVERGENCE TOWARDS POINT INTERACTIONS

The question we now discuss is: In what sense are zero range models an approximation of short range ones? We show that they arise as a norm resolvent limit of a sequence of charged particle Hamiltonians with short range interactions. In $L^2(\mathbb{R}^3)$ we introduce the following operators defined as quadratic forms [*40*]

$$H = -\Delta + V(\underline{x}) \quad (3.1)$$

$$H_\gamma(\varepsilon) = -\Delta + \frac{\gamma}{|\underline{x}|} + \lambda(\varepsilon, \gamma\varepsilon\ln\varepsilon)V(\underline{x}), \quad 0 \leqq \varepsilon < 1, \quad (3.2)$$

$$H_{\gamma,\varepsilon} = \varepsilon^{-2}U_\varepsilon H_{\varepsilon\gamma}(\varepsilon)U_\varepsilon^{-1} = -\Delta + \frac{\gamma}{|\underline{x}|} + \lambda(\varepsilon,\gamma\varepsilon\ln\varepsilon)\ \varepsilon^{-2}V(\underline{x}/\varepsilon), 0<\varepsilon<1 \quad (3.3)$$

where V is a real measurable function such that $e^{2a|\underline{x}|}V(\underline{x}) \in R$ (the Rollnik class) for some $a > 0$ i. e.

$$\int d^3x d^3y e^{2a(|\underline{x}|+|\underline{y}|)}\,|\,V(\underline{x})V(\underline{y})\,|\,|\,\underline{x}-\underline{y}\,|^{-2} < \infty \quad (3.4)$$

and $\lambda\,(.,.)$ is analytic (not necessarily real valued) around the origin, with $\lambda(0,0) = 1$. U_ε denotes the unitary dilation group on $L^2(\mathbb{R}^3)$

$$(U_\varepsilon h)(\underline{x}) = \varepsilon^{-3/2}h(\underline{x}/\varepsilon),\ \varepsilon > 0,\ h \in L^2(\mathbb{R}^3). \quad (3.5)$$

With the notation

$$v(\underline{x}) = |\,V(\underline{x})\,|^{1/2}, \qquad u(\underline{x}) = |\,V(\underline{x})\,|^{1/2}\ \text{sign}\ V(\underline{x})$$

we obtain from (3.3)

$$(H_{\gamma,\varepsilon}-k^2)^{-1}$$
$$= G_{\gamma,k} - \lambda(\varepsilon,\gamma\varepsilon\ln\varepsilon)\varepsilon^{-2}G_{\gamma,k}U_\varepsilon v(\lambda(\varepsilon,\gamma\varepsilon\ln\varepsilon)uG_{\varepsilon\gamma,\varepsilon k}v+1)^{-1}uU_\varepsilon^{-1}G_{\gamma,k},\ k^2 \notin \sigma(H_{\gamma,\varepsilon}). \quad (3.6)$$

From eq. (3.6) we infer that in order to find the limit of $H_{\gamma,\varepsilon}$ as $\varepsilon \to 0_+$ we have to control $(\lambda(\varepsilon,\gamma\varepsilon\ln\varepsilon)uG_{\varepsilon\gamma,\varepsilon k}v+1)^{-1}$ as $\varepsilon \to 0_+$. It turns out that similar to the short-range case $\gamma=0$ [3] [5] the behaviour of this inverse operator is governed by the spectral properties of $H=-\Delta+V(\underline{x})$ at zero energy. Thus we recall the notion of zero-energy resonance functions of H [3] [5]: If -1 is an eigenvalue of $uG_{0,0}v$ i. e.

$$uG_{0,0}v\phi_j = -\phi_j \quad \text{for some} \quad \phi_j \in L^2(\mathbb{R}^3), \qquad j = 1, \ldots, N$$

we call the functions $\psi_j(\underline{x})$

$$\psi_j(\underline{x}) = (G_{0,0}v\phi_j)(\underline{x}), \qquad j = 1, \ldots, N \tag{3.7}$$

(zero-energy) resonance functions of H and note that

$$H\psi_j = (-\Delta + V)\psi_j = 0, \qquad j = 1, \ldots, N$$

in the sense of distributions. Under hypothesis (3.4) (actually $V \in R$ and $(1+|\underline{x}|)V(\underline{x}) \in L^1(\mathbb{R}^3)$ suffices [3]) $\psi_j(\underline{x}) \in L^2_{loc}(\mathbb{R}^3)$, $j = 1, \ldots, N$ and $\psi_{j_0}(\underline{x}) \in L^2(\mathbb{R}^3)$ if and only if $(v, \phi_{j_0}) = -\int d^3x V(\underline{x})\psi_{j_0}(\underline{x}) = 0$.

With these notions in mind we now distinguish the following cases [3] [5] [26] [30] [33]:

CASE I. There exist no resonance functions ψ_j (i. e. -1 is not an eigenvalue of $uG_{0,0}v$).

CASE II. There exists precisely one resonance function ψ (i. e. -1 is a simple eigenvalue of $uG_{0,0}v$) and ψ is not in $L^2(\mathbb{R}^3)$.

CASE III. There exist $N \geqq 1$ resonance functions ψ_j, $j = 1, \ldots, N$ which are all in $L^2(\mathbb{R}^3)$.

CASE IV. There exist $N \geqq 2$ resonance functions ψ_j, $j = 1, \ldots, N$ and at least one of them is not in $L^2(\mathbb{R}^3)$.

We note that in case IV one can always choose a particular set of linear combinations of the resonance functions ψ_j such that $(v, \phi_1) \neq 0$ and $(v, \phi_j)=0, j=2, \ldots, N$. From now on we adopt this convention throughout. Thus in case III zero is an eigenvalue of H with multiplicity N whereas in case IV its multiplicity is $N-1$. We also stress that in the case of spherically symmetric potentials V, due to symmetry $(v, \phi) = -\int d^3x V(|\underline{x}|)\psi(\underline{x}) = 0$ for resonance functions ψ belonging to angular momentum $l \geqq 1$, in other words case II will only be found for $l=0$.

From the explicit representation of $G_{\gamma,k}$ in terms of Whittaker functions [23] [24] we observe that $uG_{\varepsilon\gamma,\varepsilon k}v$ is of the form

$$uG_{\varepsilon\gamma,\varepsilon k}v = \sum_{m=0}^{\infty} \hat{G}_{m0}(\gamma, k)\varepsilon^m + \gamma\varepsilon \ln \varepsilon \sum_{m=0}^{\infty} \hat{G}_{m1}(\gamma, k)\varepsilon^m. \tag{3.8}$$

Analyticity of λ amounts to

$$\lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon) = \sum_{m,n=0}^{\infty} \lambda_{mn}\varepsilon^m(\gamma\varepsilon \ln \varepsilon)^n, \qquad \lambda_{00} = 1, \tag{3.9}$$

so that after multiplication and reordering we again obtain a double series for $\lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon)uG_{\varepsilon\gamma,\varepsilon k}v$ in powers of ε and $(\gamma\varepsilon \ln \varepsilon)$:

$$\lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon)uG_{\varepsilon\gamma,\varepsilon k}v = \sum_{m,n=0}^{\infty} B_{mn}(\gamma, k)\varepsilon^m(\gamma\varepsilon \ln \varepsilon)^n. \tag{3.10}$$

For an explicit determination of the lowest order coefficients $\hat{G}_{m0}$, $\hat{G}_{m1}$, and B_{mn} we refer to the appendix. We also need the (norm convergent) expansion of

$$(uG_{0,0}v + 1 + \mu)^{-1} = \frac{P}{\mu} + T - \mu T^2 + \mu^2 T^3 + 0(\mu^3), \tag{3.11}$$

where P is the projector onto the eigenspace of $uG_{0,0}v$ to the eigenvalue -1,

$$P = -\sum_{j=1}^{N} (\tilde{\phi}_j, .)\phi_j, \qquad N = \dim P, \qquad \tilde{\phi}_j(\underline{x}) = \phi_j(\underline{x}) \operatorname{sign}(V(\underline{x})) \tag{3.12}$$

(from now on the normalization $(\tilde{\phi}_j, \phi_l) = -\delta_{jl}$ will be used) and T is the so called reduced resolvent [*27*]

$$T = n - \lim_{\mu\to 0} (uG_{0,0}v + 1 + \mu)^{-1}(1 - P). \tag{3.13}$$

For a proof of expansion (3.11) with P and T given by (3.12) and (3.13) compare [*3*] lemma 3.1.

With these preliminaries we are now in position to control the behaviour of $\varepsilon(\lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon)uG_{\varepsilon\gamma,\varepsilon k}v + 1)^{-1}$ as $\varepsilon \to 0_+$ according to cases I-IV:

LEMMA 3.1. — Assume case I (i. e. $P=0$). Then $(\lambda uG_{\varepsilon\gamma,\varepsilon k}v+1)^{-1}$ is analytic in ε, $(\gamma\varepsilon \ln \varepsilon)$ around the origin and we get the following expansion (valid in norm)

$$\varepsilon(\lambda uG_{\varepsilon\gamma,\varepsilon k}v + 1)^{-1} = \varepsilon[T - \varepsilon TB_{10}T - (\gamma\varepsilon \ln \varepsilon)TB_{01}T + 0((\varepsilon \ln \varepsilon)^2)]. \tag{3.14}$$

Proof. — Define

$$B(\varepsilon, k) = \lambda uG_{\varepsilon\gamma,\varepsilon k}v - uG_{0,0}v = \varepsilon B_{10}(k) + (\gamma\varepsilon \ln \varepsilon)B_{01} + 0((\varepsilon \ln \varepsilon)^2). \tag{3.15}$$

Then

$$\varepsilon(\lambda uG_{\varepsilon\gamma,\varepsilon k}v + 1)^{-1} = \varepsilon[1 + (1 + A_{00})^{-1}B(\varepsilon)]^{-1}(1 + A_{00})^{-1} = \varepsilon[1 + TB(\varepsilon)]^{-1}T, \tag{3.16}$$

which immediately proves (3.14). □

LEMMA 3.2. — Assume case II (i. e. $P = -(\tilde{\phi}, .)\phi$, $(v, \phi) \neq 0$).

A) if $\lambda_{01} = -|(v, \phi)|^2/4\pi$ then

$$\begin{aligned}
&\varepsilon(\lambda u G_{\varepsilon\gamma,\varepsilon k} v + 1)^{-1} \\
&= \langle B_{10} \rangle^{-1} (\tilde{\phi}, .)\phi + \varepsilon T - \varepsilon \langle B_{10} \rangle^{-1} [(T^* B_{10}^* \tilde{\phi}, .)\phi + (\tilde{\phi}, .) T B_{10} \phi] \\
&\quad - \gamma\varepsilon \ln \varepsilon \langle B_{10} \rangle^{-1} [(T^* B_{01}^* \tilde{\phi}, .)\phi + (\tilde{\phi}, .) T B_{01} \phi] - \\
&\quad - \langle B_{10} \rangle^{-2} [\varepsilon \langle B_{20} \rangle + \gamma\varepsilon \ln \varepsilon \langle B_{11} \rangle + \varepsilon(\gamma \ln \varepsilon)^2 \langle B_{02} \rangle](\tilde{\phi}, .)\phi + \\
&\quad + \langle B_{10} \rangle^{-2} [\varepsilon \langle B_{10} T B_{10} \rangle + \gamma\varepsilon \ln \varepsilon (\langle B_{10} T B_{01} \rangle + \langle B_{01} T B_{10} \rangle) \\
&\quad + \varepsilon(\gamma \ln \varepsilon)^2 \langle B_{01} T B_{01} \rangle](\tilde{\phi}, .)\phi + 0(\varepsilon^2 (\ln \varepsilon)^3). \qquad (3.17)
\end{aligned}$$

B) If $\lambda_{01} \neq -|(v, \phi)|^2/4\pi$ then

$$\begin{aligned}
\varepsilon(\lambda u G_{\varepsilon\gamma,\varepsilon k} v + 1)^{-1} &= (\gamma \ln \varepsilon)^{-1} \langle B_{01} \rangle^{-1} (\tilde{\phi}, .)\phi \\
&\quad - (\gamma \ln \varepsilon)^{-2} \langle B_{01} \rangle^{-2} \langle B_{10} \rangle (\tilde{\phi}, .)\phi + 0((\ln \varepsilon)^{-3}). \qquad (3.18)
\end{aligned}$$

Here we used the notation $\langle B \rangle = (\tilde{\phi}, B\phi)$ for bounded operators B. We also note that the expansions (3.17), (3.18) and all the following expansions in this section are valid in norm.

Proof. — With the help of (3.11) and (3.15) we get

$$\begin{aligned}
&\varepsilon(\lambda u G_{\varepsilon\gamma,\varepsilon k} v + 1)^{-1} = \varepsilon \{ 1 + (1 + B_{00} + \mu)^{-1} [B(\varepsilon) - \mu] \}^{-1} (1 + B_{00} + \mu)^{-1} \\
&= \varepsilon \{ PB(\varepsilon) + \mu - \mu P + \mu T B(\varepsilon) + 0(\mu^2) \}^{-1} (P + \mu T + 0(\mu^2)) \\
&= \varepsilon \{ 1 + \mu (PB(\varepsilon) + \mu)^{-1} [-P + TB(\varepsilon) + 0(\mu)] \}^{-1} (PB(\varepsilon) + \mu)^{-1} (P + \mu T + 0(\mu^2)). \qquad (3.19)
\end{aligned}$$

From $P = -(\tilde{\phi}, .)\phi$ we get

$$\mu(PB(\varepsilon) + \mu)^{-1} = 1 - \langle (B(\varepsilon) + \mu) \rangle^{-1} (B(\varepsilon)^* \tilde{\phi}, .)\phi \qquad (3.20)$$

and consequently

$$\begin{aligned}
\varepsilon(\lambda u G_{\varepsilon\gamma,\varepsilon k} v + 1)^{-1} &= \varepsilon \{ 1 + TB(\varepsilon) - \langle B(\varepsilon) \rangle^{-1} (B(\varepsilon)^* T^* B(\varepsilon)^* \tilde{\phi}, .)\phi + 0(\mu) \}^{-1} . \\
&\quad . [\langle B(\varepsilon) \rangle^{-1} (\tilde{\phi}, .)\phi + T - \langle B(\varepsilon) \rangle^{-1} (T^* B(\varepsilon)^* \tilde{\phi}, .)\phi + 0(\mu)]. \qquad (3.21)
\end{aligned}$$

Since $\mu > 0$ was arbitrary we get

$$\begin{aligned}
\varepsilon(\lambda u G_{\varepsilon\gamma,\varepsilon k} v + 1)^{-1} &= \{ 1 + TB(\varepsilon) - \langle B(\varepsilon) \rangle^{-1} (B(\varepsilon)^* T^* B(\varepsilon)^* \tilde{\phi}, .)\phi \}^{-1} . \\
&\quad . \varepsilon [\langle B(\varepsilon) \rangle^{-1} (\tilde{\phi}, .)\phi + T - \langle B(\varepsilon) \rangle^{-1} (T^* B(\varepsilon)^* T^* \tilde{\phi}, .)\phi] \\
&= \{ 1 - TB(\varepsilon) + \langle B(\varepsilon) \rangle^{-1} (B(\varepsilon)^* T^* B(\varepsilon)^* \tilde{\phi}, .)\phi + 0(\varepsilon^2 (\ln \varepsilon)^3) \} . \\
&\quad . \varepsilon [\langle B(\varepsilon) \rangle^{-1} (\tilde{\phi}, .)\phi + T - \langle B(\varepsilon) \rangle^{-1} (T^* B(\varepsilon)^* T^* \tilde{\phi}, .)\phi]. \qquad (3.22)
\end{aligned}$$

Now assume $\lambda_{01} = -|(v, \phi)|^2/4\pi$ which is equivalent to $\langle B_{01} \rangle = 0$. Then

$$\begin{aligned}
\langle B(\varepsilon) \rangle^{-1} = \varepsilon^{-1} \langle B_{10} \rangle^{-1} [1 - \langle B_{10} \rangle^{-1} (\varepsilon \langle B_{20} \rangle + \gamma\varepsilon \ln \varepsilon \langle B_{11} \rangle \\
+ \varepsilon(\gamma \ln \varepsilon)^2 \langle B_{02} \rangle) + 0(\varepsilon^2 (\ln \varepsilon)^3)] \qquad (3.23)
\end{aligned}$$

and (3.17) follows. Next suppose $\lambda_{01} \neq -|(v, \phi)|^2/4\pi$ ($\Leftrightarrow \langle B_{01} \rangle \neq 0$). Then

$$\langle B(\varepsilon) \rangle^{-1} = (\gamma\varepsilon \ln \varepsilon)^{-1} \langle B_{01} \rangle^{-1} [1 - (\gamma \ln \varepsilon)^{-1} \langle B_{01} \rangle^{-1} \langle B_{10} \rangle + 0((\ln \varepsilon)^{-2})] \quad (3.24)$$

which yields (3.18). □

LEMMA 3.3. — Assume case III

$$\left(\text{i. e.} \quad P = -\sum_{j=1}^{N} (\tilde{\phi}_j, .)\phi_j, \quad (v, \phi_j) = 0, \quad j = 1, \ldots, N \right).$$

A) If $\lambda_{01} = 0$ and the matrix $(\tilde{\phi}_j, B_{10}(k)\phi_l)$ is non singular we get

$$\begin{aligned}
\varepsilon(\lambda u G_{\varepsilon\gamma,\varepsilon k} v + 1)^{-1} &= \sum_{j,l=1}^{N} (\langle B_{10} \rangle)_{jl}^{-1} (\tilde{\phi}_l, .)\phi_j + \varepsilon T \\
&- \varepsilon \sum_{j,l=1}^{N} (\langle B_{10} \rangle)_{jl}^{-1} [(T B_{10}^* \tilde{\phi}_l, .)\phi_j + (\tilde{\phi}_l, .) T B_{10} \phi_j] \\
&- \gamma\varepsilon \ln \varepsilon \sum_{j,l=1}^{N} (\langle B_{10} \rangle)_{jl}^{-1} [(T^* B_{01}^* \tilde{\phi}_l, .)\phi_j + (\tilde{\phi}_l, .) T B_{01} \phi_j] \\
&+ \sum_{j,l,m,n=1}^{N} (\langle B_{10} \rangle)_{jl}^{-1} [\varepsilon \langle B_{10}^2 \rangle_{lm} + \gamma\varepsilon \ln \varepsilon \langle (B_{10} B_{01} + B_{01} B_{10}) \rangle_{lm} \\
&+ \varepsilon(\gamma \ln \varepsilon)^2 \langle B_{01}^2 \rangle_{lm}] (\langle B_{10} \rangle)_{mn}^{-1} (\tilde{\phi}_n, .)\phi_j \\
&+ \sum_{j,l,m,n=1}^{N} (\langle B_{10} \rangle)_{jm}^{-1} [\varepsilon \langle B_{20} \rangle_{mn} + \gamma\varepsilon \ln \varepsilon \langle B_{11} \rangle_{mn} \\
&+ \varepsilon(\gamma \ln \varepsilon)^2 \langle B_{02} \rangle_{mn}] (\langle B_{10} \rangle)_{nl}^{-1} (\phi_l, .)\phi_j + 0(\varepsilon^2 (\ln \varepsilon)^3) . \quad (3.25)
\end{aligned}$$

Here we used $\langle B \rangle_{jl} = (\tilde{\phi}_j, B\phi_l)$ and $(\langle B \rangle)_{jl}^{-1}$, the inverse matrix of $\langle B \rangle_{jl}$, where B is some bounded operator.

B) If $\lambda_{01} \neq 0$ we obtain

$$\begin{aligned}
\varepsilon(\lambda u G_{\varepsilon\gamma,\varepsilon k} v + 1)^{-1} &= (\gamma \ln \varepsilon)^{-1} \sum_{j,l=1}^{N} (\langle B_{01} \rangle)_{jl}^{-1} (\tilde{\phi}_l, .)\phi_j \\
&- (\gamma \ln \varepsilon)^{-2} \sum_{j,l,m,n=1}^{N} (\langle B_{01} \rangle)_{jm}^{-1} \langle B_{10} \rangle_{mn} (\langle B_{01} \rangle)_{nl}^{-1} (\tilde{\phi}_l, .)\phi_j + 0((\ln \varepsilon)^{-3}).
\end{aligned} \quad (3.26)$$

Proof. — Inserting

$$\mu(\mathrm{PB}(\varepsilon)+\mu)^{-1}=1-\sum_{j,l=1}^{\mathrm{N}}(\langle(\mathrm{B}(\varepsilon)+\mu)\rangle)_{jl}^{-1}(\mathrm{B}(\varepsilon)^*\tilde{\phi}_l,.)\phi_j \qquad (3.27)$$

into (3.19), we obtain after straightforward manipulations similar to that before and after (3.21)

$$\varepsilon(\lambda u\mathrm{G}_{\varepsilon\gamma,\varepsilon k}v+1)^{-1}=\{1-\mathrm{TB}(\varepsilon)+\sum_{j,l=1}^{\mathrm{N}}(\langle\mathrm{B}(\varepsilon)\rangle)_{jl}^{-1}(\mathrm{B}(\varepsilon)^*\mathrm{T}^*\mathrm{B}(\varepsilon)^*\tilde{\phi}_l,.)\phi_j$$

$$+0(\varepsilon^2(\ln\varepsilon)^3)\}.\varepsilon\left[\sum_{j,l=1}^{\mathrm{N}}(\langle\mathrm{B}(\varepsilon)\rangle)_{jl}^{-1}(\tilde{\phi}_l,.)\phi_j+\mathrm{T}-\sum_{j,l=1}^{\mathrm{N}}(\langle\mathrm{B}(\varepsilon)\rangle)_{jl}^{-1}(\mathrm{T}^*\mathrm{B}(\varepsilon)^*\tilde{\phi}_l,.)\phi_j\right]. \qquad (3.28)$$

Now assume $\lambda_{01}=0$ and $\langle \mathrm{B}_{10}\rangle_{jl}$ to be non singular. Then

$$\varepsilon(\langle\mathrm{B}(\varepsilon)\rangle)_{jl}^{-1}=(\langle\mathrm{B}_{10}\rangle)_{jl}^{-1}-\sum_{m,n=1}^{\mathrm{N}}(\langle\mathrm{B}_{10}\rangle)_{jm}^{-1}[\varepsilon\langle\mathrm{B}_{20}\rangle_{mn}$$
$$+\gamma\varepsilon\ln\varepsilon\langle\mathrm{B}_{11}\rangle_{mn}+\varepsilon(\gamma\ln\varepsilon)^2\langle\mathrm{B}_{02}\rangle_{mn}](\langle\mathrm{B}_{10}\rangle)_{nl}^{-1}+0(\varepsilon^2(\ln\varepsilon)^3) \qquad (3.29)$$

and (3.25) follows. On the other hand if $\lambda_{01}\neq 0$ then

$$\varepsilon(\langle\mathrm{B}(\varepsilon)\rangle)_{jl}^{-1}$$
$$=(\gamma\ln\varepsilon)^{-1}(\langle\mathrm{B}_{01}\rangle)_{jl}^{-1}-(\gamma\ln\varepsilon)^{-2}\sum_{m,n=1}^{\mathrm{N}}(\langle\mathrm{B}_{01}\rangle)_{jm}^{-1}\langle\mathrm{B}_{10}\rangle_{mn}(\langle\mathrm{B}_{01}\rangle)_{nl}^{-1}+0((\ln\varepsilon)^{-3}) \qquad (3.30)$$

and (3.26) holds. □

REMARK 3.1. — In the case $\lambda_{01}=0$ and $(\tilde{\phi}_j, \mathrm{B}_{10}(k)\phi_l)$ singular we expect in general non existence of $n-\lim_{\varepsilon\to 0_+}(\lambda u\mathrm{G}_{\varepsilon\gamma,\varepsilon k}v+1)^{-1}$ but we conjecture

$$(v, \varepsilon(\lambda u\mathrm{G}_{\varepsilon\gamma,\varepsilon k}v+1)^{-1}u) = o(1) \quad \text{as} \quad \varepsilon\to 0_+.$$

For a careful analysis of this point in the short-range case $\gamma=0$ see [*3*], Theorem 3.3.

LEMMA 3.4. — Assume case IV

$$\left(\text{i. e. } \mathrm{P} = -\sum_{j=1}^{\mathrm{N}}(\tilde{\phi}_j,.)\phi_j,\quad (v,\phi_1)\neq 0,\quad (v,\phi_j)=0,\quad j=2,\ldots,\mathrm{N}\right)$$

and $\gamma\neq 0$ (for $\gamma=0$ cf. [*3*], Theorem 3.4).

A) If $\lambda_{01} = -|(v, \phi_1)|^2/4\pi$ then

$$\varepsilon(\lambda u G_{\varepsilon\gamma,\varepsilon k} v + 1)^{-1} = \langle B_{10} \rangle_{11}^{-1}(\tilde{\phi}_1, .)\phi_1 + (\lambda_{01}\gamma \ln \varepsilon)^{-1} \sum_{j=2}^{N} (\tilde{\phi}_j, .)\phi_j -$$
$$-(\lambda_{01}\gamma \ln \varepsilon)^{-1} \langle B_{10} \rangle_{11}^{-1} \sum_{j=2}^{N} [\langle B_{10} \rangle_{1j}(\tilde{\phi}_j, .)\phi_1 + \langle B_{10} \rangle_{j1}(\tilde{\phi}_1, .)\phi_j]$$
$$+(\lambda_{01}\gamma \ln \varepsilon)^{-1} \langle B_{10} \rangle_{11}^{-2} \sum_{j=2}^{N} |\langle B_{10} \rangle_{1j}|^2 (\tilde{\phi}_1, .)\phi_1 + 0((\ln \varepsilon)^{-2}). \tag{3.31}$$

B) If $\lambda_{01} \neq -|(v, \phi_1)|^2/4\pi$ then

$$\varepsilon(\lambda u G_{\varepsilon\gamma,\varepsilon k} v + 1)^{-1} = (\gamma \ln \varepsilon)^{-1} \sum_{j,l=1}^{N} (\langle B_{01} \rangle)_{jl}^{-1}(\tilde{\phi}_l, .)\phi_j$$
$$-(\gamma \ln \varepsilon)^{-2} \sum_{j,l,m,n=1}^{N} (\langle B_{01} \rangle)_{jm}^{-1} \langle B_{10} \rangle_{mn} (\langle B_{01} \rangle)_{nl}^{-1} (\tilde{\phi}_l, .)\phi_j + 0((\ln \varepsilon)^{-3}). \tag{3.32}$$

Proof. — From (3.28) we get

$$\varepsilon(\lambda u G_{\varepsilon\gamma,\varepsilon k} v + 1)^{-1} = \sum_{j,l=1}^{N} (\langle (B_{10} + \gamma \ln \varepsilon B_{01}) \rangle)_{jl}^{-1}(\tilde{\phi}_l, .)\phi_j + 0(\varepsilon (\ln \varepsilon)^2. \tag{3.33}$$

If $\langle B_{01} \rangle_{jl}$ is singular, or equivalently $\lambda_{01} = -|(v, \phi_1)|^2/4\pi$, then

$$(\langle (B_{10} + \gamma \ln \varepsilon B_{01} \rangle)_{jl}^{-1} = \langle B_{10} \rangle_{11}^{-1} \delta_{j1}\delta_{l1} + (\lambda_{01}\gamma \ln \varepsilon)^{-1} [\delta_{jl} - \delta_{j1}\delta_{l1}]$$
$$-(\lambda_{01}\gamma \ln \varepsilon)^{-1} \langle B_{10} \rangle_{11}^{-1} [\delta_{j1} \langle B_{10} \rangle_{1l}(1 - \delta_{l1}) + (1 - \delta_{j1}) \langle B_{10} \rangle_{j1}\delta_{l1}]$$
$$-(\lambda_{01}\gamma \ln \varepsilon)^{-1} \langle B_{10} \rangle_{11}^{-2} \sum_{m=2}^{N} |\langle B_{10} \rangle_{1m}|^2 \delta_{j1}\delta_{l1} + 0((\ln \varepsilon)^{-2}) \tag{3.34}$$

and (3.31) follows. If $\langle B_{01} \rangle_{jl}$ is non singular, i. e. $\lambda_{01} \neq -|(v, \phi_1)|^2/4\pi$, then

$$(\langle (B_{10} + \gamma \ln \varepsilon B_{01}) \rangle)_{jl}^{-1}$$
$$= (\gamma \ln \varepsilon)^{-1} (\langle B_{01} \rangle)_{jl}^{-1} - (\gamma \ln \varepsilon)^{-2} \sum_{m,n=1}^{N} (\langle B_{01} \rangle)_{jm}^{-1} \langle B_{10} \rangle_{mn} (\langle B_{01} \rangle)_{nl}^{-1} + 0((\ln \varepsilon)^{-3}) \tag{3.35}$$

and (3.32) holds. □

With these facts we are able to state the main result of this section:

THEOREM 3.1. — Let $\gamma \neq 0$, then $H_{\gamma,\varepsilon}$ converges in norm resolvent sense to H_α^c as $\varepsilon \to 0_+$

$$n - \lim_{\varepsilon \to 0_+} (H_{\gamma,\varepsilon} - k^2)^{-1} = (H_\alpha^c - k^2)^{-1}, \qquad k^2 \notin \sigma(H_\alpha^c)$$

where α is given by [1]

$$\alpha = \begin{cases} \infty \text{ in case I}, \\ \infty \text{ in case II if } \lambda_{01} \neq -|(v,\phi)|^2/4\pi, \\ -[\lambda_{10}+\gamma(\phi, v\ln(|\gamma|x_+/2)v\phi)/4\pi]/|(v,\phi)|^2 \\ \qquad\qquad \text{in case II if } \lambda_{01} = -|(v,\phi)|^2/4\pi, \\ \infty \text{ in case III if } \lambda_{01} \neq 0 \text{ or } \lambda_{01}=0 \text{ and } (\tilde{\phi}_j, B_{10}(k)\phi_l) \\ \qquad\qquad \text{is non singular}, \\ \infty \text{ in case IV if } \lambda_{01} \neq -|(v,\phi_1)|^2/4\pi, \\ -[\lambda_{10}+\gamma(\phi_1, v\ln(|\gamma|x_+/2)v\phi_1)/4\pi]/|(v,\phi_1)|^2 \\ \qquad\qquad \text{in case IV if } \lambda_{01} = -|(v,\phi_1)|^2/4\pi. \end{cases} \tag{3.36}$$

In particular if $\lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon)$ is real then $E^c_\alpha < 0$ is an eigenvalue of H^c_α if and only if there exists a sequence $E_{\gamma,\varepsilon}$ of eigenvalues of $H_{\gamma,\varepsilon}$ that converges to E^c_α as $\varepsilon \to 0_+$. (We remark that $H^c_{\alpha=\infty} = H^c$, the ordinary Coulomb Hamiltonian. For a complete discussion of the short range case $\gamma = 0$ see [3], Th. 2.1).

Proof. — A change of variables in (3.6) yields

$$(H_{\gamma,\varepsilon}-k^2)^{-1} = G_{\gamma,k} - \lambda(\varepsilon, \gamma\varepsilon\ln\varepsilon)A_{\gamma,\varepsilon}(k)\varepsilon(\lambda(\varepsilon,\gamma\varepsilon\ln\varepsilon)uG_{\varepsilon\gamma,\varepsilon k}v+1)^{-1}C_{\gamma,\varepsilon}(k) \tag{3.37}$$

where the kernels of $A_{\gamma,\varepsilon}(k)$ and $C_{\gamma,\varepsilon}(k)$ are given by

$$A_{\gamma,\varepsilon}(k, \underline{x}, \underline{y}) = G_{\gamma,k}(\underline{x}, \varepsilon\underline{y})v(\underline{y}), \quad C_{\gamma,\varepsilon}(k, \underline{x}, \underline{y}) = u(\underline{x})G_{\gamma,k}(\varepsilon\underline{x}, \underline{y}). \tag{3.38}$$

Obviously

$$\lim_{\varepsilon\to 0_+} G_{\gamma,k}(\underline{x}, \varepsilon\underline{y}) = G_{\gamma,k}(\underline{x}, \underline{0}) = g_{\gamma,k}(\underline{x}) \tag{3.39}$$

pointwise. From [*23*] we have

$$|G_{\gamma,k}(\underline{x}, \underline{y})| \leqq C_\gamma(k)\,|\underline{x}-\underline{y}|^{-1}e^{-(\mathrm{Im}\,k)|\underline{x}-\underline{y}|}(1+|\underline{x}-\underline{y}|)^{-\theta(-\gamma)\gamma(\mathrm{Im}\,k)/2|k|^2} \tag{3.40}$$

$\mathrm{Im}\,k > 0$, $k \neq -i\gamma/2n$, $n = 1, 2, \ldots$ if $\gamma < 0$; $\theta(-\gamma) = \begin{cases} 0, & \gamma \geqq 0, \\ 1, & \gamma < 0, \end{cases}$

and we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0_+} \|A_{\gamma,\varepsilon}(k)\|_{HS} &= \|A_{\gamma,0}(k)\|_{HS}, \\ \lim_{\varepsilon\to 0_+} \|C_{\gamma,\varepsilon}(k)\|_{HS} &= \|C_{\gamma,0}(k)\|_{HS} \end{aligned} \tag{3.41}$$

where $\| \ \|_{HS}$ is the Hilbert-Schmidt norm and

$$A_{\gamma,0}(k) = (v, .)g_{\gamma,k}, \quad C_{\gamma,0}(k) = (\overline{g_{\gamma,k}}, .)u. \tag{3.42}$$

---

[1] The integral operators $v\ \ln(|\gamma|x_+/2)\ v$ are defined in the appendix.

Since by the bound (3.40) and by dominated convergence also

$$w - \lim_{\varepsilon\to 0_+} A_{\gamma,\varepsilon}(k) = A_{\gamma,0}(k), \qquad w - \lim_{\varepsilon\to 0_+} C_{\gamma,\varepsilon}(k) = C_{\gamma,0}(k) \tag{3.43}$$

holds, we actually infer ([42], Th. 2.16)

$$\lim_{\varepsilon\to 0_+} \| A_{\gamma,\varepsilon}(k) - A_{\gamma,0}(k) \|_{HS} = 0, \qquad \lim_{\varepsilon\to 0_+} \| C_{\gamma,\varepsilon}(k) - C_{\gamma,0}(k) \|_{HS} = 0. \tag{3.44}$$

Thus denoting

$$n - \lim_{\varepsilon\to 0_+} \varepsilon(\lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon) u G_{\varepsilon\gamma,\varepsilon k} v + 1)^{-1} = D_{\gamma,0}(k)$$

we obtain

$$n - \lim_{\varepsilon\to 0_+} (H_{\gamma,\varepsilon} - k^2)^{-1} = G_{\gamma,k} - (v, D_{\gamma,0}(k)u)(\overline{g_{\gamma,k}}, \cdot) g_{\gamma,k}\,. \tag{3.45}$$

Inserting $D_{\gamma,0}(k)$ from Lemmas (3.1)-(3.4), (3.36) results. □

REMARK 3.2. — *a*) If λ is real, or if $V \leqq 0$ ($V \geqq 0$) and $\operatorname{Im}\lambda \geqq 0$ ($\operatorname{Im}\lambda \leqq 0$) then $iH_{\gamma,\varepsilon}$ generates a contraction semigroup $e^{-itH_{\gamma,\varepsilon}}$, $t \geqq 0$ and Theorem 3.1 implies in particular strong convergence of $e^{-itH_{\gamma,\varepsilon}}$ to $e^{-itH^c_\alpha}$ as $\varepsilon \to 0_+$.

b) For the sake of simplicity we assumed $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$. It is obvious from the expansion (3.8) and from the proofs of Lemmas 3.1-3.4 that $V \in R$ and $(1 + |\underline{x}|)^n V(\underline{x}) \in L^1(\mathbb{R}^3)$ for some power n (depending on the order of the expansion involved) suffices for all results of this section to hold.

c) Case II if $\lambda_{01} = -|(v, \phi)|^2/4\pi$ and case IV if $\lambda_{01} = -|(v, \phi_1)^2/4\pi$ represent a variant of Klauder's phenomenon [29] [41]. In fact let us assume in addition to $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$ (but cf. Remark 2*b*)) that $V(\underline{x})$ is continuous and monotonously decreasing outside some fixed sphere of radius ρ centered at the origin. Then obviously

$$\lim_{\varepsilon\to 0_+} \lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon)\varepsilon^{-2} V(\underline{x}/\varepsilon) = 0 \qquad \text{for all} \qquad |\underline{x}| \neq 0$$

but

$$n - \lim_{\varepsilon\to 0_+} (H_{\gamma,\varepsilon} - k^2)^{-1} = (H^c_\alpha - k^2)^{-1} \neq (H^c - k^2)^{-1}.$$

4. SCATTERING THEORY

We first introduce the on-shell scattering amplitude $f_{\gamma,\varepsilon}(k, \underline{\omega}, \underline{\omega}')$ and scattering operators $S_{\gamma,\varepsilon}$, $S_{\gamma,\varepsilon}(k)$ corresponding to $H_{\gamma,\varepsilon}$. Throughout this section we assume such conditions on λ and V that $S_{\gamma,\varepsilon}$ are bounded operators (cf. Remark 3.2*a*). Similar to (2.8)-(2.10) we have

$$f_{\gamma,\varepsilon}(k, \underline{\omega}, \underline{\omega}') = f^c(k, \underline{\omega}, \underline{\omega}') + f^{sc}_{\gamma,\varepsilon}(k, \underline{\omega}, \underline{\omega}'), \quad k > 0 \tag{4.1}$$

where $f^{sc}_{\gamma,\varepsilon}(k, \underline{\omega}, \underline{\omega}')$ is given by [*18*]

$$f^{sc}_{\gamma,\varepsilon}(k, \underline{\omega}, \underline{\omega}') = -(4\pi)^{-1}\lambda(\varepsilon, \gamma\varepsilon\ln\varepsilon)(v\Psi^{(+)}_{c,\varepsilon\gamma}(\varepsilon k, \underline{\omega}, \underline{r}), \varepsilon(\lambda(\varepsilon, \gamma\varepsilon\ln\varepsilon)u\mathrm{G}_{\varepsilon\gamma,\varepsilon k}v+1)^{-1}u\Psi^{(-)}_{c,\varepsilon\gamma}(\varepsilon k, \underline{\omega}', \underline{r})) \tag{4.2}$$

and

$$\mathrm{S}_{\gamma,\varepsilon} = \mathrm{S}^c + \mathrm{T}^{sc}_{\gamma,\varepsilon}, \tag{4.2}$$

$$\mathrm{S}_{\gamma,\varepsilon}(k) = \mathrm{S}^c(k) + \mathrm{T}^{sc}_{\gamma,\varepsilon}(k), \quad k > 0, \tag{4.4}$$

$$(\mathrm{T}^{sc}_{\gamma,\varepsilon}(k)h)(\underline{\omega}) = -(2\pi i)^{-1}k\int d\omega' f^{sc}_{\gamma,\varepsilon}(k, \underline{\omega}, \underline{\omega}')h(\underline{\omega}'), \quad k > 0. \tag{4.5}$$

In (4.2) $\Psi^{(\pm)}_{c,\gamma}$ denote the Coulomb wave functions

$$\Psi^{(-)}_{c,\gamma}(k, \underline{\omega}, \underline{r}) = e^{-\pi\gamma/4k}\Gamma(1 + i\gamma/2k)e^{ik\underline{\omega}\cdot\underline{r}}{}_1\mathrm{F}_1(-i\gamma/2k; 1; ik(r - \underline{\omega}\cdot\underline{r})),$$
$$\Psi^{(+)}_{c,\gamma}(k, \underline{\omega}, \underline{r}) = \overline{\Psi^{(-)}_{c,\gamma}}(k, -\underline{\omega}, \underline{r}), \quad r = |\underline{r}|. \tag{4.6}$$

For λ real, $\mathrm{S}_{\gamma,\varepsilon}$ and $\mathrm{S}_{\gamma,\varepsilon}(k)$ are unitary as a consequence of the assumption (3.4) [*40*]. With the help of

$$\Psi^{(\mp)}_{c,\varepsilon\gamma}(\varepsilon k, \underline{\omega}, \underline{r}) = e^{-\pi\gamma/4k}\Gamma(1 \pm i\gamma/2k)[1 + i\varepsilon k\underline{\omega}\cdot\underline{r} + \varepsilon\gamma(r \mp \underline{\omega}\cdot\underline{r})/2 + 0(\varepsilon^2)] \tag{4.7}$$

and Lemmas 3.1-3.4 we get the following expansions of $f^{sc}_{\gamma,\varepsilon}$:

THEOREM 4.1. — Assume case I. Then

$$-4\pi e^{\pi\gamma/2k}\Gamma(1+i\gamma/2k)^{-2}f^{sc}_{\gamma,\varepsilon}(k, \underline{\omega}, \underline{\omega}') = \varepsilon(v, (u\mathrm{G}_{0,0}v+1)^{-1}u) + 0(\varepsilon^2\ln\varepsilon). \tag{4.8}$$

THEOREM 4.2. — Assume case II.

A) if $\lambda_{01} = -|(v, \phi)|^2/4\pi$ then

$$\begin{aligned}
&-4\pi e^{\pi\gamma/2k}\Gamma(1+i\gamma/2k)^{-2}f^{sc}_{\gamma,\varepsilon}(k, \underline{\omega}, \underline{\omega}') \\
&= \langle \mathrm{B}_{10}\rangle^{-1}|(v, \phi)|^2 + (\varepsilon\lambda_{10} + \gamma\varepsilon\ln\varepsilon\lambda_{01})\langle \mathrm{B}_{10}\rangle^{-1}|(v, \phi)|^2 \\
&+ i\varepsilon k\langle \mathrm{B}_{10}\rangle^{-1}[(v, \phi)(\phi, \underline{\omega}'\cdot\underline{r}v) - (\underline{\omega}\cdot\underline{r}v, \phi)(\phi, v)] + \varepsilon(v, \mathrm{T}u) \\
&+ \varepsilon\gamma 2^{-1}\langle \mathrm{B}_{10}\rangle^{-1}[(v, \phi)(\phi, (r - \underline{\omega}'\cdot\underline{r})v) + ((r + \underline{\omega}\cdot\underline{r})v, \phi)(\phi, v)] \\
&- \varepsilon\langle \mathrm{B}_{10}\rangle^{-1}[(v, \phi)(\tilde{\phi}, \mathrm{B}_{10}\mathrm{T}u) + (v, \mathrm{T}\mathrm{B}_{10}\phi)(\phi, v)] \\
&- \gamma\varepsilon\ln\varepsilon\langle \mathrm{B}_{10}\rangle^{-1}[(v, \phi)(\tilde{\phi}, \mathrm{B}_{01}\mathrm{T}u) + (v, \mathrm{T}\mathrm{B}_{01}\phi)(\phi, v)] \\
&- \langle \mathrm{B}_{10}\rangle^{-2}|(v, \phi)|^2[\varepsilon\langle \mathrm{B}_{20}\rangle + \gamma\varepsilon\ln\varepsilon\langle \mathrm{B}_{11}\rangle + \varepsilon(\gamma\ln\varepsilon)^2\langle \mathrm{B}_{02}\rangle] \\
&+ \langle \mathrm{B}_{10}\rangle^{-2}|(v, \phi)|^2[\varepsilon\langle \mathrm{B}_{10}\mathrm{T}\mathrm{B}_{10}\rangle + \gamma\varepsilon\ln\varepsilon\langle(\mathrm{B}_{10}\mathrm{T}\mathrm{B}_{01} + \mathrm{B}_{01}\mathrm{T}\mathrm{B}_{10})\rangle \\
&+ \varepsilon(\gamma\ln\varepsilon)^2\langle \mathrm{B}_{01}\mathrm{T}\mathrm{B}_{01}\rangle] + 0(\varepsilon^2(\ln\varepsilon)^3).
\end{aligned} \tag{4.9}$$

B) if $\lambda_{01} \neq -|(v, \phi)|^2/4\pi$ then

$$\begin{aligned}
&-4\pi e^{\pi\gamma/2k}\Gamma(1 + i\gamma/2k)^{-2}f^{sc}_{\gamma,\varepsilon}(k, \underline{\omega}, \underline{\omega}') \\
&\quad = (\gamma\ln\varepsilon)^{-1}\langle \mathrm{B}_{01}\rangle^{-1}|(v, \phi)|^2 \\
&\quad - (\gamma\ln\varepsilon)^{-2}\langle \mathrm{B}_{01}\rangle^{-2}\langle \mathrm{B}_{10}\rangle|(v, \phi)|^2 + 0((\ln\varepsilon)^{-3}).
\end{aligned} \tag{4.10}$$

THEOREM 4.3. — Assume case III. If $\lambda_{01} \neq 0$ or $\lambda_{01}=0$ and $(\tilde{\phi}_j, B_{10}(k)\phi_l)$ is non singular then

$$-4\pi e^{\pi\gamma/2k}\Gamma(1+i\gamma/2k)^{-2} f^{sc}_{\gamma,\varepsilon}(k, \underline{\omega}, \underline{\omega}') = \varepsilon(v, Tu) + 0((\varepsilon \ln \varepsilon)^2). \quad (4.11)$$

THEOREM 4.4. — Assume case IV and $\gamma \neq 0$.

A) If $\lambda_{01} = -|(v, \phi_1)|^2/4\pi$ then

$$-4\pi e^{\pi\gamma/2k}\Gamma(1+i\gamma/2k)^{-2} f^{sc}_{\gamma,\varepsilon}(k, \underline{\omega}, \underline{\omega}') = \langle B_{10}\rangle^{-1}_{11} |(v, \phi_1)|^2$$
$$+ (\lambda_{01}\gamma \ln \varepsilon)^{-1} \langle B_{10}\rangle^{-2}_{11} |(v, \phi_1)|^2 \sum_{j=2}^{N} |\langle B_{10}\rangle_{1j}|^2 + 0((\ln \varepsilon)^{-2}). \quad (4.12)$$

B) If $\lambda_{01} \neq -|(v, \phi_1)|^2/4\pi$ then

$$-4\pi e^{\pi\gamma/2k}\Gamma(1+i\gamma/2k)^{-2} f^{sc}_{\gamma,\varepsilon}(k, \underline{\omega}, \underline{\omega}') = (\gamma \ln \varepsilon)^{-1} (\langle B_{01}\rangle)^{-1}_{11} |(v, \phi_1)|^2$$
$$-(\gamma \ln \varepsilon)^{-2} |(v, \phi_1)|^2 \sum_{m,n=1}^{N} (\langle B_{01}\rangle)^{-1}_{1m} \langle B_{10}\rangle_{mn} (\langle B_{01}\rangle)^{-1}_{n1} + 0((\ln \varepsilon)^{-3}). \quad (4.13)$$

REMARK 4.1. — *a*) For an explicit computation of the coefficients $\langle B_{mn}\rangle$ in various cases we refer to the appendix.

b) In all cases one finds (just as in the case II A) of Th. 4.1) that the angle dependent terms are suppressed by a factor of ε.

Insertion of these results into (4.5) yields for the leading term of the t-matrix:

THEOREM 4.5. — Let $k > 0$. Then

$$T^{sc}_{\gamma,\varepsilon}(k) = \varepsilon(2\pi i)^{-1} k e^{-\pi\gamma/2k}\Gamma(1+i\gamma/2k)^2 (v, (uG_{0,0}v+1)^{-1}u)(Y_{0,0}, .)Y_{0,0} + 0(\varepsilon^2 \ln \varepsilon)$$
in case I. (4.14)

$$T^{sc}_{\gamma,\varepsilon}(k) = (2\pi i)^{-1} k e^{-\pi\gamma/2k}\Gamma(1+i\gamma/2k)^2 \langle B_{10}(k)\rangle^{-1} |(v, \phi)|^2 (Y_{0,0}, .)Y_{0,0} + 0(\varepsilon (\ln \varepsilon)^2)$$
in case II if $\lambda_{01} = -|(v, \phi)|^2/4\pi$, (4.15)

$$T^{sc}_{\gamma,\varepsilon}(k) = (\gamma \ln \varepsilon)^{-1}(2\pi i)^{-1} k e^{-\pi\gamma/2k}\Gamma(1+i\gamma/2k)^2 \langle B_{01}\rangle^{-1} |(v, \phi)|^2 (Y_{0,0}, .)Y_{0,0} + 0((\ln \varepsilon)^{-2})$$
in case II if $\lambda_{01} \neq -|(v, \phi)|^2/4\pi$. (4.16)

$$T^{sc}_{\gamma,\varepsilon}(k) = \varepsilon(2\pi i)^{-1} k e^{-\pi\gamma/2k}\Gamma(1+i\gamma/2k)^2 (v, Tu)(Y_{0,0}, .)Y_{0,0} + 0((\varepsilon \ln \varepsilon)^2)$$
in case III if $\lambda_{01} \neq 0$ or $\lambda_{01}=0$ and $(\tilde{\phi}_j, B_{10}(k)\phi_l)$ is non singular. (4.17)

$$T^{sc}_{\gamma,\varepsilon}(k) = (2\pi i)^{-1} k e^{-\pi\gamma/2k}\Gamma(1+i\gamma/2k)^2 \langle B_{10}(k)\rangle^{-1}_{11} |(v, \phi_1)|^2 (Y_{0,0}, .)Y_{0,0} + 0((\ln \varepsilon)^{-1})$$
in case IV if $\lambda_{01} = -|(v, \phi_1)|^2/4\pi$, $\gamma \neq 0$, (4.18)

$$
\begin{aligned}
& \mathrm{T}^{sc}_{\gamma,\varepsilon}(k) \\
& =(\gamma \ln \varepsilon)^{-1}(2\pi i)^{-1} k e^{-\pi\gamma/2k}\Gamma(1+i\gamma/2k)^2(\langle \mathrm{B}_{01}\rangle)_{11}^{-1} \mid (v, \phi_1) \mid^2 (\mathrm{Y}_{0,0}, \cdot)\mathrm{Y}_{0,0} \\
& \quad + 0((\ln \varepsilon)^{-2}) \quad \text{in case IV if } \lambda_{01} \neq -\mid (v, \phi_1) \mid^2/4\pi, \quad \gamma \neq 0. \quad (4.19)
\end{aligned}
$$

Concerning continuity of the S-matrix we have

THEOREM 4.6. — Assume such conditions on λ and V that $\mathrm{S}_{\gamma,\varepsilon}$ are uniformly bounded with respect to ε. Then

$$ n - \lim_{\varepsilon\to 0_+} \mathrm{S}_{\gamma,\varepsilon}(k) = \mathrm{S}^c_\alpha(k) \quad \text{in} \quad \mathrm{L}^2(\mathrm{S}^{(2)}), \quad k > 0 $$

and

$$ s - \lim_{\varepsilon\to 0_+} \mathrm{S}_{\gamma,\varepsilon} = \mathrm{S}^c_\alpha \quad \text{in} \quad \mathrm{L}^2(\mathbb{R}^3) $$

where α is given by (3.36) according to cases I-IV.

Proof. — Identical to that of Theorem 5.3 in [*12*]. □

Of course the assumption in Theorem 4.6 is trivially fulfilled if $e^{-it\mathrm{H}_{\gamma,\varepsilon}}$ is a contraction i. e. for real as well as for optical (purely absorbing) interactions. The important point of the theorem is that it shows in which sense the scattering of charged particles with a short range force is approximated by models with Coulomb plus zero range interaction.

Next we introduce the concept of a Coulomb modified scattering length for arbitrary (non spherically symmetric) short-range perturbations $\mathrm{V}(\underline{x})$. If V is spherically symmetric, we infer from (2.15) that

$$ \lim_{k\to 0_+} \frac{e^{\pi\gamma/k}-1}{\pi\gamma/k} \frac{e^{2i\delta^{0,sc}}(k)-1}{2ik} = \begin{cases} -a^{0,sc}, & \gamma \geqq 0, \\ \dfrac{-a^{0,sc}}{1+i\pi\gamma a^{0,sc}}, & \gamma \leqq 0 \end{cases} \quad (4.20) $$

where $a^{0,sc}$ denotes the Coulomb modified s-wave scattering length for the Hamiltonian (defined as quadratic form)

$$ -\Delta + \frac{\gamma}{|\underline{x}|} + \mathrm{V}(|\underline{x}|), \qquad e^{2a|\underline{x}|}\mathrm{V}(|\underline{x}|) \in \mathrm{R} \quad \text{for some } a > 0. \quad (4.21) $$

Eq. (4.20) immediately suggests the following definition of the Coulomb modified scattering length a^{sc} for an arbitrary non spherically symmetric interaction $\mathrm{V}(\underline{x})$ in (4.21). Approaching the threshold $k=0$ from above we take care to avoid a set $\mathscr{E}(\gamma)$ – the set of all $k^2 \geqq 0$ such that $u\mathrm{G}_{\gamma,k}v$ has an eigenvalue -1— where the energy spectrum may be singular [*40*]. In particular this is a precaution to avoid the k^2 corresponding to possible positive energy bound states. Assume now $k_m \underset{m\to\infty}{\longrightarrow} 0$, $k_m^2 \notin \mathscr{E}(\gamma)$, then

$$ \lim_{m\to\infty} \frac{e^{\pi\gamma/k_m}-1}{\pi\gamma/k_m} e^{-2i\delta^{0,c}}(k_m)(16\pi^2)^{-1} \int d\omega d\omega' f^{sc}(k_m, \underline{\omega}, \underline{\omega}') = \begin{cases} -a^{sc}, & \gamma \geqq 0, \\ \dfrac{-a^{sc}}{1+i\pi\gamma a^{sc}}, & \gamma \leqq 0, \end{cases} \quad (4.22) $$

defines a^{sc}, where [*18*]

$$f^{sc}(k, \underline{\omega}, \underline{\omega}') = -(4\pi)^{-1}(v\Psi^{(+)}_{c,\gamma}(k, \underline{\omega}, \underline{r}), (uG_{\gamma,k}v+1)^{-1}u\Psi^{(-)}_{c,\gamma}(k, \underline{\omega}', \underline{r})), \quad k^2 \notin \mathcal{E}(\gamma) \tag{4.23}$$

is the Coulomb modified scattering amplitude associated with

$$-\Delta + \frac{\gamma}{|\underline{x}|} + V(\underline{x}).$$

The following theorem provides on explicit formula for a^{sc}:

THEOREM 4.7. — Let $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$ and suppose 0, $k_m^2 \notin \mathcal{E}(\gamma)$, $k_m \xrightarrow[m\to\infty]{} 0$. Then

$$\begin{cases} a^{sc} \quad , & \gamma \geqq 0 \\ \dfrac{a^{sc}}{1 + i\pi\gamma a^{sc}}, & \gamma \leqq 0 \end{cases}$$

$$= -\lim_{m\to\infty} \frac{e^{\pi\gamma/k_m} - 1}{\pi\gamma/k_m} e^{-2i\delta^{0,c}(k_m)}(16\pi^2)^{-1} \int d\omega d\omega' f^{sc}(k_m, \underline{\omega}, \underline{\omega}')$$

$$= \begin{cases} (4\pi)^{-1}((r\gamma)^{-1/2}I_1(2\sqrt{r\gamma})v, (uG_{\gamma,0}v+1)^{-1}(r\gamma)^{-1/2}I_1(2\sqrt{r\gamma})u), \ \gamma \geqq 0, \\ (4\pi)^{-1}((r|\gamma|)^{-1/2}J_1(2\sqrt{r|\gamma|})v, (uG_{\gamma,0}v+1)^{-1}(r|\gamma|)^{-1/2}J_1(2\sqrt{r|\gamma|}u), \ \gamma \leqq 0, \end{cases} \tag{4.24}$$

where $J_1(z)$ ($I_1(z)$) denotes the (modified) Bessel function of order one [*1*].

In the special case $\gamma = 0$ we recover the well known expression

$$a^s = -\lim_{m\to\infty} (16\pi^2)^{-1} \int d\omega d\omega' f^s(k_m, \underline{\omega}, \underline{\omega}')$$

$$= -\lim_{m\to\infty} f^s(k_m, \underline{\omega}, \underline{\omega}') = (4\pi)^{-1}(v, (uG_{0,0}v + 1)^{-1}u) \tag{4.25}$$

(Here a^s and f^s denote the corresponding short-range quantities ($\gamma = 0$) for $H = -\Delta + V(\underline{x})$).

This result is a generalization of the corresponding one in the short range case. A detailed treatment of the Coulomb case will be given elsewhere [*4*].

REMARK 4.2. — Note that for $\gamma < 0$, $V \in R \cap L^1(\mathbb{R}^3)$ is certainly sufficient in order to guarantee $|a^{sc}| < \infty$ whereas for $\gamma > 0$ we roughly need $V \in R$ and $e^{(4\gamma|\underline{x}|)^{1/2}}V(\underline{x}) \in L^1(\mathbb{R}^3)$.

5. APPLICATIONS

In this section we study the relation between the effective range expansions for charged and neutral particles, as well as the shifts of Coulomb levels induced by additional short range interactions. For concreteness we envisage nucleon-nucleon scattering and mesic atoms.

We consider the Hamiltonians

$$H_{\gamma,\varepsilon} = -\Delta + \frac{\gamma}{|\underline{x}|} + \lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon)\varepsilon^{-2}V(\underline{x}/\varepsilon) \tag{5.1}$$

and

$$H_{\varepsilon} = -\Delta + \tilde{\lambda}(\varepsilon)\varepsilon^{-2}V(x/\varepsilon), \quad 0 < \varepsilon < 1, \quad \tilde{\lambda}(\varepsilon) = \sum_{m=0}^{\infty} \tilde{\lambda}_m \varepsilon^m, \quad \tilde{\lambda}_0 = 1 \tag{5.2}$$

and apply Theorem 4.7 to (5.1) and (5.2):

THEOREM 5.1. — Denote by $a^{sc}_{\gamma,\varepsilon}(a^s_\varepsilon)$ the scattering length corresponding to $H_{\gamma,\varepsilon}(H_\varepsilon)$. Then, if $\lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon)uG_{\varepsilon\gamma,0}v$ has no eigenvalue -1

$$\left\{ \begin{array}{ll} a^{sc}_{\gamma,\varepsilon} , & \gamma \geqq 0, \\ \dfrac{a^{sc}_{\gamma,\varepsilon}}{1+i\pi\gamma a^{sc}_{\gamma,\varepsilon}}, & \gamma \leqq 0, \end{array} \right\} = (4\pi)^{-1}\lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon) .$$

$$\cdot \left\{ \begin{array}{l} ((\varepsilon\gamma r)^{-1/2}I_1(2\sqrt{\varepsilon\gamma r})v, \varepsilon(\lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon)uG_{\varepsilon\gamma,0}v+1)^{-1}(\varepsilon\gamma r)^{-1/2}I_1(2\sqrt{\varepsilon\gamma r})u), \ \gamma \geqq 0, \\ ((\varepsilon|\gamma|r)^{-1/2}J_1(2\sqrt{\varepsilon|\gamma|r})v, \varepsilon(\lambda(\varepsilon,\gamma\varepsilon \ln\varepsilon)uG_{\varepsilon\gamma,0}v+1)^{-1}(\varepsilon|\gamma|r)^{-1/2}J_1(2\sqrt{\varepsilon|\gamma|r})u), \\ \qquad \gamma \leqq 0. \end{array} \right. \tag{5.3}$$

Moreover we have ([1]) as $\varepsilon \to 0_+$

$$a^{sc}_{\gamma,\varepsilon} = \varepsilon(4\pi)^{-1}(v, (uG_{0,0}v+1)^{-1}u) + 0(\varepsilon^2 \ln \varepsilon) \text{ in case I.} \tag{5.4}$$

$$a^{sc}_{\gamma,\varepsilon} = [4\pi\lambda_{10}|(v, \phi)|^{-2} + \gamma(\phi, v \ln(|\gamma|x_+/2)v\phi)|(v, \phi)|^{-2} - \gamma(\Psi(1)+\Psi(2))]^{-1}$$
$$+\varepsilon(\gamma \ln \varepsilon)^2(4\pi)^{-1}\langle B_{10}(0_+)\rangle^{-2}|(v, \phi)|^2 .$$
$$\left\{ \begin{array}{l} -\langle B_{02}\rangle + \langle B_{01}TB_{01}\rangle, \ \gamma > 0 \\ [1-i\gamma \langle B_{10}(0_+)\rangle^{-1} |(v, \phi)|^2/4]^{-2}[-\langle B_{02}\rangle + \langle B_{01}TB_{01}\rangle], \ \gamma < 0 \end{array} \right.$$
$$+0(\varepsilon \ln \varepsilon) \text{ in case II if } \lambda_{01} = -|(v, \phi)|^2/4\pi, \tag{5.5}$$

$$a^s_\varepsilon = [4\pi\tilde{\lambda}_1 |(v, \phi)|^{-2}]^{-1} + 0(\varepsilon) \text{ in case II if } \gamma = 0 \text{ and } \tilde{\lambda}_1 \neq 0, \tag{5.6}$$

$$a^{sc}_{\gamma,\varepsilon} = (\gamma \ln \varepsilon)^{-1}(4\pi)^{-1}\langle B_{01}\rangle^{-1}|(v, \phi)|^2 + 0((\ln \varepsilon)^{-2})$$
$$\text{in case II if } \lambda_{01} \neq -|(v, \phi)|^2/4\pi, \tag{5.7}$$

([1]) The integral operators $v \, ln(|\gamma|x_+/2) \, v$ are defined in the appendix.

$$\langle B_{10}(0_+)\rangle = \lambda_{10} - \gamma\left[\Psi(1)+\Psi(2) + \begin{Bmatrix} 0 \;, \gamma \geqq 0 \\ -i\pi, \gamma < 0 \end{Bmatrix}\right] |(v,\phi)|^2/4\pi + \gamma(\phi, v\ln(|\gamma|x_+/2)v\phi)/4\pi . \quad (5.8)$$

$$a^{sc}_{\gamma,\varepsilon} = \varepsilon(4\pi)^{-1}(v, Tu) + 0((\varepsilon \ln \varepsilon)^2)$$

in case III if $\lambda_{01} \neq 0$ or $\lambda_{01} = 0$ and $(\tilde{\phi}_j, B_{10}(0_+)\phi_l)$ non singular. (5.9)

$$a^{sc}_{\gamma,\varepsilon} = [4\pi\lambda_{10}|(v,\phi_1)|^{-2} + \gamma(\phi_1, v\ln(|\gamma|x_+/2)v\phi_1)|(v,\phi_1)|^{-2} - \gamma(\Psi(1)+\Psi(2))]^{-1}$$

$$-(\gamma\ln\varepsilon)^{-1}.\langle B_{10}(0_+)\rangle_{11}^{-2}\sum_{j=2}^{N}|\langle B_{10}(0_+)\rangle_{1j}|^2 \begin{cases} 1, \gamma > 0 \\ [1 - i\gamma\langle B_{10}(0_+)\rangle_{11}^{-1}|(v,\phi_1)|^2/4]^{-2}, & \gamma < 0 \end{cases}$$

$+0((\ln\varepsilon)^{-2})$ in case IV if $\lambda_{01} = -|(v,\phi_1)|^2/4\pi$, (5.10)

$a^s_\varepsilon = [4\pi\tilde{\lambda}_1|(v,\phi_1)|^{-2}]^{-1} + 0(\varepsilon)$ in case IV if $\gamma = 0$ and $\tilde{\lambda}_1 \neq 0$, (5.11)

$a^{sc}_{\gamma,\varepsilon} = (\gamma\ln\varepsilon)^{-1}(4\pi)^{-1}(\langle B_{01}\rangle)_{11}^{-1}|(v,\phi_1)|^2 + 0((\ln\varepsilon)^{-2})$, $\gamma \neq 0$
in case IV if $\lambda_{01} \neq -|(v,\phi_1)|^2/4\pi$. (5.12)

Proof. — Inserting (1)

$$B_{10}(0_+, \underline{x}, \underline{y}) = \lambda_{10}(uG_{0,0}v)(\underline{x},\underline{y}) - \gamma\left[\Psi(1)+\Psi(2) + \begin{Bmatrix} 0 \;, \gamma \geqq 0 \\ -i\pi, \gamma < 0 \end{Bmatrix}\right]u(\underline{x})v(\underline{y})/4\pi$$
$$+\gamma u(\underline{x})\ln(|\gamma|x_+/2)v(\underline{y})/4\pi \quad (5.13)$$

into Lemmas 3.1-3.4 and remarking that

$$(\varepsilon|\gamma|r)^{-1/2}\begin{Bmatrix} I_1(2\sqrt{\varepsilon\gamma r}), & \gamma \geqq 0 \\ J_1(2\sqrt{\varepsilon|\gamma|r}), & \gamma \leqq 0 \end{Bmatrix} = 1 + 0(\varepsilon)$$

we obtain (5.4), (5.5), (5.7)-(5.10), and (5.13). (5.6) and (5.11) follow from Lemmas 4.2 and 4.4 of [3]. □

At this point it is easy to discuss the physical content of our parameters λ_{10} and ε occuring in $H_{\gamma,\varepsilon}$ and H_ε. Assuming for a moment that V is spherically symmetric the (Coulomb modified) effective range expansion (2.15) and formula (3.3) immediately lead to

$$a^{sc}_{\gamma,\varepsilon} = \varepsilon a^{sc}_{\varepsilon\gamma}(\varepsilon), \qquad r^{sc}_{\gamma,\varepsilon} = \varepsilon r^{sc}_{\varepsilon\gamma}(\varepsilon) \quad (5.14)$$

where $a^{sc}_{\gamma,\varepsilon}$, $r^{sc}_{\gamma,\varepsilon}$ denote the Coulomb modified scattering length and effective range associated with $H_{\gamma,\varepsilon}$ and $a^{sc}_{\varepsilon\gamma}(\varepsilon)$, $r^{sc}_{\varepsilon\gamma}(\varepsilon)$ denote the corresponding quantities for $H_{\varepsilon\gamma}(\varepsilon) = \varepsilon^2 U_\varepsilon^{-1} H_{\gamma,\varepsilon} U_\varepsilon$. Similarly for a^s_ε, r^s_ε and $a^s(\varepsilon)$, $r^s(\varepsilon)$ the (short range) quantities associated with H_ε and $H(\varepsilon) = \varepsilon^2 U_\varepsilon^{-1} H_\varepsilon U_\varepsilon$ (put $\gamma = 0$ in (5.14))

$$a^s_\varepsilon = \varepsilon a^s(\varepsilon), \qquad r^s_\varepsilon = \varepsilon r^s(\varepsilon). \quad (5.15)$$

(1) The integral operators $v\, ln(|\gamma|x_+/2)\, v$ are defined in the appendix.

Looking at Theorem 5.1 we infer

$$a^{sc}_{\gamma,\varepsilon} = \varepsilon a^{sc}_{\varepsilon\gamma}(\varepsilon) = \begin{cases} \varepsilon a^s + 0(\varepsilon^2 \ln \varepsilon) \text{ in case I,} \\ a^{sc}(\alpha(\lambda_{10})) + 0(\varepsilon (\ln \varepsilon)^2) \text{ in case II if } \lambda_{01} = -|(v, \phi)|^2/4\pi, \\ 0((\ln \varepsilon)^{-1}) \text{ in case II if } \lambda_{01} \neq -|(v, \phi)|^2/4\pi, \\ \varepsilon a^s + 0(\varepsilon^2 (\ln \varepsilon)^3) \text{ in case III if } \lambda_{01} \neq 0 \text{ or } \lambda_{01} = 0 \\ \qquad \text{and } (\tilde{\phi}_j, B_{10}(0_+)\phi_l) \text{ non singular,} \\ a^{sc}(\alpha(\lambda_{10})) + 0((\ln \varepsilon)^{-1}) \\ \qquad \text{in case IV if } \lambda_{01} = -|(v, \phi_1)|^2/4\pi, \ \gamma \neq 0, \\ 0((\ln \varepsilon)^{-1}) \text{ in case IV if } \lambda_{01} \neq -|(v, \phi_1)|^2/4\pi, \ \gamma \neq 0, \end{cases} \quad (5.16)$$

and (cf. Lemmas 4.1-4.4 of [3])

$$a^s_\varepsilon = \varepsilon a^s(\varepsilon) = \begin{cases} \varepsilon a^s + 0(\varepsilon^2) \text{ in case I,} \\ a^s(\tilde{\alpha}(\tilde{\lambda}_1)) + 0(\varepsilon) \text{ in case II,} \\ \varepsilon a^s + 0(\varepsilon^2) \text{ in case III, } \quad \text{if } \lambda_{10} \neq 0, \\ a^s(\tilde{\alpha}(\tilde{\lambda}_1)) + 0(\varepsilon) \text{ in case IV} \end{cases} \quad (5.17)$$

where $\tilde{\alpha}(\tilde{\lambda}_1) = \alpha(\tilde{\lambda}_1)|_{\gamma=0}$ ($\alpha(\tilde{\lambda}_1)$ given by Theorem 3.1) and a^s represents the scattering length of $H = -\Delta + V(|\underline{x}|)$. Similarly we get

$$\begin{aligned} r^{sc}_{\gamma,\varepsilon} &= \varepsilon r^{sc}_{\varepsilon\gamma}(\varepsilon) = \varepsilon r^s + o(\varepsilon) \\ r^s &= \varepsilon r^s(\varepsilon) \ = \varepsilon r^s + o(\varepsilon) \end{aligned} \quad \text{in all cases} \quad (5.18)$$

where r^s denotes the effective range parameter of H. (5.16)-(5.18) clearly indicate the particular interest of case II with $\lambda_{01} = -|(v, \phi)|^2/4\pi$ (or case IV with $\lambda_{01} = -|(v, \phi_1)|^2/4\pi$): Varying λ_{10} and ε we are able to control and adjust the scattering length $a^{sc}_{\gamma,\varepsilon}$ (a^s_ε if $\gamma = 0$) and effective range $r^{sc}_{\gamma,\varepsilon}$ (r^s_ε if $\gamma = 0$) of $H_{\gamma,\varepsilon}$ (H_ε if $\gamma = 0$). Note that in all circumstances the effective range $r^{sc}_{\gamma,\varepsilon}$ (r^s_ε if $\gamma = 0$) is of order ε as $\varepsilon \to 0_+$.

Obviously this interpretation is valid irrespective of any symmetry assumptions on V. In fact for an arbitrary non spherically symmetric interaction V (5.14), (5.16), and (5.17) follow directly from (4.24) without using (2.15). In the same way (5.15) and (5.18) follow from general considerations in [4].

Now we derive a relation between the « proton-proton » and « neutron-neutron » scattering length. For that purpose we choose case II and

$$\lambda_{01} = -|(v, \phi)|^2/4\pi$$

(case IV and $\lambda_{01} = -|(v, \phi_1)^2/4\pi$ would lead to similar results). According to (5.18) and the discussion following it we choose ε_0 in such a way that $\varepsilon_0 r^s$

coincides with the known numerical value r_0 of the nuclear effective range and introduce the Hamiltonian H_{pp} modelling the p-p interaction by

$$H_{pp} = H_{\gamma,\varepsilon=\varepsilon_0}, \qquad \gamma > 0 \tag{5.19}$$

i. e.,

$$H_{pp} = -\Delta + \frac{\gamma}{|\underline{x}|} + V_s(\underline{x}), \quad V_s(\underline{x}) = \lambda(\varepsilon_0, \gamma\varepsilon_0 \ln \varepsilon_0)\varepsilon_0^{-2}V(\underline{x}/\varepsilon_0) \tag{5.20}$$

and we assume in addition

$$\lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon) = 1 + \lambda_{10}\varepsilon - \gamma(\varepsilon \ln \varepsilon) |(v, \phi)|^2/4\pi, \qquad \lambda_{10} \in \mathbb{R}. \tag{5.21}$$

For the n-n interaction we define

$$H_{nn} = H_{\varepsilon=\varepsilon_0} \tag{5.22}$$

i. e.,

$$H_{nn} = -\Delta + \tilde{V}_s(\underline{x}), \quad \tilde{V}_s(\underline{x}) = \tilde{\lambda}(\varepsilon_0)\varepsilon_0^{-2}V(\underline{x}/\varepsilon_0) \tag{5.23}$$

where now

$$\tilde{\lambda}(\varepsilon) = 1 + \tilde{\lambda}_1\varepsilon, \qquad \tilde{\lambda}_1 \in \mathbb{R}. \tag{5.24}$$

Under the additional hypothesis of charge symmetry i. e. $V_s(\underline{x}) = \tilde{V}_s(\underline{x})$ we obtain

$$\lambda_{10} = \tilde{\lambda}_1 + \gamma(\ln \varepsilon_0) |(v, \phi)|^2/4\pi, \qquad \varepsilon_0 = r_0/r^s \tag{5.25}$$

and using (5.5), (5.6), and (5.25) we get the approximate relation[1]

$$\frac{1}{a^{pp}} = \frac{1}{a^{nn}} + \gamma[\ln(\gamma r_0) + 2C - 1 + (\phi, v \ln(x_+/2r^s)v\phi)/|(v, \phi)|^2] \tag{5.26}$$

where $a^{pp} = a^{sc}_{\gamma,\varepsilon_0}$ and $a^{nn} = a^{s}_{\varepsilon_0}$ denote the p-p and n-n scattering length respectively. As already remarked (5.26) is an approximation to the extent that higher order terms in (5.5) and (5.6) have been neglected. The standard reference (J. D. Jackson and J. M. Blatt [*25*]) suppresses the model dependence inherent in (5.26). Also our results are valid for arbitrary non spherically symmetric interactions V. A quantitative investigation of the model dependent relation between charged and neutral scattering lengths will be given elsewhere with a view towards a comparison of experimental findings [*16*] and the postulate of charge symmetry.

Finally we turn to the computation of level shifts in mesic atoms [*13*] [*15*] [*34*] [*44*].

We consider spherically symmetric potentials $V(|\underline{x}|)$ fulfilling

$$\int_0^R dr r |V(r)| < \infty \qquad \text{for some } R > 0 \tag{5.27}$$

in addition to (3.4). Denoting by $k_{n,0}(\lambda_{10})$, $n = 1, 2, \ldots$ the solutions of

$(\tilde{\phi}, B_{10}(k)\phi) = 0$ in case II if $\lambda_{01} = -|(v, \phi)|^2/4\pi$ (or the solutions of $(\tilde{\phi}_1, B_{10}(k)\phi_1) = 0$ in case IV if $\lambda_{01} = -|(v, \phi_1)|^2/4\pi$) we have

THEOREM 5.2. — A) assume case I (i. e. no zero energy resonances or bound states of $H = -\Delta + V(|\underline{x}|)$) then the level shifts $\Delta E^{n,l}_{\gamma,\varepsilon} = E^{n,l}_{\gamma,\varepsilon} - E^{n,l}_c$, where $E^{n,l}_c = -\gamma^2/4(n+l)^2$, $\gamma < 0$, are the Coulomb levels, $E^{n,l}_{\gamma,\varepsilon}$ the eigenvalues of $H_{\gamma,\varepsilon} = -\Delta + \dfrac{\gamma}{|\underline{x}|} + \lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon)\varepsilon^{-2}V(|\underline{x}|/\varepsilon)$, are given by

$$\frac{\Delta E^{n,l}_{\gamma,\varepsilon}}{E^{n,l}_c} = \begin{cases} a^{0,sc}_{\gamma,\varepsilon} 2\gamma/n + 0((a^{0,sc}_{\gamma,\varepsilon})^2), \quad l=0, \quad n=1,2,\ldots, \\ a^{l,sc}_{\gamma,c} 2^{1-2l}\gamma^{2l+1}\left(\sum_{m=1}^{l}[m^{-2}-(n+l)^{-2}]\right)/(n+l) + 0((a^{l,sc}_{\gamma,\varepsilon})^2), n, \; l=1,2,\ldots, \end{cases}$$

$$= \begin{cases} \varepsilon a^{0,s} 2\gamma/n + 0(\varepsilon^2 \ln \varepsilon), \quad l=0, \quad n=1,2,\ldots, \\ \varepsilon^{2l+1} a^{l,s} 2^{1-2l}\gamma^{2l+1}\left(\sum_{m=1}^{l}[m^{-2}-(n+l)^{-2}]\right)/(n+l) + o(\varepsilon^{2l+1}), \; n, l=1,2\ldots, \end{cases} \tag{5.28}$$

where $a^{l,sc}_{\gamma,\varepsilon}$ and $a^{l,s}$ are the partial wave scattering lengths of $H_{\gamma,\varepsilon}$ and H $(a^{0,s} = (4\pi)^{-1}(v, (uG_{0,0}v + 1)^{-1}u))$.

B) Assume case II (i. e. a zero energy resonance of H in the s-wave but no zero energy bound states in higher waves) then

$$\frac{\Delta E^{n,0}_{\gamma,\varepsilon}}{E^{n,0}_c} = ((k^{n,0}(\lambda_{10}))^2 - E^{n,0}_c)/E^{n,0}_c + 0(\varepsilon \ln \varepsilon)$$
$$\text{if } \lambda_{01} = -|(v, \phi)|^2/4\pi, \quad n=1,2,\ldots, \tag{5.29}$$

$$\frac{\Delta E^{n,0}_{\gamma,\varepsilon}}{E^{n,0}_c} = a^{0,sc}_{\gamma,\varepsilon} 2\gamma/n + 0((a^{0,sc}_{\gamma,\varepsilon})^2)$$
$$= (\ln \varepsilon)^{-1} 2\gamma \langle B_{10}(0_+) \rangle^{-1} |(v, \phi)|^2/4\pi n + 0((\ln \varepsilon)^{-2})$$
$$\text{if } \lambda_{01} \neq -|(v, \phi)|^2/4\pi, \quad n=1,2,\ldots, \tag{5.30}$$

and for $l \geqq 1$, $\Delta E^{n,l}_{\gamma,\varepsilon}/E^{n,l}_c$ coincides with (5.28).

C) Assume case III (i. e. zero energy bound states of H in partial waves $l \geqq 1$, for instance in $l = l_0$) and $\lambda_{01} \neq 0$ or $\lambda_{01} = 0$ and $(\tilde{\phi}_j, B_{10}(0_+)\phi_l)$ non singular. Then

$$\frac{\Delta E^{n,0}_{\gamma,\varepsilon}}{E^{n,0}_c} = a^{0,sc}_{\gamma,\varepsilon} 2\gamma/n + 0((a^{0,sc}_{\gamma,\varepsilon})^2) = \varepsilon a^{0,s} 2\gamma/n + 0((\varepsilon \ln \varepsilon)^2), \quad n=1,2,\ldots \tag{5.31}$$

and for $l \geqq 1$, $l \neq l_0$, $\Delta E^{n,l}_{\gamma,\varepsilon}/E^{n,l}_c$ coincides with (5.28).

D) Assume case IV (i. e. a zero energy resonance of H in the s-wave and zero energy bound states for some $l_0 \geqq 1$). Then

$$\frac{\Delta E^{n,0}_{\gamma,\varepsilon}}{E^{n,0}_c} = ((k^{n,0}(\lambda_{10}))^2 - E^{n,0}_c)/E^{n,0}_c + 0(\varepsilon \ln \varepsilon)$$

$$\text{if } \lambda_{01} = -|(v, \phi)|^2/4\pi, \quad n=1, 2, \ldots, \quad (5.32)$$

$$\frac{\Delta E^{n,0}_{\gamma,\varepsilon}}{E^{n,0}_c} = (\ln \varepsilon)^{-1} 2\gamma(\langle B_{01} \rangle)^{-1}_{11} |(v, \phi_1)|^2/4\pi n + 0((\ln \varepsilon)^{-2})$$

$$\text{if } \lambda_{01} \neq -|(v, \phi_1)|^2/4\pi, \quad n=1, 2, \ldots, \quad (5.33)$$

and for $l \geqq 1$, $l \neq l_0$, $\Delta E^{n,l}_{\gamma,\varepsilon}/E^{n,l}_c$ coincides with (5.28).

Proof. — Let $E^{n,l}_{\gamma,\varepsilon} = (k^{n,l}_{\gamma,\varepsilon})^2$ then by (3.39) $k^{n,l}_{\gamma,\varepsilon}$ are the solutions of

$$\lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon) u G_{\varepsilon\gamma,\varepsilon k} v\phi = -\phi, \qquad \phi \in L^2(\mathbb{R}^3).$$

In case II if $\lambda_{01} = -|(v, \phi)|^2/4\pi$ and in case IV if $\lambda_{01} = -|(v, \phi_1)|^2/4\pi$, $k^{n,0}_{\gamma,\varepsilon}$ are analytic with respect to the variables ε, $\varepsilon \ln \varepsilon$ (they are given by simple zeros of the modified Fredholm determinant

$$D_2(1 + \lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon u G_{\varepsilon\gamma,\varepsilon k} v)) \ [6]$$

and we get

$$k^{n,0}_{\gamma,\varepsilon} = k^{n,0}(\lambda_{10}) + 0(\varepsilon \ln \varepsilon)$$

proving (5.29) and (5.32). In the remaining case we use the Coulomb modified effective range approximation for $l \geqq 0$ [*46*]

$$k^{2l} \prod_{m=1}^{l} [1 + (\gamma/2km)^2] [\pi\gamma/k(e^{\pi\gamma/k} - 1)^{-1}(k \cot \delta^{l,sc}(k) - ik) + \gamma(\Psi(1 + i\gamma/2k) + ik/\gamma + \ln(2k/i|\gamma|))]$$

$$= -(a^{l,sc})^{-1} + r^{l,sc}k^2/2 + 0(k^4), \quad l=0, 1, \ldots. \quad (5.34)$$

Since in these cases $H_{\gamma,\varepsilon}$ converges to H^c in norm resolvent sense we have

$$k^{n,l}_{\gamma,\varepsilon} = -i\gamma/2(n+l) + iv_{n,l}(\varepsilon), \quad v_{n,l}(\varepsilon) \underset{\varepsilon \to 0_-}{=} o(1), \quad l=0, 1, \ldots, \quad n=1, 2, \ldots \quad (5.35)$$

Inserting (5.35) into (5.34) with $\cot \delta^{l,sc} = i$ we get

$$(\gamma/2)^{2l}\left(\sum_{m=1}^{l} [m^{-2} - (n+l)^{-2}]\right)\gamma\Psi[1 - (n+l) - 2(n+l)^2 v_{n,l}(\varepsilon)/\gamma] \underset{\varepsilon \to 0_+}{=} -(a^{l,sc}_{\gamma,\varepsilon})^{-1} + 0(1)$$

and thus

$$v_{n,l}(\varepsilon) = -a^{l,sc}_{\gamma,\varepsilon}(\gamma/2)^{2l}\left(\prod_{m=1}^{l} [m^{-2} - (n+l)^{-2}]\right)\gamma^2/2(n+l)^2 + 0((a^{l,sc}_{\gamma,\varepsilon})^2). \quad (5.36)$$

Since

$$\Delta E^{n,l}_{\gamma,\varepsilon} = -4(n+l)v_{n,l}(\varepsilon)E^{n,l}_c/\gamma + 0((v_{n,l}(\varepsilon))^2),$$

$$\Delta E^{n,l}_{\gamma,\varepsilon}/E^{n,l}_c = a^{l,sc}_{\gamma,\varepsilon} 2^{1-2l}\gamma^{2l+1}\left(\prod_{m=1}^{l} [m^{-2} - (n+l)^{-2}]\right)/(n+l) + 0((a^{l,sc}_{\gamma,\varepsilon})^2) \quad (5.37)$$

follows. Insertion of $a^{sc}_{\gamma,\varepsilon} = a^{0,sc}_{\gamma,\varepsilon}$ from Theorem 5.1 proves our assertions for $l=0$. Using (3.2) and (3.3) we obtain (analogously to (5.16))

$$a^{l,sc}_{\gamma,\varepsilon} = \varepsilon^{2l+1} a^{l,s} + o(\varepsilon^{2l+1}), \qquad l \geqq 1, \qquad l \neq l_0$$

completing the prof. □

REMARK 5.2. — Small (negative) level shifts are obtained in the limit of large negative $\operatorname{Re} \lambda_{10}$. If $|\operatorname{Im} \lambda_{10}|$ remains bounded in (5.29) we have

$$k^{n,0}(\lambda_{10}) = -i\gamma/2n + i\mu_n(\lambda_{10}), \qquad \mu_n(\lambda_{10}) \underset{\operatorname{Re}\lambda_{10} \to -\infty}{=} o(1), \quad n = 1, 2, \ldots \tag{5.38}$$

and as in (5.36)

$$\mu_n(\lambda_{10}) = -a^{sc}(\alpha(\lambda_{10}))\gamma^2/2n^2 + 0(\lambda_{10}^{-2}) = -[4\pi\lambda_{10}|(v, \phi)|^{-2}]^{-1}\gamma^2/2n^2 + 0(\lambda_{10}^{-2})$$
$$= -a^s(\tilde{\alpha}(\lambda_{10}))\gamma^2/2n^2 + 0(\lambda_{10}^{-2})$$

and thus

$$\frac{\Delta E^{n,0}_{\gamma,\varepsilon}}{E^{n,0}_c} = \{ [4\pi\lambda_{10}|(v, \phi)|^{-2}]^{-1} + 0(\lambda_{10}^{-2}) \} 2\gamma/n + 0(\varepsilon \ln \varepsilon)$$
$$= \{ a^s(\tilde{\alpha}(\lambda_{10})) + 0(\lambda_{10}^{-2}) \} 2\gamma/n + 0(\varepsilon \ln \varepsilon), \quad n = 1, 2, \ldots \tag{5.39}$$

where $\tilde{\alpha}(\lambda_{10}) = \alpha(\lambda_{10})|_{\gamma=0}$ and $\alpha(\lambda_{10})$ is determined from (3.36). A similar result is valid for (5.32). Small positive deviations are in fact « large shifts ». As $\operatorname{Re} \lambda_{10}$ approaces $+\infty$ the $n+1^{st}$ level approaches the n^{th} Coulomb level from above, while the ground state becomes extremely strongly bound.

REMARK 5.3. — For $l=0$, the first term in (5.28) may be obtained simply by using Rayleigh-Schrödinger perturbation theory [*15*]. This argument and the method of proof in Theorem 5.2 show that (5.28), (5.29), and (5.32) are true for non spherically symmetric interactions V as far as the ground state shift is concerned.

Therefore, in order to describe complex level shifts in mesic atoms we take case II with $\lambda_{01} = -|(v, \phi)|^2/4\pi$ (or case IV with $\lambda_{01} = -|(v, \phi_1)^2/4\pi$) and choose ε_0 such that $\varepsilon_0 r^s$ coincides with the effective range of the hadronic interaction. The Hamiltonian H_{at} describing the mesic atom is then defined to be

$$H_{at} = H_{\gamma,\varepsilon=\varepsilon_0}, \qquad \gamma < 0 \tag{5.40}$$

i. e.,

$$H_{at} = -\Delta + \frac{\gamma}{|\underline{x}|} + V_s(|\underline{x}|), \qquad V_s(|\underline{x}|) = \lambda(\varepsilon_0, \gamma\varepsilon_0 \ln \varepsilon_0)\varepsilon_0^{-2} V(|\underline{x}|/\varepsilon_0) \tag{5.41}$$

where in addition λ obeys

$$\lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon) = 1 + \lambda_{10}\varepsilon - \gamma(\varepsilon \ln \varepsilon)|(v, \phi)|^2/4\pi, \quad \lambda_{10} \in \mathbb{C}. \tag{5.42}$$

Next we have to choose λ_{10} in such a way that the ground state shift in (5.29) coincides with that observed in nature. Writing

$$\Delta E^{n,l} = \delta E^{n,l} - \frac{i}{2}\Gamma^{n,l}, \qquad n = 1, 2, \ldots, \; l = 0, 1, \ldots$$

where $\delta E^{n,l}$ denotes the energy shift and $\Gamma^{n,l}$ abbreviates the line broadening (width) due to absorption we get the following approximation for the s-wave shifts and widths

$$\Delta E^{n,0} = \frac{(-\gamma)^3 \,|(v, \phi)|^2}{8\pi n^3 \operatorname{Re}\lambda_{10}} - \frac{i}{2}\frac{(-\gamma)^3\,|(v, \phi)|^2 \operatorname{Im}\lambda_{10}}{4\pi n^3 (\operatorname{Re}\lambda_{10})^2} + 0\left((\operatorname{Re}\lambda_{10})^{-2}\right) \quad (5.43)$$

neglecting higher corrections in (5.39). Note that a fit of the ground state level shift $E^{1,0}$ to the experimental value eliminates all free parameters from (5.43), for all $n > 1$. In the same approximation higher shifts are given by (5.28) (cf. also [*15*])

$$\Delta E^{n,l} = (\varepsilon_0/2)^{2l+1}(-\gamma)^{2l+3} a^{l,s}\left(\prod_{m=1}^{l}[m^{-2} - (n+l)^{-2}]\right)/(n+l)^3, \; n, \, l \geqq 1. \quad (5.44)$$

Since $\Delta E^{n,l}$ in (5.44) is real, $\Gamma^{n,l} = 0$ in this approximation i. e. absorption effects only appears in higher corrections if $l \geqq 1$ [*44*] as long as V is real. The fact that $a^s(\tilde{\alpha}(\lambda_{10})) = [4\pi\lambda_{10}\,|(v, \phi)|^{-2}]^{-1}$ in (5.39) represents the leading term of the scattering length of the short range Hamiltonian $-\Delta + V_s(|\underline{x}|)$ (i. e. $\gamma = 0$ in H_{at}) is in agreement with previous discussions [*13*] [15].

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APPENDIX

Expansion coefficients for the Greens function.

First we recall

$$uG_{\varepsilon\gamma,\varepsilon k}v = \sum_{m=0}^{\infty} \hat{G}_{m0}(\gamma, k)\varepsilon^m + \gamma\varepsilon \ln \varepsilon \sum_{m=0}^{\infty} \hat{G}_{m1}(\gamma, k)\varepsilon^m , \tag{A.1}$$

$$\lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon)uG_{\varepsilon\gamma,\varepsilon k}v = \sum_{m,n=0}^{\infty} B_{mn}(\gamma, k)\varepsilon^m(\gamma\varepsilon \ln \varepsilon)^n , \tag{A.2}$$

$$\lambda(\varepsilon, \gamma\varepsilon \ln \varepsilon) = \sum_{m,n=0}^{\infty} \lambda_{mn}\varepsilon^m(\gamma\varepsilon \ln \varepsilon)^n, \quad \lambda_{00} = 1 . \tag{A.3}$$

The lowest order coefficients are given by integral operators with kernels

$$B_{00}(\underline{x}, \underline{y}) = \hat{G}_{00}(\underline{x}, \underline{y}) = u(\underline{x})(4\pi \,|\, \underline{x} - \underline{y} \,|)^{-1}v(\underline{y}) , \tag{A.4}$$

$$\begin{aligned} B_{10}(\gamma, k, \underline{x}, \underline{y}) &= \lambda_{10}\hat{G}_{00}(\underline{x}, \underline{y}) + \hat{G}_{10}(\gamma, k, \underline{x}, \underline{y}) \\ &= \lambda_{10}u(\underline{x})(4\pi \,|\, \underline{x} - \underline{y} \,|)^{-1}v(\underline{y}) + \gamma F(i\gamma/2k)u(\underline{x})v(\underline{y})/4\pi \\ &\quad + \gamma u(\underline{x}) \ln \left[|\,\gamma\,| (|\,\underline{x}\,| + |\,\underline{y}\,| + |\,\underline{x} - \underline{y}\,|)/2) \right] v(\underline{y})/4\pi, \end{aligned} \tag{A.5}$$

$$B_{01}(\underline{x}, \underline{y}) = \hat{G}_{01}(\underline{x}, \underline{y}) + \lambda_{01}\hat{G}_{00}(\underline{x}, \underline{y}) = u(\underline{x})v(\underline{y})/4\pi + \lambda_{01}u(\underline{x})(4\pi \,|\, \underline{x} - \underline{y} \,|)^{-1}v(\underline{y}) . \tag{A.6}$$

Expansion coefficients for the scattering amplitude.

In case II A (Theorem 4.2) we have

$$\langle B_{10}(\gamma, k) \rangle = \lambda_{10} + \gamma F(i\gamma/2k) \,|\, (v, \phi) \,|^2/4\pi + \gamma(\phi, v \ln (|\,\gamma\,|\, x_+/2)v\phi)/4\pi , \tag{A.7}$$

$$\begin{aligned} \langle B_{20}(\gamma, k) \rangle &= \lambda_{20} + \lambda_{10}\gamma F(i\gamma/2k) \,|\, (v, \phi) \,|^2/4\pi + \lambda_{10}\gamma(\phi, v \ln (|\,\gamma\,|\, x_+/2)v\phi)/4\pi \\ &+ \gamma^2 \ln (2k/i \,|\, \gamma \,|)(\phi, v(x_+ + x_-)v\phi)/16\pi + \gamma^2(\phi, v \ln (|\gamma|\, x_+/2)(x_+ + x_-)v\phi)/16\pi \\ &- k^2 \{ (1 + i\gamma/2k)(i\gamma/2k)[\Psi(3) - \Psi(1) - 1/2 - (1 + i\gamma/2k)^{-1}] - 1/4 \} \left(\phi, v \frac{x_- \,.\, x_+}{x_+ - x_-} v\phi \right) /2\pi \\ &- k^2 \{ -(i\gamma/2k)^2[\Psi(1 + i\gamma/2k) - \Psi(1) - \Psi(2) + 1/2]/2 + 1/8 + i\gamma/2k \} \left(\phi, v \frac{x_-^2}{x_+ - x_-} v\phi \right) /2\pi \\ &- k^2 \{ (i\gamma/2k)^2[\Psi(2 + i\gamma/2k) - \Psi(2) - \Psi(3)]/2 + (i\gamma/4k)[\Psi(1) - \Psi(3) + (1 + i\gamma/2k)^{-1}] + 1/8 \} \,. \\ &\qquad . \left(\phi, v \frac{x_+^2}{x_+ - x_-} v\phi \right) /2\pi, \end{aligned} \tag{A.8}$$

$$\begin{aligned} \langle B_{11}(\gamma, k) \rangle &= \lambda_{11} + \gamma(\phi, v(x_+ + x_-)v\phi)/16\pi + \lambda_{10} \,|\, (v, \phi) \,|^2/4\pi + \lambda_{01}\gamma(\phi, v \ln (|\gamma|\, x_+/2)v\phi)/4\pi \\ &\quad + \lambda_{01}\gamma F(i\gamma/2k) \,|\, (v, \phi) \,|^2/4\pi , \end{aligned} \tag{A.9}$$

$$\langle B_{02} \rangle = \lambda_{02} + \lambda_{01} \,|\, (v, \phi) \,|^2/4\pi \tag{A.10}$$

where $vH(x_+, x_-)v$ denote Hilbert-Schmidt operators with kernels

$$v(\underline{x})H(x_+(\underline{x}, \underline{y}),\quad x_-(\underline{x}, \underline{y}))v(\underline{y}),\quad x_\pm(\underline{x}, \underline{y}) = |\underline{x}| + |\underline{y}| \pm |\underline{x} - \underline{y}|.$$

In case II B (Theorem (4.2) we have

$$\langle B_{01} \rangle = \lambda_{01} + |(v, \phi)|^2/4\pi. \tag{A.11}$$

In case IV A(Theorem 4.4) we have

$$\langle B_{10}(\gamma, k)_{11} = (\tilde{\phi}_1, B_{10}(\gamma, k)\phi_1) \quad (\text{replace } \phi \to \phi_1 \text{ in (A.7)})$$
$$\langle B_{10}(\gamma, k) \rangle_{1j} = \gamma(\phi_1, v \ln(|\gamma| x_+/2)v\phi_j)/4\pi, \quad j=2, \ldots, N. \tag{A.12}$$

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