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Section A :
Physique théorique.

Lower bounds to feynman integrals and class H^n (*)

by

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SUMMARY. — Lower bounds to the absolute value of Feynman integrands are derived by introducing in the process a class of functions called the H^n functions defined in terms of *lower* power asymptotic coefficients. If the so-called power counting criterion is satisfied, then it is shown that the integrals of the absolute value of the integrands belong to such a class H^n (with a different n) as well. When a certain condition on the lower power asymptotic coefficient is satisfied then usual evaluation of such integrals as *iterated* integrals may lead to ambiguous and non-unique results. The above results hold rigorously for scalar theories and up to an « almost surely » restriction in the general cases. This analysis is expected to have applications in the self-consistency problems of non-renormalizable theories.

RÉSUMÉ. — On obtient des bornes inférieures pour la valeur absolue des intégrands de Feynman en introduisant dans la démonstration une classe de fonctions, appelée la classe H^n, définie en terme de coefficients asymptotiques inférieurs. Si le critère du « comptage des puissances » est satisfait, l'intégrale de la valeur absolue des intégrands appartient aussi à la classe H^n (avec un n différent). Quand le coefficient asymptotique inférieur satisfait un certain critère, on démontre que la méthode habituelle d'évaluation de ces intégrales comme intégrales itérées est ambiguë et donne des résultats non uniques. Les résultats obtenus sont applicables

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rigoureusement aux théories scalaires, et « presque sûrement » dans les cas généraux. Cette analyse devrait avoir des applications dans le problème de consistance des théories non-renormalisables.

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1. INTRODUCTION

The basic ingredient in Lagrangian field theory is the so-called Feynman integral. Basic computations done in quantum field theory involve the evaluation of Feynman integrals as *iterated integrals.* A detailed knowledge of the structure of Feynman integrals is extremely useful not only for evaluating « elementary » type of such integrals but also to study the asymptotic behaviour [*1*] [*3*] in various regions of more complicated type of Feynman integrals which cannot be evaluated by elementary and direct means in closed forms. To this end the so-called class B_n functions [*4*] property [*5*] has been very useful. Most studies and rules dealing with consistency problems related to Feynman integrals are based on derivations of upper bound values of Feynman integrands and integrals and hence generally lead to sufficiency conditions for the existence of the corresponding integrals and for the correctness of the results embodied in the underlying analysis. Unfortunately the formulation of necessary conditions for existence problems in this analysis is not always possible but is certainly advisable. In this paper we want to take a modest step in this latter direction. To this end we introduce a class of functions called the class H^n of functions and we show that Feynman integrands belong to such a class by deriving in the process *lower bound* values to the absolute value of the integrands. If the so-called power counting criterion [*6*] is satisfied then we show that the integrals of the absolute value of the integrands belong to class H^n (with a different n) by deriving, inductively, lower bounds to the integrals. The importance of deriving lower bounds to Feynman integrals was already emphasized over twenty years ago [*6*]. As a corollary to our basic theorem we give a criterion, which if satisfied, shows that the evaluation of Feynman integrals as iterated ones may generally give ambiguous and non-unique results. This is quite important as Feynman integrals are treated as iterated integrals. The class H^n of functions is defined in Sect. 2. In Sect. 3 we establish the class H^n property of Feynman integrands. Our basic results related to the integrals of the absolute Feynman integrands are given in Sect. 4. The above results hold rigorously for scalar theories and up to an « almost surely » restriction (Sect. 5) in the general cases. It is expected that our analysis will be useful in establishing rigorously that certain field theories are non-renormalizable.

2. DEFINITION OF CLASS H^n

Consider two functions f and g of k real variables $x_1, \ldots, x_k$. If we may find real positive constants $b_1 > 1, \ldots, b_k > 1$, and we may find a strictly positive constant C independent of $x_1, \ldots, x_k$, such that for

$$|x_1| \geq b_1, \ldots, |x_k| \geq b_k, \tag{2.1}$$

we have

$$|f(x_1, \ldots, x_k)| \geq C\,|g(x_1, \ldots, x_k)|, \tag{2.2}$$

then we denote the relation (2.2) as

$$f(x_1, \ldots, x_k) = \underline{0}(g(x_1, \ldots, x_k)), \tag{2.3}$$

and use the notation $|x_1| \to \infty, \ldots, |x_k| \to \infty$, independently.

DEFINITION OF CLASS H^n. — A function $f(\vec{P})$ with $\vec{P} \in \mathbb{R}^n$ is said to belong to a class H^n, if and only if, for each non-zero subspace $S \subset \mathbb{R}^n$ there exists a coefficient (a real number) $\underline{\alpha}(S)$, such that for each choice of $k \leq n$ independent vectors $\vec{L}_1, \ldots, \vec{L}_k$ and a bounded region W in $\mathbb{R}^n$ we have

$$f(\vec{P}) = \underline{0}\left\{ \eta_1^{\underline{\alpha}(\{\vec{L}_1\})} \ldots \eta_k^{\underline{\alpha}(\{\vec{L}_1, \ldots, \vec{L}_k\})} \right\}, \tag{2.4}$$

where

$$\vec{P} = \vec{L}_1\eta_1 \ldots \eta_k + \vec{L}_2\eta_2 \ldots \eta_k + \ldots \vec{L}_k\eta_k + \vec{C}, \tag{2.5}$$

and $\vec{C} \in W$. By definition, by the condition (2.4) it is meant that there exist real numbers $b_1 > 1, \ldots, b_k > 1$ such that $\eta_1 \geq b_1, \ldots, \eta_k \geq b_k$. For a subspace S spanned by a set of independent vectors $\vec{L}_1, \ldots, \vec{L}_r$ we use the standard notation: $S = \{\vec{L}_1, \ldots, \vec{L}_r\}$. The coefficients $\underline{\alpha}(S)$ will be called *lower power asymptotic coefficients* of f.

3. CLASS H^n PROPERTY OF FEYNMAN INTEGRANDS

A Feynman integrand has the very general structure in the form:

$$I(P, K, \mu, \varepsilon) = \frac{Y(P, K, \mu, \varepsilon)}{\prod_{l=1}^{L} [Q_l^2 + \mu_l^2 - i\varepsilon(\vec{Q}_l^2 + \mu_l^2)]}, \qquad \varepsilon > 0, \tag{3.1}$$

where

$$\begin{aligned} K &= (k_1^0, \ldots, k_n^3) \\ P &= (p_1^0, \ldots, p_m^3) \quad , \\ \mu &= (\mu^1, \ldots, \mu^\rho) \end{aligned} \tag{3.2}$$

with K denoting the set of integration variables, P denoting the set of the components of independent external momenta of the graph in question, and μ denotes the set of masses available in the theory, and we assume that the $\mu^i > 0$ are kept fixed for $i = 1, \ldots, \rho$. In (3.1) we have adopted the $i\varepsilon$ prescription first introduced in [7]. Y is a polynomial in the elements in (3.2), in ε, and may be even a polynomial in the $(\mu^i)^{-1}$ as well. The polynomial dependence of Y on the $(u^i)^{-1}$ is well known for higher spin fields. Each μ_l in (3.1) coincides with one of the elements in the set μ. The Q_l's is (3.1) are of the form:

$$Q_l = \sum_{j=1}^{n} a_{lj}k_j + \sum_{j=1}^{m} b_{lj}p_j. \tag{3.3}$$

The Euclidean version of (3.1) is defined by

$$I_E(P, K, \mu) = \frac{Y_E(P, K, \mu, 0)}{\prod_{l=1}^{L} [Q_{lE}^2 + \mu_l^2]}, \tag{3.4}$$

by replacing the Minkowski metric $g_{\mu\nu}$ by the Euclidean one:

$$\eta_{\mu\nu} = \text{diag}\,[1, 1, 1, 1],$$

and setting $\varepsilon = 0$ in (3.1). In particular $Q_{lE}^2 = \vec{Q}_l^2 + Q_l^{0^2}$ denotes the Euclidean version of $Q_l^2 = \vec{Q}_l^2 - Q_l^{0^2}$. Up to Sect. 5 we restrict our analysis to scalar theories, and in Sect. 5, we explain how all of the results proved in the paper hold true under an « almost surely » restriction to be discussed later. Accordingly up to Sect. 5, y and y_E in (3.1) and (3.4), respectively, will be taken as constants.

Let

$$\vec{P} \in \mathbb{R}^{4n+4m} \tag{3.5}$$

and suppose that the elements in the sets K and P may be written as some linear combinations of the (standard) components of the vector $\vec{P}$. Suppose that $\vec{P}$ is of the form:

$$\vec{P} = \sum_{i=1}^{k} \vec{L}_i \eta_1 \ldots \eta_i + \vec{C}, \tag{3.6}$$

where $1 \leq k \leq 4n + 4m$, $\vec{L}_1, \ldots, \vec{L}_k$ are k independent vectors in $\mathbb{R}^{4n+4m}$, and $\vec{C}$ is confined to a finite region in $\mathbb{R}^{4n+4m}$. Since the $(4n + 4m)$ independent variables in the sets K and P may be written as some linear combinations z_i, $i = 1, \ldots, 4n + 4m$, of the components of the vector $\vec{P}$, we note that the z_i depend on the parameters of $\eta_1, \ldots, n_k$, i. e.,

$$z_i = z_i(\eta_1, \ldots, \eta_k), \tag{3.7}$$

with
$$k_1^0 = z_1, \ldots, k_n^3 = z_{4n}, \qquad p_1^0 = z_{4n+1}, \ldots, p_m^3 = z_{4n+4m}. \tag{3.8}$$

We introduce vectors $\vec{V}_1, \ldots, \vec{V}_{4L}$ in $\mathbb{R}^{4n+4m}$ such that
$$\vec{V}_j \cdot \vec{P} = Q(j), \qquad j = 1, \ldots, 4L, \tag{3.9}$$
where
$$Q(1) = Q_1^0, \ldots, Q(4) = Q_1^3, \ldots, Q(4L) = Q_L^3. \tag{3.10}$$
The correctness of (3.9) follows from the fact that the Q_l^σ are some linear combinations ((3.3)) of the z_i ((3.8)), and the latter in turn are some linear combinations of the components of the vector $\vec{P} \in \mathbb{R}^{4n+4m}$.

We first state the following lemma.

LEMMA. — Let $x_1 > 0, \ldots, x_n > 0$, $a_i \in \mathbb{R}^1$ (finite with arbitrary signs), with $a_0 \neq 0$, and consider the expression
$$G(x_1, \ldots, x_n) = \left[a_0 + a_1 x_1^{-1} + \ldots + a_n x_1^{-1} \ldots x_n^{-1}\right].$$
Then we may find constants $b_1 > 1, \ldots, b_n > 1$, $M_0 \geq m_0 > 0$, such that for $x_1 \geq b_1, \ldots, x_n \geq b_n$ we have
$$m_0 \leq \zeta G(x_1, \ldots, x_n) \leq M_0, \tag{3.11}$$
where
$$\zeta = \operatorname{sign} a_0. \tag{3.12}$$
A proof of this lemma is given in [5].

Suppose that for j fixed with value in the set $[1, \ldots, 4L]$:
$$\vec{V}_j \cdot \vec{L}_1 = 0, \ldots, \vec{V}_j \cdot \vec{L}_{r(j)-1} = 0, \qquad \vec{V}_j \cdot \vec{L}_{r(j)} = c_{jr} \neq 0. \tag{3.13}$$
Then we may write
$$\vec{V}_j \cdot \vec{P} = c_{jr} \eta_{r(j)} \ldots \eta_k + \ldots + c_{jk} \eta_k + c_j, \qquad c_{jr} \neq 0, \tag{3.14}$$
where $c_j = \vec{V}_j \cdot \vec{C}$. We may also rewrite (3.14) as
$$Q(j) = \eta_{r(j)} \ldots \eta_k [c_{jr} + \ldots + c_j (\eta_{r(j)} \ldots \eta_k)^{-1}]. \tag{3.15}$$
From an analysis similar to the one in [5] we may then find constants $b_1 > 1, \ldots, b_k > 1$, by the application of the above lemma, such that for
$$\eta_1 \geq b_1, \ldots, \eta_k \geq b_k, \tag{3.16}$$
we have the following inequalities
$$C_2 \eta_1^{\underline{\alpha}_1} \ldots \eta_k^{\underline{\alpha}_k} \leq \frac{1}{\prod_{l=1}^{L} (Q_{lE}^2 + \mu_l^2)} \leq C_1 \eta_1^{\underline{\alpha}_1} \ldots \eta_k^{\underline{\alpha}_k}, \tag{3.17}$$
where C_1 and C_2 are strictly positive constants. The exponents $(-\underline{\alpha}_r)$

coincide, respectively, with the *degrees* of $\prod_{l=1}^{L}(Q_{lE}^2 + \mu_l^2)$ with respect to the parameters η_r, and are of the form:

$$\underline{\alpha}_r = -2 \sum_{l=1}^{L} \Delta_l^{(r)}, \tag{3.18}$$

where $\Delta_l^{(r)} = 1$, if and only if, for the fixed l in question:

$$\begin{gathered} \vec{V}_{4(l-1)+1}\cdot\vec{L}_1 \neq 0 \quad \text{and/or} \ldots \text{and/or} \quad \vec{V}_{4(l-1)+4}\cdot\vec{L}_1 \neq 0 \\ \text{and/or} \\ \vec{V}_{4(l-1)+1}\cdot\vec{L}_2 \neq 0 \quad \text{and/or} \ldots \text{and/or} \quad \vec{V}_{4(l-1)+4}\cdot\vec{L}_r \neq 0, \end{gathered} \tag{3.19}$$

and $\Delta_l^{(r)} = 0$ otherwise. The conditions in (3.19) may be equivalently stated as having at least one of the vectors $\vec{V}_{4(l-1)+1}, \ldots, \vec{V}_{4(l-1)+4}$, for each l, not orthogonal to the subspace $\{\vec{L}_1, \ldots, \vec{L}_r\}$ and we may rewrite $\underline{\alpha}_r$ in (3.18) as

$$\underline{\alpha}_r = \underline{\alpha}(\{\vec{L}_1 \ldots \vec{L}_r\}). \tag{3.20}$$

Finally by using the inequality [7],

$$\frac{1}{[Q_l^2 + \mu_l^2 - i\varepsilon(\vec{Q}_l^2 + \mu_l^2)]} \geq \frac{1}{\sqrt{1+\varepsilon^2}}\cdot\frac{1}{(Q_{lE}^2 + \mu_l^2)}, \tag{3.21}$$

we see from the *left-hand-side* of the inequality in (3.17) that

$$|\,I(P, K, \mu, \varepsilon)\,| \geq A_1 \eta_1^{\underline{\alpha}(\{\vec{L}_1\})} \ldots \eta_k^{\underline{\alpha}(\{\vec{L}_1,\ldots,\vec{L}_k\})}, \tag{3.22}$$

$$|\,I_E(P, K, \mu)\,| \geq A_2 \eta_1^{\underline{\alpha}(\{\vec{L}_1\})} \ldots \eta_k^{\underline{\alpha}(\{\vec{L}_1,\ldots,\vec{L}_k\})}, \tag{3.23}$$

where A_1, A_2 are some positive numbers, thus establishing the fact that I and I_E belong to class H^n. The cases when Y and Y_E are, in general, not constants will be taken up in Sect. 5.

4. H^n PROPERTY OF THE INTEGRALS : SCALAR THEORIES

Consider a one dimensional integral

$$\int_I |f| = \int_{-\infty}^{\infty} dy\, |f(\vec{P} + \vec{L}y)|, \qquad \{\vec{L}\} = I, \tag{4.1}$$

where $f(\vec{P} + \vec{L}y) \in H^n$, and

$$\vec{P} = \sum_{i=1}^{k} \vec{L}_i \eta_1 \ldots \eta_i + \vec{C}, \qquad k \leq n-1 \tag{4.2}$$

with $\vec{L}_1, \ldots, \vec{L}_k$ k independent vectors. As in [6] we may write the infinite interval $(-\infty, \infty)$ as the union of a *finite* number of intervals. These intervals may be so chosen as they do not overlap [8] in the following form:

$$\int_{-\infty}^{\infty} dy\, |f(\vec{P}+\vec{L}y)| = \sum_{\pm} \sum_{\substack{r=0 \\ (i_1\ldots i_r)}}^{k} \int_{J^{\pm}_{i_1\ldots i_r}} dy\, |f(\vec{P}+\vec{L}y)| + \\ + \sum_{(i_1\ldots i_k)} \int_{J^0_{i_1\ldots i_k}} dy\, |f(\vec{P}+\vec{L}y)|, \quad (4.3)$$

where the intervals $J^{+}_{i_1\ldots i_r}$, $J^{-}_{i_1\ldots i_r}$, $(r = 0, 1, \ldots, k)$, $J^0_{i_1\ldots i_k}$ are non-overlapping and are of the form:

$$J^{\pm}_{i_1\ldots i_r} = \{ y : y = U_{i_1}\eta_1 \ldots \eta_k + \ldots + U_{i_1\ldots i_r}\eta_r \ldots \eta_k \pm |z|\eta_{r+1}\ldots\eta_k ; \\ b_0(i_1 \ldots i_r) \leq |z| \leq \eta_r \lambda^{\pm}_{i_1\ldots i_r} \}, \quad (4.4)$$

$r = 1, \ldots, k$, for some $b_0(i_1 \ldots i_r) > 1$, $\lambda^{\pm}_{i_1\ldots i_r} < 1$, $U_{i_1}, \ldots, U_{i_1\ldots i_r} \in \mathbb{R}^1$, and for $r = 0$

$$J^{\pm} = \{ y : y = \pm |z| \eta_1 \ldots \eta_k,\ |z| \geq b_0 \}, \quad (4.5)$$

for some $b_0 > 1$,

$$J^0_{i_1\ldots i_k} = \{ y : y = U_{i_1}\eta_1 \ldots \eta_k + \ldots + U_{i_1\ldots i_k}\eta_1 \ldots \eta_k \pm |z|,\ |z| \leq \eta_k b_0(i_1 \ldots i_k) \}, \quad (4.6)$$

for some $b_0(i_1 \ldots i_k) > 1$. The parameters $i_1, \ldots, i_k$ vary over a finite set of integers.

On each of the intervals through (4.4)-(4.6) we may then use the class H" property of f. Suppose that $|f|$ is locally integrable. Then on each of the intervals $J^{\pm}_{i_1\ldots i_r}$, $r = 1, \ldots, k$, $J^0_{i_1\ldots i_k}$ the integrals over them give finite contributions since these intervals are bounded. On the intervals $J^{\pm}$ we have

$$\int_{J^{\pm}} dy\, |f(\vec{P}+\vec{L}y)| \geq C \int_{b_0}^{\infty} d|z| (|z|)^{\underline{\alpha}(\{\vec{L}\})} \prod_{i=1}^{k} \eta_i^{\underline{\alpha}(\{\vec{L},\vec{L}_1,\ldots,\vec{L}_i\})+1}. \quad (4.7)$$

Accordingly for $C \neq 0$, the integral on the left-hand-side of (4.7), and hence also the integral in (4.3), *diverges* if

$$\underline{\alpha}(\{\vec{L}\}) + 1 \geq 0. \quad (4.8)$$

On the other hand if

$$\underline{\alpha}(\{\vec{L}\}) + 1 < 0, \quad (4.9)$$

then the integral on the right-hand-side of (4.7) *converges*. By an analysis similar to the one in [7] we obtain in this case

$$\int_{-\infty}^{\infty} dy\, |f(\vec{P}+\vec{L}y)| \geq C_0 \eta_1^{\underline{\alpha}_1(\{\vec{L}_1\})} \ldots \eta_k^{\underline{\alpha}_1(\{\vec{L}_1,\ldots,\vec{L}_k\})}, \quad (4.10)$$

i. e., $\int_{\mathrm{I}} |f| \in \mathrm{H}^{n-1}$, where

$$\underline{\alpha}_{\mathrm{I}}(\mathrm{S}) = \min_{\Lambda(\mathrm{I})\mathrm{S}'=\mathrm{S}} [\underline{\alpha}(\mathrm{S}') + \dim \mathrm{S}' - \dim \mathrm{S}], \tag{4.11}$$

with $\Lambda(\mathrm{I})$ denoting the *projection* operation on a complement E of I in $\mathbb{R}^n$, disjoint from I, *along* the subspace I [*6*] [*9*]: $\mathbb{R}^n = \mathrm{I} \oplus \mathrm{E}$. dim S denotes the dimension of the subspace S. Note that in (4.11) we have $\min\limits_{\Lambda(\mathrm{I})\mathrm{S}'=\mathrm{S}}$ rather than $\max\limits_{\Lambda(\mathrm{I})\mathrm{S}'=\mathrm{S}}$. We note that the relation (4.9) may be equivalently rewritten as

$$\max_{\mathrm{S}\subset\mathrm{I}} [\underline{\alpha}(\mathrm{S}) + \dim \mathrm{S}] < 0, \tag{4.12}$$

since I itself is the only non-zero subspace of I.

In the sequal $\alpha(\mathrm{S})$ will denote ordinary power asymptotic coefficients of f [*6*]. We denote by $\underline{\alpha}_{\mathrm{I}}$ *lower* power asymptotic coefficients of the integral $|f|_{\mathrm{I}} = \int_{\mathrm{I}} |f|$. The symbol f will denote any of the Feynman integrands I, I_{E}.

THEOREM 4.1. — *If*

$$\max_{\mathrm{S}\subset\mathrm{I}} [\alpha(\mathrm{S}) + \dim \mathrm{S}] < 0, \tag{4.13}$$

then $|f|_{\mathrm{I}} \in \mathrm{H}^{4m}$, *where* $\dim \mathrm{I} = 4n$, *with lower power asymptotic coefficients:*

$$\underline{\alpha}_{\mathrm{I}}(\mathrm{S}) = \min_{\Lambda(\mathrm{I})\mathrm{S}'=\mathrm{S}} [\underline{\alpha}(\mathrm{S}') + \dim \mathrm{S}' - \dim \mathrm{S}], \tag{4.14}$$

$\mathrm{S} \subset \mathrm{E}$, $\mathbb{R}^{4n+4m} = \mathrm{I} \oplus \mathrm{E}$.

The proof is by induction. As an induction hypothesis suppose that the theorem is true whenever $\dim \mathrm{I} < 4n$. Let $4n - 1 = k'$. Then we prove that the theorem is true for $\dim \mathrm{I} = k' + 1$. First we recall the situation for $\dim \mathrm{I} = 1$. Quite generally we note from (3.17) that we always have

$$\underline{\alpha}(\mathrm{S}) \leq \alpha(\mathrm{S}). \tag{4.15}$$

Hence the condition (4.13) implies (4.12) and therefore the theorem is true for $\dim \mathrm{I} = 1$.

In general let $\mathrm{I} = \mathrm{I}_1 \oplus \mathrm{I}_2$, where $\dim \mathrm{I}_2 = k'$, $\dim \mathrm{I}_1 = 1$. We note that (4.13) implies from (4.15) that

$$\max_{\mathrm{S}\subset\mathrm{I}} [\underline{\alpha}(\mathrm{S}) + \dim \mathrm{S}] \leq \max_{\mathrm{S}\subset\mathrm{I}} [\alpha(\mathrm{S}) + \dim \mathrm{S}] < 0, \tag{4.16}$$

and

$$\max_{\mathrm{S}\subset\mathrm{I}_i} [\alpha(\mathrm{S}) + \dim \mathrm{S}] \leq \max_{\mathrm{S}\subset\mathrm{I}} [\alpha(\mathrm{S}) + \dim \mathrm{S}] < 0, \tag{4.17}$$

for $i = 1, 2$. Eq. (4.15) also implies from (4.17) that

$$\max_{\mathrm{S}\subset\mathrm{I}_i} [\underline{\alpha}(\mathrm{S}) + \dim \mathrm{S}] < 0 \tag{4.18}$$

The power counting criterion (4.13) ((4.16)) implies that we may integrate $\int |f|$ as in $\int_{\mathrm{I}_1}\left(\int_{\mathrm{I}_2}|f|\right)$ since its value is unique by Fubini-Tonelli's Theorem [*10*]. According to the induction hypothesis we have $|f|_{\mathrm{I}_2} \in \mathrm{H}^{4n+4m-k'}$ with lower power asymptotic coefficients:

$$\underline{\alpha}_{\mathrm{I}_2}(\mathrm{S}') = \min_{\Lambda(\mathrm{I}_2)\mathrm{S}''=\mathrm{S}'} [\underline{\alpha}(\mathrm{S}'') + \dim \mathrm{S}'' - \dim \mathrm{S}'], \tag{4.19}$$

where $\mathbb{R}^{k'+1} = \mathrm{I}_2 \oplus \mathrm{E}_2$, $\mathrm{S}' \subset \mathrm{E}_2$. Eq. (4.16) in particular implies that

$$\max_{\Lambda(\mathrm{I}_2)\mathrm{S}=\{\vec{\mathrm{L}}\}} [\underline{\alpha}(\mathrm{S}) + \dim \mathrm{S}] \leq \max_{\mathrm{S}\subset \mathrm{I}} [\underline{\alpha}(\mathrm{S}) + \dim \mathrm{S}] < 0, \tag{4.20}$$

where $\mathrm{I}_1 \equiv \{\vec{\mathrm{L}}\}$. On the other hand we may bound the expression on the extreme left-hand-side of (4.20) from below by

$$\min_{\Lambda(\mathrm{I}_2)\mathrm{S}=\{\vec{\mathrm{L}}\}} [\underline{\alpha}(\mathrm{S}) + \dim \mathrm{S}] = \underline{\alpha}_{\mathrm{I}_2}(\{\vec{\mathrm{L}}\}) + 1 < 0 \tag{4.21}$$

where we have used the relation in (4.19). Since $\dim \mathrm{I}_1 = \dim\{\vec{\mathrm{L}}\} = 1$ and (4.21) is true, we may use our previous analysis in one dimension to conclude; by using in the process (4.13), that:

$$\int_{\mathrm{I}_1}\left(\int_{\mathrm{I}_2}|f|\right) = \int_{\mathrm{I}}|f| \in \mathrm{H}^{4n+4m-k'-1} = \mathrm{H}^{4m}, \tag{4.22}$$

with lower power asymptotic coefficients

$$\underline{\alpha}_{\mathrm{I}}(\mathrm{S}) = \min_{\Lambda(\mathrm{I}_1)\mathrm{S}'=\mathrm{S}} \{ \min_{\Lambda(\mathrm{I}_2)\mathrm{S}''=\mathrm{S}'} [\underline{\alpha}(\mathrm{S}'') + \dim \mathrm{S}'' - \dim \mathrm{S}'] + \dim \mathrm{S}' - \dim \mathrm{S} \}, \tag{4.23}$$

or

$$\underline{\alpha}_{\mathrm{I}}(\mathrm{S}) = \min_{\Lambda(\mathrm{I})\mathrm{S}''=\mathrm{S}} [\underline{\alpha}(\mathrm{S}'') + \dim \mathrm{S}'' - \dim \mathrm{S}]. \tag{4.24}$$

This completes the proof of the theorem.

COROLLARY. — Suppose that

$$\max_{\mathrm{S}\subset \mathrm{I}_2} [\alpha(\mathrm{S}) + \dim \mathrm{S}] < 0, \tag{4.25}$$

for some $\mathrm{I}_2 \subset \mathrm{I}$. If there exists a one dimensional subspace I_1 of I disjoint from I_2 such that

$$[\underline{\alpha}_{\mathrm{I}_2}(\mathrm{I}_1) + 1] \geq 0, \tag{4.26}$$

then the following iterated integral

$$\int_{\mathrm{I}_1}\left(\int_{\mathrm{I}_2}|f|\right) \tag{4.27}$$

diverges.

The proof of the corollary follows immediately from the above theorem.

The corollary leads to questions of ambiguities in the evaluation of the integrals as iterated integrals and hence it embodies powerful results. In the next section we study the generalization of our results to the cases when Y and Y_E in (3.1) and (3.4), respectively, are not necessarily constants.

5. EXTENSION TO THE GENERAL CASES AND APPLICATION

In this section we discuss how the analysis carried out in the bulk of the paper is modified to treat the situations when Y and Y_E in (3.1) and (3.4), respectively, are not constants. An application of the main analysis will be then pointed out.

Since Y and Y_E are some polynomials in $z_1, \ldots, z_{4n+4m}$, as defined in (3.8), we may then find, in general, positive constants D_1, D_2 such that

$$|\,Y(P, K, \mu, \varepsilon)\,| \geq D_1, \tag{5.1}$$

$$|\,Y_E(P, K, \mu)\,| \geq D_2. \tag{5.2}$$

Accordingly unless the polynomials Y, Y_E vanish for vectors $\vec{P}$ as given in (2.5) by some detailed cancellations we then see that I and I_E « almost surely » belong to class H^{4n+4m}, in the general cases as well. The proof of Theorem 4.1, given in Sect. 4, then remains intact in such cases except that the statement of the theorem should be slightly modified by saying that if (4.13) is true then almost surely $|\,f\,|_I \in H^{4m}$. The same conclusion is reached for the corollary of Sect. 4, where it should be stated that under the conditions of the corollary, the iterated integral in (4.27) almost surely diverges. The almost sure divergence as opposed to a sure divergence follows from the fact that a « miraculous » cancellation may occur which « makes » a constant C in (2.2) equal to zero for $|\,f\,|_{I_2}$.

Non-renormalizable theories in the strict sense are generally defined as those theories in which the naive degree of divergence associated with Feynman graphs increases with the order of perturbation theory. As the present paper develops lower bounds to Feynman integrals, the analysis may be useful in the study (at least for scalar theories in Euclidean space, prior to applications of subtractions) of the non-existence of a certain class of Feynman integrals as iterated integrals. This ultimately may provide a test to discriminate, rigorously, between renormalizable and non-renormalizable theories in the strict sense. In a forthcoming report the analysis of this paper will be applied to study such self-consistency problems related to non-renormalizable theories.

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