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A completeness theorem relative to one-dimensional Schrödinger equations with energy-dependent potentials (**)

by

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ABSTRACT. — The one-dimensional Schrödinger equations:

$$y^{\pm''} + [k^2 - (u(x) \pm kq(x))]y^{\pm} = 0, \quad x \in \mathbb{R}, \quad k \in \mathbb{C},$$

are considered as matrix eigenvalue equations:

$$HY^{\pm} = (\pm k)Y^{\pm}$$

where $H = \begin{pmatrix} 0 & 1 \\ -\dfrac{d^2}{dx^2} + u & q \end{pmatrix}, \qquad Y^{\pm} = (y^{\pm}, (\pm k)y^{\pm})^{T}.$

We prove that if $F_1^{\pm}(k, x)$, resp. $F_2^{\pm}(k, x)$, is a particular vector eigen function of H resp. H* (the « adjoint » operator of H), then all functions of a certain class $\mathscr{E}$ can be uniquely expanded through $F_1^{\pm}(k, x)$ and $F_2^{\pm}(k, x)$.

RÉSUMÉ. — On considère les équations de Schrödinger à une dimension :

$$y^{\pm''} + [k^2 - (u(x) \pm kq(x))]y^{\pm} = 0, \quad x \in \mathbb{R}, \quad k \in \mathbb{C},$$

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écrites sous la forme d'équations matricielles aux valeurs propres :

$$HY^{\pm} = (\pm k)Y^{\pm}$$

$$\text{où } H = \begin{pmatrix} 0 & 1 \\ -\dfrac{d^2}{dx^2} + u & q \end{pmatrix}, \qquad Y^{\pm} = (y^{\pm}, (\pm k)y^{\pm})^{T}.$$

Nous montrons que si $F_1^{\pm}(k, x)$, resp. $F_2^{\pm}(k, x)$, est une fonction propre particulière de H, resp. de H* (l'opérateur « adjoint » de H), associée à la valeur propre $(\pm k)$, alors toute fonction vectorielle appartenant à une certaine classe $\mathscr{E}$ peut être développée de façon unique à l'aide de $F_1^{\pm}(k, x)$ et de $F_2^{\pm}(k, x)$.

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1. INTRODUCTION

We deal with the one-dimensional Schrödinger equations:

$$y^{+''} + [k^2 - (u(x) + kq(x))]y^{+} = 0, \quad x \in \mathbb{R}, \quad k \in \mathbb{C}, \tag{1.1}$$

$$y^{-''} + [k^2 - (u(x) - kq(x))]y^{-} = 0, \quad x \in \mathbb{R}, \quad k \in \mathbb{C}, \tag{1.2}$$

which it is easy to write both, in a single expression:

$$y^{\pm''} + [k^2 - (u(x) \pm kq(x))]y^{\pm} = 0, \quad x \in \mathbb{R}, \quad k \in \mathbb{C}, \tag{1.3}$$

where $u(x)$ and $q(x)$ belong to a large class of potentials.

These equations have already been considered by M. Jaulent and C. Jean [*1*] in order to study the corresponding inverse problem. Application of this work to other inverse scattering problems occuring in absorbing media has been treated in [*3*]. Furthermore, M. Jaulent and I. Miodek [*4*] have obtained a class of non linear evolution equations associated with (1.3).

In this paper, we prove a completeness theorem relative to (1.3). For this, we consider the equations (1.3) as matrix eigenvalue equations:

$$HY^{\pm} = (\pm k)Y^{\pm}, \tag{1.4}$$

$$\text{where} \quad H = \begin{pmatrix} 0 & 1 \\ -\dfrac{d^2}{dx^2} + u(x) & q(x) \end{pmatrix}, \qquad Y^{\pm} = (y^{\pm}, (\pm k)y^{\pm})^{T}, \tag{1.5}$$

to which we associate the « adjoint » matrix eigenvalue equations:

$$H^*Z^{\pm} = (\pm k)Z^{\pm} \tag{1.6}$$

where $$H^* = \begin{pmatrix} 0 & -\frac{d^2}{dx^2} + u(x) \\ 1 & q(x) \end{pmatrix}, \quad Z^{\pm} = ((\pm k - q)y^{\pm}, y^{\pm})^T ; \quad (1.7)$$

the « scalar product » of two vector functions $F(x)$ and $G(x)$ is defined by

$$\langle F(x), G(x) \rangle = \int_{-\infty}^{\infty} F^T(x)G(x)dx = \int_{-\infty}^{\infty} (f_1(x)g_1(x)+f_2(x)g_2(x))dx, \quad (1.8)$$

$f_1(x)$ and $f_2(x)$ resp. $g_1(x)$ and $g_2(x)$ being the components of $F(x)$ resp. $G(x)$, and « T » meaning « transposed ».

In section 2, we state the definitions, the properties and the relations relative to the fundamental solutions of (1.3), particularly to $f_1^{\pm}(k,x)$ resp. $f_2^{\pm}(k,x)$, the Jost solutions at $+\infty$ resp. at $-\infty$.

In section 3, we study interesting orthogonality properties of

$$F_1^{\pm}(k,x) = (f_1^{\pm}(k,x), (\pm k)f_1^{\pm}(k,x))^T$$

and

$$F_2^{\pm}(k,x) = ((\pm k - q(x))f_2^{\pm}(k,x), f_2^{\pm}(k,x))^T$$

which are vector eigenfunctions respectively of (1.4) and (1.6) associated with the eigenvalue $(\pm k)$.

In section 4, starting from (1.3) and using the Green's function, we prove that all the vector functions of a certain class ξ may be expressed through $F_1^{\pm}(k,x)$ and $F_2^{\pm}(k,x)$.

In section 5, we show that this expression is unique.

More precisely, we establish the following theorem:

Let ξ be the class of four times continuously differentiable vector complex valued functions $\Phi(x)$ defined in $\mathbb{R}$ such that $x\Phi(x)$ and the first four derivatives are integrable in $\mathbb{R}$. Then, $\Phi(x)$ is uniquely written as:

$$\begin{aligned}\Phi(x) = \int_{-\infty}^{\infty} dk\alpha^+(k)F_1^+(k,x)\langle F_2^+(k,y), \Phi(y)\rangle \\ + \int_{-\infty}^{\infty} dk\alpha^-(k)F_1^-(k,x)\langle F_2^-(k,y), \Phi(y)\rangle \\ + \sum_{n=1}^{N^+} \beta^+(k_n^+)F_1^+(k_n^+,x)\langle F_2^+(k_n^+,y), \Phi(y)\rangle \\ + \sum_{n=1}^{N^-} \beta^-(k_n^-)F_1^-(k_n^-,x)\langle F_2^-(k_n^-,y), \Phi(y)\rangle, \quad (1.9)\end{aligned}$$

where $$\alpha^{\pm}(k) = \frac{1}{(\pm k)4\pi C^{\pm}_{12}(k)}, \qquad \beta^{\pm}(k^{\pm}_n) = \frac{i}{(\pm k)2\dot{C}^{\pm}_{12}(k^{\pm}_n)}, \tag{1.10}$$

$$C^{\pm}_{12}(k) = \frac{1}{2ik} W[f_1^{\pm}(k,x), f_2^{\pm}(k,x)], \tag{1.11}$$

$$k^{\pm}_n \text{ is a simple zero of } C^{\pm}_{12}(k), \ \text{Im}\, k < 0\,. \tag{1.12}$$

We recall that $W[f, g]$ is the wronskian of f and g. We note $\dot{r}(k) = \frac{d}{dk} r(k)$, and later on, we apply the same notation to all the partial derivatives with respect to k.

2. FUNDAMENTAL SOLUTIONS OF (1.3). PROPERTIES AND RELATIONS

In this paragraph, we state all the results relative to the equations (1.3) which will be used throughout the following study. We suppose that $u(x)$ and $q(x)$ as complex valued functions on $\mathbb{R}$ satisfy:

H_1 : $u(x)$ is a twice continuously differentiable function such that $x^2u(x)$ and the first two derivatives are integrable in $\mathbb{R}$.

H_2 : $q(x)$ is a three times continuously differentiable function such that $xq(x)$ and the first three derivatives are integrable in $\mathbb{R}$.

We only recall the definitions, the properties and the relations which have already been proved in [1]. The results especially established for this paper are given without proof because the technics used are rather standard. For more details, we refer to [5].

$f_1^{\pm}(k, x)$ and $f_2^{\pm}(k, x)$, the Jost solutions of (1.3) defined by

$$\lim_{x\to\infty} e^{ikx} f_1^{\pm}(k,x) = 1, \qquad \lim_{x\to-\infty} e^{-ikx} f_2^{\pm}(k,x) = 1, \tag{2.1}$$

are unique and continuous as functions of k for $\text{Im}\, k \leq 0$, analytic for $\text{Im}\, k < 0$ and derivable for $k \in \mathbb{R}^*$ (cf. [5], Appendix A); they obey the bounds:

$$|f_1^{\pm}(k,x)| \leq C_{x_0} e^{bx}, \quad b = \text{Im}\, k \leq 0, \quad x \geq x_0, \tag{2.2}$$

$$|f_2^{\pm}(k,x)| \leq C_{x_0} e^{-bx}, \quad b = \text{Im}\, k \leq 0, \quad x \leq x_0, \tag{2.3}$$

where x_0 is any fixed real number and C_{x_0} is, for given x_0, a positive constant; moreover, $f_1^{\pm\prime}(k,x)$, $\dot{f}_1^{\pm}(k,x)$, $\dot{f}_1^{\pm\prime}(k,x)$ resp. $f_2^{\pm\prime}(k,x)$, $\dot{f}_2^{\pm}(k,x)$ and $\dot{f}_2^{\pm\prime}(k,x)$ are continuous for $k \in \mathbb{R}^*$, and their behaviour, for $\text{Im}\, k \leq 0$, $k \neq 0$, when $x \to \infty$ resp. $x \to -\infty$ is given by: (cf. [5], Appendix A).

$$f_1^{\pm}(k,x) = e^{-ikx} + r_1^{\pm}(k,x), \ \lim_{x\to\infty} r_1^{\pm}(k,x) = 0 \text{ uniformly for } k, \ \text{Im}\, k \leq 0, \tag{2.4}$$

$$f_1^{\pm\prime}(k,x) = -ike^{-ikx} + ks_1^{\pm}(k,x), \tag{2.5}$$

where $\lim_{x\to\infty} s_1^\pm(k,x) = 0$ uniformly for $|k| \geq K, K \neq 0,$ (2.6)

$\lim_{x\to\infty} ks_1^\pm(k,x) = 0$ uniformly for $|k| \leq K',$ (2.7)

$\dot{f}_1^\pm(k,x) = -ixe^{-ikx} + \dot{r}_1^\pm(k,x), \lim_{x\to\infty} \dot{r}_1^\pm(k,x) = 0$

uniformly for $|k| \geq K, K \neq 0,$ (2.8)

$\dot{f}_1^{\pm\prime}(k,x) = (-i - kx)e^{-ikx} + \dot{r}_1^{\pm\prime}(k,x), \lim_{x\to\infty} \dot{r}_1^{\pm\prime}(k,x) = 0$

uniformly for $K \leq |k| \leq K', K \neq 0,$ (2.9)

$f_2^\pm(k,x) = e^{ikx} + r_2^\pm(k,x), \lim_{x\to-\infty} r_2^\pm(k,x) = 0$ uniformly for k, $\text{Im}\, k \leq 0,$ (2.10)

$f_2^{\pm\prime}(k,x) = ike^{ikx} + ks_2^\pm(k,x),$ (2.11)

where $\lim_{x\to-\infty} s_2^\pm(k,x) = 0$ uniformly for $|k| \geq K, K \neq 0,$ (2.12)

$\lim_{x\to-\infty} ks_2^\pm(k,s) = 0$ uniformly for $|k| \leq K',$ (2.13)

$\dot{f}_2^\pm(k,x) = ixe^{ikx} + \dot{r}_2^\pm(k,x), \lim_{x\to-\infty} \dot{r}_2^\pm(k,x) = 0$

uniformly for $|k| \geq K, K \neq 0,$ (2.14)

$\dot{f}_2^{\pm\prime}(k,x) = (i - kx)e^{ikx} + \dot{r}_2^{\pm\prime}(k,x), \lim_{x\to-\infty} \dot{r}_2^{\pm\prime}(k,x) = 0$

uniformly for $K \leq |k| \leq K', K \neq 0,$ (2.15)

where K and K′ are arbitrary positive constants.

Other solutions of (1.3) exist: $g_1^\pm(k,x)$ and $g_2^\pm(k,x)$ such that:

$$\lim_{x\to\infty} e^{-ikx} g_1^\pm(k,x) = 1, \qquad \lim_{x\to-\infty} e^{ikx} g_2^\pm(k,x) = 1, \tag{2.16}$$

are not defined uniquely for $\text{Im}\, k < 0$ however such solutions can be defined (cf. [2], chap. I, section 4); let us remark that:

$$g_1^\pm(k,x) = f_1^\mp(-k,x), \qquad k \in \mathbb{R}, \qquad x \in \mathbb{R}, \tag{2.17}$$

$$g_2^\pm(k,x) = f_2^\mp(-k,x), \qquad k \in \mathbb{R}, \qquad x \in \mathbb{R}, \tag{2.18}$$

The functions $f_1^\pm(k,x)$, $f_2^\pm(k,x)$, $g_1^\pm(k,x)$ and $g_2^\pm(k,x)$, for $k \neq 0$, are related by:

$$f_2^\pm(k,x) = C_{11}^\pm(k) f_1^\pm(k,x) + C_{12}^\pm(k) g_1^\pm(k,x), \tag{2.19}$$

$$f_1^\pm(k,x) = C_{22}^\pm(k) f_2^\pm(k,x) + C_{21}^\pm(k) g_2^\pm(k,x), \tag{2.20}$$

where

$$C_{12}^\pm(k) = C_{21}^\pm(k) = \frac{1}{2ik} W[f_1^\pm(k,x), f_2^\pm(k,x)] \tag{2.21}$$

$$C_{11}^\pm(k) = \frac{1}{2ik} W[f_2^\pm(k,x), g_1^\pm(k,x)], \tag{2.22}$$

$$C_{22}^\pm(k) = \frac{-1}{2ik} W[f_1^\pm(k,x), g_2^\pm(k,x)]; \tag{2.23}$$

it follows from (2.17) and (2.18) through (2.22) and (2.23) that:

$$C_{11}^{\pm}(k) = -C_{22}^{\mp}(-k), \qquad \text{for} \qquad k \in \mathbb{R}^*. \tag{2.24}$$

The functions $C_{11}^{\pm}(k)$ ($k \in \mathbb{R}^*$) and $C_{12}^{\pm}(k)$ ($\operatorname{Im} k \leq 0$, $k \neq 0$) are continuous, continuously differentiable for $k \in \mathbb{R}^*$ while $C_{12}^{\pm}(k)$ is analytic for $\operatorname{Im} k < 0$.

We now make a supplementary assumption H_3:

$$\left[\begin{array}{l} C_{12}^{\pm}(k) \neq 0 \quad \text{for} \quad k \in \mathbb{R}^*; \qquad \lim\limits_{\substack{k \to 0 \\ k \in \mathbb{R}^*}} [k C_{12}^{\pm}(k)]^{-1} = a; \\ \text{the zeros of } C_{12}^{\pm}(k) \ (\operatorname{Im} k < 0) \text{ are simple.} \end{array}\right.$$

One can prove that $C_{12}^{\pm}(k)$ ($\operatorname{Im} k < 0$) have each a finite number of zeros, $N^{\pm}$, located at the points $k_n^{\pm}$ ($n = 1, \ldots, N^{\pm}$). The corresponding functions $f_1^{\pm}(k_n^{\pm}, x)$ are (modulo normalization) the only $L^2(\mathbb{R})$ solutions of (1.3) for $\operatorname{Im} k \leqslant 0$ and are « the bound states ». Let us remark that $[C_{12}^{\pm}(k)]^{-1}$ can be continuously extended to $k = 0$ and obviously:

$$\lim_{k \to 0} [C_{12}^{\pm}(k)]^{-1} = 0. \tag{2.25}$$

It is useful to know the behaviour of $f_1^{\pm}(k, x)$, $f_2^{\pm}(k, x)$, $C_{11}^{\pm}(k)$ and $C_{12}^{\pm}(k)$ when $|k| \to \infty$. It is not difficult to show (cf. [5], Appendix B) that we have, for $x \in \mathbb{R}$, $|k| \geq K$, $K \neq 0$:

$$f_1^{\pm}(k, x) = e^{-ikx} f_1^{\pm}(x) + \frac{e^{-ikx}}{k} g_1^{\pm}(x) + \frac{e^{-ikx}}{k^2} h_1^{\pm}(x) + \frac{e^{-ikx}}{k^3} t_1^{\pm}(k, x), \tag{2.26}$$

$$f_2^{\pm}(k, x) = e^{ikx} f_2^{\pm}(x) + \frac{e^{ikx}}{k} g_2^{\pm}(x) + \frac{e^{ikx}}{k^2} h_2^{\pm}(x) + \frac{e^{ikx}}{k^3} t_2^{\pm}(k, x) \tag{2.27}$$

where

$$f_1^{\pm}(x) = e^{\pm \int_x^{\infty} \frac{q(t)}{2i} dt}, \tag{2.28}$$

$$f_2^{\pm}(x) = e^{\pm \int_{-\infty}^{x} \frac{q(t)}{2i} dt}, \tag{2.29}$$

$g_1^{\pm}(x)$ and $g_2^{\pm}(x)$ are twice continuously differentiable, $g_1^{\pm}(x)$, $g_2^{\pm}(x)$, $g_1^{\pm\prime}(x)$, $g_2^{\pm\prime}(x)$, $g_1^{\pm\prime\prime}(x)$ and $g_2^{\pm\prime\prime}(x)$ are bounded in $\mathbb{R}$, $h_1^{\pm}(x)$ and $h_2^{\pm}(x)$ are continuously differentiable, $h_1^{\pm}(x)$, $h_2^{\pm}(x)$, $h_1^{\pm\prime}(x)$ and $h_2^{\pm\prime}(x)$ are bounded in $\mathbb{R}$, $t_1^{\pm}(k, x)$ and $t_2^{\pm}(k, x)$ are bounded for $x \in \mathbb{R}$ and $|k| \geq K$, $K \neq 0$.

Taking into account (2.22), (2.23), (2.27), (2.4), (2.5), (2.10) and (2.11), we deduce from (2.26) and (2.27):

$$C_{11}^{\pm}(k) = 0\left(\frac{1}{k}\right), \quad |k| \to \infty, \quad k \in \mathbb{R}^*, \tag{2.30}$$

$$C_{12}^{\pm}(k) = f_1^{\pm}(-\infty) + 0\left(\frac{1}{k}\right), \quad |k| \to \infty, \quad \operatorname{Im} k \leq 0. \tag{2.31}$$

If we now consider $\varphi(x)$ a three times continuously differentiable function defined in $\mathbb{R}$ such that $x\varphi(x)$ and the first three derivatives are integrable in R, we can prove (cf. [5], Appendix C) from the derivability of $f_i^{\pm}(k, x)$ for $k \in \mathbb{R}^*$ that:

$$\int_{-\infty}^{\infty} f_i^{\pm}(k, x)\varphi(x)dx \qquad \text{is derivable for} \qquad k \in \mathbb{R}^*,$$

and from (2.26) and (2.27) that:

$$\int_{-\infty}^{\infty} f_i^{\pm}(k, x)\varphi(x)dx = 0\left(\frac{1}{k^3}\right), \qquad |k| \to \infty, \qquad i = 1, 2. \tag{2.32}$$

3. EIGENFUNCTIONS OF (1.2) AND (1.4)

Let us consider $F_1^{\pm}(k, x)$ and $F_2^{\pm}(k, x)$ defined by:

$$F_1^{\pm}(k, x) = (f_1^{\pm}(k, x), (\pm\, k)f_1^{\pm}(k, x))^T,\ x \in \mathbb{R},\ \text{Im}\, k \leq 0, \tag{3.1}$$

$$F_2^{\pm}(k, x) = ((\pm\, k - q(x))f_2^{\pm}(k, x),\ f_2^{\pm}(k, x))^T,\ x \in \mathbb{R},\ \text{Im}\, k \leq 0. \tag{3.2}$$

Using the definition of H (1.5) and H* (1.7), we easily find that:

$$HF_1^{\pm}(k, x) = (\pm\, k)F_1^{\pm}(k, x), \quad x \in \mathbb{R}, \quad \text{Im}\, k \leq 0, \tag{3.3}$$

$$H^*F_2^{\pm}(k, x) = (\pm\, k)F_2^{\pm}(k, x), \quad x \in \mathbb{R}, \quad \text{Im}\, k \leq 0. \tag{3.4}$$

It is then interesting to evaluate the « scalar product » of $F_1^{\pm}(k, x)$ and $F_2^{\pm}(k', x)$. We find that this product exists only for certain values of k and k' for which $F_1^{\pm}(k, x)$ and $F_2^{\pm}(k', x)$ are « conjugated ». Let us prove precisely proposition 1:

PROPOSITION 1. — $\langle F_1^{\pm}(k, x), F_2^{\pm}(k', x) \rangle$ exists and

$$\langle HF_1^{\pm}(k, x), F_2^{\pm}(k', x) \rangle = \langle F_1^{\pm}(k, x), H^*F_2^{\pm}(k', x) \rangle$$

in the following cases:

(a) $k = k' = k_n^{\pm},\ \langle F_1^{\pm}(k_n^{\pm}, x), F_2^{\pm}(k_n^{\pm}, x) \rangle = \mp\, 2ik_n^{\pm}\dot{C}_{12}^{\pm}(k_n^{\pm}),$ (3.5)

(b) $k = k_n^{\pm} \neq k' = k_m^{\pm},\ \langle F_1^{\pm}(k_n^{\pm}, x), F_2^{\pm}(k_m^{\pm}, x) \rangle = 0,$ (3.6)

(c) $k \in \mathbb{R}^*,\ k' = k_m^{\pm}$ or $k = k_n^{\pm},\ k' \in \mathbb{R}^*,$

$$\langle F_1^{\pm}(k, x), F_2^{\pm}(k_m^{\pm}, x) \rangle = \langle F_1^{\pm}(k_n^{\pm}, x), F_2^{\pm}(k', x) \rangle = 0, \tag{3.7}$$

where $k_n^{\pm}$ has been defined by (1.12).

Proof. — To prove the property *a*), we multiply on the left each member of (3.3) by $\dot{F}_2^{\pm}(k, x)^T$, we obtain, for $\text{Im}\, k \leqslant 0,\ k \neq 0$:

$$\dot{F}_2^{\pm}(k, x)^T HF_1^{\pm}(k, x) - (\pm\, k)\dot{F}_2^{\pm}(k, x)^T F_1^{\pm}(k, x) = 0. \tag{3.8}$$

Differentiating (3.4) with respect to k and multiplying on the left by $F_1^{\pm}(k, x)^T$, we find:

$$F_1^{\pm}(k, x)^T H^* F_2^{\pm}(k, x) - (\pm k) F_1^{\pm}(k, x)^T F_2^{\pm}(k, x) - (\pm 1) F_1^{\pm}(k, x)^T \dot{F}_2^{\pm}(k, x) = 0. \quad (3.9)$$

Subtracting (3.9) from (3.8), we obtain:

$$(\pm 1) F_1^{\pm}(k, x)^T \dot{F}_2^{\pm}(k, x) = F_1^{\pm}(k, x)^T H^* \dot{F}_2^{\pm}(k, x) - \dot{F}_2^{\pm}(k, x)^T H F_1^{\pm}(k, x); \quad (3.10)$$

and then, using the definitions (1.5), (1.7), (3.1) and (3.2), we find:

$$(\pm 1) \langle F_1^{\pm}(k, x), F_2^{\pm}(k, x) \rangle = \lim_{x \to \infty} \Delta^{\pm}(k, x) - \lim_{x \to -\infty} \Delta^{\pm}(k, x), \quad (3.11)$$

where $$\Delta^{\pm}(k, x) = f_1^{\pm\prime}(k, x) \dot{f}_2^{\pm}(k, x) - f_1^{\pm}(k, x) \dot{f}_2^{\pm\prime}(k, x). \quad (3.12)$$

Starting from (2.19) and (2.20), with the help of (2.4), (2.5), (2.8) and (2.9) we can deduce for $k \in \mathbb{R}^*$, and $x \to \infty$;

$$f_2^{\pm}(k, x) \sim C_{11}^{\pm}(k) e^{-ikx} + C_{12}^{\pm}(k) e^{ikx}, \quad (3.13)$$

$$\dot{f}_2^{\pm}(k, x) \sim [\dot{C}_{11}^{\pm}(k) - ixC_{11}^{\pm}(k)] e^{-ikx} + [\dot{C}_{12}^{\pm}(k) + ixC_{12}^{\pm}(k)] e^{ikx}, \quad (3.14)$$

$$\dot{f}_2^{\pm\prime}(k, x) \sim [-ik(\dot{C}_{11}^{\pm}(k) - ixC_{11}^{\pm}(k)) - iC_{11}^{\pm}(k)] e^{-ikx} + [ik(\dot{C}_{12}^{\pm}(k) + ixC_{12}^{\pm}(k)) + iC_{12}^{\pm}(k)] e^{ikx}; \quad (3.15)$$

by means of (2.10), (2.11), (2.14) and (2.15) we obtain for $x \to -\infty$:

$$f_1^{\pm}(k, x) \sim C_{22}^{\pm}(k) e^{ikx} + C_{21}^{\pm}(k) e^{-ikx}, \quad (3.16)$$

$$f_1^{\pm\prime}(k, x) \sim ikC_{22}^{\pm}(k) e^{ikx} - ikC_{21}^{\pm}(k) e^{-ikx}; \quad (3.17)$$

and then taking into account (2.4), (2.5) and (2.14), (2.15), it follows:

$$\Delta^{\pm}(k, x) \sim iC_{11}^{\pm}(k) e^{-2ikx} + (2xk - i) C_{12}^{\pm}(k) - 2ik\dot{C}_{12}^{\pm}(k), \quad x \to \infty, \quad (3.18)$$

$$\Delta^{\pm}(k, x) \sim iC_{22}^{\pm}(k) e^{2ikx} + (2xk - i) C_{21}^{\pm}(k), \quad x \to -\infty; \quad (3.19)$$

hence, it is clear that $\langle F_1^{\pm}(k, x), F_2^{\pm}(k, x) \rangle$ doesn't exist for $k \in \mathbb{R}^*$. Let us now consider k non real, $\operatorname{Im} k < 0$. Starting from (2.19) and using (2.1) and (2.16), we find, for $x \to \infty$:

$$f_2^{\pm}(k, x) \sim C_{12}^{\pm}(k) e^{ikx}, \quad (3.20)$$

$$\dot{f}_2^{\pm}(k, x) \sim [\dot{C}_{12}^{\pm}(k) + ixC_{12}^{\pm}(k)] e^{ikx}, \quad (3.21)$$

$$\dot{f}_2^{\pm\prime}(k, x) \sim [ik(\dot{C}_{12}^{\pm}(k) + ixC_{12}^{\pm}(k)) + iC_{12}^{\pm}(k)] e^{ikx}; \quad (3.22)$$

with the help of (2.1) and (2.16) in (2.20), we have for $x \to -\infty$

$$f_1^{\pm}(k, x) \sim C_{21}^{\pm}(k) e^{-ikx}, \quad (3.23)$$

$$f_1^{\pm\prime}(k, x) \sim -ikC_{12}^{\pm}(k) e^{-ikx}; \quad (3.24)$$

and then:

$$\Delta^{\pm}(k, x) \sim (2xk - i) C_{12}^{\pm}(k) - 2ik\dot{C}_{12}^{\pm}(k), \quad x \to \infty, \quad (3.25)$$

$$\Delta^{\pm}(k, x) \sim (2xk - i) C_{21}^{\pm}(k), \quad x \to -\infty; \quad (3.26)$$

consequently, $\langle F_1^{\pm}(k,x), F_2^{\pm}(k,x)\rangle$ exists only for $k = k_n^{\pm}$ and its value is directly deduced from (3.25).

Now, for $k \neq k'$, let us compute $\langle F_1^{\pm}(k,x), F_2^{\pm}(k',x)\rangle$. From (3.1) and (3.2) through (1.8), it can be written as:

$$\langle F_1^{\pm}(k,x), F_2^{\pm}(k',x)\rangle = \int_{-\infty}^{\infty} (\pm k \pm k' - q(x)) f_1^{\pm}(k,x) f_2^{\pm}(k',x)dx. \quad (3.27)$$

Starting from (1.3) in which, in a first step, we substitute $y^{\pm}$ by $f_1^{\pm}(k,x)$, and in a second step, k by k' and $y^{\pm}$ by $f_2^{\pm}(k',x)$, we obtain:

$$f_1^{\pm''}(k,x) + [k^2 - (u(x) \pm kq(x))] f_1^{\pm}(k,x) = 0, \quad (3.28)$$

$$f_2^{\pm''}(k',x) + [k'^2 - (u(x) \pm k'q(x))] f_2^{\pm}(k',x) = 0; \quad (3.29)$$

then, we multiply (3.28) by $f_2^{\pm}(k',x)$ and (3.29) by $f_1^{\pm}(k,x)$ and we subtract the new resulting equations; we deduce for $k \neq k'$:

$$\int_B^A F_2^{\pm}(k',x)^T F_1^{\pm}(k,x)dx = \frac{\pm 1}{k-k'} [\nabla^{\pm}(k,k',A) - \nabla^{\pm}(k,k',B)], \quad (3.30)$$

$$\langle F_1^{\pm}(k,x), F_2^{\pm}(k',x)\rangle = \frac{\pm 1}{k-k'} [\lim_{A\to\infty} \nabla^{\pm}(k,k',A) - \lim_{B\to-\infty} \nabla^{\pm}(k,k',B)], \quad (3.31)$$

where $\quad \nabla^{\pm}(k,k',x) = f_1^{\pm}(k,x) f_2^{\pm'}(k',x) - f_1^{\pm'}(k,x) f_2^{\pm}(k',x). \quad (3.32)$

let us prove that the formula (3.30) is also valid for $k = k'$ and let us work out its value. To this end, we rewrite (3.32) in the case $x = A$ and $x = B$

$$\frac{\nabla^{\pm}(k,k',A)}{k-k'} = \frac{f_1^{\pm}(k',A) f_2^{\pm'}(k',A) - f_1^{\pm'}(k',A) f_2^{\pm}(k',A)}{k-k'}$$

$$+ f_2^{\pm'}(k',A) \frac{f_1^{\pm}(k,A) - f_1^{\pm}(k',A)}{k-k'} - f_2^{\pm}(k',A) \frac{f_1^{\pm}(k',A) - f_1^{\pm'}(k',A)}{k-k'}; \quad (3.33)$$

$$\frac{\nabla^{\pm}(k,k',B)}{k-k'} = \frac{f_1^{\pm}(k,B) f_2^{\pm'}(k,B) - f_1^{\pm'}(k,B) f_2^{\pm}(k,B)}{k-k'}$$

$$+ f_1^{\pm'}(k,B) \frac{f_2^{\pm}(k,B) - f_2^{\pm}(k',B)}{k-k'} - f_1^{\pm}(k,B) \frac{f_2^{\pm'}(k,B) - f_2^{\pm'}(k',B)}{k-k'}; \quad (3.34)$$

taking into account (2.21) and the analyticity for $\operatorname{Im} k < 0$ or the derivability for $k \in \mathbb{R}^*$ of $f_1^{\pm}(k,A)$, $f_1^{\pm'}(k,A)$, $f_2^{\pm}(k,B)$, $f_2^{\pm'}(k,B)$ and $C_{12}^{\pm}(k)$, we can assert that (3.30) is also valid when k tends to k'.

Let us now compute (3.33) when $A \to \infty$. Starting from (2.19) and making use successively of (2.4) and (2.5), (2.30) and (2.31), we find the estimates:

$$f_2^{\pm}(k',A) = C_{11}^{\pm}(k')e^{-ik'A} + C_{12}^{\pm}(k')e^{ik'A} + \tilde{r}_2^{\pm}(k',A), \quad (3.35)$$

$$f_2^{\pm'}(k',A) = C_{11}^{\pm}(k')(-ik')e^{-ik'A} + ik'C_{12}^{\pm}(k')e^{ik'A} + k'\tilde{s}_2^{\pm}(k',A), \quad (3.36)$$

where $\lim_{A\to\infty} \tilde{r}_2^{\pm}(k', A) = 0$ uniformly for $\operatorname{Im} k' \leq 0$, (3.37)

$\lim_{A\to\infty} \tilde{s}_2^{\pm}(k', A) = 0$ uniformly for $|k'| \geq K, K \neq 0$, (3.38)

$\lim_{A\to\infty} k'\tilde{s}_2^{\pm}(k', A) = 0$ uniformly for $|k'| \leq K'$; (3.39)

and finally, we find for (3.33):

$$\frac{\nabla^{\pm}(k, k', A)}{k - k'} = iC_{11}^{\pm}(k')e^{-i(k+k')A} + i(k+k')\frac{C_{12}^{\pm}(k)}{k-k'}e^{i(k'-k)A} + R_1^{\pm}(k, k', A), \quad (3.40)$$

where

$$R_1^{\pm}(k, k', A) = f_2^{\pm\prime}(k', A)\frac{r_1^{\pm}(k, A) - r_1^{\pm}(k', A)}{k - k'} + k'\tilde{s}_2^{\pm}(k', A)\frac{e^{-ikA} - e^{-ik'A}}{k - k'}$$
$$-f_2^{\pm}(k', A)\frac{r_1^{\pm\prime}(k, A) - r_1^{\pm\prime}(k', A)}{k - k'} - \tilde{r}_2^{\pm}(k', A)\frac{(-ik)e^{-ikA} - (-ik')e^{-ik'A}}{k - k'}, \quad (3.41)$$

and $\lim_{A\to\infty} R_1^{\pm}(k, k', A) = 0$ (also valid for $k \to k'$). (3.42)

To calculate (3.34) when $B \to -\infty$, we proceed in a similar way. Starting from (2.20) and applying (2.10) and (2.11), we have:

$$f_1^{\pm}(k, B) = C_{22}^{\pm}(k)e^{ikB} + C_{21}^{\pm}(k)e^{-ikB} + \tilde{r}_1^{\pm}(k, B), \quad (3.43)$$
$$f_1^{\pm\prime}(k, B) = ikC_{22}^{\pm}(k)e^{ikB} - ikC_{21}^{\pm}(k)e^{-ikB} + k\tilde{s}_1^{\pm}(k, B), \quad (3.44)$$

where $\lim_{B\to-\infty} \tilde{r}_1^{\pm}(k, B) = 0$ uniformly for $\operatorname{Im} k \leq 0$, (3.45)

$\lim_{B\to-\infty} \tilde{s}_1^{\pm}(k, B) = 0$ uniformly for $|k| \geq K, K \neq 0$. (3.46)

$\lim_{B\to-\infty} k\tilde{s}_1^{\pm}(k, B) = 0$ uniformly for $|k| \leq K'$; (3.47)

and then we obtain for (3.34):

$$\frac{\nabla^{\pm}(k, k', B)}{k - k'} = -iC_{22}^{\pm}(k)e^{i(k+k')B} + i(k+k')\frac{C_{21}^{\pm}(k)}{k-k'}e^{i(k'-k)B} + R_2^{\pm}(k, k', B), \quad (3.48)$$

where

$$R_2^{\pm}(k, k', B) = f_1^{\pm\prime}(k, B)\frac{r_2^{\pm}(k, B) - r_2^{\pm}(k', B)}{k-k'} + k\tilde{s}_1^{\pm}(k, B)\frac{e^{ikB} - e^{ik'B}}{k-k'}$$
$$-f_1^{\pm}(k, B)\frac{r_2^{\pm\prime}(k, B) - r_2^{\pm\prime}(k', B)}{k-k'} - \tilde{r}_1^{\pm}(k, B)\frac{ike^{ikB} - ik'e^{ik'B}}{k-k'} \quad (3.49)$$

and $\lim_{B\to-\infty} R_2^{\pm}(k, k', B) = 0$ (also valid for $k = k'$). (3.50)

Collecting (3.40) and (3.48), we obtain for (3.30):

$$\int_B^A F_2^{\pm}(k', x)^T F_1^{\pm}(k, x)dx = \pm i[C_{11}^{\pm}(k')e^{-i(k+k')A} + C_{22}^{\pm}(k)e^{i(k+k')B}]$$
$$\pm i\frac{(k+k')}{k-k'}[C_{12}^{\pm}(k')e^{i(k'-k)A} - C_{21}^{\pm}(k)e^{i(k'-k)B}]$$
$$+ R_1^{\pm}(k, k', A) - R_2^{\pm}(k, k', B), \qquad A \to \infty, \qquad B \to -\infty, \quad (3.51)$$

Then, we see from (3.51) that $\langle F_1^{\pm}(k, x), F_2^{\pm}(k', x) \rangle$ exists only for $k = k_n^{\pm}$, $k' \in \mathbb{R}^*$ or $k \in \mathbb{R}^*$, $k' = k_n^{\pm}$ or $k = k_n^{\pm}$, $k' = k_m^{\pm}$, $n \neq m$ (hence (3.6) and (3.7) are proved) and also for $k = -k'$, $k' \in \mathbb{R}^*$; in this case, we have:

$$\langle F_1^{\pm}(k, x), F_2^{\pm}(-k, x) \rangle = \pm i[C_{11}^{\pm}(-k) + C_{22}^{\pm}(k)], \qquad k \in \mathbb{R}^*. \quad (3.52)$$

It is also useful to know the « scalar product » of $F_1^{\pm}(k, x)$ and $F_2^{\mp}(k', x)$. In the same way, we obtain similar results which we state in proposition 2:

Proposition 2. — $\langle F_1^{\pm}(k, x), F_2^{\mp}(k', x) \rangle$ exists and

$$\langle HF_1^{\pm}(k, x), F_2^{\mp}(k', x) \rangle = \langle F_1^{\pm}(k, x), H^*F_2^{\mp}(k', x) \rangle$$

in the following cases:

(*a*) $k = k_n^{\pm}$, $k' = k_m^{\mp}$ and $\langle F_1^{\pm}(k_n^{\pm}, x), F_2^{\mp}(k_m^{\mp}, x) \rangle = 0$, (3.53)

(*b*) $k \in \mathbb{R}^*$, $k' = k_m^{\mp}$ or $k = k_n^{\pm}$, $k' \in \mathbb{R}^*$

and

$$\langle F_1^{\pm}(k, x), F_2^{\mp}(k_m^{\mp}, x) \rangle = \langle F_1^{\pm}(k_n^{\pm}, x), F_2^{\mp}(k', x) \rangle = 0. \quad (3.54)$$

Proof. — Let us compute $\langle F_1^{\pm}(k, x), F_2^{\mp}(k', x) \rangle$. We obtain various results which we only state:

$$\langle F_1^{\pm}(k, x), F_2^{\mp}(k', x) \rangle = \int_{-\infty}^{\infty} (\pm k \mp k' - q(x)) f_1^{\pm}(k, x) f_2^{\mp}(k', x)dx. \quad (3.55)$$

$$\int_B^A F_2^{\mp}(k', x)^T F_1^{\pm}(k, x)dx = \frac{\pm 1}{k-k'}[\Omega^{\pm}(k, k', A) - \Omega^{\pm}(k, k', B)], \quad k \neq -k', \quad (3.56)$$

where $$\Omega^{\pm}(k, k', x) = f_1^{\pm}(k, x) f_2^{\mp\prime}(k', x) - f_1^{\pm\prime}(k, x) f_2^{\mp}(k', x). \quad (3.57)$$

In the case where $\text{Im}\, k < 0$ and $\text{Im}\, k' < 0$ or $\text{Im}\, k < 0$ and $k' \in \mathbb{R}^*$ or $k \in \mathbb{R}^*$ and $\text{Im}\, k' < 0$, a glance at (3.51) allows us to see that:

$$\langle F_1^{\pm}(k, x), F_2^{\mp}(k', x) \rangle = \lim_{\substack{A \to \infty \\ B \to -\infty}} \left(\pm i \cdot \frac{k-k'}{k+k'}[C_{11}^{\mp}(k')e^{-i(k+k')A} + C_{22}^{\pm}(k)e^{i(k+k')B}] \right.$$
$$\left. \pm i[C_{12}^{\mp}(k')e^{i(k'-k)A} - C_{21}^{\pm}(k)e^{i(k'-k)B}] \right); \quad (3.58)$$

which proves proposition 2.

For k and k' belonging to $\mathbb{R}^*$, it is useful to know the estimate of (3.56) when A $\to \infty$ and B $\to -\infty$. We find:

$$\int_{B}^{A} F_2^{\mp}(k', x)^T F_1^{\pm}(k, x) dx = \pm i \frac{k-k'}{k+k'} \left[C_{11}^{\mp}(k') e^{-i(k+k')A} + C_{22}^{\pm}(k) e^{i(k+k')B} \right]$$
$$\pm i C_{12}^{\mp}(k') e^{i(k'-k)A} \mp i C_{21}^{\pm}(k) e^{i(k'-k)B} + S_1^{\pm}(k, k', A) - S_2^{\pm}(k, k', B), \quad (3.59)$$

where

$$S_1^{\pm}(k, k', A) = f_2^{\mp\prime}(k', A) \frac{r_1^{\pm}(k, A) - r_1^{\pm}(-k', A)}{k+k'} + k' \tilde{s}_2^{\mp}(k', A) \cdot \frac{e^{-ikA} - e^{ik'A}}{k+k'}$$
$$- f_2^{\mp}(k', A) \frac{r_1^{\pm\prime}(k, A) - r_1^{\pm\prime}(-k', A)}{k+k'} - \tilde{r}_2^{\mp}(k', A) \frac{(-ik) e^{-ikA} - (ik') e^{ik'A}}{k+k'}, \quad (3.60)$$

$$S_2^{\pm}(k, k', B) = f_1^{\pm\prime}(k, B) \frac{r_2^{\mp}(-k, B) - r_2^{\mp}(k', B)}{k+k'} + k \tilde{s}_1^{\pm}(k, B) \frac{e^{-ikB} - e^{ik'B}}{k+k'}$$
$$- f_1^{\pm}(k, B) \frac{r_2^{\mp\prime}(-k, B) - r_2^{\mp\prime}(k', B)}{k+k'} + \tilde{r}_1^{\pm}(k, B) \frac{(-ik) e^{-ikB} - (ik') e^{ik'B}}{k+k'}; \quad (3.61)$$

from (2.24) and the derivability of $S_1^{\pm}(k, k', A)$ and $S_2^{\pm}(k, k', B)$, we deduce that (3.59) is also valid for $k \to -k'$.

Note also that (3.58) has a meaning when $k' = k$, $\operatorname{Im} k \leq 0$, $k \neq 0$

$$\langle F_1^{\pm}(k, x), F_2^{\mp}(k, x) \rangle = \pm i [C_{12}^{\mp}(k) - C_{21}^{\pm}(k)]. \quad (3.62)$$

4. COMPLETENESS THEOREM : EXISTENCE

First, for the sake of simplicity, let us consider the formula (1.9) in the case where there is no bound state. Using the definitions (3.1) and (3.2), we obtain for $\Phi(x) = (\varphi_1(x), \varphi_2(x))^T$ belonging to the class $\mathscr{E}$:

$$\varphi_1(x) = \int_{-\infty}^{\infty} dk \alpha^+(k) f_1^+(k, x) \int_{-\infty}^{\infty} [(k - q(y))\varphi_1(y) + \varphi_2(y)] f_2^+(k, y) dy$$
$$+ \int_{-\infty}^{\infty} dk \alpha^-(k) f_1^-(k, x) \int_{-\infty}^{\infty} [(-k - q(x))\varphi_1(y) + \varphi_2(y)] f_2^-(k, y) dy, \quad (4.1)$$

$$\varphi_2(x) = \int_{-\infty}^{\infty} dk k \alpha^+(k) f_1^+(k, x) \int_{-\infty}^{\infty} [(k - q(y))\varphi_1(y) + \varphi_2(y)] f_2^{\pm}(k, y) dy$$
$$- \int_{-\infty}^{\infty} dk k \alpha^-(k) f_1^-(k, x) \int_{-\infty}^{\infty} [(-k - q(y))\varphi_1(y) + \varphi_2(y)] f_2^-(k, y) dy. \quad (4.2)$$

In fact, we shall prove successively:

$$\varphi_2(x) = \int_{-\infty}^{\infty} dkk\alpha^+(k)f_1^+(k,x)\int_{-\infty}^{\infty} f_2^+(k,y)\varphi_2(y)dy + \int_{-\infty}^{\infty} dkk\alpha^-(k)f_1^-(k,x)\int_{-\infty}^{\infty} f_2^-(k,y)\varphi_2(y)dy, \tag{4.3}$$

$$0 = \int_{-\infty}^{\infty} dkk\alpha^+(k)f_1^+(k,x)\int_{-\infty}^{\infty} (k - q(y))f_2^+(k,y)\varphi_1(y)dy - \int_{-\infty}^{\infty} dkk\alpha^-(k)f_1^-(k,x)\int_{-\infty}^{\infty} (-k - q(y))f_2^-(k,y)\varphi_1(y)dy, \tag{4.4}$$

$$0 = \int_{-\infty}^{\infty} dk\alpha^+(k)f_1^+(k,x)\int_{-\infty}^{\infty} f_2^+(k,y)\varphi_2(y)dy + \int_{-\infty}^{\infty} dk\alpha^-(k)f_1^-(k,x)\int_{-\infty}^{\infty} f_2^+(k,y)\varphi_2(y)dy, \tag{4.5}$$

$$\varphi_1(x) = \int_{-\infty}^{\infty} dk\alpha^+(k)f_1^+(k,x)\int_{-\infty}^{\infty} (k - q(y))\varphi_1(y)f_2^+(k,y)dy + \int_{-\infty}^{\infty} dk\alpha^-(k)f_1^-(k,x)\int_{-\infty}^{\infty} (-k - q(y))\varphi_1(y)f_2^-(k,y)dy. \tag{4.6}$$

We see that in adding (4.3) and (4.4) resp. (4.5) and (4.6), we find again (4.2) resp. (4.1).

Let us note that, when the formulas (4.1), (4.2), (4.3), (4.4), (4.5) and (4.6) will contain the part corresponding to the bound states, we shall call them « complete formulas ».

To obtain the theorem, we proceed in three steps. In the first one, we solve the one-dimensional Schrödinger equation (1.3) with second member, by using the Green's function. From the obtained results, in the second step, with integrations along a closed path contained in the lower-half of the complex k-plane, we find four relations between $\varphi_i(x)$ ($i = 1, 2$) and the Green's function. And then, in the last part, using algebraic relations, we deduce the « complete formulas » (4.3), (4.4), (4.5) and (4.6).

First, let $p(x)$ be a continuous function defined in $\mathbb{R}$ and integrable and let us now consider the equations:

$$\psi^{\pm''}(x) + [k^2 - (u(x) \pm kq(x))]\psi^{\pm}(x) = \rho(x), \quad \operatorname{Im} k \leq 0, \quad x \in \mathbb{R}, \tag{4.7}$$

which we solve by the constant variation method. Let $G^{\pm}(k, x, y)$

$$(\operatorname{Im} k \leq 0, \quad k \neq 0, \quad k \neq k_n^{\pm}, \quad x \in \mathbb{R}, \quad y \in \mathbb{R})$$

be the Green's function:

$$G^{\pm}(k,x,y) = \begin{cases} \dfrac{f_1^{\pm}(k,y)f_2^{\pm}(k,x)}{2ikC_{12}^{\pm}(k)}, & y > x, \\ \dfrac{f_1^{\pm}(k,x)f_2^{\pm}(k,y)}{2ikC_{12}^{\pm}(k)}, & y < x, \end{cases} \tag{4.8}$$

where $f_1^{\pm}(k, x)$, $f_2^{\pm}(k, x)$ and $C_{12}^{\pm}(k)$ have been defined respectively by (2.1) and (2.9); clearly, $G^{\pm}(k, x, y)$ are, for fixed x and y, continuous as function of k for $\operatorname{Im} k \leq 0$, $k \neq 0$, $k \neq k_n^{\pm}$ and analytic for $\operatorname{Im} k < 0$, $k \neq k_n^{\pm}$, and verify the bound:

$$|G^{\pm}(k,x,y)| \leq \frac{C}{2|k|\,|C_{12}^{\pm}(k)|}, \qquad \forall y \in \mathbb{R}, \qquad \forall x \in I, \tag{4.9}$$

where I is an arbitrary real compact and C is, for fixed k, a constant depending uniquely of I.

The solution $\psi^{\pm}(x)$ of (4.7) are then given by:

$$\psi^{\pm}(x) = -\int_{-\infty}^{\infty} G^{\pm}(k,x,y)p(y)dy, \ x \in \mathbb{R}, \ \operatorname{Im} k \leq 0, \ k \neq 0, \ k \neq k_n^{\pm}. \tag{4.10}$$

Now, from the result (4.10), we are going to establish the four following relations:

$$\varphi(x) = \frac{1}{i\pi} \lim_{R \to \infty} \int_{\substack{|k|=R \\ \operatorname{Im} k < 0}} dk \int_{-\infty}^{\infty} [k - (\pm\, q(y))] G^{\pm}(k,x,y)\varphi(y)dy, \tag{4.11}$$

$$\varphi(x) = -\frac{1}{i\pi} \lim_{R \to \infty} \int_{\substack{|k|=R \\ \operatorname{Im} k < 0}} kdk \int_{-\infty}^{\infty} G^{\pm}(k,x,y)\varphi(y)dy, \tag{4.12}$$

$$0 = \lim_{R \to \infty} \int_{\substack{|k|=R \\ \operatorname{Im} k < 0}} dk \int_{-\infty}^{\infty} G^{\pm}(k,x,y)\varphi(y)dy, \tag{4.13}$$

$$0 = \lim_{R \to \infty} \left[\int_{\substack{|k|=R \\ \operatorname{Im} k < 0}} kdk \left(\int_{-\infty}^{\infty} (k - q(y)) G^{+}(k,x,y)\varphi(y)dy - \int_{-\infty}^{\infty} (k + q(y)) G^{-}(k,x,y)\varphi(y)dy \right) \right], \tag{4.14}$$

where the integrals converge uniformly for $x \in I$ and $\varphi(x)$ is a four times continuously differentiable function such that $x\varphi(x)$ and the first four derivatives are integrable in $\mathbb{R}$.

For this, let us set:

$$p^{\pm}(x) = \varphi''(x) + [k^2 - (u(x) \pm kq(x))]\varphi(x), \tag{4.15}$$

and apply the result (4.10), we have:

$$\varphi(x) = -\int_{-\infty}^{\infty} G^{\pm}(k, x, y)[\varphi''(y) - u(y)\varphi(y)]dy - \int_{-\infty}^{\infty} [k^2 - (\pm kq(y))]G^{\pm}(k, x, y)\varphi(y)dy. \quad (4.16)$$

We divide (4.16) by k and we integrate each member of the resulting formula along a half circle $|k| = R$ contained in the lower-half of the complex k-plane.

Thanks to (4.9) we can apply a Jordan's lemma to prove that:

$$\lim_{R\to\infty}\int_{|k|=R} \frac{dk}{k}\int_{-\infty}^{\infty} G^{\pm}(k, x, y)[\varphi''(y)-u(y)\varphi(y)]dy=0, \quad \text{uniformly for } x\in I, \quad (4.17)$$

and then we obtain (4.11).

If we divide (4.16) by k^2 and we integrate each member of the resulting formula along a half circle $|k| = R$, $\operatorname{Im} k < 0$, we find:

$$\int_{|k|=R} \frac{\varphi(x)}{k^2}dk = -\int_{|k|=R} \frac{dk}{k^2}\int_{-\infty}^{\infty} G^{\pm}(k, x, y)[\varphi''(y) - u(y)\varphi(y)]dy - \int_{|k|=R} dk\int_{-\infty}^{\infty} G^{\pm}(k, x, y)\varphi(y)dy \pm \int_{|k|=R} \frac{dk}{k}\int_{-\infty}^{\infty} G^{\pm}(k, x, y)\varphi(y)dy. \quad (4.18)$$

If $R \to \infty$, using (4.9) to apply a Jordan's lemma, we prove (4.13). Since $q(x)$ verifies the condition H_2, (4.13) is also valid when we replace $\varphi(y)$ by $q(y)\varphi(y)$ and accordingly, from (4.11), we can deduce (4.12).

In order to find (4.14), we start again from (4.16) and we subtract (4.16) corresponding to + from (4.16) corresponding to − ; we have:

$$0 = \int_{-\infty}^{\infty} G^{+}(k, x, y)[\varphi''(y)-u(y)\varphi(y)]dy+\int_{-\infty}^{\infty} (k^2-kq(y))G^{+}(k, x, y)\varphi(y)dy - \int_{-\infty}^{\infty} G^{-}(k, x, y)[\varphi''(y)-u(y)\varphi(y)]dy-\int_{-\infty}^{\infty} (k^2+kq(y))G^{-}(k, x, y)\varphi(y)dy; \quad (4.19)$$

we integrate (4.19) along a half circle $|k| = R$, $\operatorname{Im} k < 0$; $u(y)$ verifying H_1 and $\varphi(y)$ being four times differentiable, we remark that (4.13) is also valid when we replace $\varphi(y)$ by $[\varphi''(y)-u(y)\varphi(y)]$ and then we obtain (4.14).

Let us now establish the « complete formulas » (4.2) and (4.1). For this, we consider $\Phi(x) = (\varphi_1(x), \varphi_2(x))^T$ belonging to the class $\mathscr{E}$ and we expand the expressions (4.12) and (4.14) resp. (4.11) and (4.13) to obtain the « complete formulas » (4.3) and (4.4) resp. (4.6) and (4.5) and therefore (4.2) resp. (4.1).

Starting from (4.12) where $\varphi(x)$ has been replaced by $\varphi_2(x)$, we apply the residues method to the integral. By virtue of the definition (4.8) of $G^{\pm}(k, x, y)$ and the condition H_3, we find:

$$\varphi_2(x) = \frac{i}{\pi} \lim_{R\to\infty} \int_{-R}^{+R} k dk \int_{-\infty}^{\infty} G^{\pm}(k, x, y)\varphi_2(y)dy + i \sum_{n=1}^{N^{\pm}} \frac{1}{\dot{C}_{12}^{\pm}(k_n^{\pm})} \int_{-\infty}^{\infty} C^{\pm}(k_n^{\pm}, x, y)\varphi_2(y)dy, \tag{4.20}$$

where
$$C^{\pm}(k, x, y) = 2ikC_{12}^{\pm}(k)G^{\pm}(k, x, y); \tag{4.21}$$

note too that the integral of the first term of (4.20) converges uniformly for $x \in I$.

Now, to compute (4.20), we substitute $G^{\pm}(k, x, y)$ through (4.8) and using the relations (2.7) and (2.5) and taking into account that $C_{12}^{\pm}(k_n^{\pm})=0$, we obtain:

$$\begin{aligned}\varphi_2(x) = \frac{1}{2\pi} \lim_{R\to\infty} \int_{-\infty}^{\infty} dk \Bigg(\int_{-\infty}^{x} f_1^{\pm}(k, x) \left[\frac{C_{11}^{\pm}(k)}{C_{12}^{\pm}(k)} f_1^{\pm}(k, y) + f_1^{\mp}(-k, y) \right] \varphi_2(y) dy \\ + \int_{x}^{\infty} f_1^{\pm}(k, y) \left[\frac{C_{11}^{\pm}(k)}{C_{12}^{\pm}(k)} f_1^{\pm}(k, x) + f_1^{\mp}(-k, x) \right] \varphi_2(y) dy \Bigg) \\ + i \sum_{n=1}^{N^{\pm}} \frac{C_{11}^{\pm}(k_n^{\pm})}{\dot{C}_{12}^{\pm}(k_n^{\pm})} \int_{-\infty}^{\infty} f_1^{\pm}(k_n^{\pm}, x) f_1^{\pm}(k_n^{\pm}, y)\varphi_2(y)dy\end{aligned} \tag{4.22}$$

we then add the formula (4.22) corresponding to + and the formula (4.22) corresponding to − where we have exchanged k by $-k$; and by virtue of (2.19), we can write:

$$\begin{aligned}\varphi_2(x) = \int_{-\infty}^{\infty} \frac{dk}{4\pi C_{12}^{+}(k)} f_1^{+}(k, x) \int_{-\infty}^{\infty} f_2^{+}(k, y)\varphi_2(y)dy \\ + \int_{-\infty}^{\infty} \frac{dk}{4\pi C_{12}^{-}(k)} f_1^{-}(k, x) \int_{-\infty}^{\infty} f_2^{-}(k, y)\varphi_2(y)dy \\ + \sum_{n=1}^{N^{+}} \frac{i}{2\dot{C}_{12}^{+}(k_n^{+})} f_1^{+}(k_n^{+}, x) \int_{-\infty}^{\infty} f_2^{+}(k_n^{+}, y)\varphi_2(y)dy \\ + \sum_{n=1}^{N^{-}} \frac{i}{2\dot{C}_{12}^{-}(k_n^{-})} f_1^{-}(k_n^{-}, x) \int_{-\infty}^{\infty} f_2^{-}(k_n^{-}, y)\varphi_2(y)dy,\end{aligned} \tag{4.23}$$

where the integrals converge uniformly for $x \in I$ and where we can exchange f_1^{+} resp. f_1^{-} by f_2^{+} and f_2^{-};

a glance at (1.9) and (1.10) allows us to recognize the « complete formula » (4.3). To obtain (4.4), we start from (4.14) where $\varphi(x)$ has been replaced by $\varphi_1(x)$ and similar computations drive us to the « complete formula » (4.4):

$$0 = \int_{-\infty}^{\infty} \frac{dk}{4\pi C_{12}^{+}(k)} f_1^{+}(k, x) \int_{-\infty}^{\infty} (k - q(y)) f_2^{+}(k, y)\varphi_1(y)dy$$

$$- \int_{-\infty}^{\infty} \frac{dk}{4\pi C_{12}^{-}(k)} f_1^{-}(k, x) \int_{-\infty}^{\infty} (k + q(y)) f_2^{-}(k, y)\varphi_1(y)dy$$

$$+ \sum_{n=1}^{N^+} \frac{i}{2\dot{C}_{12}^{+}(k_n^{+})} f_1^{+}(k_n^{+}, x) \int_{-\infty}^{\infty} (k_n^{+} - q(y)) f_2^{+}(k_n^{+}, y)\varphi_1(y)dy$$

$$- \sum_{n=1}^{N^-} \frac{i}{2\dot{C}_{12}^{-}(k_n^{-})} f_1^{-}(k_n^{-}, x) \int_{-\infty}^{\infty} (k_n^{-} + q(y)) f_2^{-}(k_n^{-}, y)\varphi_1(y)dy, \quad (4.24)$$

where the integrals converge uniformly for $x \in I$ and where we can exchange f_1^{+} resp. f_1^{-} by f_2^{+} resp. f_2^{-}.

In order to establish the « complete formula » (4.5), we first consider (4.13) where $\varphi(x)$ has been replaced by $\varphi_2(x)$. The condition H_3 involves that $k = 0$ is not a pole of $G^{\pm}(k, x, y)$, and we have:

$$0 = \lim_{R\to\infty} \int_{-R}^{+R} G^{\pm}(k, x, y)\varphi_2(y)dy$$

$$+ \pi \sum_{n=1}^{N^{\pm}} \frac{1}{k_n^{\pm}\dot{C}_{12}^{\pm}(k_n^{\pm})} \int_{-\infty}^{\infty} C^{\pm}(k_n^{\pm}, x, y)\varphi_2(y)dy. \quad (4.25)$$

We subtract (4.25) corresponding to $-$ form (4.25) corresponding to $+$ and taking into account that $f_1^{\pm}(0, x) = f_1^{\mp}(0, x)$, $f_2^{\pm}(0, y) = f_2^{\mp}(0, y)$, we find:

$$0 = \lim_{R\to\infty} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} G^{+}(k, x, y)\varphi_2(y)dy - \lim_{R\to\infty} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} G^{-}(k, x, y)\varphi_2(y)dy$$

$$+ \pi \sum_{n=1}^{N^+} \frac{1}{k_n^{+}\dot{C}_{12}^{+}(k_n^{+})} \int_{-\infty}^{\infty} C^{+}(k_n^{+}, x, y)\varphi_2(y)dy$$

$$- \pi \sum_{n=1}^{N^-} \frac{1}{k_n^{-}\dot{C}_{12}^{-}(k_n^{-})} \int_{-\infty}^{\infty} C^{-}(k_n, x, y)\varphi_2(y)dy; \quad (4.26)$$

and then, multiplying by $\frac{i}{2\pi}$ and expliciting (4.26), we obtain the « complete formula » (4.5):

$$0 = \int_{-\infty}^{\infty} \frac{dk}{4\pi k C_{12}^{+}(k)} f_1^{+}(k, x) \int_{-\infty}^{\infty} f_2^{+}(k, y)\varphi_2(y)dy$$
$$- \int_{-\infty}^{\infty} \frac{dk}{4\pi k C_{12}^{-}(k)} f_1^{-}(k, x) \int_{-\infty}^{\infty} f_2^{-}(k, y)\varphi_2(y)dy$$
$$+ \sum_{n=1}^{N^+} \frac{i}{2k_n^+ \dot{C}_{12}^{+}(k_n^+)} f_1^{+}(k_n^+, x) \int_{-\infty}^{\infty} f_2^{+}(k_n^+, y)\varphi_2(y)dy$$
$$- \sum_{n=1}^{N^-} \frac{i}{2k_n^- \dot{C}_{12}^{-}(k_n^-)} f_1^{-}(k_n^-, x) \int_{-\infty}^{\infty} f_2^{-}(k_n^-, y)\varphi_2(y)dy, \qquad (4.27)$$

where the integrals converge uniformly for $x \in I$ and where we can exchange f_1^+ resp. f_1^- by f_2^+ resp. f_2^-.

To obtain (4.6), we remark that (4.27) is also valid when we replace $\varphi_2(y)$ by $-q(y)\varphi_1(y)$ and that (4.23) is also valid when we replace $\varphi_2(y)$ by $\varphi_1(y)$. Adding the two resulting relations, we have the « complete formula » (4.6) which we seek.

In fact, from (2.32), (2.2) and (2.31) it follows that the integrals of (1.9) converge uniformly for $x \geq x_0$.

5. COMPLETENESS THEOREM : UNIQUENESS

First, let us remark that $\alpha^{\pm}(k) \langle F_2^{\mp}(k, y), \Phi(y) \rangle$ is continuous for $k \in \mathbb{R}$ and derivable for $k \in \mathbb{R}^*$ and has the following behaviour when $|k| \to \infty$:

$$\alpha^{\pm}(k) \ \langle F_2^{\mp}(k, y), \Phi(y) \rangle = 0\left(\frac{1}{k^3}\right). \qquad (5.1)$$

To prove the uniqueness of (1.9), we consider that $\Phi(x)$ is written as:

$$\Phi(x) = \int_{-\infty}^{\infty} dk \tilde{\alpha}^{+}(k) F_1^{+}(k, x) + \int_{-\infty}^{\infty} dk \tilde{\alpha}^{-}(k) F_1^{-}(k, x)$$
$$+ \sum_{n=1}^{N^+} \tilde{\beta}_1^{+}(k_n^+) F_1^{+}(k_n^+, x) + \sum_{n=1}^{N^-} \tilde{\beta}_1^{-}(k_n^-) F_1^{-}(k_n^-, x), \qquad (5.2)$$

where $\tilde{\alpha}^{\pm}(k)$ is a continuous function in $\mathbb{R}$, derivable in $\mathbb{R}^*$ and has the behaviour of (5.1) when $|k| \to \infty$ and $\tilde{\beta}_1^{\pm}(k_n^{\pm})$ is a constant.

Let us now consider the « scalar product » $\langle F_2^+(k', x), \Phi(x) \rangle$ where k' belongs to $\mathbb{R}^*$. From (5.2) and with the help of (3.7), and (3.54) it follows:

$$\langle F_2^+(k', x), \Phi(x) \rangle = \int_{-\infty}^{\infty} dx F_2^+(k', x)^T \int_{-\infty}^{\infty} dk \tilde{\alpha}^+(k) F_1^+(k, x) + \int_{-\infty}^{\infty} dx F_2^+(k', x)^T \int_{-\infty}^{\infty} dk \tilde{\alpha}^-(k) F_1^-(k, x). \quad (5.3)$$

Remarking that the integrals of (5.2) converge uniformly for $x \geq x_0$ for any x_0, we can write:

$$\langle F_2^+(k', x), \Phi(x) \rangle = \lim_{\substack{A \to \infty \\ B \to -\infty}} \int_{-\infty}^{\infty} dk \tilde{\alpha}^+(k) \int_B^A F_2^+(k', x)^T F_1^+(k, x) dx + \lim_{\substack{A \to \infty \\ B \to -\infty}} \int_{-\infty}^{\infty} dk \tilde{\alpha}^-(k) \int_B^A F_2^+(k', x)^T F_1^-(k, x) dx. \quad (5.4)$$

Let us make explicit the first integral of the right-hand side of (5.4) by applying (3.51). We find that:

$$\lim_{\substack{A \to \infty \\ B \to -\infty}} \int_{-\infty}^{\infty} dk \tilde{\alpha}^+(k) \int_B^A F_2^+(k', x)^T F_1^+(k, x) dx =$$

$$= \lim_{\substack{A \to \infty \\ B \to -\infty}} \int_{-\infty}^{\infty} dk \tilde{\alpha}^+(k) i \left[C_{11}^+(k') e^{-i(k+k')A} + C_{22}^+(k) e^{i(k'+k)B} \right]$$

$$+ \lim_{\substack{A \to \infty \\ B \to -\infty}} \int_{-\infty}^{\infty} dk \tilde{\alpha}^+(k) i(k + k') \frac{e^{i(k'-k)A} C_{12}^+(k') - e^{i(k'-k)B} C_{21}^+(k)}{k - k'}$$

$$+ \lim_{\substack{A \to \infty \\ B \to -\infty}} \int_{-\infty}^{\infty} dk \tilde{\alpha}^+(k) \left[R_1^+(k, k', A) - R_2^+(k, k', B) \right]. \quad (5.5)$$

According to the Riemann Lebesgue's theorem, the first term of (5.5) is equal to zero; a glance to (3.41), (3.37), (3.38), (3.39) and to (3.49), (3.45), (3.46), (3.47) allows us to assert that the third term is also equal to zero.

Let us now consider the second term T_2 of (5.5) rewritten as:

$$T_2 = \lim_{\substack{A \to \infty \\ B \to -\infty}} 2ik' \int_{-\infty}^{\infty} dk \tilde{\alpha}^+(k) e^{ikA} \cdot e^{ik'B} \frac{C_{12}^+(k') e^{ik'(A-B)} - C_{12}^+(k) e^{ik(A-B)}}{k - k'}$$

$$+ \lim_{\substack{A \to \infty \\ B \to -\infty}} i \int_{-\infty}^{\infty} dk \tilde{\alpha}^+(k) e^{-ikA} \cdot e^{ik'B} \left[C_{12}^+(k') e^{ik'(A-B)} - C_{12}^+(k) e^{ik(A-B)} \right]. \quad (5.6)$$

Because of the Riemann Lebesgue's theorem, the second term of (5.6) is equal to zero.

And then:

$$\mathrm{T}_2 = \lim_{\substack{\mathrm{A}\to\infty \\ \mathrm{B}\to-\infty}} 2ik' \int_{-\infty}^{\infty} dk\tilde{\alpha}^+(k)e^{-ik\mathrm{A}}e^{ik'\mathrm{B}} \frac{\mathrm{C}_{12}^+(k')e^{ik'(\mathrm{A}-\mathrm{B})} - \mathrm{C}_{12}^+(k)e^{ik(\mathrm{A}-\mathrm{B})}}{k-k'}$$
$$= \lim_{\mathrm{A}\to\infty} 2ik'\mathrm{C}_{12}^+(k')e^{ik'\mathrm{A}}\left[\int_{-\infty}^{\infty} dke^{-ik\mathrm{A}} \frac{\tilde{\alpha}^+(k)-\tilde{\alpha}^+(k')}{k-k'} + \tilde{\alpha}^+(k')\int_{-\infty}^{\infty} dk\frac{e^{-ik\mathrm{A}}}{k-k'}\right]$$
$$+ \lim_{\mathrm{B}\to-\infty} 2ik'e^{ik'\mathrm{B}}\left[\int_{-\infty}^{\infty} dke^{-ik\mathrm{B}} \frac{\tilde{\alpha}^+(k')\mathrm{C}_{12}^+(k')-\tilde{\alpha}^+(k)\mathrm{C}_{12}^+(k)}{k-k'} - \mathrm{C}_{12}^+(k')\tilde{\alpha}^+(k')\int_{-\infty}^{\infty} dk\frac{e^{-ik\mathrm{B}}}{k-k'}\right]. \quad (5.7)$$

Taking into account that $\alpha^+(k)$ and $\mathrm{C}_{12}^+(k)$ are derivable for $k \in \mathbb{R}^*$, we can apply the Riemann Lebesgue's theorem to the first and the third integrals of (5.7) which vanish.

Because A is a positive number and B a negative number, we have:

$$\int_{-\infty}^{\infty} dk\frac{e^{-ik\mathrm{A}}}{k-k'} = -\pi i e^{-ik'\mathrm{A}}, \int_{-\infty}^{\infty} dk\frac{e^{-ik\mathrm{B}}}{k-k'} = \pi i e^{-ik'\mathrm{B}}, \quad (5.8)$$

and finally:

$$\mathrm{T}_2 = 4\pi k'\tilde{\alpha}^+(k')\mathrm{C}_{12}^+(k'), \quad (5.9)$$

and hence, for $k' \in \mathbb{R}^*$, we have:

$$\int_{-\infty}^{\infty} dx\mathrm{F}_2^+(k',x)^{\mathrm{T}} \int_{-\infty}^{\infty} dk\tilde{\alpha}^+(k)\mathrm{F}_1^+(k,x) = 4\pi k'\tilde{\alpha}^+(k')\mathrm{C}_{12}^+(k'). \quad (5.10)$$

Let us now consider the second term of the right-hand side of (5.3). With the help of (3.59), we find similarly:

$$\int_{-\infty}^{\infty} dx\mathrm{F}_2^+(k',x)^{\mathrm{T}} \int_{-\infty}^{\infty} dk\tilde{\alpha}^-(k)\mathrm{F}_1^-(k,x) = 2\pi k'\tilde{\alpha}^-(-k')[\mathrm{C}_{11}^+(k')+\mathrm{C}_{22}^-(-k')]=0, \quad (5.11)$$

The addition of (5.10) and (5.11) yields:

$$\langle \mathrm{F}_2^+(k',x), \Phi(x) \rangle = 4\pi k'\tilde{\alpha}^+(k')\mathrm{C}_{12}^+(k'), \quad \text{for} \quad k' \in \mathbb{R}^*, \quad (5.12)$$

and hence $\quad \tilde{\alpha}^+(k') = \alpha^+(k')\langle \mathrm{F}_2^+(k',x), \Phi(x) \rangle, \quad k' \in \mathbb{R}, \quad (5.13)$

because $\tilde{\alpha}^+(k')$ and $\alpha^+(k')$ are continuous in $\mathbb{R}$.

Considering the « scalar product » of $\langle \mathrm{F}_2^-(k',x), \Phi(x) \rangle$, we can prove similarly that $\tilde{\alpha}^-(k') = \alpha^-(k')\langle \mathrm{F}_2^-(k',x), \Phi(x) \rangle$, for $k' \in \mathbb{R}$.

To show the uniqueness of the coefficients of $\mathrm{F}_1^+(k_n^+, x)$ and $\mathrm{F}_1^-(k_n^-, x)$

in (1.9), we calculate $\langle F_2^+(k_n^+, x), \Phi(x) \rangle$ by means of (5.2). Because of (3.51), (3.6) and (3.53), we find:

$$\begin{aligned} \langle F_2^+(k_m^+, x), \Phi(x) \rangle = \lim_{\substack{A\to\infty \\ B\to-\infty}} \int_{-\infty}^{\infty} & dk\tilde{\alpha}^+(k)[iC_{11}^+(k_m^+)e^{-i(k+k_m^+)A} \\ & + iC_{22}^+(k)e^{i(k+k_m^+)B} - i\frac{k+k_m^+}{k-k_m^+}C_{21}^+(k)e^{i(k_m^+-k)B} \\ & + R_1^+(k, k_m^+, A) - R_2^+(k, k_m^+, B)] \\ & + \tilde{\beta}^+(k_m^+)\langle F_2^+(k_m^+, x), F_1^+(k_m^+, x)\rangle. \end{aligned} \tag{5.14}$$

It is obvious that the integral is equal to zero and applying (3.5), it follows that

$$\tilde{\beta}_1^+(k_m^+) = \frac{-1}{2ik_m^+\dot{C}_{12}^+(k_m^+)}\langle F_2^+(k_m^+, x), \Phi(x)\rangle, \tag{5.15}$$

and hence

$$\tilde{\beta}^+(k_m^+) = \beta^+(k_m^+)\langle F_2^+(k_m^+, x), \Phi(x)\rangle. \tag{5.16}$$

Lastly, in a similar way, the computation of $\langle F_2^-(k_m^-, x), \Phi(x)\rangle$ by means of (5.2) yields the equality of $\tilde{\beta}^-(k_m^-)$ and $\beta^-(k_m^-)\langle F_2^-(k_m^-, x), \Phi(x)\rangle$.

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