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Section A :
Physique théorique.

Deformations of the algebra of functions on hermitian symmetric spaces resulting from quantization

by

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ABSTRACT . — A$*$-product on a symplectic manifold, in the sense of A. Lichnerowicz, is the first element in what should be called Weyl's quantization on the manifold.

Following some papers of F. A. Berezin, on Hermitian Symmetric spaces M the covariant symbols $A(z; \bar{z})$, $B(z; \bar{z})$ and Weyl's covariant symbols $\overset{\circ}{\mathscr{A}}(z; \bar{z})$, $\overset{\circ}{\mathscr{B}}(z; \bar{z})$, of operators $\hat{A}$, $\hat{B}$ on a suitable Hilbert space, can be defined.

The covariant symbol $A * B$ of the operator $\hat{A} \cdot \hat{B}$ is an exponential integral with positive phase and $\hbar^{-1}$ as real parameter. We give an expression to find recussivelly each term of the asymptotic development, as powers of $\hbar$, of this symbol. In this way we obtain a nontrivial formal deformation of the associative algebra $C^\infty(M)$. This deformation is not a Vey deformation at any order, just as in the Wick quantization on $\mathbb{C}^n$. Each 2-cochain of the deformation is invariant by the group $I_0(M)$.

Similarly we consider the contravariant Weyl's symbol $\overset{\circ}{\mathscr{A}} *'' \overset{\circ}{\mathscr{B}}$ of the operator $\hat{A} \cdot \hat{B}$. (In the Weyl's quantization on $\mathbb{C}^n$ this symbol is just the Moyal product of $\overset{\circ}{\mathscr{A}}$ and $\overset{\circ}{\mathscr{B}}$). In this case the deformation obtained is a Vey deformation at the first order. It is no longer a Vey deformation at the second order.

RÉSUMÉ. — Un produit $*$ sur une variété symplectique, au sens de A. Lichnerowicz, est le premier élément de ce que serait la quantization de Weyl sur la variété.

En suivant des travaux de F. A. Berezin, sur les espaces hermitiens symétriques M, on peut définir les symboles covariants $A(z;\bar{z})$, $B(z;\bar{z})$ et les symboles de Weyl contravariants $\overset{\circ}{\mathscr{A}}(z;\bar{z})$, $\overset{\circ}{\mathscr{B}}(z;\bar{z})$ des opérateurs $\hat{A}$, $\hat{B}$ sur un certain espace de Hilbert.

Le symbole covariant $A * B$, de l'opérateur $\hat{A}\cdot\hat{B}$ est une intégrale exponentielle à phase positive et paramètre réel $\hbar^{-1}$. Dans ce papier, nous donnons une expression pour calculer par récurrence chaque terme du développement asymptotique, en puissance de $\hbar$, de ce symbole. Nous obtenons ainsi une déformation formelle non triviale de l'algèbre associative $C^{\infty}(M)$. La déformation n'est pas une déformation de Vey, tout comme dans la quantization de Wick sur $\mathbb{C}^n$. Chaque 2-cochaine de la déformation est invariante par le groupe $I_0(M)$.

Analoguement nous considérons le symbole de Weyl contravariant $\overset{\circ}{\mathscr{A}} *'' \overset{\circ}{\mathscr{B}}$ de l'opérateur $\hat{A}\cdot\hat{B}$. (Dans la quantization de Weyl sur $\mathbb{C}^n$ ce symbole n'est autre que le produit de Moyal de $\overset{\circ}{\mathscr{A}}$ et $\overset{\circ}{\mathscr{B}}$). Dans ce cas la déformation obtenue est une déformation de Vey au premier ordre et ne l'est plus au second ordre.

INTRODUCTION

1) This paper propose to begin to explore the general definition of Quantization on symplectic manifolds, proposed by F. A. Berezin in the main references [*1*] [*2*] [*3*], from the point of view of $*$-producs.

$*$-products were introduced by A. Lichnerowicz, M. Flato, D. Sternheimer, F. Bayen and C. Fronsdal in Ref. [*4*] and also by J. Vey in Ref. [*5*]. Further they have been studied mainly by S. Gutt and M. Cahen [*6*] [*7*], A. Lichnerowicz [*8*] [*9*] [*10*] and C. Fronsdal [*11*]. See also Refs. [*28*] [*29*] [*30*] [*31*].

A $*$-product on a symplectic manifold is the first element in what should be called Weyl's quantization on the manifold.

In Refs. [*1*] [*2*] [*3*] the general theory is developed mostly on Kähler manifolds, homogeneous Kähler manifolds, and the (Euclidean) symmetric bounded domains and its duals (on Hermitian Symmetric spaces).

In this paper we restrict ourselves to the simplest, non trivial Kähler manifold: Hermitian symmetric spaces of rank 1. Since the calculations and conclusions are qualitatively similar in higher dimensions and in compact case, we will restrict ourselves still further, and will work in Poincare's disk.

2) The *covariant* and the *contravariant* symbols of an operator in a suitable Hilbert space of analytic functions on the pertinent symplectic manifold are defined in Refs. [*1*] [*12*]. They are a suitable generalization of the Wick and anti-Wick symbols in the usual quantization on $\mathbb{C}^n$. A composition law, call it $*$, can thus be defined for two covariant symbols A, B. The composite symbol $A * B$ will be the covariant symbol of the corresponding composite operators. The expression of $A * B$ on suitable Kähler manifolds is an exponential integral with strictly positive phase, and its real parameter is the inverse of Planck's constant.

3) For Poincare's disk (and consequently for any Hermitian symmetric space), we apply the classical Laplace method to find the asymptotic development of that exponential integral. We do this with the help of some explicit formulae given in a paper by E. Combet Ref. [*13*]. This development is in fact a *non trivial* (from the first order) deformation (in the differentiable cohomologie [*5*] [*8*]) of the associative algebra of functions on the manifold. Below we provide an expression to find recursively any 2-cochain of this deformation. All 2-cochain are bidifferential operators without constant term.

The calculated deformation *is invariant* in the usual sense, under the action of the connected component $I_0(M)$ of the isometry group of the manifold. *The usefulness of this deformation to obtain directly from the symbol the spectral properties of the self-adjoint operators of the corresponding unitary representation of* $I_0(M)$, (*see below*) *is thereby assured* [*14*] [*15*]. See Ref. [*27*].

The above-mentioned deformation is not a Vey's deformation [*5*] [*8*] at any order, and the condition of parity on the 2-cochains which is contained in the definition of a $*$-product [*8*] is not satisfied. Consequently, on Poincare's disk, *this rule of quantization* is not the analogue of Weyl's quantization on the canonical manifold $(\mathbb{R}^{2n}, W_0)$. It is, however, the analogue of the *normal rule of quantization* Ref. [*16*].

4) *We now expound the definition of the analogue of Weyl's quantization on symmetric spaces given in Ref.* [*3*]; thereafter we state the formal deformation of the algebra $C^\infty(D^1)$ that we obtain from it.

4_1) The usual Weyl quantization is given by the formula

$$W(\overset{\circ}{\mathscr{A}}) = \int_{\mathbb{R}^{2n}} (F\overset{\circ}{\mathscr{A}})(\xi)U(\xi)d\xi$$

where $U(\xi)$ is a projective representation of the group $\mathbb{R}^{2n}$ with factor $\exp i\frac{2}{\hbar}\omega(\xi;\eta)$, and where F stands for the symplectic Fourier Transform [14] [17] [18]. $\overset{\circ}{\mathscr{A}}$ is called the *contravariant Weyl symbol* of the operator $\hat{A}$. Formally we can also write

$$W(\overset{\circ}{\mathscr{A}}) = \int_{\mathbb{R}^{2n}} \overset{\circ}{\mathscr{A}}(\xi)(FU)(\xi)d\xi \tag{1}$$

Operator $(FU)(\xi)$ has a real meaning: it is the unitary operator corresponding to *the reflexion* around the point $\xi \in \mathbb{R}^{2n}$ in a suitable projective unitary representation of the group engendered by $\mathbb{R}^{2n}$ and by the reflexions around every point of $\mathbb{R}^{2n}$. (This group can be viewed as the semidirect product of $\mathbb{R}^{2n}$ and Z_2. See Ref. [17]).

On Poincare's disk (and thus on any other homogeneous Kähler manifold as in Ref. [1]) the isometry group acts on the covariant symbols of of bounded operators and therefore on the C*-algebra engendered by these operators. In fact, each isometry acts in this way as an automorphism; but it must of course be an inner automorphism Ref. [19]. In this way, a projective unitary representation of the (connected) isometry group is obtained.

The presence of *reflexions* around each point is the basic property of symmetric spaces, and they are elements of $I_0(M)$. The foregoing unitary representation assigns to each reflexion a unitary operator $\hat{U}(x)$ which can be used to define the operator

$$W(\overset{\circ}{\mathscr{A}}) = C(\hbar) \int_{M} \overset{\circ}{\mathscr{A}}(x)\hat{U}(x)d\mu(x) \tag{2}$$

($d\mu(x)$ is the invariant measure on M, and $C(\hbar)$ a suitable factor). *This is the operator which, (on symmetric spaces, where a symplectic Fourier transform cannot be defined), will take the place of operator* (1) *in the canonical manifold* $(\mathbb{R}^{2n}; \omega_0)$.

On space $(\mathbb{R}^{2n}; \omega_0)$ the contravariant Weyl symbol $\overset{\circ}{\mathscr{A}} *' \overset{\circ}{\mathscr{B}}$ of the operator $W(\overset{\circ}{\mathscr{A}})\cdot W(\overset{\circ}{\mathscr{B}})$ is precisely the Moyal $*$-product.

4_2) In this paper, we will find the asymptotic development of the symbol $\overset{\circ}{\mathscr{A}} *' \overset{\circ}{\mathscr{B}}$ on the Poincare disk (and therefore on Hermitian symmetric spaces). It is a non trivial formal deformation (in the differentiable cohomology) of the usual associative algebra $C^\infty(D^1)$. We provide an expression to find recursively any 2-cochain in the deformation. 2-cochains are defined by bidifferential operators without constant term. *The deformation is an infinitesimal associative Vey deformation. It is no longer a Vey deformation at the second order.*

In Refs. [7] [20], S. Gutt and M. Cahen ontained many important results about $*$-products on the manifolds considered here.

J. Vey in Ref. [5] p. 453, was able to find universal formulae for a $*$-product *up to order four.*

1. SYMPLETIC STRUCTURE OF $D^1(\mathbb{C})$

a) On Poincare's disk [*21*] [*22*],

$$D^1(\mathbb{C}) = \{ Z \in \mathbb{C} \mid |Z| < 1 \}$$

the (connected) group of matrices

$$SU(1;1) \left\{ g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \; a, b \in \mathbb{C}; \; |a|^2 - |b|^2 = 1 \right\}$$

acts transitively in according with

$$Z \to g(Z) = \frac{aZ + b}{\bar{b}Z + \bar{a}}.$$

The isotropy group at the point $Z = 0$ is

$$K = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \qquad -\pi \leqslant \theta \leqslant \pi \right\}$$

We then have

$$D^1(\mathbb{C}) = SU(1;1)/K.$$

b) $D^1(\mathbb{C})$ is a bounded domain [*23*] [*24*]. Its Bergman Kernel is

$$K(Z;\bar{v}) = (1 - Z\bar{v})^{-2}$$

and the metric

$$ds^2 = \frac{\partial^2 \log K(Z;\bar{Z})}{\partial Z \cdot \partial \bar{Z}} dZ \otimes d\bar{Z} = 2(1 - Z\cdot\bar{Z})^{-2} dZ \otimes d\bar{Z}$$

is a Kählerian matric [*23*]. The associated fundamental closed non-singular two-form is

$$F = (1 - Z\cdot\bar{Z})^{-2} dZ \wedge d\bar{Z}$$

$(D^1; F)$ is then a symplectic manifold where the Poisson bracket of two functions $f(Z;\bar{Z})\cdot g(Z;\bar{Z})$ in $C^\infty(D^1)$ is

$$P(f;g) = i(1 - Z\cdot\bar{Z})^2(\partial_Z f\cdot\partial_{\bar{Z}} g - \partial_{\bar{Z}} f\cdot\partial_Z g) \tag{3}$$

c) Since the geodesic symmetrie (reflexion) at the origin given by S_0: $Z \to -Z$ is an element of $SU(1;1)$, the isometry gS_0g^{-1} is a reflexion at the point $g\cdot 0$. Seeing that $SU(1;1)$ acts transitively, $D^1(\mathbb{C})$ is a globally symmetric space [*22*].

2. QUANTIZATION ON $D^1(\mathbb{C})$ [*3*]

a) $\hbar$ will be Planck's constant, and $\omega = \hbar^{-1}$. Let $H_\omega(D^1)$ be the Hilbert space of analytic functions $f, g, \ldots$ on $D^1(\mathbb{C})$ with the scalar product

$$(f;g) = (\omega - 1)\int f(Z)\cdot\bar{g}(Z)(1 - Z\cdot\bar{Z})^\omega d\mu(Z,\bar{Z}), \qquad (\omega > 1); \tag{4}$$

where

$$d\mu(Z;\overline{Z}) = \frac{1}{2\pi i}\frac{dZ \wedge d\overline{Z}}{(1 - Z\cdot\overline{Z})^2}$$

is the invariant measure on D^1.

Let $L^2_\omega(D^1)$ be the vector space of measurable functions for which the scalar product (4) is convergent. $H_\omega(D^1)$ is closed in $L^2_\omega(D^1)$.

b) On the space $L^2(D^1)$, the kernel

$$L_\omega(Z;\overline{v}) = (1 - Z\cdot\overline{v})^\omega$$

have properties similar to these of the Bergman kernel on space $L^2_2(D^1)$. That is $L_\omega(Z,\overline{v})$ is the kernel of the orthogonal projection of $L^2_\omega(D^1)$ on $H_\omega(D^1)$, and it has the reproducing property on $H_\omega(D^1)$. If we write $\phi_{\overline{v}}(Z) = L_\omega(Z;\overline{v})$, we have

$$\phi_{\overline{v}} \in H_\omega(D^1) \qquad \text{and} \qquad (f;\phi_{\overline{v}}) = f(v) \tag{5}$$

c) If $\hat{A}$ is a bounded operator on $H_\omega(D^1)$, the covariant symbol of $\hat{A}$ is defined by [2]

$$A(Z;\overline{v}) = \frac{(\hat{A}\phi_{\overline{v}};\phi_{\overline{Z}})}{(\phi_{\overline{v}};\phi_{\overline{Z}})} \tag{6}$$

$A(Z;\overline{v})$ is analytic in Z and $\overline{v}$; $A(Z;\overline{Z})$ is therefore analytic in $x = \mathrm{Re}(Z)$, $y = \mathrm{Im}(Z)$, and $A(Z;\overline{v})$ is an analytic continuation of $A(Z;\overline{Z})$. From this follows that the correspondence between covariant symbols and operators is *bijective* [3].

From property(5) and definition (6), it follows that the operator $\hat{A}$ on $H_\omega(D^1)$ is defined by

$$(\hat{A}f)(Z) = (\omega - 1)\int A(Z;\overline{v})f(v)\left(\frac{1 - v\cdot\overline{v}}{1 - Z\cdot\overline{v}}\right)^\omega d\mu(v;\overline{v}) \tag{7}$$

If $\hat{A}$ and $\hat{B}$ are two operators whose covariant symbols are respectively $A(Z;\overline{v})$, $B(Z;\overline{v})$, the covariant symbol $(A * B)(Z;\overline{v})$ of the operator $\hat{A}\cdot\hat{B}$ is the analytic continuation of the function

$$(A * B)(Z;\overline{Z}) = (\omega - 1\int A(Z,\overline{v})B(v;\overline{Z})\cdot\left[\frac{(1 - Z\cdot\overline{Z})(1 - v\cdot\overline{v})}{(1 - Z\overline{v})(1 - v\cdot\overline{Z})}\right]^\omega d\mu(v;\overline{v}) \tag{8}$$

One may define the operator T_ω by the expression

$$(T_\omega f)(Z;\overline{Z}) = (\omega - 1)\int f(v;\overline{v})\left[\frac{(1 - Z\cdot\overline{Z})(1 - v\cdot\overline{v})}{(1 - Z\cdot\overline{v})(1 - v\cdot\overline{Z})}\right]^\omega d\mu(v;\overline{v}) \tag{9}$$

d) The group $SU(1;1)$ acts on covariant symbols $A(Z;\overline{Z})$ according to the formula

$$g \in SU(1;1); \qquad (gA)(Z;\overline{Z}) = A(g^{-1}Z;\overline{g^{-1}Z}) \tag{10}$$

Checking directly on (8), we find that g is an automorphism of the algebra of covariant symbols for the $*$-composition law:

$$g(\mathrm{A} * \mathrm{B}) = (g\mathrm{A}) * (g\mathrm{B}) \tag{11}$$

From this, it follows that an automorphism of the C*-algebra of bounded operators on $\mathrm{H}_\omega(\mathrm{D}^1)$ is determined by g. This automorphism must be interior [*19*], that is, must to be of the form

$$\hat{\mathrm{A}} \to \mathrm{U}_g \hat{\mathrm{A}} \mathrm{U}_g^{-1}.$$

If in (7), $\hat{\mathrm{A}}$ is self-adjoint, $\mathrm{A}(\mathrm{Z}; \overline{\mathrm{Z}})$ is real, and by (10) $(g\mathrm{A})(\mathrm{Z}; \overline{\mathrm{Z}})$ is also real. The operator $\mathrm{U}_g \hat{\mathrm{A}} \mathrm{U}_g^{-1}$ is therefore self-adjoint; that is

$$\mathrm{U}_g^* \mathrm{U}_g \hat{\mathrm{A}} = \hat{\mathrm{A}} \mathrm{U}_g^* \mathrm{U}_g$$

and this *for every self-adjoint* $\hat{\mathrm{A}}$. Then $\mathrm{U}_g^* \mathrm{U}_g = \lambda \mathrm{I}$; $\lambda \in \mathbb{C}$. U_g is thus unitary up to an (arbitrary) scalar factor. If we choose one of the form $e^{i\varphi}$, we obtain a projective unitary representation

$$g \to \hat{\mathrm{U}}_g \tag{12}$$

of the group SU(1; 1) on the space $\mathrm{H}_\omega(\mathrm{D}^1)$. It turns out to be irreductible [*3*]. The explicit action of the operator $\hat{\mathrm{U}}_g$ is

$$(\hat{\mathrm{U}}_g f)(\mathrm{Z}) = e^{i\varphi} f\left(\frac{\bar{a}\mathrm{Z} - b}{-\bar{b}\mathrm{Z} + \bar{a}}\right) \cdot (-\bar{b}\mathrm{Z} + \bar{a})^{-\omega} \qquad f \in \mathrm{H}_\omega(\mathrm{D}^1). \tag{13}$$

The covariant symbol of the operator $\hat{\mathrm{U}}_g$ is easily calculated from (6) and (13), namely

$$\mathrm{U}_g(\mathrm{Z}; \overline{\mathrm{Z}}) = e^{i\varphi} \left(\frac{1 - \mathrm{Z}\cdot\overline{\mathrm{Z}}}{a - \bar{a}\mathrm{Z}\cdot\overline{\mathrm{Z}} + b\overline{\mathrm{Z}} - \bar{b}\mathrm{Z}} \right)^{\omega} \tag{14}$$

e) By (2), the covariant symbol $\mathrm{A}(\mathrm{Z}; \overline{\mathrm{Z}})$ of an operator $\hat{\mathrm{A}}$ whose contravariant Weyl symbol is $\mathring{\mathscr{A}}(\mathrm{Z}; \overline{\mathrm{Z}})$, is given by

$$\mathrm{A}(\mathrm{Z}; \overline{\mathrm{Z}}) = (\omega - 1) \int \mathring{\mathscr{A}}(v; \bar{v}) \mathrm{U}_{v,\bar{v}}(\mathrm{Z}; \overline{\mathrm{Z}}) d\mu(v; \bar{v}), \tag{15}$$

where $\mathrm{U}_{v,\bar{v}}(\mathrm{Z}; \overline{\mathrm{Z}})$ is the covariant symbol of the unitary operator $\hat{\mathrm{U}}_{v,\bar{v}}$ which corresponds by representation (12) to the geodesic symmetry $g_{(v,\bar{v})} \in \mathrm{SU}(1; 1)$, around the point $v \in \mathrm{D}^1$.

The reflexion around the point $\mathrm{Z} = 0$ is given by the element of SU(1; 1)

$$\mathrm{S}_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

If $g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$ is a generic element of SU(1; 1),

$$g_{\left(\frac{b}{\overline{a}};\frac{\overline{b}}{a}\right)} = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} \overline{a} & -b \\ -\overline{b} & a \end{pmatrix},$$

is the reflexion around the point $v = \frac{b}{\overline{a}}$. That is

$$g_{(v,\overline{v})} = \begin{pmatrix} \dfrac{i(1+v\cdot\overline{v})}{1-v\cdot\overline{v}} & \dfrac{-2iv}{1-v\cdot\overline{v}} \\ \dfrac{2i\overline{v}}{1-v\cdot\overline{v}} & \dfrac{-i(1+v\cdot\overline{v})}{1-v\cdot\overline{v}} \end{pmatrix}$$

Expression (14) thus yields

$$U_{v,\overline{v}}(Z;\overline{Z}) = \left[\frac{(1-Z\cdot\overline{Z})(1-v\cdot\overline{v})}{(1-Z\cdot\overline{v})(1-v\cdot\overline{Z})}\right]^{\omega}\left[1+\frac{Z-v}{1-\overline{Z}v}\cdot\frac{\overline{Z}-\overline{v}}{1-Z\cdot\overline{v}}\right]^{-\omega} \tag{16}$$

up to a factor of the form $e^{i\varphi}$. We arbitrarily choose $\varphi = 0$.

Using (16) and (15), we can define the operator

$$(S_\omega\mathring{\mathscr{A}})(Z;\overline{Z}) = (\omega-1)\int\mathring{\mathscr{A}}(v;\overline{v})\left[\frac{(1-Z\cdot\overline{Z})(1-v\cdot\overline{v})}{(1-Z\cdot\overline{v})(1-v\cdot\overline{Z})}\right]^{\omega}\left[1+\frac{Z-v}{1-\overline{Z}v}\frac{\overline{Z}-\overline{v}}{1-Z\cdot\overline{v}}\right]^{-\omega} d\mu(v;\overline{v}) \tag{17}$$

And then $A(Z;\overline{Z}) = (S_\omega\mathring{\mathscr{A}})(Z;\overline{Z})$.

Operator S_ω^{-1} exists, and it is thus certain that the correspondence $A \Leftrightarrow \mathring{\mathscr{A}}$ is bijective.

f) In the canonical manifold $(\mathbb{R}^{2n};\omega_0)$, the composition law for Wick symbols or for contravariant Weyl symbols has an asymptotic development as powers of ω^{-1}. It is therefore natural to look for the developments of the analogous composition laws on the manifold we are considering.

3. THE OPERATORS T_ω AND S_ω

a) The transformation $v \to W$ given by

$$v = \frac{Z-W}{1-\overline{Z}W},$$

is an element of SU(1; 1). For any function $f(v;\overline{v})$, one may define:

$$(\tau_{Z\overline{Z}}f)(W;\overline{W}) = f\left(\frac{Z-W}{1-\overline{Z}\cdot W};\frac{\overline{Z}-\overline{W}}{1-Z\cdot\overline{W}}\right) \tag{18}$$

Expression (9) and (17) thus become respectively

$$(\mathrm{T}_\omega f)(\mathrm{Z};\overline{\mathrm{Z}}) = (\omega - 1)\int(\tau_{\mathrm{Z}\overline{\mathrm{Z}}}f)(\mathrm{W};\overline{\mathrm{W}})(1 - \mathrm{W}\cdot\overline{\mathrm{W}})^\omega d\mu(\mathrm{W};\overline{\mathrm{W}}) \quad (19)$$

$$(\mathrm{S}_\omega f)(\mathrm{Z};\overline{\mathrm{Z}}) = (\omega - 1)\int(\tau_{\mathrm{Z}\overline{\mathrm{Z}}}f)(\mathrm{W};\overline{\mathrm{W}})\left[\frac{1 - \mathrm{W}\cdot\overline{\mathrm{W}}}{1 + \mathrm{W}\cdot\overline{\mathrm{W}}}\right]^\omega d\mu(\mathrm{W};\overline{\mathrm{W}}) \quad (20)$$

b) In expression (19), we make the change of variable

$$\mathrm{W} = \sqrt{1 - e^{-\rho^2}}\cdot e^{i\theta};\qquad \theta\in[0;2\pi];\qquad \mathrm{W}\in\mathrm{D}^1 \quad (21)$$

Then

$$d\mu(\mathrm{W};\overline{\mathrm{W}}) = \frac{1}{\pi}\cdot\rho\cdot e^{-\rho^2}\cdot d\rho\wedge d\theta$$

And the expression (19) becomes

$$(\mathrm{T}_\omega f)(\mathrm{Z};\overline{\mathrm{Z}})$$
$$= \frac{1}{\pi}(\omega-1)\int(\tau_{\mathrm{Z}\overline{\mathrm{Z}}}f)\left(\sqrt{1-e^{-\rho^2}}e^{i\theta};\sqrt{1-e^{-\rho^2}}e^{-i\theta}\right)\cdot e^{-\rho^2\omega}\cdot\rho\cdot e^{\rho^2}\cdot d\rho\cdot d\theta.$$

That is

$$(\mathrm{T}_\omega f)(\mathrm{Z};\overline{\mathrm{Z}}) = \frac{1}{\pi}(\omega-1)\int_0^\infty e^{-\omega\rho^2}g_2(\mathrm{Z};\overline{\mathrm{Z}};\rho)d\rho \quad (22)$$

where

$$g_2(\mathrm{Z};\overline{\mathrm{Z}};\rho) = \frac{1}{2}\frac{d}{d\rho}(e^{\rho^2})\cdot\int_0^{2\pi}(\tau_{\mathrm{Z}\overline{\mathrm{Z}}}f)\left(\sqrt{1-e^{-\rho^2}}e^{i\theta};\sqrt{1-e^{-\rho^2}}e^{-i\theta}\right)d\theta \quad (23)$$

c) Similarly, in expression (20), we make the change

$$\mathrm{W} = \sqrt{\frac{1 - e^{-\rho^2}}{1 + e^{-\rho^2}}}\cdot e^{i\theta};\qquad \rho\in[0;\infty[;\ \theta\in[0;\ 2\pi[\quad (24)$$

We then obtain

$$(\mathrm{S}_\omega f)(\mathrm{Z};\overline{\mathrm{Z}}) = \frac{1}{\pi}(\omega - 1)\int_0^\infty e^{-\omega\rho^2}g_2'(\mathrm{Z};\overline{\mathrm{Z}};\rho)d\rho \quad (25)$$

Where

$$g_2'(\mathrm{Z};\overline{\mathrm{Z}};\rho)$$
$$= \frac{1}{4}\frac{d}{d\rho}(e^{\rho^2})\cdot\int_0^{2\pi}(\tau_{\mathrm{Z}\overline{\mathrm{Z}}}f)\left(\left(\frac{1-e^{-\rho^2}}{1+e^{-\rho^2}}\right)^{1/2}e^{i\theta};\left(\frac{1-e^{-\rho^2}}{1+e^{-\rho^2}}\right)^{1/2}e^{-i\theta}\right)d\theta \quad (26)$$

Remark. — The change of variables (21), (24) are C^∞. At the points with $\rho\neq 0$ this is trivial, but not for $\rho = 0$. In fact, this follows from the Morse Lemma [*25*], p. 6, [*13*], p. 6, applied to the function, $\psi(\mathrm{R}) = \ln(1-\mathrm{R}^2)$; $0\leqslant\mathrm{R}<1$ in case (21) and to the function $\psi(\mathrm{R}) = \ln\frac{1-\mathrm{R}^2}{1+\mathrm{R}^2}$ in case (22). At $\mathrm{R} = 0$ these functions have a non degenerate critical point

$$(\varphi'(0) = \psi'(0) = 0;\ \varphi''(0),\ \psi''(0)\neq 0)\qquad\text{with}\qquad(\varphi(0) = \psi(0) = 0).$$

4. ASYMPTOTIC DEVELOPMENT OF T_ω AND S_ω

a) According to the proposition on the p. 5 in Ref. [*13*], we have

$$\int_0^\infty e^{-\omega\rho^2} g_2(Z;\overline{Z};\rho)d\rho \underset{\omega\to\infty}{\cong} \frac{1}{2}\sum_{K=0}^{\infty} \frac{\Gamma\left(\frac{K+1}{2}\right)}{K!} g_2^{(K)}(Z;\overline{Z};0)\omega^{-\frac{K+1}{2}},$$

Where

$$g_2^{(K)}(Z;\overline{Z};0) = \frac{d^K}{d\rho^K} g_2(Z;\overline{Z};\rho)\Big|_{\rho=0}$$

But from [*13*], p. 6 and [*26*], p. 71-72, we see that

$$g_2^{(2N)}(Z;\overline{Z};0) = 0; \qquad N = 1, 2, 3, \ldots$$

Then we have

$$\int_0^\infty e^{-\omega\rho^2} g_2(Z;\overline{Z};\rho)d\rho \underset{\omega\to\infty}{\cong} \frac{1}{2}\sum_N \frac{\Gamma(N+1)}{(2N+1)!} g_2^{(2N+1)}(Z;\overline{Z};0)\omega^{-N-1},$$

And finally, for expression (22), we obtain

$$\pi(T_\omega f)(Z;\overline{Z}) = (\omega-1)\int_0^\infty e^{-\omega\rho^2} g_2(Z;\overline{Z};\rho)d\rho \underset{\omega\to\infty}{\cong}$$

$$\underset{\omega\to\infty}{\cong} \frac{1}{2} g_2^{(1)}(Z;\overline{Z};0) + \sum_N G_2(Z;\overline{Z};N)\omega^{-N-1} \quad (27)$$

where

$$G_2(Z;\overline{Z};N) = \frac{1}{2}\left[\frac{\Gamma(N+2)}{(2N+3)!} g_2^{(2N+3)}(Z;\overline{Z};0) - \frac{\Gamma(N+1)}{(2N+1)!} g_2^{(2N+1)}(Z;\overline{Z};0)\right] \quad (28)$$

$N = 0, 1, 2, \ldots$

b) In the same way, we can obtain an analogous asymptotic development for expression (25). That is

$$\pi(S_\omega f)(Z;\overline{Z}) = (\omega-1)\int e^{-\omega\rho^2} g'_2(Z;\overline{Z};\rho)d\rho \underset{\omega\to\infty}{\cong}$$

$$\underset{\omega\to\infty}{\cong} \frac{1}{2} g_2'^{(1)}(Z;\overline{Z};0) + \sum_N G'_2(Z;\overline{Z};N)\omega^{-N-1}, \quad (29)$$

where

$$G'_2(Z;\overline{Z};N) = \frac{1}{2}\left[\frac{\Gamma(N+2)}{(2N+3)!} g_2'^{(2N+3)}(Z;\overline{Z};0) - \frac{\Gamma(N+1)}{(2N+1)!} g_2'^{(2N+1)}(Z;\overline{Z};0\right]$$

$$N = 0, 1, 2, \ldots \quad (30)$$

c) The problem is now to find the fonctions

$$g_2^{(1)}(Z;\overline{Z};0); \qquad G_2(Z;\overline{Z};N); \qquad g_2'^{(1)}(Z;\overline{Z};0); \qquad G_2'(Z;\overline{Z};N)$$

If we try to find it directly from expressions (23), (28) *and* (26), (30) *we will be unables to do so. We must use the following device.*

5. SOME USEFUL RELATIONS

a) In expression (9) for operator T_ω, we write

$$K(Z,\overline{Z}\mid v,\overline{v}) = \frac{(1 - Z\cdot\overline{Z})(1 - v\cdot\overline{v})}{(1 - Z\cdot\overline{v})(1 - v\cdot\overline{Z})}$$

We have the properties

$$\begin{aligned}
&1) && K(Z;\overline{Z}\mid v;\overline{v}) = K(v,\overline{v}\mid Z;\overline{Z})\\
&2) && \Delta_{Z,\overline{Z}}K^{\omega}(Z;\overline{Z}\mid v;\overline{v}) = -\omega^2K^{\omega+1}(Z;\overline{Z}\mid v;\overline{v}) + \omega(\omega-1)K^{\omega}(Z;\overline{Z}\mid v;\overline{v})\\
&3) && \Delta_{v,\overline{v}}K^{\omega}(Z;\overline{Z}\mid v;\overline{v}) = \Delta_{Z,\overline{Z}}K^{\omega}(Z;\overline{Z}\mid v;\overline{v}) \qquad (31)\\
&4) && \Delta T_\omega = T_\omega\Delta
\end{aligned}$$

Relation 4) follows from the definition of T_ω, and the self-adjoint property of Δ. Relation 3) follows from relation 2), which its itself follows by straightforward calculation.

From relations 2) and 4), we obtain one of the must useful relations for this paper

$$T_\omega\Delta = -\omega(\omega - 1)T_{\omega+1} + (\omega - 1)^2T_\omega \qquad (32)$$

b) In expression (17) for the operator S_ω, we write

$$V(v,\overline{v}\mid Z;\overline{Z}) = \left[\frac{(1-Z\cdot\overline{Z})(1-v\cdot\overline{v})}{(1-Z\cdot\overline{v})(1-\overline{Z}v)}\right]\left[1 + \frac{Z - v}{1 - \overline{Z}v}\,\frac{\overline{Z} - \overline{v}}{1 - Z\cdot\overline{v}}\right]^{-1}$$

From (19) we obtain

$$(\tau_{Z\overline{Z}}V(.,.\mid Z,\overline{Z}))(W;\overline{W}) = \left[\frac{1 - W\cdot\overline{W}}{1 + W\cdot\overline{W}}\right] \qquad (33)$$

We have the properties

$$\begin{aligned}
&1') && V(v;\overline{v}\mid Z;\overline{Z}) = V(Z;\overline{Z}\mid v;\overline{v})\\
&2') && \Delta_{W,\overline{W}}\left(\frac{1 - W\cdot\overline{W}}{1 + W\cdot\overline{W}}\right)^{\omega} = \omega(\omega - 1)\left(\frac{1 - W\cdot\overline{W}}{1 + W\cdot\overline{W}}\right)^{\omega}\\
&&& \qquad - \omega(\omega + 1)\left(\frac{1 - W\cdot\overline{W}}{1 + W\cdot\overline{W}}\right)^{\omega+2}
\end{aligned}$$

3′) $\Delta_{v,\bar{v}}V^{\omega}(v;\bar{v}\,|\,Z;\overline{Z}) = \omega(\omega-1)V^{\omega}(v;\bar{v}\,|\,Z;\overline{Z}) - \omega(\omega+1)V^{\omega+2}(v;\bar{v}\,|\,Z;\overline{Z})$

4′) $\Delta_{v,\bar{v}}V^{\omega}(v;\bar{v}\,|\,Z;\overline{Z}) = \Delta_{Z,\overline{Z}}V^{\omega}(v;\bar{v}\,|\,Z;\overline{Z})$

5′) $\Delta S_{\omega} = S_{\omega}\Delta.$

Relation 5′) follows from definition (17) of S_{ω}; relation 4′) and the self-adjointess of Δ. Relation 4′) in turn follows from relation 3′), which itself follows from relation 2′), (33), and the self-adjointness of Δ. Relation 2′) follows by a direct calculation.

The other basic relation for this paper follows from (5′), (3′) and (17).

$$S_{\omega}\Delta = \omega(\omega-1)S_{\omega} - \omega(\omega-1)S_{\omega+2} \tag{35}$$

6. AN EXPRESSION TO FIND $G_2(Z;\ \bar{Z};\ N)$, $N = 0, 1, 2,$ BY RECURRENCE

By identifying the asymptotic development of the two members in relation (32), we will obtain an expression to find by recurrence $G_2(Z;\overline{Z};N)$, $N = 0, 1, 2, \ldots.$

a) From relation (22), a similar one for $T_{\omega+1}$, and (32) we have

$$\pi(T_{\omega}\Delta f)(Z;\overline{Z}) = -\,\omega^2(\omega-1)\int_0^{+\infty} e^{-\omega\rho^2}g_1(Z;\overline{Z};\rho)d\rho + \omega(\omega-1)^2\int_0^{+\infty} e^{-\omega\rho^2}g_2(Z;\overline{Z};\rho)d\rho \tag{36}$$

Where $g_2(Z;\overline{Z};\rho)$ is given by (23) and

$$g_1(Z;\overline{Z};\rho) = e^{-\rho^2}g_2(Z;\overline{Z};\rho) \tag{37}$$

From expression (27) and the Taylor development of $e^{-\rho^2}$ around $\rho = 0$, we deduce the following

$$(\omega-1)\int_0^{+\infty} e^{-\omega\rho^2}g_1(Z;\overline{Z};\rho)d\rho \underset{\omega\to\infty}{\simeq} \underset{\omega\to\infty}{\simeq} \frac{1}{2}\,g_1^{(1)}(Z;\overline{Z};0) + \sum_{N=0}^{\infty} G_1(Z;\overline{Z};N)\omega^{-N-1} \tag{38}$$

where

$$G_1(Z;\overline{Z};N) = \frac{1}{2}\left[\frac{\Gamma(N+2)}{(2N+3)!}\,g_1^{(2N+3)}(Z;\overline{Z};0) - \frac{\Gamma(N+1)}{(2N+1)!}\,g_1^{(2N+1)}(Z;\overline{Z};0)\right] \tag{39}$$

and

$$g_1^{(K)}(Z;\overline{Z};0) = \frac{d^K}{d\rho^K}\,g_1(Z;\overline{Z};\rho)\bigg|_{\rho=0}.$$

Substituting expressions (27) and (38) in (36) and ordering by the powers of ω^{-1} we obtain

$$\begin{aligned}\pi(\mathrm{T}_\omega \Delta f)(\mathrm{Z};\overline{\mathrm{Z}}) \underset{\omega\to\infty}{\simeq} & \left[\frac{1}{2} g_2^{(1)}(\mathrm{Z};\overline{\mathrm{Z}};0) - \frac{1}{2} g_1^{(1)}(\mathrm{Z};\overline{\mathrm{Z}};0)\right]\omega^2 + \\ & + \left[\mathrm{G}_2(\mathrm{Z};\overline{\mathrm{Z}};0) - \mathrm{G}_1(\mathrm{Z};\overline{\mathrm{Z}};0) - \frac{1}{2} g_2^{(1)}(\mathrm{Z};\overline{\mathrm{Z}};0)\right]\omega + \\ & + [\mathrm{G}_2(\mathrm{Z};\overline{\mathrm{Z}};1) - \mathrm{G}_1(\mathrm{Z};\overline{\mathrm{Z}};0) - \mathrm{G}_2(\mathrm{Z};\overline{\mathrm{Z}};0)] + \\ & + \sum_{\mathrm{N}=0}^{\infty} [\mathrm{G}_2(\mathrm{Z};\overline{\mathrm{Z}};\mathrm{N}+2) - \mathrm{G}_1(\mathrm{Z};\overline{\mathrm{Z}};\mathrm{N}+1) - \mathrm{G}_2(\mathrm{Z};\overline{\mathrm{Z}};\mathrm{N}+1)]\omega^{-\mathrm{N}-1}\end{aligned} \tag{40}$$

The left-hand side of expression (40) has the following asymptotic development

$$\begin{aligned}\pi(\mathrm{T}_\omega(\Delta f))(\mathrm{Z};\overline{\mathrm{Z}}) &= (\omega - 1)\int_0^{+\infty} e^{-\omega\rho^2} h_0(\mathrm{Z};\overline{\mathrm{Z}};\rho)d\rho \underset{\omega\to\infty}{\simeq} \\ & \underset{\omega\to\infty}{\simeq} \frac{1}{2} h_0^{(1)}(\mathrm{Z};\overline{\mathrm{Z}};0) + \sum_{\mathrm{N}=0}^{\infty} \mathrm{H}_0(\mathrm{Z};\overline{\mathrm{Z}};\mathrm{N})\omega^{-\mathrm{N}-1}\,;\end{aligned} \tag{41}$$

where

$$h_0(\mathrm{Z};\overline{\mathrm{Z}};\rho) = \frac{1}{2}\frac{d}{d\rho}(e^{\rho^2})\int_0^{2\pi} (\tau_{\mathrm{Z}\overline{\mathrm{Z}}}\Delta f)(\sqrt{1-e^{-\rho^2}}e^{i\theta};\sqrt{1-e^{-\rho^2}}e^{-i\theta})d\theta$$

$$\mathrm{H}_0(\mathrm{Z};\overline{\mathrm{Z}};\mathrm{N}) = \frac{1}{2}\left[\frac{\Gamma(\mathrm{N}+2)}{(2\mathrm{N}+3)!} h_0^{(2\mathrm{N}+3)}(\mathrm{Z};\overline{\mathrm{Z}};0) - \frac{\Gamma(\mathrm{N}+1)}{(2\mathrm{N}+1)!} h_0^{(2\mathrm{N}+1)}(\mathrm{Z};\overline{\mathrm{Z}};0)\right] \tag{42}$$

and

$$h_0^{(\mathrm{K})}(\mathrm{Z};\overline{\mathrm{Z}};0) = \frac{d^{\mathrm{K}}}{d\rho^{\mathrm{K}}} h_0(\mathrm{Z};\overline{\mathrm{Z}};\rho)\Big|_{\rho=0}.$$

b) By identifying the two developments (40) and (41), we obtain the relations

a) $g_2^{(1)}(\mathrm{Z};\overline{\mathrm{Z}};0) - g_1^{(1)}(\mathrm{Z};\overline{\mathrm{Z}};0)$; terms in ω^2

b) $\mathrm{G}_2(\mathrm{Z};\overline{\mathrm{Z}};0) - \mathrm{G}_1(\mathrm{Z};\overline{\mathrm{Z}};0) = \frac{1}{2} g_2^{(1)}(\mathrm{Z};\overline{\mathrm{Z}};0)$; terms in ω

c) $\mathrm{G}_2(\mathrm{Z};\overline{\mathrm{Z}};1) - \mathrm{G}_1(\mathrm{Z};\overline{\mathrm{Z}};1) - \mathrm{G}_2(\mathrm{Z};\overline{\mathrm{Z}}'0) = \frac{1}{2}\mathrm{h}_0^{(1)}(\mathrm{Z};\overline{\mathrm{Z}};0)$; termsin ω^0

d) $\mathrm{G}_2(\mathrm{Z};\ \overline{\mathrm{Z}};\ \mathrm{N}+2) - \mathrm{G}_1(\mathrm{Z};\ \overline{\mathrm{Z}};\ \mathrm{N}+2) - \mathrm{G}_1(\mathrm{Z};\ \overline{\mathrm{Z}};\ \mathrm{N}+1) = \mathrm{H}_0(\mathrm{Z};\ \overline{\mathrm{Z}};\ \mathrm{N})$ terms in $\omega^{-\mathrm{N}-1}$, $\mathrm{N} \geqslant 0$.

We now have

LEMMA 1.

1) $$\left(\frac{d}{d\rho}\right)^{2\mathrm{N}+1} e^{-\rho^2}\Big|_{\rho=0} = 0$$

2) $$\left(\frac{d}{d\rho}\right)^{2N} e^{-\rho^2}\Big|_{\rho=0} = \frac{(2N)!}{N!}(-1)^N$$

3) $$g_1^{(2N+1)}(Z;\overline{Z};0) = \sum_{L=0}^{N} \frac{(2N+1)!}{(2L+1)!(N-L)!}(-1)^L g_2^{(2L+1)}(Z;\overline{Z};0).$$ ■

c) The relations *a*) and *b*) are identities. This can be seen by a direct calculation, considering property 3) N = 1 in Lemma 1.

RELATION *c*)

The term $h_0^{(1)}(Z;\overline{Z};0)$ is known

$$h_0^{(1)}(Z;\overline{Z};0) = \frac{d}{d\rho}\left[\frac{1}{2}\frac{d}{d\rho}(e^{\rho^2})\int_0^{2\pi}(\tau_{Z\overline{Z}}\Delta f)(\sqrt{1-e^{-\rho^2}}e^{i\theta};\sqrt{1-e^{-\rho^2}}e^{-i\theta})d\theta\right]_{\rho=0}$$

We thus obtain

$$h_0^{(1)}(Z;\overline{Z};0) = 2\pi(\Delta f)(Z;\overline{Z}) \tag{43}$$

Similarly, from (23) we obtain

$$g_2^{(1)}(Z;\overline{Z};0) = 2\pi f(Z;\overline{Z}) \tag{44}$$

Then term $G_1(Z;\overline{Z};1)$ is defined from $g_1^{(5)}(Z;\overline{Z};0)$ and $g_1^{(3)}(Z;\overline{Z};0)$. Using Lemma 1, 3), N = 1, 2, we can write these derivatives as functions of $g_2^{(K)}(Z;\overline{Z};0)$; $K \leq 5$. We then obtain for the left-hand side of relation *c*)

$$G_2(Z;\overline{Z};1) - G_1(Z;\overline{Z};1) - G_2(Z;\overline{Z};0) = G_2(Z;\overline{Z};0).$$

Then from relation *c*) and the above we obtain

$$G_2(Z;\overline{Z};0) = \frac{1}{2}h_0^{(1)}(Z;\overline{Z};0)$$

And using (43), we obtain

$$G_2(Z;\overline{Z};0) = \pi(\Delta f)(Z;\overline{Z}) \tag{45}$$

We now proceed by induction on N in the relation *d*).

RELATION *d*) N = 0

We have

$$G_2(Z;\overline{Z};2) - G_1(Z;\overline{Z};2) - G_2(Z;\overline{Z};1) = H_0(Z;\overline{Z};0) \tag{46}$$

$H_0(Z;\overline{Z};0)$ can be obtained from the expression (45) for $G_2(Z;\overline{Z};0)$ by the changing f into Δf (see (42)). We then have

$$H_0(Z;\overline{Z};0) = \pi(\Delta^2 f)(Z;\overline{Z}) \tag{47}$$

The terms $G_2(Z;\overline{Z};2)$ and $G_2(Z;\overline{Z};1)$ are defined by means of the derivatives $g_2^{(K)}(Z;\overline{Z};0)$, $K \leq 7$; but $G_1(Z;\overline{Z};2)$ contains $g_1^{(7)}(Z;\overline{Z};0)$, $g_1^{(5)}(Z;\overline{Z};0)$. We then write these with the help of Lemma 1, 3), $N = 2, 3$ and rearrange the terms on the left-hand side of (46). We then obtain

$$G_2(Z;\overline{Z};2)-G_1(Z;\overline{Z};2)-G_2(Z;\overline{Z};1)=2G_2(Z;\overline{Z};1)-2G_2(Z;\overline{Z};0) \quad (48)$$

Relation (46), (45), (47) and (48) then allow us to find

$$G_2(Z;\overline{Z};1) = \pi\left[\frac{1}{2}(\Delta^2 + 2\Delta)f\right](Z;\overline{Z}) \quad (49)$$

RELATION *d*) $N = 1$

We have

$$G_2(Z;\overline{Z};3)-G_1(Z;\overline{Z};3)-G_2(Z;\overline{Z};2)=H_0(Z;\overline{Z};1) \quad (50)$$

$H_0(Z;\overline{Z};1)$ is obtained from the expression of $G_0(Z;\overline{Z};1)$ by changing f into Δf. Thus

$$H_0(Z;\overline{Z};1) = \pi\left[\frac{1}{2}(\Delta^3 + 2\Delta^2)f\right](Z;\overline{Z}) \quad (51)$$

The derivatives $g_1^{(K)}(Z;\overline{Z};0)$; $K \leq 9$ in $G_1(Z;\overline{Z};3)$ are written as functions of $g_2^{(K)}(Z;\overline{Z};0)$; $K \leqslant 9$ using Lemma 1, 3) $N = 3, 4$. We thus obtain the equality

$$\begin{aligned} &G_2(Z;\overline{Z};3) - G_1(Z;\overline{Z};3) - G_2(Z;\overline{Z};2) \\ &\qquad = 3G_2(Z;\overline{Z};2) - 5G_2(Z;\overline{Z};1) + 2G_2(Z;\overline{Z};0) \quad (52)\end{aligned}$$

Relations (49), (50), (51), (52) then allow us to find

$$G_2(Z;\overline{Z};2) = \pi\left[\frac{1}{6}(\Delta^3 + 7\Delta^2 + 6\Delta)f\right](Z;\overline{Z}) \quad (53)$$

RELATION *d*) $N = 2$

In the same way, we obtain

$$\begin{aligned} &G_2(Z;\overline{Z};4) - G_1(Z;\overline{Z};4) - G_2(Z;\overline{Z};3) = \\ &\quad = 4G_2(Z;\overline{Z};3) - 9G_2(Z;\overline{Z};2) + 7G_2(Z;\overline{Z};1) - 2G_2(Z;\overline{Z};0) \quad (54)\end{aligned}$$

And from (53), by changing f into Δf

$$H_0(Z;\overline{Z};2) = \pi\left[\frac{1}{6}(\Delta^4 + 7\Delta^3 + 6\Delta^2)f\right](Z;\overline{Z}) \quad (55)$$

Relations $d)$ $N = 2$, (53), (54) and (55) then allows to obtain

$$G_2(Z;\overline{Z};3) = \pi\left[\frac{1}{4.6}(\Delta^4 + 16\Delta^3 + 148\Delta^2 + 24\Delta)f\right](Z;\overline{Z}) \qquad (56)$$

$d)$ In general,

PROPOSITION 2. — If $N \geqslant 0$ we have

$$G_2(Z;\overline{Z};N+2)-G_1(Z;\overline{Z};N+2)-G_2(Z;\overline{Z};N+1)=(N+2)G_2(Z;\overline{Z};N+1)+$$
$$+\sum_{K=0}^{N}(-1)^K\frac{1}{(2+K)!}\frac{\Gamma(N+2)}{\Gamma(N+1-K)}(N+4+K)G_2(Z;\overline{Z};N-K).$$

Proof. — This is the same as to say

$$G_1(Z;\overline{Z};N+2)=\sum_{K=0}^{N+2}(-1)^K\frac{1}{K!}\frac{\Gamma(N+2)}{\Gamma(N+3-K)}(N+2-K)G_2(Z;\overline{Z};N+2-K)$$

This relation can be checked by writing both right – and left – hand sides as functions of the derivatives $g_2^{(K)}(Z;\overline{Z};0)$; $K \leqslant N + 2$ with the help of Lemma 1.3). ■

Relation $d)$ for any N allows us to find

PROPOSITION 3. — For $N \geqslant 0$, we have

$$G_2(Z;\overline{Z};N+1) =$$
$$=\frac{1}{N+2}\sum_{K=0}^{N}(-1)^K\frac{1}{(2+K)!}\frac{\Gamma(N+2)}{\Gamma(N+1-K)}(N+4+K)G_2(Z;\overline{Z};N-K)$$
$$+\frac{1}{N+2}H_0(Z;\overline{Z};N)\,;$$

where $H_0(Z;\overline{Z};N)$ is obtained from the expression of $G_2(Z;\overline{Z};N)$ by changing f into Δf. ■

(57) is then an expression enabling us to find by recurrence $G_2(Z;\overline{Z};N)$; $N = 0, 1, 2, \ldots$.

$e)$ Introduce the operators $G_2(N)$ by means of

$$(G_2(N)f)(Z;\overline{Z}) = G_2(Z;\overline{Z};N)$$

and

$$(g_2^{(1)}f)(Z;\overline{Z}) = g_2^{(1)}(Z;\overline{Z};0)\,.$$

We can then obtain from (45), ..., (56)

$$g_2^{(1)} = 2\pi I$$
$$G_2(0) = \pi\Delta$$
$$G_2(1) = \pi\frac{1}{2}(\Delta^2 + 2\Delta)$$
$$G_2(2) = \pi\frac{1}{6}(\Delta^3 + 7\Delta^2 + 6\Delta)$$
$$G_2(3) = \pi\frac{1}{24}(\Delta^4 + 16\Delta^3 + 48\Delta^2 + 24\Delta).$$

From expression (57), we conclude that $\frac{1}{\pi}$ $G_2(N)$ *is a polynomial in* Δ *of order* $N+1$ *without constant term and with rational coefficients.*

We will write

$$T_\omega \underset{\omega\to\infty}{\simeq} I + \sum_{N=0}^{\infty}\left(\frac{1}{\pi}G_2(N)\right)\omega^{-N-1} \tag{59}$$

7. THE ASYMPTOTIC DEVELOPMENT OF $A_1 * A_2$ AS A FORMAL DEFORMATION OF $C^\infty(D^1)$

The asymptotic development of $(A_1 * A_2)(Z;\overline{Z})$ follows from that of $T_\omega f$ by writing in latter

$$f(v;\overline{v}) = (A_2(\,.\,;\overline{Z})A_1(Z;\,.\,))(v;\overline{v}) \tag{60}$$

for Z, $\overline{Z}$ fixed. (Cf. relations (8) and (9)). Substituting function (60) in the expression of $G_2(Z;\overline{Z};N)$, we obtain in particular

$$g_2^{(1)}(Z;\overline{Z};0) = 2\pi A_1(Z;\overline{Z})A_2(Z;\overline{Z})$$

$$G_2(Z;\overline{Z};0) = \pi(1 - Z\cdot\overline{Z})^2\,\frac{\partial A_2(Z;\overline{Z})}{\partial Z}\cdot\frac{\partial A_1(Z;\overline{Z})}{\partial \overline{Z}}$$

$$\begin{aligned} G_2(Z;\overline{Z};1) = \pi\Big[& 2Z\cdot\overline{Z}(1 - Z\cdot\overline{Z})^2\,\frac{\partial A_2(Z;\overline{Z})}{\partial Z}\cdot\frac{\partial A_1(Z;\overline{Z})}{\partial \overline{Z}} - \\ & - \overline{Z}(1 - Z\cdot\overline{Z})^3\,\frac{\partial A_2(Z;\overline{Z})}{\partial Z}\cdot\frac{\partial^2 A_1(Z\cdot\overline{Z})}{\partial \overline{Z}^2} - \\ & - Z(1 - Z\cdot\overline{Z})^3\,\frac{\partial^2 A_2(Z;\overline{Z})}{\partial Z^2}\cdot\frac{\partial A_1(Z\cdot\overline{Z})}{\partial \overline{Z}} + \\ & + \frac{1}{4}(1 - Z\cdot\overline{Z})^5\,\frac{\partial^2 A_2(Z;\overline{Z})}{\partial Z^2}\cdot\frac{\partial^2 A_1(Z;\overline{Z})}{\partial \overline{Z}^2}\Big] \end{aligned} \tag{61}$$

and thus

$$(A * B)(Z;\overline{Z}) = M_0(A;B) + \hbar M_1(A;B) + \hbar^2 M_2(A;B) + \dots$$

From (57), we can see that M_N, $N = 0, 1, 2, \dots$ are differentiable Chevalley 2-cochains such that $M_N(A; B) = 0$, if A or B is a constant. We have in particular

M_0 = usual function product.

$$M_1 = (1 - Z\cdot\overline{Z})^2 \frac{\partial}{\partial \overline{Z}} \otimes \frac{\partial}{\partial Z}$$

$$M_2 = 2Z\cdot\overline{Z}(1-Z\cdot\overline{Z})^2 \frac{\partial}{\partial \overline{Z}} \otimes \frac{\partial}{\partial Z} - \overline{Z}(1-Z\cdot\overline{Z})^3 \frac{\partial^2}{\partial \overline{Z}^2} \otimes \frac{\partial}{\partial Z} -$$
$$- Z(1-Z\cdot\overline{Z})^3 \frac{\partial}{\partial \overline{Z}} \otimes \frac{\partial^2}{\partial Z^2} + \frac{1}{4}(1 - Z\cdot\overline{Z})^4 \frac{\partial^2}{\partial \overline{Z}^2} \otimes \frac{\partial^2}{\partial Z^2}.$$

8. AN EXPRESSION TO FIND $G_2(Z; \bar{Z}; N)$, $N = 0, 1, 2,$ BY RECURRENCE

By identifying the asymptotic development of the two members in relation (35), we will obtain an expression to find by recurrence $G'_2(Z;\overline{Z};N)$, $N = 0, 1, 2, \dots$.

a) From (29) and the Taylor development of $e^{-2\rho^2}$ we obtain

$$\pi(S_{\omega+2}f)(Z;\overline{Z}) = (\omega + 1)\int_0^\infty e^{-\omega\rho^2} g'_1(Z;\overline{Z};\rho)d\rho \underset{\omega\to\infty}{\simeq}$$
$$\underset{\omega\to\infty}{\simeq} \frac{\omega+1}{\omega-1}\left[\frac{1}{2} g_1'^{(1)}(Z;\overline{Z};0) + \sum_{N=0}^{\infty} G'_1(Z;\overline{Z};N)\omega^{-N-1}\right] \tag{62}$$

where $g'_2(Z;\overline{Z};0)$ is given by (26) and where

$$g'_1(Z;\overline{Z};\rho) = e^{-2\rho^2} g'_2(Z;\overline{Z};\rho); \qquad g_1'^{(K)}(Z;\overline{Z};0) = \left(\frac{d}{d\rho}\right)^K g'_1(Z;\overline{Z};\rho)\bigg|_{\rho=0}$$

$$G'_1(Z;\overline{Z};N) = \frac{1}{2}\left[\frac{\Gamma(N+2)}{(2N+3)!} g_1'^{(2N+3)}(Z;\overline{Z};0) - \frac{\Gamma(N+1)}{(2N+1)!} g_1'^{(2N+1)}(Z;\overline{Z};0)\right]$$

Similarly from (25) and (26) we obtain

$$\pi(S_\omega \Delta f)(Z;\overline{Z}) = (\omega - 1\int e^{-\omega\rho^2} h'_0(Z;\overline{Z};\rho)d\rho \underset{\omega\to\infty}{\simeq}$$
$$\underset{\omega\to\infty}{\simeq} \frac{1}{2} h_0'^{(1)}(Z;\overline{Z};0) + \sum_{N=0}^{\infty} H'_0(Z;\overline{Z};N)\omega^{-N-1} \tag{63}$$

where

$$h'_0(Z;\overline{Z};\rho) = = \frac{1}{4}\frac{d}{d\rho}(e^{\rho^2}) \int_0^{2\pi} (\tau_{Z\overline{Z}}\Delta f)\left(\left(\frac{1-e^{-\rho^2}}{1+e^{-\rho^2}}\right)^{1/2} e^{i\theta};\ \left(\frac{1-e^{-\rho^2}}{1+e^{-\rho^2}}\right)^{1/2} e^{-i\theta}\right)d\theta \quad (64)$$

and

$$H'_0(Z;\overline{Z};N) = \frac{1}{2}\left[\frac{\Gamma(N+2)}{(2N+3)!}h_0'^{(2N+3)}(Z;\overline{Z};0) - \frac{\Gamma(N+1)}{(2N+1)!}h'^{(2N+1)}(Z;\overline{Z};0)\right]$$

b) Now we transport the development (29), (62) and (64) into relation (35), and we obtain the relations

a′) $g_1'^{(1)}(Z;\overline{Z};0)-g_1'^{(1)}(Z;\overline{Z};0)=0$, terms in ω^2

b′) $G'_2(Z;\overline{Z};0)-G'_1(Z;\overline{Z};0)-\frac{1}{2}(g_2'^{(1)}(Z;\overline{Z};0)+g_1'^{(1)}(Z;\overline{Z};0))=0$, terms in ω

c′) $G'_2(Z;\overline{Z};1)-G'_2(Z;\overline{Z};0)-G'_1(Z;\overline{Z};1)-G'_1(Z;\overline{Z};0)-$
$-\frac{1}{2}h'_0(Z;\overline{Z};0)=0$, terms in ω^0

d′) $G'_2(Z;\overline{Z};N+2)-G'_2(Z;\overline{Z};N+1)-G'_1(Z;\overline{Z};N+2)-$
$-G'_1(Z;\overline{Z};N+1)-H'_0(Z;\overline{Z};N)=0$, terms in ω^{-N-1}; $N=0,1,2,\ldots$.

LEMMA 4. — We have

1′) $$\left(\frac{d}{d\rho}\right)^{2N+1} e^{-2\rho^2}\Big|_{\rho=0} = 0$$

2′) $$\left(\frac{d}{d\rho}\right)^{2N} e^{-2\rho^2}\Big|_{\rho=0} = \frac{(2N)!}{N!}(-2)^N$$

3′) $$g_1'^{(2K+1)}(Z;\overline{Z};0)=\sum_{L=0}^{K}\frac{(2K+1)!}{(2L+1)!(K-L)!}(-2)^{K-L}g_2'^{(2L+1)}(Z;\overline{Z};0).$$ ■

c) Relations *a′*) *b′*) are identities. This can be seen by a direct direct calculation, using 3′), K = 1 in lemma 4.

RELATION *c′*)

We write the derivatives $g_1'^{(K)}(Z;\overline{Z};0)$; $K \leqslant 3$ as functions of derivatives $g_2'^{(K)}(Z;\overline{Z};0)$; $K \leqslant 3$ with the help of 3′), K = 1, in Lemma 4. The left-hand side of relation *c′*) is then

$$G'_2(Z;\overline{Z};1)-G'_2(Z;\overline{Z};0)-G'_1(Z;\overline{Z};1)-G'_1(Z;\overline{Z};0)=2G'_2(Z;\overline{Z};0) \quad (65)$$

And relation c') allows us to obtain

$$G'_2(Z;\overline{Z};0) = \frac{1}{4}\, h'^{(1)}_0(Z;\overline{Z};0) \tag{66}$$

From (26) and (64), it follows that

$$g'^{(1)}_2(Z;\overline{Z};0) = \pi f(Z;\overline{Z}); \qquad h'^{(1)}_0(Z;\overline{Z};0) = \pi(\Delta f)(Z;\overline{Z}) \tag{67}$$

And then

$$G'_2(Z;\overline{Z};0) = \pi\left(\frac{1}{4}\Delta f\right)(Z;\overline{Z}) \tag{68}$$

We proceed now by induction on N in relation d').

RELATION d') $N = 0$

With the help of 3′) we write the derivatives $g'^{(K)}_1(Z;\overline{Z};0)$ in the terms $G'_1(Z;\overline{Z};2)$ and $G'_1(Z;\overline{Z};1)$, as functions of $g'^{(K)}_2(Z;\overline{Z};0)$. Then the left-hand-side of d') $N = 0$, becomes

$$\begin{aligned} G'_2(Z;\overline{Z};2) - G'_2(Z;\overline{Z};1) - G'_1(Z;\overline{Z};2) - G'_1(Z;\overline{Z};1) &= \\ = 4G'_2(Z;\overline{Z};1) - 6G'_2(Z;\overline{Z};0) \end{aligned} \tag{69}$$

But $H'_0(Z;\overline{Z};0)$ can be obtained from the relation (68) by changing f into Δf. That is

$$H'_0(Z;\overline{Z};0) = \pi\left(\frac{1}{4}\Delta^2 f\right)(Z;\overline{Z}) \tag{70}$$

And from relation d'), $N = 0$, (69) and (70) we obtain

$$G'_2(Z;\overline{Z};1) = \pi\left(\frac{1}{16}(\Delta^2 + 6\Delta)f\right)(Z;\overline{Z}) \tag{71}$$

RELATION d') $N = 1$

We proceed similarly. We obtain for the left-hand-side of this relation

$$\begin{aligned} G'_2(Z;\overline{Z};3)-G'_2(Z;\overline{Z};2)-G'_1(Z;\overline{Z};3)-G'_1(Z;\overline{Z};2) &= \\ =6G'_2(Z;\overline{Z};2)-16G'_2(Z;\overline{Z};1)+12G'_2(Z;\overline{Z};0)\,. \end{aligned} \tag{72}$$

The term $H'_0(Z;\overline{Z};1)$ is obtained from the expression of $G'_2(Z;\overline{Z};1)$ by changing f into Δf. We then have

$$H'_0(Z;\overline{Z};1) = \pi\left(\frac{1}{16}(\Delta^3+6\Delta^2)f\right)(Z;\overline{Z}) \tag{73}$$

And finally from relation d'), $N = 1$, (72) and (73), it follows that

$$G'_2(Z;\overline{Z};2)=\pi\left(\frac{1}{6.16}(\Delta^3 + 22\Delta^2 + 48\Delta)f\right)(Z;\overline{Z}) \tag{74}$$

RELATION d') $N = 2$

Analogously we obtain

$$\begin{aligned} G'_2(Z;\overline{Z};4) - G'_2(Z;\overline{Z};3) - G'_1(Z;\overline{Z};4) - G'_1(Z;\overline{Z};3) = \\ = 8G'_2(Z;\overline{Z};3) - 30G'_2(Z;\overline{Z};2) + \\ + 44G'_2(Z;\overline{Z};1) - 24G'_2(Z;\overline{Z};0) \end{aligned} \tag{75}$$

Also from (74)

$$H'_2(Z;\overline{Z};2) = \pi\left(\frac{1}{6.16}(\Delta^4 + 22\Delta^3 + 48\Delta^2)f\right)(Z;\overline{Z}) \tag{76}$$

And finally the relation d') $N = 2$, (75) and (76) allow us to obtain

$$G'_2(Z;\overline{Z};3) = \pi\left(\frac{1}{8.6.16}(\Delta^4 + 52\Delta^3 + 444\Delta^2 + 432\Delta)f\right)(Z;\overline{Z}) \tag{77}$$

d) In general

PROPOSITION 5. — If $N \geqslant 0$ we have

$$\begin{aligned} &G'_2(Z;\overline{Z};N+2) - G'_2(Z;\overline{Z};N+1) - G'_1(Z;\overline{Z};N+2) - G'_1(Z;\overline{Z};N+1) = \\ &= 2(N+2)G'_2(Z;\overline{Z};N+1) + \\ &+ \sum_{K=0}^{N}(-1)^{K+1}\frac{2^{K+1}}{(2+K)!}\frac{(N+1)!}{(N-K)!}(2(N+3)+K)G'_2(Z;\overline{Z};N-K). \end{aligned}$$

Proof. — By identification of the left-hand-right-hand sides after sustitution of $g'^{(K)}_1(Z;\overline{Z};0)$ as functions of $g'^{(K)}_2(Z;\overline{Z};0)$ with the help of 3') in Lemma 4. ■

Relation d'), $N \geqslant 0$ enables us to find $G'_2(Z;\overline{Z};N+1)$ by recurrence.

PROPOSITION 6. — For any $N \geqslant 0$, we have

$$\begin{aligned} &G'_2(Z;\overline{Z};N+1) = \\ &= \frac{1}{N+2}\sum_{K=0}^{N}(-1)^K\frac{2^K}{(2+K)!}\frac{\Gamma(N+2)}{\Gamma(N+1-K)}(2N+6+K)G'_2(Z;\overline{Z};N-K) + \\ &+ \frac{1}{2(N+2)}H'_0(Z;\overline{Z};N); \end{aligned}$$

Where $H'_0(Z;\overline{Z};N)$ is obtained from the expression of $G'_2(Z;\overline{Z};N)$ after the substitution of f for Δf. ■

e) Let us now define the operators

$$\begin{aligned} (g'^{(1)}_2 f)(Z;\overline{Z}) &= g'^{(1)}_2(Z;\overline{Z};0) \\ (G'_2(N)f)(Z;\overline{Z}) &= G'_2(Z;\overline{Z};N) \end{aligned} \tag{80}$$

We then obtain in particular

$$g_2'^{(1)} = \pi \mathrm{I}\,; \quad G_2'(0) = \pi \frac{1}{4}\Delta\,; \quad G_2'(1) = \pi \frac{1}{16}(\Delta^2 + 6\Delta)$$

$$G_2'(2) = \pi \frac{1}{6.16}(\Delta^3 + 22\Delta^2 + 48\Delta) \tag{81}$$

$$G_2'(3) = \pi \frac{1}{8.6.16}(\Delta^4 + 52\Delta^3 + 444\Delta^2 + 432\Delta)$$

From (79), we conclude that $\frac{1}{\pi} G_2'(N)$ is a polynomial in Δ of order $N + 1$ without constant term and with rational coefficients.

We will write

$$S_\omega \underset{\omega\to\infty}{\simeq} \frac{1}{2}\mathrm{I} + \sum_{N=0}^{\infty} \left(\frac{1}{\pi} G_2'(N)\right)\omega^{-N-1} \tag{82}$$

9. THE ASYMPTOTIC DEVELOPMENT OF $\overset{\circ}{\mathscr{A}} * \overset{\circ}{\mathscr{B}}$

a) Let $\hat{A}$, $\hat{B}$ be two bounded operators on space $H_\omega(D^1)$; $A(Z; \bar{Z})$; $B(Z; \bar{Z})$ its covariant symbols; and $\overset{\circ}{\mathscr{A}}(Z; \bar{Z})$, $\overset{\circ}{\mathscr{B}}(Z; \bar{Z})$ its contravariant Weyl symbols. From relations (8), (17), we see that the function

$$\overset{\circ}{\mathscr{A}} *' \overset{\circ}{\mathscr{B}} = S_\omega^{-1}(S_\omega \overset{\circ}{\mathscr{A}} * S_\omega \overset{\circ}{\mathscr{B}}), \tag{83}$$

is the contravariant Weyl symbol of the operator $\hat{A} \cdot \hat{B}$. The associative composition law $*'$ for these symbols is defined in this way.

b) We now want to find asymptotic development of (83) when $\omega \to +\infty$. We first of all need to find the asymptotic development of S_ω^{-1}. We write

$$S_\omega^{-1} \underset{\omega\to\infty}{\simeq} K_0 + \sum_{N=0} a(N)\omega^{-N-1}, \tag{84}$$

and we try to solve for S_ω^{-1} in the equations

$$S_\omega \cdot S_\omega^{-1} \underset{\omega\to\infty}{\simeq} 1$$

$$S_\omega^{-1} \cdot S_\omega \underset{\omega\to\infty}{\simeq} 1$$

We obtain

$$K_0 = 2\mathrm{I}\,; \qquad a(0) = -4\left(\frac{1}{\pi} G_2'(0)\right)$$

$$a(L+1) = -4\left(\frac{1}{\pi} G_2'(L+1)\right) - \sum_{R=0}^{L} 2a(L-R)\left(\frac{1}{\pi} G_2'(R)\right) \quad L = 0, 1, 2, \ldots \tag{85}$$

And from (81) we have

$$\mathrm{K}_0 = 2\mathrm{I}\,; \qquad a(0) = -\Delta\,; \qquad a(1) = \frac{1}{4}(\Delta^2 - 6\Delta)$$

$$a(2) = \frac{1}{24}(-\Delta^3 + 14\Delta^2 - 48\Delta) \tag{86}$$

$$a(3) = \frac{1}{8.6.4}(\Delta^4 - 20\Delta^3 + 156\Delta^2 - 432\Delta)$$

c) From definition (9) of T_ω we have

$$\mathrm{S}_\omega\mathring{\mathscr{A}} * \mathrm{S}_\omega\mathring{\mathscr{B}} = \mathrm{T}_\omega((\mathrm{S}_\omega\mathring{\mathscr{B}})(\,.\,;\overline{\mathrm{Z}})\cdot(\mathrm{S}_\omega\mathring{\mathscr{A}})(\mathrm{Z};\,.\,))(\mathrm{Z};\overline{\mathrm{Z}}) \tag{87}$$

We will then employ the following calculation

$$\begin{aligned}
&(\mathrm{S}_\omega\mathring{\mathscr{B}})(v;\overline{\mathrm{Z}})\cdot(\mathrm{S}_\omega\mathring{\mathscr{A}})(\mathrm{Z};\overline{v}) \underset{\omega\to+\infty}{\simeq} \\
&\underset{\omega\to+\infty}{\simeq} \left(\frac{1}{2}\mathring{\mathscr{B}}(v;\overline{\mathrm{Z}}) + \sum_{\mathrm{N}=0}\left(\frac{1}{\pi}\mathrm{G}_2'(\mathrm{N})\mathring{\mathscr{B}}\right)(v;\overline{\mathrm{Z}})\omega^{-\mathrm{N}-1}\right)\cdot \\
&\cdot\left(\frac{1}{2}\mathring{\mathscr{A}}(\mathrm{Z};\overline{v}) + \sum_{\mathrm{R}=0}\left(\frac{1}{\pi}\mathrm{G}_2'(\mathrm{R})\mathring{\mathscr{A}}\right)(\mathrm{Z};\overline{v})\omega^{-\mathrm{R}-1}\right) \underset{\omega\to+\infty}{\simeq} \\
&\underset{\omega\to+\infty}{\simeq} \mathrm{H}_{00}(\mathring{\mathscr{A}};\mathring{\mathscr{B}})(v;\overline{v}\mid\mathrm{Z};\overline{\mathrm{Z}}) + \mathrm{H}_{01}(\mathring{\mathscr{A}};\mathring{\mathscr{B}})(v;\overline{v}\mid\mathrm{Z};\overline{\mathrm{Z}})\omega^{-1} + \\
&+ \sum_{\mathrm{L}=0}^{\infty}\mathrm{H}_{11}(\mathrm{L};\mathring{\mathscr{A}};\mathring{\mathscr{B}})(v;\overline{v}\mid\mathrm{Z};\overline{\mathrm{Z}})\omega^{-\mathrm{L}-2}
\end{aligned} \tag{88}$$

Where

$$\mathrm{H}_{00}(\mathring{\mathscr{A}};\mathring{\mathscr{B}})(v;\overline{v}\mid\mathrm{Z};\overline{\mathrm{Z}}) = \frac{1}{4}\mathring{\mathscr{A}}(\mathrm{Z};\overline{v})\mathring{\mathscr{B}}(v;\overline{\mathrm{Z}})$$

$$\begin{aligned}
&\mathrm{H}_{01}(\mathring{\mathscr{A}};\mathring{\mathscr{B}})(v;\overline{v}\mid\mathrm{Z};\overline{\mathrm{Z}}) \\
&= \frac{1}{2}\left[\mathring{\mathscr{B}}(v;\overline{\mathrm{Z}})\left(\frac{1}{\pi}\mathrm{G}_2'(0)\mathring{\mathscr{A}}\right)(\mathrm{Z};\overline{v}) + \mathring{\mathscr{A}}(\mathrm{Z};\overline{v})\left(\frac{1}{\pi}\mathrm{G}_2'(0)\mathring{\mathscr{B}}\right)(v;\overline{\mathrm{Z}})\right]
\end{aligned}$$

$$\begin{aligned}
&\mathrm{H}_{11}(\mathring{\mathscr{A}};\mathring{\mathscr{B}};\mathrm{L})(v;\overline{v}\mid\mathrm{Z};\overline{\mathrm{Z}}) \\
&= \frac{1}{2}\mathring{\mathscr{B}}(v;\overline{\mathrm{Z}})\left(\frac{1}{\pi}\mathrm{G}_2'(\mathrm{L}+1)\mathring{\mathscr{A}}\right)(\mathrm{Z};\overline{v}) + \frac{1}{2}\mathring{\mathscr{A}}(\mathrm{Z};\overline{v})\left(\frac{1}{\pi}\mathrm{G}_2'(\mathrm{L}+1)\mathring{\mathscr{B}}\right)(v;\overline{\mathrm{Z}}) + \\
&+ \sum_{\mathrm{R}=0}^{\mathrm{L}}\left(\frac{1}{\pi}\mathrm{G}_2'(\mathrm{R})\mathring{\mathscr{A}}\right)(\mathrm{Z};\overline{v})\left(\frac{1}{\pi}\mathrm{G}_2'(\mathrm{L}-\mathrm{R})\mathring{\mathscr{B}}\right)(v;\overline{\mathrm{Z}}) \quad \mathrm{L} = 0, 1, 2, \ldots
\end{aligned}$$

Then using (88), (89) and (59) expression (87)

$$(\mathrm{S}_\omega\mathring{\mathscr{A}} * \mathrm{S}_\omega\mathring{\mathscr{B}})(\mathrm{Z};\overline{\mathrm{Z}}) \simeq \sum_{\mathrm{L}=0}^{\infty}\mathrm{A}_\mathrm{L}(\mathrm{Z};\overline{\mathrm{Z}})\omega^{-\mathrm{L}} \tag{90}$$

where

$$A_0(Z;\overline{Z}) = H_{00}(\mathring{\mathscr{A}};\mathring{\mathscr{B}})(Z;\overline{Z}\mid Z;\overline{Z})$$

$$A_1(Z;\overline{Z}) = H_{01}(\mathring{\mathscr{A}};\mathring{\mathscr{B}})(Z;\overline{Z}\mid Z;\overline{Z}) + \left(\frac{1}{\pi}G_2(0)H_{00}(\mathring{\mathscr{A}};\mathring{\mathscr{B}})\right)(Z;\overline{Z}\mid Z;\overline{Z})$$

$$A_2(Z;\overline{Z}) = H_{11}(0;\mathring{\mathscr{A}};\mathring{\mathscr{B}})(Z;\overline{Z}\mid Z;\overline{Z}) + \left(\frac{1}{\pi}G_2(1)H_{00}(\mathring{\mathscr{A}};\mathring{\mathscr{B}})\right)(Z;\overline{Z}\mid Z;\overline{Z}) + \\ + \left(\frac{1}{\pi}G_2(0)H_{01}(\mathring{\mathscr{A}};\mathring{\mathscr{B}})\right)(Z;\overline{Z}\mid Z;\overline{Z})$$

$$\begin{aligned} A_{L+3}(Z;\overline{Z}) = {} & H_{11}(L+1;\mathring{\mathscr{A}};\mathring{\mathscr{B}})(Z;\overline{Z}\mid Z;\overline{Z}) + \\ & + \left(\frac{1}{\pi}G_2(L+2)H_{00}(\mathring{\mathscr{A}};\mathring{\mathscr{B}})\right)(Z;\overline{Z}\mid Z;\overline{Z}) + \\ & + \left(\frac{1}{\pi}G_2(L+1)H_{01}(\mathring{\mathscr{A}};\mathring{\mathscr{B}})\right)(Z;\overline{Z}\mid Z;\overline{Z}) + \\ & + \sum_{R=0}^{L}\left(\frac{1}{\pi}G_2(R)H_{11}(L-R;\mathring{\mathscr{A}};\mathring{\mathscr{B}})\right)(Z;\overline{Z}\mid Z;\overline{Z}) \cdot L = 0,1,2,\ldots \end{aligned} \tag{91}$$

d) By applying the asymptotic development (84) to the left-hand side of expression (90), we obtain

$$\mathring{\mathscr{A}} * \mathring{\mathscr{B}} \simeq C_0(\mathring{\mathscr{A}};\mathring{\mathscr{B}}) + \sum_{L=0}^{\infty} C_{L+1}(\mathring{\mathscr{A}};\mathring{\mathscr{B}})\omega^{-L-1}, \tag{92}$$

where

$$C_0(\mathring{\mathscr{A}};\mathring{\mathscr{B}}) = 2A_0$$

$$C_{L+1}(\mathring{\mathscr{A}};\mathring{\mathscr{B}}) = 2A_{L+1} + \sum_{N=0}^{L} a(N)A_{L-N} \quad L = 0,1,2,\ldots \tag{93}$$

A_L and $a(N)$ being given by expressions (91) and (85) respectively.

10. THE ASYMPTOTIC DEVELOPMENT OF $\mathring{\mathscr{A}} *'' \mathring{\mathscr{B}}$ AS A DEFORMATION OF $C^\infty(D^1)$

a) From expressions (93), (91), (89), (57) we deduce that C_0 and C_{L+1}, $L = 0, 1, 2, \ldots$ are differentiable Chevalley 2-cochains such that $C_{L+1}(\mathring{\mathscr{A}};\mathring{\mathscr{B}}) = 0$ if $\mathring{\mathscr{A}}$ or $\mathring{\mathscr{B}}$ is a constant.

b) We need to know the explicit form of C_1 and C_2 .We have from (91), (89), (81)

$$A_0(Z;\overline{Z}) = \frac{1}{4}\mathring{\mathscr{B}}(Z;\overline{Z})\cdot\mathring{\mathscr{A}}(Z;\overline{Z})$$

$$A_1(Z;\overline{Z}) = \frac{1}{8}\mathring{\mathscr{B}}\cdot\Delta\mathring{\mathscr{A}} + \mathring{\mathscr{A}}\cdot\Delta\mathring{\mathscr{B}}) + \frac{1}{4}(1-Z\cdot\overline{Z})^2\partial_Z\mathring{\mathscr{B}}\cdot\partial_{\overline{Z}}\mathring{\mathscr{A}}$$

$$\begin{aligned} A_2(Z;\overline{Z}) = {} & \frac{1}{32}(\mathring{\mathscr{B}}(\Delta^2 + 2\Delta)\mathring{\mathscr{A}} + \mathring{\mathscr{A}}(\Delta^2+2\Delta)\mathring{\mathscr{B}} + 2\Delta\mathring{\mathscr{A}} \infty \Delta\mathring{\mathscr{B}}) \\ & + \frac{1}{2}Z\cdot\overline{Z}(1-Z\cdot\overline{Z})^2\partial_Z\mathring{\mathscr{B}}\cdot\partial_{\overline{Z}}\mathring{\mathscr{A}} - \frac{1}{4}\overline{Z}(1-Z\cdot\overline{Z})^3\partial_Z\mathring{\mathscr{B}}\cdot\partial^2_{\overline{Z}^2}\mathring{\mathscr{A}} - \\ & - \frac{1}{4}\overline{Z}(1-Z\cdot\overline{Z})^3\partial^2_{\overline{Z}^2}\mathring{\mathscr{B}}\cdot\partial_{\overline{Z}}\mathring{\mathscr{A}} + \frac{1}{8}(1-Z\cdot\overline{Z})^4\partial^2_{\overline{Z}^2}\mathring{\mathscr{B}}\cdot\partial^2_{\overline{Z}^2}\mathring{\mathscr{A}} - \\ & - \frac{1}{4}Z(1-Z\cdot\overline{Z})\cdot\partial_Z\mathring{\mathscr{B}}\cdot\Delta\mathring{\mathscr{A}} + \frac{1}{8}(1-Z\cdot\overline{Z})^2\partial_Z\mathring{\mathscr{B}}\cdot\Delta(\partial_{\overline{Z}} - \mathring{\mathscr{A}}) - \\ & - \frac{1}{4}\overline{Z}(1-Z\cdot\overline{Z})\cdot\partial_{\overline{Z}}\mathring{\mathscr{A}}\cdot(\Delta\mathring{\mathscr{B}}) + \frac{1}{8}(1-Z\cdot\overline{Z})^2\partial_{\overline{Z}}\mathring{\mathscr{A}}\cdot\Delta(\partial_Z\mathring{\mathscr{B}}). \end{aligned}$$

From(93) and (94), we obtain

$$C_0(\mathring{\mathscr{A}};\mathring{\mathscr{B}}) = \frac{1}{4}\mathring{\mathscr{A}}(Z;\overline{Z})\cdot\mathring{\mathscr{B}}(Z;\overline{Z}) \tag{95}$$

$$\begin{aligned} C_1(\mathring{\mathscr{A}};\mathring{\mathscr{B}}) & = 2A(1) + a(0)A(0) = \\ & = \frac{1}{4}(\mathring{\mathscr{B}}\cdot\Delta\mathring{\mathscr{A}} + \Delta\mathring{\mathscr{B}}\cdot\mathring{\mathscr{A}}) + \frac{1}{2}(1-Z\cdot\overline{Z})^2\partial_Z\mathring{\mathscr{B}}\cdot\partial_{\overline{Z}}\mathring{\mathscr{A}} - \frac{1}{4}\Delta(\mathring{\mathscr{A}}\cdot\mathring{\mathscr{B}})\,; \end{aligned}$$

but

$$\begin{aligned} \Delta(\mathring{\mathscr{A}}\cdot\mathring{\mathscr{B}}) = {} & (\Delta\mathring{\mathscr{A}})\cdot\mathring{\mathscr{B}} + \mathring{\mathscr{A}}\cdot(\Delta\mathring{\mathscr{B}}) + \\ & + (1-Z\cdot\overline{Z})^2\partial_Z\mathring{\mathscr{A}}\cdot\partial_{\overline{Z}}\mathring{\mathscr{B}} + (1-Z\cdot\overline{Z})^2\partial_{\overline{Z}}\mathring{\mathscr{A}}\cdot\partial_Z\mathring{\mathscr{B}}. \end{aligned}$$

From which we obtain finally

$$C_1(\mathring{\mathscr{A}};\mathring{\mathscr{B}}) = -\frac{1}{4}(1 - Z\cdot\overline{Z})^2(\partial_Z\mathring{\mathscr{A}}\cdot\partial_{\overline{Z}}\mathring{\mathscr{B}} - \partial_{\overline{Z}}\mathring{\mathscr{A}}\cdot\partial_Z\mathring{\mathscr{B}}) = \frac{i}{4}P(\mathring{\mathscr{A}};\mathring{\mathscr{B}}), \tag{96}$$

by the definition of Poisson's bracket (3).

$$\begin{aligned} C_2(\mathring{\mathscr{A}};\mathring{\mathscr{B}}) & = 2A_2 + a(0)A_1 + a(1)A_0 = \\ & = \frac{1}{6}(\mathring{\mathscr{B}}\cdot(\Delta^2 + 6\Delta)\mathring{\mathscr{A}} + \mathring{\mathscr{A}}\cdot(\Delta^2 + 6\Delta)\mathring{\mathscr{B}}) - \\ & - \frac{1}{8}(1 - Z\cdot\overline{Z})^2(\partial_Z\mathring{\mathscr{B}}\cdot\partial_{\overline{Z}}(\Delta\mathring{\mathscr{A}}) + \partial_Z\mathring{\mathscr{A}}\cdot\partial_{\overline{Z}}(\Delta\mathring{\mathscr{B}})) - \frac{1}{8}(\mathring{\mathscr{B}}\cdot\Delta^2\mathring{\mathscr{A}} + \mathring{\mathscr{A}}\cdot\Delta^2\mathring{\mathscr{B}}) - \\ & - \frac{3}{8}(\Delta\mathring{\mathscr{A}})\cdot(\Delta\mathring{\mathscr{B}}) + \frac{1}{16}(\Delta^2 - 6\Delta)(\mathring{\mathscr{A}}\cdot\mathring{\mathscr{B}}) - \\ & - \frac{1}{8}(1 - Z\cdot\overline{Z})^2(\partial_{\overline{Z}}\mathring{\mathscr{A}}\cdot\partial_Z(\Delta\mathring{\mathscr{B}}) + \partial_{\overline{Z}}\mathring{\mathscr{B}}\cdot\partial_Z(\Delta\mathring{\mathscr{A}})) + \frac{1}{2}(1 - Z\cdot\overline{Z})^2\partial_Z\mathring{\mathscr{B}}\cdot\partial_{\overline{Z}}\mathring{\mathscr{A}}. \end{aligned}$$

And we have

$$C_2(\overset{\circ}{\mathscr{A}};\overset{\circ}{\mathscr{B}}) - C_2(\overset{\circ}{\mathscr{B}};\overset{\circ}{\mathscr{A}}) = \frac{i}{2}P(\overset{\circ}{\mathscr{A}};\overset{\circ}{\mathscr{B}}) \tag{97}$$

Then : C_2 *is not symmetrical.*

c) We define the associative composition law $*''$ by

$$\overset{\circ}{\mathscr{A}} *'' \overset{\circ}{\mathscr{B}} = 2(\overset{\circ}{\mathscr{A}} *' \overset{\circ}{\mathscr{B}})$$

We then have from (95), (96), (92) and $\hbar = \omega^{-1}$

$$\overset{\circ}{\mathscr{A}} *'' \overset{\circ}{\mathscr{B}} \underset{\omega \to -\infty}{\simeq} \overset{\circ}{\mathscr{A}} \cdot \overset{\circ}{\mathscr{B}} + \frac{i\hbar}{2}P(\overset{\circ}{\mathscr{A}};\overset{\circ}{\mathscr{B}}) + \\ + \sum_{N=0}^{\infty} \left(\frac{i\hbar}{2}\right)^{N+2} C'_{N+2}(\overset{\circ}{\mathscr{A}};\overset{\circ}{\mathscr{B}}) \tag{98}$$

where

$$C'_{N+2} = 2(-i)^{N+2}C_{N+2}\,.$$

We can then state

PROPOSITION 7. — The asymptotic development (98) with the parameter $\frac{ih}{2}$ is an associative formal deformation at any order of the usual product of C^∞ functions on D^1. It is an infinitesimal Vey deformation. It is no longer a Vey deformation at the second order. ■

Further results are obtained in Ref. [*27*].

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