ANNALES DE L'I. H. P., SECTION A

J. MAGNEN
R. SENEOR

A note on cluster expansions

Annales de l'I. H. P., section A, tome 38, nº 1 (1983), p. 93-98
<http://www.numdam.org/item?id=AIHPA_1983__38_1_93_0>

Ann. Inst. Henri Poincaré,
Vol. XXXVIII, n° 1, 1983, p. 93-98.

Section A :
Physique théorique.

A note on cluster expansions

by

J. MAGNEN and R. SENEOR
Centre de Physique Théorique de l'École Polytechnique,
Plateau de Palaiseau, 91128 Palaiseau, Cedex, France
« Groupe de Recherche du C. N. R. S. n° 48 »

ABSTRACT. — We make, on a particular field theory model, an estimate of the weakest long range behaviour necessary to obtain easily a convergent cluster expansion.

RÉSUMÉ. — On estime, dans un modèle particulier de Théorie des Champs, le comportement à grande distance le plus faible permettant d'obtenir facilement un développement en « clusters » convergent.

1. INTRODUCTION

We prove the existence of the infinite volume limit, with a cluster expansion, of a φ_2^4 model with a covariance

$$C(x, y) = \int d^2p \exp i(x - y) \frac{|p|^{1,5+\varepsilon}}{(p^2 + 1)^2} \qquad \text{for} \qquad 0 < \varepsilon < 1/2$$

so that

$$| C(x, y) | \leq \frac{0(1)}{| x - y |^{3,5+\varepsilon/2} + 1}$$

Such models have been considered by Federbush [*1*].

In field theory the cluster expansion is a creation of Glimm Jaffe Spencer; the first version [*3*] used only characteristic functions, and Brydges gave of it a convenient form that we use [*0*]. Here the two estimates (lemma 1 and 2) are taken from Glimm and Jaffe [*4*], [*2*].

2. THE EXPANSION

For simplicity we consider a two point function:

$$\frac{S_\Lambda(f, g)}{Z_\Lambda} = Z_\Lambda^{-1}\int \varphi(f)\varphi(g)\exp\left(-\lambda\int_\Lambda \varphi^4(x)dx\right)d\mu$$

where

$$Z_\Lambda = \int \exp\left(-\lambda\int_\Lambda \varphi^4(x)\,dx\right)d\mu \quad \text{and} \quad \varphi(f) = \int \varphi(x)\,f(x)dx$$

where Λ is a space cutoff and $d\mu$ is the Gaussian measure of covariance $C(x, y)$ and mean zero.

We consider a unit lattice $\mathcal{D}$ on R^2. The support of f (resp. g) is in $\Delta f \in \mathcal{D}$[resp. Δg].

To a square $\Delta_1 \in \mathcal{D}^*$ we associate a variable s_1 and define

$$C(s_1) = s_1 C + (1 - s_1)C_1, \qquad C_1 = \chi_{\Delta_1} C \chi_{\Delta_1} + (1 - \chi_{\Delta_1})\, C(1 - \chi_{\Delta_1})$$

where χ_{Δ_1} is the characteristic function of Δ_1.

Let $d\mu(s_1)$ be the measure of covariance $C(s_1)$ so that for some function Q of the field:

$$S(s_1) = \int Q\; d\mu(s_1)$$

A step of the expansion consists in the decomposition

$$S = S(s_1)\,|_{s_1 = 1} = S(0) + \int_0^1 \left(\frac{d}{ds_1} S(s_1)\right) ds_1$$

$$\frac{d}{ds_1}\int Q d\mu(s_1) = \sum_{\Delta_2}\int P(\Delta_1, \Delta_2) Q d\mu(s_1)$$

with (see [*4*]).

$$P(\Delta_1, \Delta_2) = \int \frac{dC(s_1)}{ds_1}(x, y)\frac{\delta}{\delta\varphi(x)}\frac{\delta}{\delta\varphi(y)}\chi_{\Delta_1}(x)\chi_{\Delta_2}(y)dxdy$$

In our case $Q = \varphi(f)\varphi(g)e^{-\int\varphi^4}$ so that, with $Q\,|_R$ = the part of Q with support in R, $C_\Delta = \chi_\Delta C \chi_\Delta$ and $S_{\sim\Delta} = \int Q\,|_{\sim\Delta} d\mu\,|_{\sim\Delta}$:

$$S = \int Q\,|_{\Delta_1} d\mu\,|_{\Delta_1} . S_{\sim\Delta} + \sum_{\Delta_2}\int_0^1 ds_1 \int P(\Delta_1, \Delta_2) Q d\mu(s_1)$$

(*) We can take $\Delta_1 = \Delta f$ or Δg, but this is not necessary.

Then we define

$$C(s_1, s_2) = s_2C(s_1) + (1 - s_2)C(s_1)_2$$

$$C(s_1)_2 = \chi_{\Delta_1\cup\Delta_2}C(s_1)\chi_{\Delta_1\cup\Delta_2} + (1 - \chi_{\Delta_1\cup\Delta_2})C(s_1)(1 - \chi_{\Delta_1\cup\Delta_2})$$

and

$$\int P(\Delta_1, \Delta_2)Q d\mu(s_1) = \int P(\Delta_1, \Delta_2)Q \Big|_{\Delta_1\cup\Delta_2} d\mu(s_1, s_2)\Big|_{\substack{s_2=0\\ \Delta_1\cup\Delta_2}} . S_{\sim\Delta_1\cup\Delta_2}$$

$$+ \sum_{j=1}^{2}\sum_{\Delta_3}\int_0^1 ds_2 P(\Delta_j, \Delta_3)P(\Delta_1, \Delta_2)Q d\mu(s_1, s_2)$$

and so forth; we obtain

$$S = \int \varphi(f)e^{-\int_{\Delta_1}\varphi^4} d\mu\Big|_{\Delta_1} . S_{\sim\Delta_1}$$

$$+ \sum_{\Delta_2}\int ds_1 \int P(\Delta_1, \Delta_2)Q\Big|_{\Delta_1\cup\Delta_2} d\mu(s_1, s_2)\Big|_{\substack{s_2=0\\ \Delta_1\cup\Delta_2}} \times S_{\sim\Delta_1\cup\Delta_2}$$

$$+ \sum_{i=3}^{\infty}\sum_{j_1=1}^{1}\sum_{j_2=1}^{2}\cdots\sum_{j_{i-1}=1}^{i-1}\sum_{\substack{\{\Delta_j\}\\ i\geq j\geq 2}}\int P(\Delta_{j_{i-1}}, \Delta_i)P(\Delta_{j_{i-2}}, \Delta_{i-1})$$

$$\ldots P(\Delta_{j_1}, \Delta_2)Q\Big|_{\cup\Delta_i} ds_1, \ldots, ds_{i-1}d\mu_{(s_1,\ldots,s_{i-1},s_i)}\Big|_{\substack{s_i=0\\ \cup\Delta_i}} . S_{\sim\cup\Delta_i} \quad (1)$$

where $|\cup\Delta_i$ means restricted to $\Delta_1 \cup \ldots \cup \Delta_i$.

The formula giving $C(s_1, \ldots, s_i)$ is the generalisation of the one giving $C(s_1, s_2)$:

$$C(s_1, \ldots, s_i) = s_iC(s_1, \ldots, s_{i-1}) + (1 - s_i)C(s_1, \ldots, s_{i-1})_i$$

$$C(s_1, \ldots, s_{i-1})_i = \chi_{\cup\Delta_i}C(s_1, \ldots, s_{i-1})\chi_{\cup\Delta_i} + (1 - \chi_{\cup\Delta_i})C(s_1, \ldots, s_{i-1})(1 - \chi_{\cup\Delta_i})$$

with $\chi_{\cup\Delta_i} = \chi_{\Delta_1\cup\ldots\cup\Delta_i}$.

Then as noted above $C(s_1, \ldots, s_i)|_{s_i=0}$ is a direct sum, so that $d\mu|_{s_i=0}$ factorizes and then in each term of formula (1) the Schwinger function in $\Delta_1 \cup \ldots \cup \Delta_i$ is factorized out. We call such a term a tree, because the propagators $\frac{dC}{ds}$ form no cycle (by construction):

$$\frac{dC}{ds_k}(s_1, \ldots, s_k)(x, y) \quad \text{is non zero only if} \quad \begin{array}{l} x \in \Delta_1 \cup \ldots \cup \Delta_k \\ y \notin \Delta_1 \cup \ldots \cup \Delta_k \end{array}$$

so that at each step a propagator connects the tree formed by the squares of the tree (in formation) with some new square.

If we choose $s_i = 0$ then the tree is formed by $\Delta_1, \ldots, \Delta_i$ and is factorized out, as said.

Then in each term of formula (1) we expand in the same way $S_{\sim \cup \Delta_i}$ choosing a Δ'_1 in $R^2 \backslash \cup \Delta_i$ which plays the role of Δ_1, ... For given $\{j_1, \ldots, j_{i-1}\}$ in formula (1), a vertex Δ_k is said of order n_k or

$$n(\Delta_k) = \# \{ j_\alpha = k, \alpha \geq k \},$$

i. e. there are $n_k + 1$ propagators which have an extremity in Δ_k, except for Δ_1 where there are only n_1.

Finally we obtain

$$S = \Sigma \Pi \text{ trees}$$

One tree contains Δf and Δg by parity.

For $\Delta_1, \ldots, \Delta_i$ given we call $T(\Delta_1, \ldots, \Delta_i)$ the sum of the trees whose vertices are $\Delta_1, \ldots, \Delta_i$.

In the following we prove

PROPOSITION. —

$$\left| \sum_{\substack{\{\Delta k\} \\ k \neq i,j}} T(\Delta_1, \ldots, \Delta_n) \right| \leq 0(1) \frac{1}{(\text{dist}(\Delta_i, \Delta_j) + 1)^{3,5+\varepsilon/4}} e^{-nK}$$

where K is as big as we want for λ small enough.

As a conclusion each tree is exponentially small like the number of squares that it contains. Moreover we can sum over the localization of the squares of each tree.

A standard argument of statistical mechanics [*4*, Chapter 6] gives then

Final result. — For λ small enough $\lim\limits_{\Lambda \to \infty} \dfrac{S_\Lambda(f, g)}{Z_\Lambda}$ converges and is bounded by

$$\text{Cst}(f)\ \text{Cst}(g)\ (\text{dist}(\Delta f, \Delta g) + 1)^{-(3,5+\varepsilon/4)}$$

3. THE BOUNDS

The form of each tree is given by the $n(\Delta)$'s. The sum over each $n(\Delta)$ is controlled using $\sum\limits_{n(\Delta)} 2^{-n(\Delta)} \leq 0(1)$ so that $\sum\limits_{n(\Delta)} . \leq 0(1) \sup\limits_{n(\Delta)} 2^{n(\Delta)}..$

Now the sum over the order of the Δ's in the tree is taken in into account by summing over all the localizations of the vertices of the tree using $[d(\Delta_1, \Delta) + 1]^{-2-\varepsilon/4}$ as a combinatorial factor. We use

$$\sum_{\Delta} (d(\Delta_1, \Delta) + 1)^{-2-\varepsilon/4} \leqslant 0(1)$$

for the $n(\Delta_1)$ squares linked to Δ; then we iterate the process.

A chain of propagators links Δf to Δg in the tree containing both. Then using:

$$\sum_{\Delta_2} \frac{1}{(\operatorname{dist}(\Delta_1, \Delta_2) + 1)^{3,5+\varepsilon/4}} \frac{1}{(\operatorname{dist}(\Delta_2, \Delta_3) + 1)^{3.5+\varepsilon/4}} \leq \frac{0(1)}{(\operatorname{dist}(\Delta_1, \Delta_3) + 1)^{3,5+\varepsilon/4}}$$

we obtain the decrease in the distance of Δf to Δg.

Now we prove the proposition on trees.

For each vertex (localized in a square Δ) we apply the bound:

$$\int_\Delta \prod_{i=1}^{\alpha} | C(x_i, y)\varphi^{4-\alpha}(y) | \, dy \leq \left(\prod_{i=1}^{\alpha} \sup_{y \in \Delta} | C(x_i, y) | \right) \int_\Delta | \varphi^{4-\alpha}(y) | \, dy$$

So that the contribution of a square Δ is [for $\Delta = \Delta_1$ the product is up to $n(\Delta_1)$]:

$$I(n(\Delta)) = \left| \left(\prod_{i=1}^{n(\Delta)+1} \int_\Delta \frac{\delta}{\delta | \varphi | (x)} dx \right) e^{-\lambda \int |\varphi|^4} \right|$$

LEMMA 1. —

$$I(n(\Delta)) \leq 0(1) \lambda^{\frac{n(\Delta)}{4}} n(\Delta)^{\frac{3}{4} n(\Delta)} 0(1)^{n(\Delta)}$$

Proof. — The $\frac{\delta}{\delta | \varphi |}$'s derive either the exponentiel or already produced vertices. We use the Holder inequality:

$$\int_\Delta (\lambda^{1/4} | \varphi(x) |)^\alpha dx \leq \left(\lambda \int_\Delta \varphi(x)^4 \right)^{\alpha/4} \qquad \alpha = 1, 2, 3,$$

then just by bookkeeping one obtains:

$$I(n) \leq 0(1)^n \lambda^{\frac{n}{4}} \sup_{n'} \left(\left[\lambda \int_\Delta | \varphi(x) |^4 \right]^{\frac{4n'-n}{4}} e^{-\lambda \int_\Delta \varphi^4} n^{(n-n')} \right)$$

where n' is the number of vertices created by the $\frac{\delta}{\delta\varphi}$'s and $n^{(n-n')}$ bounds the number of terms produced by the derivations not acting on the exponential.

Using $x^l e^{-x} \leq l^l$ $(x > 0)$, one obtains the lemma.

— We have still a non used decrease of the propagators, and if there is $n(\Delta)$ distinct cubes then there is at least $\left(1 - \frac{\varepsilon}{100}\right) n(\Delta)$ cubes which are such that their distance to Δ is bigger than $0(1)\varepsilon^{1/2} n(\Delta)^{1/2}$ so that:

LEMMA 2. — Let $\Delta_1, \ldots, \Delta_{n(\Delta)}$ be the squares linked to Δ and $x_i \in \Delta_i$:

$$\prod_{i=1}^{n(\Delta)} (C(x_i, y) (\text{dist}(\Delta, \Delta_i) + 1)^{2+\varepsilon/4}) \leq \prod_{i=1}^{n(\Delta)} (\text{dist}(\Delta, \Delta_i) + 1)^{-3/2-3\varepsilon/4}$$
$$\leq 0(1)\varepsilon^{-n(\Delta)} n(\Delta)^{-3/4 n(\Delta)}$$

The two lemmas prove then the bound of the proposition.

Remark. — Such an analysis can be made on a lot of models using the positivity of the potential or its equivalent.

REFERENCES

[0] D. BRYDGES, *Comm. Math. Phys.*, t. **58**, 1978, p. 313. See also.
D. BRYDGES, P. FEDERBUSH, *J. Math. Phys.*, t. **19**, 1978, p. 2064.
[1] P. FEDERBUSH, *Commun. Math. Phys.*, t. **81**, 1981, p. 327.
[2] J. GLIMM, A. JAFFE, *Fortschr. Physik*, t. **21**, 1973, p. 327.
[3] J. GLIMM, A. JAFFE, T. SPENCER, *Ann. Math.*, t. **100**, 1974, p. 585.
[4] J. GLIMM, A. JAFFE, T. SPENCER, *Lecture Notes in Physics*, t. **25**; G. VELO, A. WIGHTMAN, Springer, 1973.

(Manuscrit reçu le 25 mai 1982)