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An alternate constructive approach to the φ_3^4 quantum field theory, and a possible destructive approach to φ_4^4 (*)

by

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ABSTRACT. — I study the construction of φ^4 quantum field theories by means of lattice approximations. It is easy to prove the existence of the continuum limit (by subsequences); the key question is whether this limit is something other than a (generalized) free field. I use correlation inequalities, infrared bounds and field equations to investigate this question. For space-time dimension d less than four, I give a simple proof that the continuum-limit theory is indeed nontrivial; it relies, however, on a conjectured correlation inequality closely related to the Γ_6 conjecture of Glimm and Jaffe. Moreover, the Euclidean invariance of the continuum theory is an open question within the present approach. For space-time dimension d greater than or equal to four, I argue — but do not prove — that the continuum limit is inevitably a (generalized) free field, irrespective of the choice of charge renormalization. The argument is based on old ideas of Landau and Pomeranchuk, improved through the use of correlation inequalities applied to the exact field equations.

RÉSUMÉ. — On étudie la construction des théories φ^4 des champs quantifiés au moyen de l'approximation du réseau. On démontre facilement

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l'existence de la limite du continu (par sous-suites); la question sérieuse est de savoir si cette limite est nécessairement un champ libre (généralisé). Pour examiner cette question on utilise les inégalités de corrélation, les bornes infrarouges, et les équations de champ. Pour la dimension d'espace-temps $d < 4$, on donne une démonstration simple de la non-trivialité de la limite du continu; cette démonstration repose cependant sur une inégalité de corrélation non encore démontrée, celle-ci ayant un rapport étroit avec la conjecture Γ_6 de Glimm et Jaffe. De plus, l'invariance euclidienne de la théorie continue est une question non résolue dans le cadre de la présente méthode. Pour la dimension d'espace-temps $d \geq 4$, on présente des raisonnements qui, sans pour autant aboutir à une démonstration rigoureuse, suggèrent que la limite du continu est inévitablement un champ libre (généralisé), quel que soit le choix de la renormalisation de charge. Le raisonnement se fonde sur des idées anciennes de Landau et Pomeranchuk, précisées ici par l'utilisation des inégalités de corrélation appliquées aux équations de champ exactes.

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1. INTRODUCTION

The proof of the existence and nontriviality of the φ_3^4 quantum field theory [1-5''''] is one of the great triumphs of modern mathematical physics, a *tour de force* of intricate and exceedingly difficult mathematical argument. I should like to suggest, however, that φ_3^4 need not be as complicated as it has heretofore seemed. In this paper, I shall propose an alternate program for the construction of the φ_3^4 theory, one which I believe to be simple and intuitive. However, I should warn at the outset that in this paper I only begin the proposed study; I do not complete it. In particular, much of the present work relies on a conjectured correlation inequality (Conjecture 3.2) which, though very likely true, has not yet been proven. Moreover, the Euclidean invariance of the continuum limit is an open question within the present approach.

The extraordinary complexity of the φ_3^4 theory arises, of course, from its ultraviolet divergences. Unlike the $P(\varphi)_2$ theory, in which all ultraviolet divergences are cancelled by the simple expedient of Wick ordering [6], the φ_3^4 theory requires additional « infinite » mass and vacuum-energy renormalizations. The conventional constructions of the φ_3^4 theory are based on guessing explicitly the required « infinite » counterterms — that is, on guessing explicitly the required dependence of the bare mass and bare vacuum energy as a function of the ultraviolet cutoff — and then on demonstrating explicitly that these counterterms do, in fact, suffice to yield the desired finite (and controllable) result in the limit of infinite cutoff. It is this latter step which is, of course, at the heart of the difficulty: one must explicitly demonstrate complicated cancellations among formally infinite quantities.

What I should like to point out, however, is that there is no real need to know *explicitly* the infinite counterterms: for the construction of the renormalized theory, it suffices to know simply that such counterterms do exist. Furthermore, in the approach proposed here, the vacuum-energy renormalization never arises. Thus, the proof of existence of the continuum (infinite-cutoff) φ_3^4 theory is reduced to the existence of the needed mass renormalization, a question which is already largely settled [7]. Indeed, I shall argue that the proof of existence of the continuum φ^4 theory is in fact a very simple matter [8] in *any* space-time dimension; what is really difficult is not the existence but the nontriviality. That is, the central problem is to show that the continuum φ^4 theory, obtained as a limit of cutoff φ^4 theories, is something other than a (generalized) free field. And I shall argue that even this problem is not really so difficult in space-time dimension $d < 4$.

More specifically, the proposed approach is as follows:

(1) Begin with the φ^4 theory on a finite lattice, and take first the infinite-

volume limit. This procedure [9] has the advantage that translation invariance and other symmetries of the infinite-volume lattice theory can be exploited in the study of the ultraviolet problem; we need not worry about finding the correct renormalization counterterms in *finite* volume ([1]).

(2) Perform the desired mass, field-strength and charge renormalizations [7] [12], and take the lattice spacing to zero. One can always extract at least a convergent subsequence [8]. This limiting theory satisfies all the Osterwalder-Schrader axioms [6] [13] [14] except perhaps Euclidean (rotation) invariance.

(3) The key problem is to show that the continuum-limit theory is non-Gaussian, or equivalently [12b] [15], that the dimensionless renormalized 4-point coupling constants of the lattice theories are bounded away from zero as the continuum limit is approached. To do this — and this is the main new idea of the present paper — I follow the intuition suggested by perturbation theory, namely, that the renormalized coupling constant is given by $g = g_0 + O(g_0^2)$, where g_0 is the bare coupling constant. (Here the leading term is given simply by the tree graph.) If it can be shown that this is really true — that the non-leading contributions are $O(g_0^2)$, *uniformly in the lattice spacing* as it tends to zero — then it will immediately follow that g is nonzero in the continuum limit for sufficiently small but nonzero g_0. To bound the non-leading terms, I exploit the field equations, correlation inequalities and infrared bounds. But the heuristic idea is simply that all radiative corrections, except for the mass renormalization, are ultraviolet convergent in dimension $d < 4$.

In dimension $d \geq 4$, on the other hand, this is not the case. Indeed, the opposite is true: high orders of perturbation theory are *dominant*, even for small bare coupling, if the ultraviolet cutoff is large (i. e. if the lattice spacing is small). This observation can be made the basis of a possible approach to what I call (perhaps whimsically) *destructive quantum field theory* — that is, to proving that the only continuum limits of φ^4 lattice theories are (generalized) free fields. The basic idea was proposed by Landau, Pomeranchuk and collaborateurs [16-23] (see also [24-29]) a quarter-century ago, but it has remained unclear whether their result is merely an artifact of low-order perturbation theory. Indeed, it has become clear that the ultraviolet behavior of non-asymptotically-free theories, such as φ^4 theory in dimension $d \geq 4$, is an extremely difficult strong-coupling problem. The new ingredient proposed here is the use of correlation inequalities, applied to the exact field equations, to show that a huge class of diagrams neglected by Landau *et al.* are totally harmless: if treated exactly,

([1]) On the other hand, the conventional procedure [6] [10] [11] of taking the continuum limit *before* the infinite-volume limit has the advantage that the Euclidean invariance of the final theory is manifest — provided that one has sufficient control over the continuum limit to ensure the Euclidean *covariance* of the continuum theory in finite volume.

they would merely reinforce the Landau result. The arguments given here are far from being a complete rigorous proof, but they do provide a plausible and self-consistent picture of the dynamical mechanism by which φ^4 theory in dimension $d \geq 4$ *probably* becomes free in the continuum limit.

It should by now be clear that many of the ideas in the present paper are not novel; rather, the approach taken here is an eclectic combination of ideas of many previous workers, with perhaps a shift of emphasis. The key new ideas are in step 3 outlined above for φ_3^4, and in the corresponding argument (with the opposite conclusion!) for φ_4^4. Many of the technical estimates are based on ideas of Glimm and Jaffe [*30*], which I clarify and extend. These estimates, contained mostly in Section 2.2 and Appendix A, are likely to have applications in statistical mechanics and quantum field theory beyond those considered here (see e. g. [*31, 31'*]); therefore, they may be of interest even to those readers who remain unconvinced by the proposed approaches to φ_3^4 and φ_4^4.

The plan of this paper is as follows: In Chapter 2, I review the properties of the φ^4 lattice model and show how to take the continuum limit. The heart of the paper is Chapter 3 (and technical Appendix A), where I combine correlation inequalities and infrared bounds with a detailed study of the field equation for the 4-point function, to show that the continuum limit is non-Gaussian for weakly coupled φ_d^4 models ($d < 4$). In Chapter 4, I begin the analogous study for φ_d^4 models in dimension $d \geq 4$, and argue (but do not prove!) that the continuum limit is always Gaussian. I conclude, in Chapter 5, with discussion of some open questions. Appendix A contains a variety of estimates on the 2-point function. Appendix B applies the ideas of [*30*] [*32*] and the present paper to derive bounds on critical exponents for φ^4 lattice models studied from the viewpoint of statistical mechanics. Among the results is that, assuming Conjecture 3.2, hyperscaling fails (as expected) for φ^4 lattice models in dimension $d > 4$.

IMPORTANT NOTE.

After the completion of this work, Michael Aizenman [*229*] [*230*] succeeded in proving, by a beautiful argument quite different from the ones used here, that the continuum limit of φ^4 or Ising models, with arbitrary charge renormalization, is always a (generalized) free field for dimension $d > 4$. (In particular, hyperscaling fails.) Shortly therafter, Jürg Fröhlich [*231*] obtained the same result using the random-walk ideas of [*128-131*] [*232*] (see Section 4.1). Both of these proofs are *ab initio*, i. e. they do not require Conjecture 3.2 or any other unproved result.

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2. LATTICE φ^4 MODEL AND THE CONTINUUM LIMIT

2.1. Review of Renormalization Theory.

The φ_d^4 Euclidean field theory is defined formally by the family of Schwinger functions (complete Euclidean Green's functions)

$$S_n(x_1, \ldots, x_n) = \frac{\int \varphi(x_1) \ldots \varphi(x_n) \exp\left[- \int d^d x \mathcal{H}(\varphi) \right] \prod_x d\varphi(x)}{\int \exp\left[- \int d^d x \mathcal{H}(\varphi) \right] \prod_x d\varphi(x)} \tag{2.1}$$

with the Hamiltonian density

$$\mathcal{H}(\varphi) = \frac{\tilde{A}_0}{2}(\nabla\varphi)^2 + \frac{\tilde{B}_0}{2}\varphi^2 + \frac{\tilde{\lambda}_0}{4!}\varphi^4 . \tag{2.2}$$

Usually $\tilde{A}_0 = 1$ and $\tilde{B}_0 = m_0^2$, but we retain the more general notation for reasons that will become clear shortly. Of course, the functional integral (2.1) fails to make rigorous sense for a number of reasons [*6*] [*33*], but we can expand it formally in perturbation theory (powers of $\tilde{\lambda}_0$) as a source of insight. It then turns out [*34*] that for $d \geq 2$, the Schwinger functions develop infinities associated with the divergence of momentum integrals at large momenta (« ultraviolet divergences »). In order to save the situation, we proceed as follows:

First we modify (« regularize ») the theory so that the momentum integrations are effectively cut off at some large but finite momentum Λ; then we attempt to choose the « bare parameters » $\tilde{A}_0$, $\tilde{B}_0$ and $\tilde{\lambda}_0$ as functions of the ultraviolet cutoff Λ in such a way that the cutoff Schwinger functions converge to finite limits as $\Lambda \to \infty$. If this can be arranged, we can then study the limiting Schwinger functions in order to verify the Osterwalder-Schrader axioms [*6*] [*13*] [*14*] and other physically desirable properties. The key question of renormalization theory is: What are the possible limiting theories (if any) as $\Lambda \to \infty$ for the various choices of $\tilde{A}_0(\Lambda)$, $\tilde{B}_0(\Lambda)$ and $\tilde{\lambda}_0(\Lambda)$?

A variety of regularization procedures are available [35]; in this paper, we use (for reasons that will become clear shortly) a spatial lattice of spacing a, so that $\Lambda \approx a^{-1}$. That is, we replace the Euclidean space $\mathbb{R}^d$ by the simple (hyper-) cubic lattice $a\,\mathbb{Z}^d$, and replace (2.1)-(2.2) by a lattice approximation:

$$S_n^{(a)}(x_1, \ldots, x_n) = \frac{\int \varphi_{i_1} \cdots \varphi_{i_n} \exp\left[-\sum_i \left(\frac{B_0}{2}\varphi_i^2 + \frac{\lambda_0}{4!}\varphi_i^4\right) + \sum_{|i-j|=1} \frac{J}{2}\varphi_i\varphi_j\right] \prod_i d\varphi_i}{\int \exp\left[-\sum_i \left(\frac{B_0}{2}\varphi_i^2 + \frac{\lambda_0}{4!}\varphi_i^4\right) + \sum_{|i-j|=1} \frac{J}{2}\varphi_i\varphi_j\right] \prod_i d\varphi_i} \tag{2.3}$$

where the letters i, j, etc. denote points in $\mathbb{Z}^d$, and $x_1 = ai_1$, etc. The new bare parameters are given by

$$B_0 = \tilde{B}_0 a^d + 2d\tilde{A}_0 a^{d-2} \tag{2.4}$$

$$\lambda_0 = \tilde{\lambda}_0 a^d \tag{2.5}$$

$$J = \tilde{A}_0 a^{d-2}\,. \tag{2.6}$$

The point of this exercise is that (2.3) defines a ferromagnetic Ising model with nearest-neighbor interaction and single-spin measure

$$d\nu(\varphi) = \exp\left[-\frac{B_0}{2}\varphi^2 - \frac{\lambda_0}{4!}\varphi^4\right]d\varphi\,; \tag{2.7}$$

much is known rigorously about these models. Of course, (2.3) is still, as it stands, a formal expression, because of the infinite-volume sums; but unlike (2.1), it is not hard to make rigorous sense of (2.3) [see Section 2.2].

A HEURISTIC ASIDE.

The reader may well inquire as to the dimensions, in the sense of dimensional analysis, of the quantities introduced above. The assignment of dimensions contains considerable arbitrariness: for example, classical physics introduces mass, length and time as independent fundamental quantities, while the modern field-theoretic convention is to define $\hbar$ and c as dimensionless (and equal to unity!) and thereby to reduce the three fundamental quantities to one. Dimensional analysis should be viewed as a convenient way of summarizing the transformation properties of various quantities under one or more scalings; thus, the assignment of dimensions should depend on which scalings are of interest. We shall be interested in two types of scaling, namely $\varphi = \alpha\varphi'$ and $x = \beta x'$; hence, we introduce two fundamental dimensions, field strength (F) and length (L). Thus, by definition,

$$[\varphi(x)] = [\varphi_i] = \mathrm{F} \tag{2.8}$$

$$\left[\int d^dx \mathcal{H}(\varphi)\right] = \text{dimensionless}\,.$$

It follows that

$$\begin{array}{ll} [\tilde{\mathrm{B}}_0] = \mathrm{F}^{-2}\mathrm{L}^{-d} & [\mathrm{B}_0] = \mathrm{F}^{-2} \\ [\tilde{\mathrm{A}}_0] = \mathrm{F}^{-2}\mathrm{L}^{2-d} & [\lambda_0] = \mathrm{F}^{-4} \\ [\tilde{\lambda}_0] = \mathrm{F}^{-4}\mathrm{L}^{-d} & [\mathrm{J}] = \mathrm{F}^{-2} \\ [a] = \mathrm{L} & \end{array} \tag{2.9}$$

However, these dimension assignments should be considered as merely a heuristic shorthand for the rigorous scaling laws.

To derive such a scaling law, let us consider the change of variables $\varphi_i = \alpha\varphi'_i$ in (2.3). Then a trivial calculation shows that

$$\mathrm{S}_n^{(a)}(x_1, \ldots, x_n;\, \mathrm{B}_0, \lambda_0, \mathrm{J}) = \alpha^n \mathrm{S}_n^{(a)}(x_1, \ldots, x_n;\, \alpha^2\mathrm{B}_0, \alpha^4\lambda_0, \alpha^2\mathrm{J}) \tag{2.10}$$

for any $\alpha > 0$. Thus, of the three parameters in (2.3), only two are really significant; the third leads to a mere change of scale of the field. Only quantities invariant under the rescaling (2.10) can have physical significance. Thus we define the bare coupling in the *statistical-mechanical* normalization convention

$$\lambda_0^{\mathrm{SM}} = \lambda_0 \mathrm{J}^{-2} \tag{2.11}$$

and the bare coupling in the *field-theoretic* normalization convention

$$\lambda_0^{\mathrm{FT}} = \tilde{\lambda}_0 \tilde{\mathrm{A}}_0^{-2}\,. \tag{2.12}$$

They are related by

$$\lambda_0^{\mathrm{SM}} = \lambda_0^{\mathrm{FT}} a^{4-d}, \tag{2.13}$$

and have the dimensions

$$[\lambda_0^{\mathrm{SM}}] = \text{dimensionless} \qquad [\lambda_0^{\mathrm{FT}}] = \mathrm{L}^{d-4}\,. \tag{2.14}$$

Both are free of dimensions F, by construction.

Similarly we would like to investigate the consequences of a change of variable $x = \beta x'$. Since this is slightly subtle, we postpone it to the end of this section. Suffice it to say here that only those quantities invariant under *both* the field rescaling $\varphi = \alpha\varphi'$ and the length rescaling $x = \beta x'$ can have physical significance *in the continuum limit*. We reserve the term *dimensionless* for such quantities.

The purpose of explaining these matters in such pedantic detail is to help alleviate the widespread confusion caused by the existence of the two different normalization conventions. As practice in learning to translate between the two languages, let us consider two commonly-used ways of taking the continuum limit ($a \to 0$):

CASE 1 : λ_0^{FT} *fixed.*

This is the standard procedure in super-renormalizable field theory ($d < 4$). From the point of view of field theory, no nontrivial coupling-constant renormalization is being performed; and none need be performed, since the ultraviolet divergences are not so severe in $d < 4$. From the point of view of statistical mechanics, the theory is becoming extremely *weakly* coupled ($\lambda_0^{\mathrm{SM}} \sim a^{4-d}$); but the effects of this coupling are amplified by the *infrared* divergences of the φ^4 lattice theory near the critical point in $d < 4$ [*34*], leading to a non-Gaussian limit. (As will be seen in Section 2.3, the mass renormalization will entail a choice of lattice theories with correlation lengths $\xi \sim a^{-1}$; thus the critical point is inevitably approached as $a \to 0$. See e. g. [*9*].)

CASE 2 : λ_0 *fixed* (λ_0^{SM} *fixed or almost fixed*).

This is the standard procedure in the statistical-mechanical theory of critical phenomena: after all, here the interactions *are* fixed — by Nature! Typically we fix λ_0 and B_0 and increase J towards its critical value J_c; as $J \to J_c$, λ_0^{SM} approaches a constant, so it is for all practical purposes fixed. (Alternatively we could fix λ_0 and J and decrease B_0 towards its critical value B_{0c}; then λ_0^{SM} is strictly fixed.) In this theory it is for $d > 4$ that matters are simple: correlation functions are given by a perturbation expansion around mean field theory that is free of infrared divergences [*34*]. From the point of view of field theory this might seem surprising, since the φ_d^4 field theory for $d > 4$ is non-renormalizable, with horrendous *ultraviolet* divergences; but this is mitigated by the fact that the theory is becoming *weakly* coupled ($\lambda_0^{\mathrm{FT}} \sim a^{d-4}$). On the other hand, for $d < 4$ the theory of critical phenomena is quite complicated, by virtue of *infrared* divergences; this is reflected in field theory in the fact that the theory is becoming extremely *strongly* coupled ($\lambda_0^{\mathrm{FT}} \sim a^{d-4}$), causing perturbation theory to break down even through the internal momentum integrations are ultraviolet convergent. (Such « super-strongly-coupled » field theories have been studied by Wilson [*36*].)

Clearly it is the factor a^{4-d} in (2.13) which accounts for the interplay between ultraviolet and infrared as one translates from one viewpoint to the other. In the remainder of this paper, I shall often appeal to the field-theoretic viewpoint for heuristic and motivational purposes, but shall adhere to the statistical-mechanical normalization convention for all rigorous arguments. (This choice is simply a matter of taste.) In any case, I shall always attempt to make clear which viewpoint I am using!

Let us now return to consider what can be learned from the change of variables $x = \beta x'$. It is most natural to make this change of variables in the continuum expression (2.1); a *formal* computation yields the putative relation

$$S_n(x_1, \ldots, x_n; \tilde{A}_0, \tilde{B}_0, \tilde{\lambda}_0) = S_n(\beta^{-1}x_1, \ldots, \beta^{-1}x_n; \beta^{d-2}\tilde{A}_0, \beta^d\tilde{B}_0, \beta^d\tilde{\lambda}_0). \qquad (2.15)$$

However, this relation is valid only in those cases in which (2.1) makes sense without cutoffs, i. e. $d=1$. Otherwise this relation is meaningless. The correct definition of (2.1) requires the introduction of an additional dimensionful parameter, the ultraviolet cutoff Λ; the correct scaling law is

$$S_n(x_1, \ldots, x_n; \tilde{A}_0, \tilde{B}_0, \tilde{\lambda}_0, \Lambda) = S_n(\beta^{-1}x_1, \ldots, \beta^{-1}x_n; \beta^{d-2}\tilde{A}_0, \beta^d\tilde{B}_0, \beta^d\tilde{\lambda}_0, \beta\Lambda). \qquad (2.16)$$

However, this relation is totally uninteresting: using the lattice cutoff, for example, and recalling (2.4)-(2.6) and $\Lambda \sim a^{-1}$, we find that the left and right sides of (2.16) refer to the *same* lattice theory (B_0, λ_0, J) evaluated at the *same* lattice sites $i_1, \ldots, i_n$! Equality *here* is hardly very informative.

Since (2.16), unlike (2.10), places no constraints on the lattice-model correlation functions, the lattice quantities need not be invariant under the rescaling $x = \beta x'$ in order to have physical significance. The rescaling $x = \beta x'$ is of interest only in the continuum limit; and even there it is problematic when applied to the bare (unrenormalized) parameters, as the discussion of (2.15) shows. For a (presumed) scaling relation with respect to the *renormalized* parameters, see (2.70)-(2.71).

2.2. Properties of the φ^4 Lattice Model.

We now review the properties of the model (2.3) studied from the viewpoint of statistical mechanics. Here B_0, λ_0 and J are arbitrary parameters (with, of course, $\lambda_0 > 0$ and $J \geq 0$); we forget about (2.4)-(2.6). Thus the lattice spacing a never appears; we might as well imagine $a = 1$.

To define the model (2.3), we consider it first in finite volume $V \subset \mathbb{Z}^d$ with zero boundary conditions (2):

$$\langle \varphi_{i_1} \ldots \varphi_{i_n} \rangle_V = \frac{\displaystyle\int \varphi_{i_1} \ldots \varphi_{i_n} \exp\left[- \sum_{i \in V} \left(\frac{B_0}{2}\varphi_i^2 + \frac{\lambda_0}{4!}\varphi_i^4 \right) + \sum_{\substack{|i-j|=1 \\ i,j \in V}} \frac{J}{2}\varphi_i\varphi_j \right] \prod_{i \in V} d\varphi_i}{\displaystyle\int \exp\left[- \sum_{i \in V} \left(\frac{B_0}{2}\varphi_i^2 + \frac{\lambda_0}{4!}\varphi_i^4 \right) + \sum_{\substack{|i-j|=1 \\ i,j \in V}} \frac{J}{2}\varphi_i\varphi_j \right] \prod_{i \in V} d\varphi_i} \qquad (2.17)$$

(2) In the statistical-mechanics literature, these are often called *free* b. c.; in the field-theory literature, these are called *Dirichlet* (or *half-Dirichlet*) b. c. See the warning in [*10*, p. 206].

if all $i_1, \ldots, i_n \in V$, $\langle \varphi_{i_1} \ldots \varphi_{i_n} \rangle_V = 0$ otherwise. (We have shifted for convenience to the statistical-mechanics notation.) The infinite-volume limit $V \uparrow \mathbb{Z}^d$ can now be taken, using Griffiths' inequalities [*37*] [*38*]; the correlation functions $\langle \varphi_{i_1} \ldots \varphi_{i_n} \rangle_V$ increase monotonically to a limit $\langle \varphi_{i_1} \ldots \varphi_{i_n} \rangle$ which has the following properties [*38*]:

a) There exists a probability measure μ on $\mathbb{R}^{\mathbb{Z}^d}$ such that

$$\langle \varphi_{i_1} \ldots \varphi_{i_n} \rangle = \int \varphi_{i_1} \ldots \varphi_{i_n} d\mu(\varphi) ; \tag{2.18}$$

b) μ is invariant under all symmetries of the lattice $\mathbb{Z}^d$ (lattice translations, lattice rotations, and lattice reflections) and under the transformation $\varphi \to -\varphi$;

c) μ satisfies the Dobrushin-Lanford-Ruelle (DLR) equations [*38-43*] and the superstability bounds [*43*] [*44*] [*38*].

Moreover, μ satisfies the following correlation inequalities (among others):

d) Griffiths' first and second inequalities [*45*] :

$$\langle \varphi^A \rangle \geq 0 \tag{2.19}$$

$$\langle \varphi^A ; \varphi^B \rangle \equiv \langle \varphi^A \varphi^B \rangle - \langle \varphi^A \rangle \langle \varphi^B \rangle \geq 0 . \tag{2.20}$$

Here $A = (A_i)_{i \in \mathbb{Z}^d}$ is a multi-index, and

$$\varphi^A \equiv \prod_{i \in \mathbb{Z}^d} \varphi_i^{A_i} . \tag{2.21}$$

e) Ginibre inequalities [*45*]:

$$\langle q^A t^B \rangle \geq 0 . \tag{2.22}$$

Here $q_i = (\varphi_i - \varphi_i')/\sqrt{2}$, $t_i = (\varphi_i + \varphi_i')/\sqrt{2}$, where $\{ \varphi_i \}$ and $\{ \varphi_i' \}$ are independent sets each distributed according to the measure μ.

f) Lebowitz inequality for the 4-point function [*45*]:

$$\langle \varphi_i \varphi_j \varphi_k \varphi_l \rangle - \langle \varphi_i \varphi_j \rangle \langle \varphi_k \varphi_l \rangle - \langle \varphi_i \varphi_k \rangle \langle \varphi_j \varphi_l \rangle - \langle \varphi_i \varphi_l \rangle \langle \varphi_j \varphi_k \rangle \leq 0 . \tag{2.23}$$

g) Gaussian inequality [*46-48*]:

$$\langle \varphi_{i_1} \ldots \varphi_{i_{2n}} \rangle \leq \prod_{\mathscr{P}} \prod_{\{\alpha,\beta\} \in \mathscr{P}} \langle \varphi_{i_\alpha} \varphi_{i_\beta} \rangle , \tag{2.24}$$

where $\mathscr{P}$ is the set of all partitions of $\{ 1, \ldots, 2n \}$ into pairs. (Of course, the Lebowitz inequality (2.23) is actually a special case of this.)

h) FKG inequality [*6*] [*49*]:

$$\langle F(\varphi_{i_1}, \ldots, \varphi_{i_n}) ; G(\varphi_{i_1}, \ldots, \varphi_{i_n}) \rangle \geq 0 \tag{2.25}$$

for any *increasing* functions F, G on $\mathbb{R}^n$.

i) Newman's Lee-Yang inequalities [*15*]: Let X be any finite sum $\Sigma \alpha_i \varphi_i$ with all $\alpha_i \geq 0$, and define the cumulants u_n by

$$\langle \exp(zX) \rangle = \exp\left(\sum_{n=1}^{\infty} \frac{u_n}{n!} z^n \right). \tag{2.26}$$

Then $u_{2m-1} = 0$ and $(-1)^{m-1} u_{2m} \geq 0$; moreover, if any $u_{2m} = 0$, then in fact $u_n = 0$ for *all* $n > 2$ and X is Gaussian. (A quantitative version of this result is given in [*15*, Theorem 6].) If X is Gaussian for all such $\{ \alpha_i \}$, then μ is Gaussian.

j) Newman's exponential bound [*15*] [*50*]:

$$\left| \left\langle \exp\left(\sum_i z_i \varphi_i \right) \right\rangle \right| \leq \exp\left[\frac{1}{2} \sum_{i,j} |\operatorname{Re} z_i| \, |\operatorname{Re} z_j| \langle \varphi_i \varphi_j \rangle \right] \tag{2.27}$$

for any complex $\{ z_i \}$.

k) Schrader — Messager — Miracle-Sole inequalities [*51-54*]: The two-point function

$$G(x) = \langle \varphi_0 \varphi_x \rangle \tag{2.28}$$

is decreasing in each $| x_i |$; in particular,

$$G(x_1, \ldots, x_d) \leq G(a, 0, \ldots, 0) \tag{2.29}$$

whenever $| a | \leq | x |_\infty \equiv \max\limits_{1 \leq i \leq d} | x_i |$. A diagonal version of this inequality says (among other things) that

$$G(a, 0, \ldots, 0) \leq G(y_1, \ldots, y_d) \tag{2.30}$$

whenever $| a | \geq | y |_1 \equiv \sum_{i=1}^{d} | y_i |$.

Finally :

(*l*) μ is reflection-positive (RP) [*55*] with respect to coordinate hyperplanes through sites, coordinate hyperplanes bisecting bonds, and diagonal hyperplanes through sites. It follows [*30*] [*56*] [*57*] that the Fourier-transformed 2-point function

$$\tilde{G}(p) = \sum_x e^{-ip \cdot x} G(x) \tag{2.31}$$

satisfies the spectral representation discussed below, and the infrared bound [*58*] [*55*] [*59*]

$$0 \leq \tilde{G}(p) \leq c\delta(p) + \left[2J \sum_{i=1}^{d} (1 - \cos p_i) \right]^{-1}. \tag{2.23}$$

(For the case we consider, $c = 0$.)

Remark. — It is not entirely trivial to prove *both* of the Schrader — Messager — Miracle-Sole inequalities [(2.29) and (2.30)] for the same infinite-volume state. (2.29) holds initially for certain finite-volume *rectangular* boxes, while (2.30) holds for certain *diamond-shaped* boxes; it is not obvious how to arrange for *both* of them to hold in the infinite-volume limit. We are saved by the Griffiths inequalities, which ensure [*37*] [*38*] that the *same* infinite-volume state μ is obtained irrespective of the sequence of finite-volume boxes employed; thus both (2.29) and (2.30) do hold in the infinite-volume limit. The same remark applies to the simultaneous attainment of the three types of reflection positivity.

We now recall some properties of the 2-point function G. Of course, $G(x) \geq 0$ by the Griffiths inequality. Moreover, we shall be interested in the regime in which $G(x)$ decays exponentially as $|x| \to \infty$ (perhaps at a very slow rate). Hence we define the *susceptibility*

$$\chi = \sum_x G(x) \tag{2.33}$$

and, for each $\phi > 0$, the *correlation length of order* ϕ

$$\xi_\phi = \left(\chi^{-1} \sum_x |x|^\phi G(x)\right)^{1/\phi}. \tag{2.34}$$

(Note that, by Hölder's inequality, ξ_ϕ is increasing in ϕ.) We also define the *exponential* (or « true ») *correlation length*

$$\xi = \limsup_{x_1 \to \infty} \left(- x_1/\log G(x_1, 0, \ldots, 0)\right) \tag{2.35}$$

and the *mass gap*

$$m = \xi^{-1}. \tag{2.36}$$

(Actually, the lim sup is unnecessary; it follows from the spectral representation (2.37) that the limit (2.35) exists.)

Remark. — By the FKG inequalities, all correlation functions cluster at a rate at least m [*60*] [*31, 61*].

From now on, we assume for simplicity that $m > 0$; this will hold for all cases we need to consider. It then follows from reflection positivity (with respect to *both* coordinate hyperplanes through sites and those bisecting bonds) that G has the spectral representation [*30*] [*56*] [*57*]

$$\tilde{G}(p_1, \mathbf{p}) = c_{\mathbf{p}} + \int_{\cosh m - 1}^{\infty} \frac{d\rho_{\mathbf{p}}(a)}{1 - \cos p_1 + a} \tag{2.37}$$

for some $c_{\mathbf{p}} \geq 0$ and some measure $d\rho_{\mathbf{p}}(a) \geq 0$. [Here we have written $p = (p_1, \mathbf{p})$.] The *field strength renormalization constant* Z is then defined as the strength of the « one-particle pole » at zero spatial momentum, i. e.

$$Z = \rho_0(\{ \cosh m - 1 \}). \tag{2.38}$$

Clearly $Z \geq 0$.

Remarks. 1. — The constant c_p corresponds to the term $a = \infty$ in $d\rho_p(a)$. It is generally believed that $c_p = 0$ (except in the case $J = 0$ of uncoupled sites), but I do not know of any proof. In any case, the c_p term will have no effect on our results.

2. A representation analogous to (2.37) can be derived using *diagonal* reflection positivity [57]; we shall not need it in this paper. But see Remark 4 following Proposition A.4.

We are now prepared to prove a number of fairly easy bounds which we shall need later. The first is

$$\frac{Z}{\cosh m - 1} \leq \chi \leq (4J)^{-1} \frac{\cosh m + 1}{\cosh m - 1}, \tag{2.39}$$

which for $m \ll 1$ (i. e. $\xi \gg 1$) reduces simply to

$$\frac{2Z}{m^2} \lesssim \chi \lesssim \frac{1}{Jm^2}. \tag{2.40}$$

The lower bound in (2.39) is an immediate consequence of the spectral representation (2.37) and the definitions of Z and $\chi = \tilde{G}(0)$. To deduce the upper bound in (2.39), combine the infrared bound (2.32) [here $c=0$] with the spectral representation (2.37), and evaluate at $\boldsymbol{p} = 0$, $p_1 = \pi$:

$$2c_0 + \int_{\cosh m - 1}^{\infty} \frac{2}{2 + a} d\rho_0(a) \leq (2J)^{-1}. \tag{2.41}$$

Since $a \geq \cosh m - 1$, we have

$$\frac{1}{a} \leq \frac{1}{2} \left(\frac{\cosh m + 1}{\cosh m + 1} \right) \frac{2}{2 + a}. \tag{2.42}$$

Hence

$$\begin{aligned} \chi = \tilde{G}(0) &\leq c_0 + \frac{1}{2} \left(\frac{\cosh m + 1}{\cosh m - 1} \right) [(2J)^{-1} - 2c_0] \\ &\leq (4J)^{-1} \frac{\cosh m + 1}{\cosh m - 1} \end{aligned} \tag{2.43}$$

since $c_0 \geq 0$. This proves (2.39).

Next we prove an inequality relating the correlation lengths ξ and ξ_ϕ:

$$\text{const} \times (ZJ)^{1/\phi} \xi \leq \xi_\phi \leq \text{const} \times \xi, \tag{2.44}$$

where the constants depend on ϕ and d but on nothing else, and where the lower bound in (2.44) is actually valid only for ξ not near zero (say $\xi \geq 1$).

The proof of (2.44) is easy in concept but disgusting in details, most of which I therefore omit. To simplify matters, define first

$$\chi_\phi = \sum_x |x|^\phi G(x) \tag{2.45}$$

and

$$\hat{\chi}_\phi = \sum_x |x_1|^\phi G(x) \tag{2.46}$$

for each $\phi \geq 0$. Thus $\chi = \chi_0$ and $\xi_\phi = (\chi_\phi/\chi_0)^{1/\phi}$. Moreover,

$$\hat{\chi}_\phi \leq \chi_\phi \leq d^{\phi/2}\hat{\chi}_\phi,$$

so we might as well work with $\hat{\chi}_\phi$. By Fourier transforming (2.37), we find that

$$\hat{\chi}_\phi = \sum_{x_1} \int_{\cosh m - 1}^{\infty} |x_1|^\phi \frac{e^{-\mu|x_1|}}{\sinh \mu} d\rho_0(a), \tag{2.48}$$

where $\mu = \cosh^{-1}(1 + a)$. (For $\phi = 0$, the term c_0 in (2.37) also contributes. We neglect c_0 everywhere, since it is easy to see that it (which corresponds to $a = \infty$) can't do any harm that contributions in $d\rho_0(a)$ from large but finite a couldn't do.) Now

$$F_\phi(\mu) \equiv \sum_{x_1} |x_1|^\phi \frac{e^{-\mu|x_1|}}{\sinh \mu} \sim \begin{Bmatrix} (1 + \mu^{-(2+\phi)})e^{-2\mu} & \text{if} \quad \phi > 0 \\ (1 + \mu^{-2})e^{-\mu} & \text{if} \quad \phi = 0 \end{Bmatrix}, \tag{2.49}$$

where by $\sim$ we mean that this is the correct asymptotic behavior for both $\mu \to 0$ and $\mu \to \infty$, hence that the stated expression multiplied by suitable constants can be both an upper and a lower bound for $F_\phi(\mu)$ uniformly over the entire range $0 < \mu < \infty$. Substituting (2.49) into (2.48), we find immediately that

$$\frac{\hat{\chi}_\phi}{\hat{\chi}_0} \leq \text{const} \times m^{-\phi}\,; \tag{2.50}$$

this gives the upper bound in (2.44). Making the same substitution and discarding all but the $a = \cosh m - 1$ contribution in (2.48), we find

$$\hat{\chi}_\phi \geq \text{const} \times Z(1 + m^{-(2+\phi)})e^{-2m}. \tag{2.51}$$

Inserting the upper bound of (2.39), we find, after a little algebra, an ugly expression which reduces to the lower bound in (2.44) if we assume that ξ is not near zero.

Finally we prove the inequality

$$\chi \leq \text{const} \times J^{-1}(1 + \xi_\phi^2). \tag{2.52}$$

In view of (2.44), which says that ξ_ϕ isn't much bigger than ξ but could be much smaller, this is an improvement of the upper bound in (2.39)-(2.40).

To prove (2.52), we use (2.48) and (2.49) along with (2.41):

$$\begin{aligned}\chi = \hat{\chi}_0 &\sim \int (1 + \mu^{-2}) e^{-\mu} d\rho_0(a) \\ &\leq \left(\int (1 + \mu^{-2})^{\frac{2+\phi}{2}} e^{-\mu} d\rho_0(a) \right)^{\frac{2}{2+\phi}} \left(\int e^{-\mu} d\rho_0(a) \right)^{\frac{\phi}{2+\phi}} \\ &\leq \text{const} \times (\chi + \hat{\chi}_\phi)^{\frac{2}{2+\phi}} J^{-\frac{\phi}{2+\phi}}, \end{aligned} \tag{2.53}$$

where the key step used Hölder's inequality. A little juggling yields (2.52). This completes the list of ugly but necessary inequalities.

Remarks. 1. — The upper bound in (2.44) has been used by the present author [*61*] as a technical device in the proof of the low-temperature Josephson inequality for critical exponents. But this is probably due more to my lack of insight into the Josephson inequality than to any profundity inherent in (2.44).

2. By (2.39) we have $0 \leq ZJ \leq \text{const}$ for ξ not near zero; this is the analogue of the Källén-Lehmann bound $0 \leq Z \leq 1$ in quantum field theory. What is more difficult is to show that ZJ is bounded away from zero; indeed, this is a detailed dynamical issue, and is presumably *not* true in general. However, it is true in the *superrenormalizable* case, as we show (modulo a conjectured correlation inequality) in Section 3.3.

3. Simon [*62*] [*63*] has shown, using a new correlation inequality, that $Z \geq \sinh m$ in the spin $-\frac{1}{2}$ Ising model. A simple modification of his proof, using [*63*, Theorem 3.1], shows that $ZJ \geq \text{const} \times m$ for all models satisfying the Lebowitz inequality. However, this bound, which vanishes as the critical point is approached, is too weak for our purposes.

It is useful also to introduce the *inverse propagator*

$$\tilde{\Gamma}(p) = 1/\tilde{G}(p). \tag{2.54}$$

(Note that our definition of Γ is the negative of that used in [*30, 56, 64, 65*].) Γ has the spectral representation [*30, 56, 64, 65*]

$$\tilde{\Gamma}(p_1, \mathbf{p}) = \alpha_{\mathbf{p}}(1 - \cos p_1) + \beta_{\mathbf{p}} - \int_{\cosh m - 1}^{\infty} \left[\frac{1}{1 - \cos p_1 + a} - \frac{a}{a^2 + 1} \right] d\nu_{\mathbf{p}}(a), \tag{2.55}$$

with $\alpha_{\mathbf{p}} \geq 0$, $\beta_{\mathbf{p}}$ real, and $d\nu_{\mathbf{p}}(a) \geq 0$. Moreover,

$$Z^{-1} = \frac{\partial}{\partial(1 - \cos p_1)} \tilde{\Gamma}(p_1, \mathbf{p}) \bigg|_{p_1 = im,\, \mathbf{p} = 0} \tag{2.56}$$

$$= \alpha_0 + \int_{\cosh m - 1}^{\infty} \frac{d\nu_0(a)}{(1 - \cosh m + a)^2} \tag{2.57}$$

These formulas will be important in Section 3.3.

Finally, we study the mass renormalizability of the φ_d^4 lattice theory:

PROPOSITION 2.1. — Fix $\lambda_0 \geq 0$ and $J > 0$. Then there exists B_{0c} (if $d \geq 2$, then $B_{0c} > -\infty$) such that ξ is a continuous decreasing function of B_0 [cf. (2.3)] in the interval (B_{0c}, ∞), with $0 < \xi < \infty$ and

$$\lim_{B_0 \downarrow B_{0c}} \xi(B_0) = \infty \tag{2.58}$$

$$\lim_{B_0 \uparrow \infty} \xi(B_0) = 0 . \tag{2.59}$$

Moreover, for each $\phi > 0$, ξ_ϕ is a continuous function of B_0 in the same interval, with $0 < \xi_\phi < \infty$ and

$$\lim_{B_0 \downarrow B_{0c}} \xi_\phi(B_0) = \infty \tag{2.60}$$

$$\lim_{B_0 \uparrow \infty} \xi_\phi(B_0) = 0 . \tag{2.61}$$

In particular, ξ and ξ_ϕ assume each positive value in the region $B_{0c} < B_0 < \infty$.

Proof. — The hard part of this proposition (that dealing with ξ) has already been proven by Rosen [7] (see also [*66*] [*62*] [*63*] [*67*]). To prove the other part, recall first the definition (2.45) of χ_ϕ. Clearly $0 < \chi$, $\chi_\phi < \infty$ for $B_{0c} < B_0 < \infty$, and χ and χ_ϕ are decreasing functions of B_0 there. Then we have *formally*

$$0 \leq -\frac{\partial \chi_\phi}{\partial B_0} = \frac{1}{2} \sum_{x,y} | x |^\phi \langle \varphi_0 \varphi_x \, ; \varphi_y^2 \rangle \tag{2.62}$$

$$\leq \sum_{x,y} | x |^\phi \langle \varphi_0 \varphi_y \rangle \langle \varphi_x \varphi_y \rangle$$

$$\leq \text{const}\,(\phi) \sum_{x,y} (| y |^\phi + | x - y |^\phi) G(y) G(x-y)$$

$$= \text{const}\,(\phi) \chi_\phi \chi , \tag{2.63}$$

where the lower bound is Griffiths' inequality (2.20) and the upper bound is Lebowitz' inequality (2.23). It follows that $\partial \xi_\phi / \partial B_0$ obeys a universal bound in terms of χ and χ_ϕ, so that, in particular, ξ_ϕ is continuous. This argument is not yet a rigorous proof, because the equality in (2.62) [a « fluctuation-dissipation relation »] is as yet only formal. However, it can be proved rigorously by the methods of Bricmont and the author [*61*, Appendix]: the idea is to begin with (2.62) in finite volume (where it is trivial to make rigorous), and then use the dominated convergence theorem to pass to the infinite-volume limit. The estimate which makes this possible is precisely (2.63). [*See Note Added in Proof.*]

(2.61) follows immediately from (2.59) and (2.44). (2.60) follows from (2.52) and the fact that $\chi \to \infty$ as $B_0 \downarrow B_{0c}$ [*62, 63, 68*] (if χ didn't approach ∞ as $B_0 \downarrow B_{0c}$, then the theory at B_{0c} would have a nonzero mass gap [*62*] [*63*], contradicting (2.58) and the monotonicity of ξ). ■

Remarks. — 1. There are probably cleaner proofs of (2.60).

2. Virtually identical arguments show that we could alternatively fix λ_0 and B_0 and increase J toward a critical value J_c, obtaining identical control over ξ and ξ_ϕ.

3. *Presumably* ξ_ϕ is, like ξ, a *decreasing* function of B_0, but this has not been proven. It is a very interesting question; its proof would presumably involve a new family of correlation inequalities which, unlike previous ones, would explicitly involve the geometric structure of the lattice. In any case we do not need here the monotonicity of ξ_ϕ; if ξ_ϕ is non-monotonic, then the mass renormalization is non-unique, but that is of no importance.

4. The fact that $\xi < \infty$ for the zero-boundary-condition infinite-volume state (or more generally that $G(x) \leq O(|x|^{-(d-1+\varepsilon)})$ as $|x| \to \infty$, which entails exponential decay anyway [*62*] [*63*]) implies that this state is the *unique* regular Gibbs state [*69*]. The proof uses a result of Lebowitz and Martin-Löf [*70*] [*44*] together with the GHS inequalities [*45*].

2.3. The Continuum Limit.

It is now easy to take the continuum limit. The idea is simple: we consider a sequence of φ_d^4 lattice models with correlation lengths ξ_ϕ going to infinity, and rescale them to lattice spacings $a \sim \xi_\phi^{-1}$. Thus the rescaled field has correlation length O(1). Moreover, we rescale the magnitude of the field so as to fulfill a second normalization condition [say, that the rescaled susceptibility is also O(1)]. (By (2.10), this latter rescaling can always be accomplished by a suitable choice of parameters in the lattice model.) These two normalization conditions on the 2-point function imply that *all* of the correlation functions converge for a suitable subsequence [*8*] [*12*] [*15*] [*50*], and that the limiting 2-point function function is not identically zero. Moreover, the limiting Schwinger functions are easily seen to obey all Osterwalder-Schrader axioms except perhaps Euclidean (rotation) invariance. Thus, the only remaining question is whether the continuum-limit theory is non-trivial, i. e. whether the connected 4-point and higher-point functions are nonzero. This depends [*15*] on the vanishing or non-vanishing of a single quantity $\tilde{g}$, the dimensionless renormalized 4-point coupling constant [see (2.67) and (2.77)].

More precisely, we seek to normalize the continuum theory according to the zero-momentum (« intermediate ») normalization

$$\tilde{S}_2(p = 0) = \tilde{z}\tilde{m}^{-2} \tag{2.64}$$

$$\sum_{i=1}^{d} \frac{\partial^2}{\partial p_i^2} \tilde{S}_2(p = 0) = \tilde{z}\tilde{m}^{-4} \tag{2.65}$$

$$\tilde{S}_4^T(p = 0) = - \tilde{z}^2\tilde{m}^{-8}\tilde{\lambda} \tag{2.66}$$

Here $\tilde{z}$ and $\tilde{m}$ are arbitrary scale factors which can be adjusted at will by the trivial rescalings $\varphi \to \alpha\varphi$ and $x \to \beta x$; thus there would be no loss in taking $\tilde{z} = \tilde{m} = 1$, although we shall not do so. (The usual field-theoretic convention [*34*] [*71*] does take $\tilde{z} = 1$.) But this exhausts the trivial rescalings; the *dimensionless renormalized 4-point coupling constant of the continuum theory*

$$\tilde{g} = \tilde{m}^{d-4}\tilde{\lambda} \tag{2.67}$$

is invariant under these rescalings.

Remarks. — 1. The definitions (2.64)-(2.67) differ by a constant factor from the standard ones [*34*] [*71*]

$$\tilde{m}_{\text{STD}} = (2d)^{1/2}\tilde{m} \tag{2.68}$$

$$\tilde{g}_{\text{STD}} = (2d)^{d/2}\tilde{g}. \tag{2.69}$$

Note also that $\tilde{g}_{\text{STD}}$ is called u in references [*34*] [*71*].

2. A continuum φ^4 theory rescaled by $\varphi \to \alpha\varphi$ and $x \to \beta x$ is still a continuum φ^4 theory; this is because the scalings can be implemented in the approximating lattice theories by a change in parameters [cf. (2.10)] and/or a change in lattice spacings [cf. (2.84)]. Thus, *if* for each parameter set $(\tilde{z}, \tilde{m}, \tilde{\lambda})$ there is at most one continuum φ^4 theory (this is presumably true, although we shall be unable to prove it because of our use of subsequences), then the continuum theory satisfies the scaling laws

$$S_n(x_1, \ldots, x_n; \tilde{z}, \tilde{m}, \tilde{\lambda}) = \alpha^n S_n(x_1, \ldots, x_n; \alpha^{-2}\tilde{z}, \tilde{m}, \tilde{\lambda}) \tag{2.70}$$

and

$$S_n(x_1, \ldots, x_n; \tilde{z}, \tilde{m}, \tilde{\lambda}) = S_n(\beta^{-1}x_1, \ldots, \beta^{-1}x_n; \beta^{2-d}\tilde{z}, \beta\tilde{m}, \beta^{4-d}\tilde{\lambda}). \tag{2.71}$$

(2.70) and (2.71) can be summarized by assigning the dimensions

$$[\tilde{z}] = F^2L^{d-2} \qquad [\tilde{m}] = L^{-1} \qquad [\tilde{\lambda}] = L^{d-4} \qquad [\tilde{g}] = \text{dimensionless} \tag{2.72}$$

Thus, only the dimensionless combination $\tilde{g}$ has physical significance. Equation (2.71) is the useful analogue of (2.15)-(2.16).(2.70)-(2.71) lead to homogeneous renormalization-group equations (albeit trivial ones).

If the normalization of the lattice theories were to carry over to the continuum limit (this has to be *proved* [*12b*]), then (2.64)-(2.66) would correspond to

$$\tilde{z}\tilde{m}^{-2} \simeq a^d \chi \tag{2.73}$$

$$\tilde{z}\tilde{m}^{-4} \simeq a^{d+2} \chi \xi_2^2 \tag{2.74}$$

$$-\tilde{z}^2 \tilde{m}^{-8} \tilde{\lambda} \simeq -a^{3d} \overline{u_4} \tag{2.75}$$

where a is the lattice spacing, χ and ξ_2 are defined in (2.33)-(2.34), and $\overline{u_4}$ is the connected 4-point function [see (3.16)] at zero momentum

$$\overline{u_4} = \sum_{k_2, k_3, k_4} G_4(0, k_2, k_3, k_4), \tag{2.76}$$

all in the lattice theory. Moreover, $\tilde{g}$ corresponds to the *dimensionless renormalized 4-point coupling constant of the lattice theory*

$$g \equiv -\overline{u_4} \chi^{-2} \xi_2^{-d}, \tag{2.77}$$

i. e.

$$\tilde{g} \simeq g. \tag{2.78}$$

We shall shortly prove that the $\simeq$ sign in (2.73)-(2.75) and (2.78) can be interpreted as equality, in the sense that the continuum quantity on the left side is the limit of the lattice quantities on the right side.

To take the continuum limit, we proceed as follows: First choose the desired normalizations $\tilde{z}$ and $\tilde{m}$ for the continuum theory (this has no physical significance, and we could just as well take $\tilde{z} = \tilde{m} = 1$). Then choose a charge renormalization $g_0(a)$, where g_0 is the *dimensionless bare coupling constant*

$$g_0 = \tilde{m}^{d-4} \lambda_0^{\mathrm{FT}} = (a\tilde{m})^{d-4} \lambda_0^{\mathrm{SM}} = (a\tilde{m})^{d-4} \mathrm{J}^{-2} \lambda_0 \,; \tag{2.79}$$

this is the crucial physical choice. For example, in the superrenormalizable case ($d < 4$) we shall choose g_0 *constant* (i. e. λ_0^{FT} constant) as $a \to 0$ (see Section 2.1). Now, for each $a > 0$, pick a φ_d^4 lattice theory as follows: First imagine fixing $\mathrm{J} = 1$, and fixing λ_0 according to (2.79); then choose B_0 by Proposition 2.1 so as to achieve a correlation length

$$\xi_2 = (a\tilde{m})^{-1}. \tag{2.80}$$

This gives some susceptibility χ with $0 < \chi < \infty$; now pick a rescaling (2.10) so as to make

$$\chi = \tilde{z}\tilde{m}^{-2} a^{-d}. \tag{2.81}$$

(Such a rescaling leaves ξ_2 and g_0 unchanged.) Thus, for each $a > 0$, we find a parameter set $(\mathrm{B}_0, \lambda_0, \mathrm{J})$ satisfying the normalizations (2.73), (2.74) and (2.79).

Remark. — Since the effective ultraviolet cutoff is $\Lambda \sim a^{-1}$, (2.80) says that ξ_2 can be interpreted as $\Lambda/\tilde{m}$.

We now wish to study the convergence properties of the just-defined lattice models as $a \to 0$. First, some notation: If μ is a probability measure on $\mathbb{R}^{\mathbb{Z}^d}$ which is supported on $\mathscr{s}'(\mathbb{Z}^d)$, the space of polynomially bounded sequences (the measures for lattice φ_d^4 models certainly have this property [44]), then for any lattice spacing $a > 0$ we can define a probability measure μ^a on the Schwartz distribution space $\mathscr{S}'(\mathbb{R}^d)$, as follows:

$$\mu^a = \mathrm{P}_a \mu \,, \tag{2.82}$$

where $\mathrm{P}_a : \mathscr{s}'(\mathbb{Z}^d) \to \mathscr{S}'(\mathbb{R}^d)$ is defined by

$$\mathrm{P}_a(\{ \alpha_i \}_{i \in \mathbb{Z}^d}) = a^d \sum_{i \in \mathbb{Z}^d} \alpha_i \delta_{ai} \,. \tag{2.83}$$

(Here δ_x denotes the delta function located at the point x.) The Schwinger functions are then given by

$$\int_{\mathscr{S}'(\mathbb{R}^d)} \varphi(f_1) \dots \varphi(f_n) d\mu^a(\varphi) = a^{nd} \left\langle \left(\sum_{i \in \mathbb{Z}^d} f_1(ai)\varphi_i \right) \dots \left(\sum_{i \in \mathbb{Z}^d} f_n(ai)\varphi_i \right) \right\rangle_\mu \tag{2.84}$$

LEMMA 2.2. — Let $\{ \mu_m \}$ be a sequence of φ_d^4 lattice measures, and let $\{ a_m \} \downarrow 0$ be a sequence of strictly positive numbers. Assume that

$$\left| \int \varphi(f)\varphi(g) d\mu_m^{a_m}(\varphi) \right| \leq \mathrm{C} \, \| f \| \, \| g \| \tag{2.85}$$

uniformly in m, for some fixed $\mathscr{S}$-norm $\| \, . \, \|$. Then there exists a subsequence $\{ m_j \}$ and a probability measure ν on $\mathscr{S}'(\mathbb{R}^d)$ such that

$$\int \exp\,[\varphi(f)] d\nu(\varphi) = \lim_{j \to \infty} \int \exp\,[\varphi(f)] d\mu_{m_j}^{a_{m_j}}(\varphi) \tag{2.86}$$

and

$$\int \varphi(f_1) \dots \varphi(f_n) d\nu(\varphi) = \lim_{j \to \infty} \int \varphi(f_1) \dots \varphi(f_n) d\mu_{m_j}^{a_{m_j}}(\varphi) \tag{2.87}$$

for all (complex-valued) $f, f_1, \dots, f_n \in \mathscr{S}(\mathbb{R}^d)$. Moreover, the Schwinger functions (2.87) satisfy all the Osterwalder-Schrader axioms except perhaps Euclidean (rotation) invariance and clustering.

Proof. — See [8] [15] [50] for the proof of everything except the translation invariance of ν. To prove this, it suffices to show that

$$\int e^{i\varphi(f)} d\nu(\varphi) = \int e^{i\varphi(f_b)} d\nu(\varphi) \tag{2.88}$$

for all *real-valued* $f \in \mathscr{S}(\mathbb{R}^d)$ and all $b \in \mathbb{R}^d$, where

$$f_b(x) \equiv f(x - b) \,. \tag{2.89}$$

Now since $a_m \downarrow 0$, there exists a sequence of $b_m \in a_m \mathbb{Z}^d$ with $b_m \to b$. Thus, by (2.86) and the $\mathbb{Z}^d$-translation invariance of the lattice models, we find

$$\int [e^{i\varphi(f)} - e^{i\varphi(f_b)}]d\nu(\varphi) = \lim_{j\to\infty} \int [e^{i\varphi(f)} - e^{i\varphi(f_b)}]d\nu_j(\varphi)$$
$$= \lim_{j\to\infty} \int [e^{i\varphi(f)} - e^{i\varphi(f_{b-b_{m_j}})}]d\nu_j(\varphi), \quad (2.90)$$

where we have written $d\nu_j$ as shorthand for $d\mu_{m_j}^{a_{m_j}}$. Now using

$$| e^{is} - e^{is'} | \leq | s - s' |, \quad (2.91)$$

we bound (2.90) in absolute value by

$$\lim_{j\to\infty} \int | \varphi(f - f_{b-b_{m_j}}) | d\nu_j(\varphi) \leq \lim_{j\to\infty} \left(\int | \varphi(f - f_{b-b_{m_j}}) |^2 d\nu_j(\varphi) \right)^{1/2}. \quad (2.92)$$

Now introduce $g_j \equiv f - f_{b-b_{m_j}}$; then, whenever $| b - b_{m_j} | \leq 1$, we have

$$| g_j(x) | \leq | b - b_{m_j} | (\sup_{|y-x|\leq 1} | (\nabla f)(y) |)$$
$$\leq | b - b_{m_j} | h(x) \quad (2.93)$$

for a suitable function $h \in \mathcal{S}(\mathbb{R}^d)$ [dependent on f but independent of j]. Thus, by (2.84) and Griffiths' inequality,

$$\int \varphi(g_j)^2 d\nu_j(\varphi) = a^{2d} \sum_{k,l\in\mathbb{Z}^d} g_j(ak)g_j(al) \langle \varphi_k\varphi_l \rangle$$
$$\leq a^{2d} | b - b_{m_j} |^2 \sum_{k,l\in\mathbb{Z}^d} h(ak)h(al) \langle \varphi_k\varphi_l \rangle$$
$$= | b - b_{m_j} |^2 \int \varphi(h)^2 d\nu_j(\varphi)$$
$$\leq C \| h \|^2 | b - b_{m_j} |^2 \quad (2.94)$$

by hypothesis (2.85). This vanishes as $j \to \infty$. ■

COROLLARY 2.3. — Let $\{ \mu_m \}$ be the sequence of models [for the given $\{ a_m \}$] constructed in (2.79)-(2.81). Then (2.85) holds, and hence so do the conclusions of Lemma 2.2.

Proof. — By (2.84),

$$\int \varphi(f)\varphi(g)d\mu^a(\varphi) = a^{2d} \sum_{i,j} f(ai)g(aj) \langle \varphi_i\varphi_j \rangle, \quad (2.95)$$

and this is bounded in absolute value by

$$a^d \chi \| f \|_{2,a} \| g \|_{2,a} = \tilde{z}\tilde{m}^{-2} \| f \|_{2,a} \| g \|_{2,a}, \quad (2.96)$$

where

$$\|f\|_{2,a} \equiv \left(a^d \sum_{x \in a\mathbb{Z}^d} |f(x)|^2 \right)^{1/2}. \tag{2.97}$$

But for any $a \leq 1$,

$$\|f\|_{2,a} \leq 2^d \|f\|_{l^2(L^\infty)},$$

where

$$\|f\|_{l^2(L^\infty)} \equiv \left(\sum_{i \in \mathbb{Z}^d} \sup_{x \in C_i} |f(x)|^2 \right)^{1/2} \tag{2.98}$$

and C_i is the *unit* cube centered at i. Moreover,

$$\|f\|_{l^2(L^\infty)} \leq \text{const} \times \|(1 + |x|^2)^\alpha f\|_\infty \tag{2.99}$$

for any $\alpha > d/4$, and thus is surely an $\mathcal{S}$-norm. ■

So we can at least choose a subsequence for which all the Schwinger functions converge, in the sense of the usual (either weak or strong) topology on $\mathcal{S}'(\mathbb{R}^{nd})$. The only remaining question is: do the normalization conditions (2.64)-(2.66) turn out right? That is, do (2.73)-(2.75) and (2.78) hold? We shall show, following Schrader [*12b*] with slight variations, that under suitable conditions they do.

Consider, for example, the susceptibility condition (2.64). For each lattice spacing a, the lattice 2-point Schwinger function (in difference variables) $S_2^{(a)}(x)$ is a positive measure on $\mathbb{R}^d$ (by Griffiths' inequality) with total mass

$$\int S_2^{(a)}(x) d^d x = \tilde{z}\tilde{m}^{-2}; \tag{2.100}$$

this was arranged by construction [cf. (2.81)]. Moreover, by (2.87), we have

$$\int S_2^{(a)}(x) f(x) d^d x \to \int S_2(x) f(x) d^d x \tag{2.101}$$

for the chosen subsequence of $a \downarrow 0$, for each $f \in \mathcal{S}(\mathbb{R}^d)$. Thus, to prove (2.64), we need to extend (2.101) to hold also for the function $f \equiv 1$, which does not lie in $\mathcal{S}$. A sufficient condition is given by the following lemma:

LEMMA 2.4. — Let $\{\rho_m\}$, ρ be positive measures on $\mathbb{R}^n$, with

$$\lim_{m \to \infty} \int f(x) d\rho_m(x) = \int f(x) d\rho(x) \tag{2.102}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$. Let F be a nonnegative continuous function on $\mathbb{R}^n$ such that

$$\int F(x) d\rho_m(x) \leq C \tag{2.103}$$

uniformly in m. Then (2.102) holds also for every continuous f with is $0(\mathrm{F})$ as $|x| \to \infty$, i. e. for which the set

$$\{ x : |f(x)| \geq \varepsilon \mathrm{F}(x) \} \tag{2.104}$$

is compact for each $\varepsilon > 0$.

Proof. — First let K be any compact set in $\mathbb{R}^n$, and let $\overline{\mathrm{F}} \in \mathrm{C}_0^\infty$ be a non-negative function which is equal to 1 on K. Then $\int \overline{\mathrm{F}}(x) d\rho_m(x)$ converges to $\int \overline{\mathrm{F}}(x) d\rho(x) < \infty$ as $m \to \infty$, hence is uniformly bounded. So we can assume without loss of generality that $\mathrm{F}(x) \geq 1$ on K.

Now let $\{ \mathrm{F}_j \} \in \mathrm{C}_0^\infty$ be a sequence with $0 \leq \mathrm{F}_j \uparrow \mathrm{F}$ pointwise. Then, for each j,

$$\int \mathrm{F}_j(x) d\rho(x) = \lim_{m \to \infty} \int \mathrm{F}_j(x) d\rho_m(x) \leq \mathrm{C} ; \tag{2.105}$$

so by the monotone convergence theorem,

$$\int \mathrm{F}(x) d\rho(x) \leq \mathrm{C} . \tag{2.106}$$

Now fix $\varepsilon > 0$, and let K be a compact ball containing the set (2.104). Then there exists a function $g \in \mathrm{C}_0^\infty$ such that $|f(x) - g(x)| \leq \varepsilon$ for $x \in \mathrm{K}$, $|f(x) - g(x)| \leq f(x)$ for $x \notin \mathrm{K}$. Thus $|f(x) - g(x)| \leq \varepsilon \mathrm{F}(x)$ for all x. The claim of the theorem then follows from (2.102) together with (2.103) and (2.106). ■

We apply Lemma 2.4 with $\mathrm{F}(x) = |x|^\phi$ for some $\phi > 0$. Thus, for (2.64) to hold, it suffices that $a\xi_\phi$ be bounded as $a \to 0$, for some $\phi > 0$. But this holds, by construction [cf. (2.80)], for $\phi = 2$.

Similarly, for (2.65) to hold, we need (2.101) for $f(x) = |x|^2$; it thus suffices that $a\xi_\phi$ be bounded as $a \to 0$, for some $\phi > 2$. Using (2.80), this is equivalent to the boundedness of ξ_ϕ/ξ_2. By (2.44), it is certainly weaker than the boundedness of ξ/ξ_2. But we shall prove in Section 3.3 that ξ/ξ_2 is indeed bounded (for $d < 4$, assuming Conjecture 3.2).

Finally, we apply the same method to study (2.66)/(2.78). The lattice connected 4-point Schwinger function (in difference variables) $\mathrm{S}_4^{\mathrm{T}(a)}(x_1, x_2, x_3)$ is a negative measure on $\mathbb{R}^{3d}$, by the Lebowitz inequality. Applying Lemma 2.4 with

$$\mathrm{F}(x) = |x_1|^\alpha + |x_2|^\alpha + |x_3|^\alpha \tag{2.107}$$

for some $\alpha > 0$, we need to prove a uniform bound

$$- \int \mathrm{F}(x) \mathrm{S}_4^{\mathrm{T}(a)}(x_1, x_2, x_3) d^d x_1 d^d x_2 d^d x_3 \leq \mathrm{C} \tag{2.108}$$

uniformly in a. We mimic the proof of the Glimm-Jaffé upper bound on g [72]-[74]. By an inequality of Newman [46, Theorem 5] (which also follows [74] from the Ginibre inequality $\langle q_i q_j t_k t_l \rangle \geq 0$ and permutations), we have

$$- G_4(0, i_1, i_2, i_3) \leq 2 \min [G(i_1)G(i_2 - i_3), \quad G(i_2)G(i_1 - i_3), \quad G(i_3)G(i_1 - i_2)], \quad (2.109)$$

where G_4 is the connected 4-point function of the lattice theory [cf. (3.16)]. Thus, for example,

$$- \sum_{i_1,i_2,i_3} | i_1 |^\alpha G_4(0, i_1, i_2, i_3) \leq 2 \sum_{i_1,i_2,i_3} | i_1 |^\alpha [G(i_1)G(i_2 - i_3)G(i_2)G(i_1 - i_3)]^{1/2}$$

$$= 2(H * H * H * I)(0), \quad (2.110)$$

where

$$H(i) = G(i)^{1/2} \quad (2.111)$$

and

$$I(i) = | i |^\alpha G(i)^{1/2} . \quad (2.112)$$

By Young's inequality, (2.110) is bounded by

$$\text{const} \times \| H \|_{4/3}^3 \ \| I \|_{4/3} = \text{const} \times \left(\sum_i G(i)^{2/3} \right)^{9/4} \left(\sum_i | i |^{4\alpha/3} G(i)^{2/3} \right)^{3/4} . \quad (2.113)$$

By a simple argument using Hölder's inequality [74], this is bounded by

$$\text{const} \times \chi^2(1 + \xi_\phi^d) \quad (2.114)$$

for any $\phi > \frac{d}{2} + 2\alpha$. Combining this with (2.80) and (2.81), we find that (2.108) is bounded provided that ξ_ϕ/ξ_2 is bounded for some $\phi > d/2$. But for $d < 4$ this is trivial, since we can take $\phi = 2$! (For $d \geq 4$ it requires some extra control over the 2-point function.)

This completes the proof of existence and correct normalization of the continuum limit (modulo the bound on ξ/ξ_2 from Section 3.3). Thus, to prove that the continuum theory constructed here is non-Gaussian, i. e., that $\tilde{g} > 0$, it suffices to show that g is bounded away from zero *uniformly as the lattice spacing a tends to zero.* This is a statement about the lattice φ_d^4 theories, and it forms the subject of the next chapter.

3. DETAILED STUDY OF THE FIELD EQUATIONS

3.1. Derivation of the Field Equations.

We are now ready to attack the heart of the problem: to show that the continuum limit of the φ_d^4 lattice theory ($d < 4$) with the conventional (superrenormalizable) charge renormalization is non-Gaussian. By the results of Section 2.3, it suffices to show that g, the dimensionless renormalized 4-point coupling constant of the lattice theory, is bounded away from zero *uniformly as the critical point is approached.* To demonstrate this for sufficiently small (but nonzero) bare coupling constant g_0, we follow the intuition of field-theoretic perturbation theory: namely, that the leading (tree-graph) contribution to g is just g_0, while the non-leading contributions are $O(g_0^2)$ *uniformly as the ultraviolet cutoff* Λ *tends to infinity.* (This is because, in dimension $d < 4$, all graphs are finite once the mass renormalization has been performed.) Since perturbation theory can be obtained as the formal iterative « solution » of the exact field equations, it is natural to make use of these equations. We emphasize that we always work with the *lattice* theory defined by (2.3) and (2.17) ff.; in this framework, the field equations are *rigorous theorems* relating the various correlation functions.

The field equations can be derived by integration by parts in the functional integral defining the theory [*75*] [*64*]. To make this rigorous, consider first the finite-volume theory (2.17) written as follows:

$$\langle F(\varphi) \rangle_V = \frac{\displaystyle\int F(\varphi) \exp\left[-\frac{1}{2} \sum_{i,j \in V} K_{ij} \varphi_i \varphi_j - \sum_{i \in V} V(\varphi_i) + \sum_{i \in V} h_i \varphi_i \right] \prod_{i \in V} d\varphi_i}{\displaystyle\int \exp\left[-\frac{1}{2} \sum_{i,j \in V} K_{ij} \varphi_i \varphi_j - \sum_{i \in V} V(\varphi_i) + \sum_{i \in V} h_i \varphi_i \right] \prod_{i \in V} d\varphi_i} \tag{3.1}$$

where

$$K_{ij} = \alpha \delta_{ij} - J \delta_{ij}^{NN} = J\left(-\Delta + \frac{\alpha - 2dJ}{J} \right)_{ij} \tag{3.2}$$

$$V(\varphi) = \frac{\lambda_0}{4!} \varphi^4 + \frac{1}{2}(B_0 - \alpha)\varphi^2 \tag{3.3}$$

and α is an arbitrary number strictly greater than $2dJ$. We have introduced the magnetic field h_i for reasons that will become clear shortly. Using the identity

$$\int \frac{\partial}{\partial \varphi_i} \left[F(\varphi) \exp\left[-\frac{1}{2} \sum_{i,j \in V} K_{ij} \varphi_i \varphi_j - \sum_{i \in V} V(\varphi_i) + \sum_{i \in V} h_i \varphi_i \right] \right] \prod_{i \in V} d\varphi_i = 0, \tag{3.4}$$

we conclude that

$$\left\langle \frac{\partial F}{\partial \varphi_i} - F(\varphi) \left[\sum_{j \in V} K_{ij} \varphi_j + V'(\varphi_i) - h_i \right] \right\rangle_V = 0 \tag{3.5}$$

for each $i \in V$. Now consider an infinite-volume theory defined formally by

$$\langle F(\varphi) \rangle = \frac{\displaystyle\int F(\varphi) \exp\left[-\frac{1}{2} \sum_{i,j} K_{ij}\varphi_i\varphi_j - \sum_i V(\varphi_i)\right] \prod_i d\varphi_i}{\displaystyle\int \exp\left[-\frac{1}{2} \sum_{i,j} K_{ij}\varphi_i\varphi_j - \sum_i V(\varphi_i)\right] \prod_i d\varphi_i} \tag{3.6}$$

and satisfying the requisite DLR equations. Then, for any finite volume V, the *conditional* expectation of $\langle \, . \, \rangle$, conditioned on the spins outside V, is given by (3.1) with

$$h_i = - \sum_{j \notin V} K_{ij}\varphi_j . \tag{3.7}$$

Thus the conditional expectation satisfies (3.5), and $\langle \, . \, \rangle$ satisfies

$$\left\langle \frac{\partial F}{\partial \varphi_i} - F(\varphi)\left[\sum_j K_{ij}\varphi_j + V'(\varphi_i) \right] \right\rangle = 0 \tag{3.8}$$

for each $i \in \mathbb{Z}^d$, and for each well-behaved function F of a finite number of spins (for example, a finite product). This is the basic field equation.

Note now that since we have taken $\alpha > 2dJ$, the infinite matrix K is invertible; let C be its inverse. Then (3.8) implies

$$\langle \varphi_i F(\varphi) \rangle = \sum_j C_{ij}\left[\left\langle \frac{\partial F}{\partial \varphi_j} \right\rangle - \langle V'(\varphi_j)F(\varphi) \rangle \right]. \tag{3.9}$$

(It is easy to see that the formal manipulation of infinite sums here is legitimate, at least if $\langle \, . \, \rangle$ satisfies the superstability bounds.) However, (3.9) is not yet suitable for our purposes, because it contains the *bare* propagator C, which is divergent due to infinite mass renormalization in dimension $d \geq 3$. (Actually, since we have not bothered to Wick order, it is divergent in $d \geq 2$.) We therefore wish to rewrite (3.9) using the *interacting* propagator

$$G_{ij} = \langle \varphi_i \varphi_j \rangle . \tag{3.10}$$

We proceed as follows [*64*]: Setting $F(\varphi) = \varphi_j$ in (3.9), we obtain

$$G_{ij} = C_{ij} - \sum_k C_{ik} \langle V'(\varphi_k)\varphi_j \rangle . \tag{3.11}$$

Inserting this into (3.9), we find

$$\langle \varphi_i F(\varphi) \rangle = \sum_j G_{ij}\left[\left\langle \frac{\partial F}{\partial \varphi_j} \right\rangle - \langle V'(\varphi_j)F(\varphi) \rangle \right]$$
$$+ \sum_{j,k} C_{ik} \langle V'(\varphi_k)\varphi_j \rangle \left[\left\langle \frac{\partial F}{\partial \varphi_j} \right\rangle - \langle V'(\varphi_j)F(\varphi) \rangle \right]. \quad (3.12)$$

Assume now that the state $\langle\ .\ \rangle$ is translation-invariant. In that case, the second term in (3.12) can be rewritten as

$$\sum_{j,k} \langle V'(\varphi_i)\varphi_j \rangle C_{jk}\left[\left\langle \frac{\partial F}{\partial \varphi_k} \right\rangle - \langle V'(\varphi_k)F(\varphi) \rangle \right]$$
$$= \sum_j \langle V'(\varphi_i)\varphi_j \rangle \langle \varphi_j F(\varphi) \rangle$$
$$= \sum_{j,k,l} G_{ij} \langle V'(\varphi_j)\varphi_k \rangle \Gamma_{kl} \langle \varphi_l F(\varphi) \rangle, \quad (3.13)$$

where we have used (3.9) and introduced the inverse propagator $\Gamma = G^{-1}$ [*30*] [*64*]. (We assume that the state $\langle\ .\ \rangle$ is ergodic, so that $G_{ij} \to 0$ as $|i - j| \to \infty$; then Γ is well-defined [*30*]. Note that our definition of Γ is the negative of that used by Glimm and Jaffe [*30*] [*64*].) Inserting (3.13) into (3.12), we conclude that

$$\langle \varphi_i F(\varphi) \rangle = \sum_j G_{ij}\left[\left\langle \frac{\partial F}{\partial \varphi_j} \right\rangle - \langle V'(\varphi_j)(1 - P_1)F(\varphi) \rangle \right], \quad (3.14)$$

where we have introduced for notational convenience the projection operator

$$P_1 = \sum_{k,l} |\, \varphi_k \rangle \Gamma_{kl} \langle \varphi_l \,| . \quad (3.15)$$

(3.14) is the mass-renormalized field equation: compared to (3.9), the replacement of C by G is compensated by the term involving P_1. Note that the bare mass never occurs in (3.14): C is explicitly absent, and the term $(B_0 - \alpha)\varphi_j$ in $V'(\varphi_j)$ is annihilated by the projector $1 - P_1$. Thus we can work with (3.14) and never again worry about the mass renormalization.

Our basic tool in this paper is the field equation for the connected 4-point function. Taking $F(\varphi) = \varphi_{j_1}\varphi_{j_2}\varphi_{j_3}$ in (3.14) and rewriting in terms of the *connected* correlation functions

$$G_4(k_1, k_2, k_3, k_4) = \langle \varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4} \rangle - G_{k_1k_2}G_{k_3k_4}$$
$$- G_{k_1k_3}G_{k_2k_4} - G_{k_1k_4}G_{k_2k_3} \quad (3.16)$$

and

$$G_6(k_1, \ldots, k_6) = \langle \varphi_{k_1} \ldots \varphi_{k_6} \rangle - \sum_{\{\alpha,\beta\} \subset \{1,\ldots 6\}} G_{k_\alpha k_\beta} \left\langle \prod_{\gamma \neq \alpha,\beta} \varphi_{k_\gamma} \right\rangle + 2 \sum_{\text{pairings}} G_{k_\alpha k_\beta} G_{k_\gamma k_\delta} G_{k_\varepsilon k_\zeta}, \tag{3.17}$$

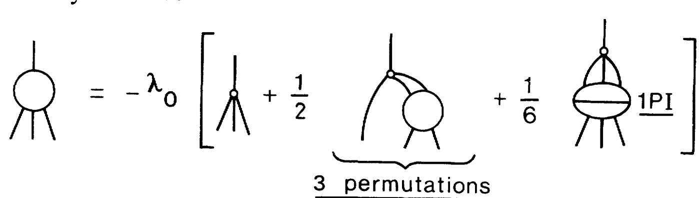
we find, after a bit of algebraic manipulation,

$$G_4(i, j_1, j_2, j_3) = -\lambda_0 \left[\sum_k G_{ik} G_{kj_1} G_{kj_2} G_{kj_3} + \frac{1}{2} \sum_k G_{ik} G_{kj_1} G_4(k, k, j_2, j_3) + \text{permutations} + \frac{1}{6} \sum_k G_{ik} G_6^{1\text{PI}}(k, k, k \mid j_1, j_2, j_3) \right], \tag{3.18}$$

where we have introduced the connected 6-point function one-particle-irreducible in a single channel

$$G_6^{1\text{PI}}(k, k, k \mid j_1, j_2, j_3) = G_6(k, k, k, j_1, j_2, j_3) - \sum_{l,m} G_4(k, k, k, l) \Gamma_{lm} G_4(m, j_1, j_2, j_3). \tag{3.19}$$

Pictorially we have



(3.20)

Here the lines are *full* propagators G, and the blobs are *connected* correlation functions G_4 and $G_6^{1\text{PI}}$.

To summarize the argument thus far:

THEOREM 3.1. — Any translation-invariant, ergodic, even measure satisfying the superstability bounds and the DLR equations for the model (2.3) satisfies the field equations (3.8), (3.9), (3.11), (3.14) and (3.18).

Remarks. — 1. The reader should convince herself, using Feynman graphs, that (3.20) is correct – in particular, that the 1PI nature of the last term is what converts bare to interacting propagators on the upper leg.

2. These equations have also been obtained by Johnson [*76*], cf. his

equation (22); in fact, he goes much further, and attempts to eliminate the bare *charge* as well as the bare mass, in favor of the fully renormalized quantities. Although his work is extremely interesting, we take a different approach. Unlike Johnson, we do not consider the field equations as continuum equations in search of a solution, but as true stratements (one among many) concerning an already-defined theory: the φ^4 *lattice* model.

3. It should be noted that the field equations do *not* contain the full content of the theory. This is easily seen, for example, in the $d = 0$ theory, where the field equations are three-term recursion relations for the expectations $\langle \varphi^{2n} \rangle$. Although the theory is uniquely defined, the field equations have a one-parameter family of solutions labelled, for example, by the arbitrary choice of $\langle \varphi^2 \rangle$. Presumably only one of these solutions satisfies Nelson-Symanzik positivity, namely the correct one given by

$$\langle \varphi^{2n} \rangle = \frac{\int \varphi^{2n} \exp\left[-\frac{B_0}{2}\varphi^2 - \frac{\lambda_0}{4!}\varphi^4 \right] d\varphi}{\int \exp\left[-\frac{B_0}{2}\varphi^2 - \frac{\lambda_0}{4!}\varphi^4 \right] d\varphi}$$

4. DLR-like equations for the correlation functions—which contain *more* information than the field equations used here—have been derived by Suzuki [*77*] and Schwabl [*78*], among others. It would be very interesting to make use of these equations in the present context, but I have not found a way to do so.

3.2. Proof of a Lower Bound on the Renormalized Coupling.

The idea of the proof is as follows: The first term on the right side of (3.20) is explicitly nonzero, since it is proportional to λ_0. Hence it suffices to show that the second and third terms are $O(\lambda_0^2)$, *uniformly as the critical point is approached.* We shall handle the third term by a (conjectured) correlation inequality. We handle the second term by iterating (3.20) once; this strategem works precisely because the leading radiative correction to the 4-point function is ultraviolet *convergent* in dimension $d < 4$. In other words, the tree graph really is the dominant term on the right side of (3.20), for $d < 4$.

Note first that the connected 4-point function with two arguments tied together takes on a particularly simple form in the Gaussian ($\lambda_0 = 0$) and Ising ($\lambda_0 = \infty$) limits:

$$G_4(k, k, j_2, j_3) = \begin{Bmatrix} 0 & \text{(Gaussian)} \\ -2G_{kj_2}G_{kj_3} & \text{(Ising)} \end{Bmatrix}. \qquad (3.21)$$

The Gaussian value of zero is obvious; the Ising value follows from the fact that, in an Ising model, φ takes only the two values $+c$ and $-c$, so that

$$\langle \varphi_k^2 \varphi_{j_2} \varphi_{j_3} \rangle = c^2 \langle \varphi_{j_2} \varphi_{j_3} \rangle = \langle \varphi_k^2 \rangle \langle \varphi_{j_2} \varphi_{j_3} \rangle ; \qquad (3.22)$$

combining this with (3.16) gives (3.21). Moreover, in *any* φ^4 theory, this 4-point function is rigorously bounded between its Ising and Gaussian values, i. e.

$$-2G_{kj_2}G_{kj_3} \leq G_4(k, k, j_2, j_3) \leq 0 ; \qquad (3.23)$$

these are just the Griffiths and Lebowitz inequalities, respectively.

Remark. — In the continuum φ^4 theory in dimension $d \geq 4$, the rigorous bound (3.23) is *violated* in every order of perturbation theory, since the tying together of two arguments produces an ultraviolet divergence. Thus, perturbation theory (even *renormalized* perturbation theory) is an extremely unreliable guide for $d \geq 4$. Rather, the ultraviolet behavior of G_4 in the continuum φ^4 field theory must be considerably *softer* than that predicted by perturbation theory. (Of course, it may be *so* soft as to be identically zero, i. e. a (generalized) free field!) A similar rigorous result (which refers, however, to the *amputated* vertex function) has been derived by Lehmann, Symanzik and Zimmermann [*79*] (see also [*80*]-[*83*]) for the Yukawa theory, and by Evans [*84*] for quantum electrodynamics. These bounds are also violated in every order of perturbation theory, as is the Källén-Lehmann positivity condition for the 2-point function. The meaning of this violation of (3.23) will be discussed further in Section 4.3.

Similarly, the connected, partially-1PI 6-point function with *three* arguments tied together reduces to a simple form in the Gaussian and Ising cases:

$$G_6^{1PI}(k, k, k \mid j_1, j_2, j_3) = \begin{cases} 0 & \text{(Gaussian)} \\ 12G_{kj_1}G_{kj_2}G_{kj_3} & \text{(Ising)} \end{cases} . \qquad (3.24)$$

The Gaussian value is obvious; the Ising value follows, by a straightforward computation using (3.17) and (3.19), from the identities $\varphi^2 = c^2$ and $\varphi^3 = c^2\varphi$. Hence it is natural to conjecture:

CONJECTURE 3.2. — In any φ^4 lattice theory,

$$0 \leq G_6^{1PI}(k, k, k \mid j_1, j_2, j_3) \leq 12G_{kj_1}G_{kj_2}G_{kj_3} . \qquad (3.25)$$

This conjecture is supported by the analogy with (3.23)—and more generally by the idea that one expects all φ^4 theories to « lie between » the Gaussian and Ising extremes. However, I have at present no idea how to prove Conjecture 3.2; this problem is discussed further in Chapter 5.

Remarks. — 1. One can also introduce [*64*] the connected (unamputated) 6-point function which is one-particle-irreducible in *all* channels, denoted Γ_6.

A similar computation then shows that

$$\Gamma_6(k, k, k, j_1, j_2, j_3) = \left\{ \begin{array}{ll} 0 & \text{(Gaussian)} \\ -24 G_{kj_1} G_{kj_2} G_{kj_3} & \text{(Ising)} \end{array} \right\}. \quad (3.26)$$

(This has been noticed in [*38*] [*85*] for the special case $j_1 = j_2 = j_3$.) It is then natural to conjecture that

$$-24 G_{kj_1} G_{kj_2} G_{kj_3} \leq \Gamma_6(k, k, k, j_1, j_2, j_3) \leq 0 \quad (3.27)$$

for all φ^4 theories. The weaker conjecture

$$\Gamma_6(k, k, k, j, j, j) \leq 0 \quad (3.28)$$

has been employed by Glimm and Jaffe [*64*] [*30*] to derive some rather strong information on the field-strength renormalization; see also Section 3.3 below. There is some numerical evidence for (3.28) in dimension $d = 1$ [*86*].

2. (3.25) and (3.27) are true in the lowest non-trivial order of perturbation theory [$O(\lambda_0^2)$ for (3.25), $O(\lambda_0^3)$ for (3.27)].

We are now prepared to analyze (3.18)/(3.20) and to show, assuming Conjecture 3.2, that the renormalized coupling constant g is bounded away from zero. By the Lebowitz inequality, $G_4(i, j_1, j_2, j_3)$ is pointwise negative; we first show that it is *not too* negative. Since the second term in the brackets in (3.18)/(3.20) is pointwise negative (by the Lebowitz and Griffiths inequalities), it can be discarded. Using the upper bound in (3.25), we conclude that

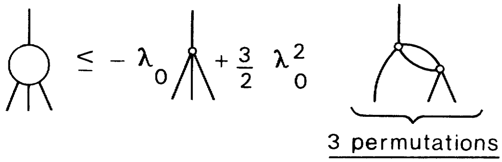
$$G_4(i, j_1, j_2, j_3) \geq -3\lambda_0 \sum_k G_{ik} G_{kj_1} G_{kj_2} G_{kj_3}. \quad (3.29)$$

Thus, the connected 4-point function is bounded by three times its tree-graph value.

Now we show that $G_4(i, j_1, j_2, j_3)$ is *sufficiently* negative. By the lower bound in (3.25), the last term in (3.18)/(3.20) can be discarded. So we need only look at the second term in the brackets in (3.18)/(3.20); we bound it using (3.29) with two arguments tied together. Thus

$$G_4(i, j_1, j_2, j_3) \leq -\lambda_0 \sum_k G_{ik} G_{kj_1} G_{kj_2} G_{kj_3}$$
$$+ \frac{3}{2}\lambda_0^2 \sum_{k,l} G_{ik} G_{kj_1} G_{kl}^2 G_{lj_2} G_{lj_3} + \text{permutations}. \quad (3.30)$$

Pictorially,



(3.31)

Now the heuristic idea is that the right-most graph in (3.31) is ultraviolet convergent for $d < 4$, so that this term is really $O(\lambda_0^2)$ [or more precisely, $O(g_0^2)$] *uniformly in the ultraviolet cutoff* Λ. Of course, this idea is not quite right, because the lines in (3.31) are *interacting* propagators G. But we can still get the desired bounds. Evaluating (3.30)/(3.31) at zero momentum (i. e. summing over j_1, j_2, j_3), we get

$$-\overline{u_4} \geq \lambda_0 \chi^4 - \frac{9}{2} \lambda_0^2 \chi^4 \sum_x G(x)^2 . \tag{3.32}$$

But by the universal bounds proven in Appendix A (Propositions A.4 and A.5), we have

$$\sum_x G(x)^2 \leq c_1 J^{-2}(1 + J\chi)^{2-\frac{d}{2}} \tag{3.33}$$

for $d < 4$, for a suitable (universal) constant c_1. Hence

$$g \equiv \frac{-\overline{u_4}}{\chi^2 \xi_2^d} \geq \frac{\lambda_0 \chi^2}{\xi_2^d}\left[1 - \frac{9}{2} c_1 \lambda_0 J^{-2}(1 + J\chi)^{2-\frac{d}{2}}\right]. \tag{3.34}$$

Now recall that we are using the conventional superrenormalizable charge renormalization, in which λ_0^{FT} and hence g_0 are held fixed. By (2.79) and (2.80),

$$\lambda_0 = J^2 a^{4-d} \lambda_0^{FT} = J^2 \xi_2^{d-4} g_0 , \tag{3.35}$$

so

$$g \geq g_0 \left(\frac{J\chi}{\xi_2^2}\right)^2 \left[1 - \frac{9}{2} c_1 g_0 \left(\frac{1 + J\chi}{\xi_2^2}\right)^{2-\frac{d}{2}}\right] \tag{3.36}$$

$$\geq g_0 \left(\frac{J\chi}{\xi_2^2}\right)^2 [1 - c_2 g_0] \tag{3.37}$$

by (2.52), for a suitable (universal) constant c_2 and for ξ_2 not near zero. In Section 3.3 we shall obtain a non-zero uniform lower bound on $J\chi/\xi_2^2$. It then follows that if we choose

$$0 < g_0 < c_2^{-1} , \tag{3.38}$$

we find a nonzero lower bound on g, uniform as the critical point is approached. This is precisely what we set out to prove.

Remarks. — 1. The bound (3.29) implies a critical-exponent inequality which entails the failure of hyperscaling for φ^4 models in dimension $d > 4$; see Appendix B. This has recently been proven also by Aizenman [*229, 230*] and Fröhlich [*231*] without using conjecture 3.2.

2. We didn't really need the full strength of Conjecture 3.2; it would have sufficed to have instead

$$-c_1 G_{kj_1} G_{kj_2} G_{kj_3} \leq G_6^{1PI}(k, k, k \mid j_1, j_2, j_3) \leq c_2 G_{kj_1} G_{kj_2} G_{kj_3} \quad (3.39)$$

for some *universal* constants $c_1 < 6$ and $c_2 < \infty$. Moreover, we need the lower bound in (3.39) only at zero momentum (i. e. summed over j_1, j_2, j_3). Since the upper bound is used with two legs tied together [see (3.29) → (3.30)], not all the legs can be at zero momentum; but it suffices to know the inequality, say, summed over j_2 and j_3. See, however, Section 3.3, equation (3.45) ff.

3. The argument can also be made to work on the basis of the $\Gamma_6 \leq 0$ conjecture of Glimm and Jaffe [*30*] [*64*]. However, this is somewhat tricky, because the extra terms involving Γ may not have a definite sign in *position space* (cf. [*30*, Proposition 3.2]). One must proceed cleverly so as to arrange that all Γ lines carry zero momentum (for we do know that $\tilde{\Gamma}(0) \geq 0$). However, this is hardly worth the effort; it shows that the « natural » conjecture for this problem concerns G_6^{1PI}, not Γ_6. See also Section 3.3 and Remark 4 there.

4. The argument fails for $d \geq 4$ because the right-most graph in (3.31) is ultraviolet *divergent*; in other words, (3.33) does not hold. Indeed, the divergence of this graph can be made the basis of a possible approach to $\varphi_d^4(d \geq 4)$ yielding a conclusion exactly opposite to that obtained for $d < 4$; see Section 4.2.

3.3. **Bounds on the Field-Strength Renormalization.**

In order to complete the argument of the preceding section, we need to derive a uniform lower bound on $J\chi/\xi_2^2$. This quantity is essentially the field-strength renormalization constant in the « intermediate renormalization »; it is thus closely related to the physical field-strength renormalization constant Z. [See also (2.40) and (2.44).] In this section we shall in fact prove a uniform lower bound on both $J\chi/\xi_2^2$ and ZJ. Unlike the bounds proven in Section 2.2, however, this one is a deep dynamical issue: we shall need to use the field equation for the inverse propagator Γ, as well as Conjecture 3.2.

Note first that

$$\frac{\xi_2^2}{J\chi} = \frac{\sum_x |x|^2 G(x)}{J\left[\sum_x G(x)\right]^2} = \frac{-\left.\frac{\partial^2 \tilde{G}}{\partial p^2}\right|_{p=0}}{J\tilde{G}(0)^2}$$

$$= J^{-1} \left.\frac{\partial^2 \tilde{\Gamma}}{\partial p^2}\right|_{p=0} = -J^{-1} \sum_x |x|^2 \Gamma(x), \qquad (3.40)$$

where we have used the fact that $\partial\tilde{\Gamma}/\partial p\,|_{p=0} = 0$ by symmetry. Hence we need to study Γ.

To derive the field equation for Γ [*64*], we apply the operator $C^{-1}\Gamma = \Gamma C^{-1}$ to (3.11):

$$(C^{-1})_{ij} = \Gamma_{ij} - \sum_k \Gamma_{ik} \langle V'(\varphi_k)\varphi_j \rangle$$

$$= \Gamma_{ij} - \sum_k \Gamma_{ik} \langle \varphi_k V'(\varphi_j) \rangle \qquad (3.41)$$

by reflection invariance. Inserting (3.14) [with $F(\varphi) = V'(\varphi_j)$], we find

$$(C^{-1})_{ij} = \Gamma_{ij} - \langle V''(\varphi_j) \rangle \delta_{ij} + \langle V'(\varphi_i)(1 - P_1)V'(\varphi_j) \rangle \quad (3.42)$$

Recalling that C^{-1} is just the matrix K defined in (3.2), we conclude that

$$\Gamma_{ij} = [\alpha + \langle V''(\varphi_j) \rangle]\delta_{ij} - J\delta_{ij}^{NN} - \langle V'(\varphi_i)(1 - P_1)V'(\varphi_j) \rangle \quad (3.43)$$

$$= \left[B_0 + \frac{\lambda_0}{2} \langle \varphi_j^2 \rangle\right]\delta_{ij} - J\delta_{ij}^{NN} - \frac{\lambda_0^2}{36} \langle \varphi_i^3(1 - P_1)\varphi_j^3 \rangle . \quad (3.44)$$

(Note that the arbitrary number α has disappeared from (3.44), as it had better.) This is the field equation for Γ. Note that, unlike (3.14), this equation is not entirely free of references to the bare mass term B_0; so it might be thought that it is of no use to us when the mass renormalization is infinite. (To be sure, the ultraviolet divergence in B_0 must be cancelled by a similar divergence in $\langle \varphi_j^2 \rangle$, leaving a finite remainder; but this observation does us no good, since we have no control over this cancellation. Indeed, our entire approach is based on *renouncing* the attempt to control explicitly the mass renormalization.) But the crucial fact is that the uncontrolled mass renormalization affects only the $i = j$ term in Γ_{ij}; by (3.40), this does not contribute to the field-strength renormalization. (In the language of field theory, this is simply the fact that the proper self-energy part may be infinite while its second derivative with respect to momentum is finite.)

To bound (3.44) for $i \neq j$, we first expand the last term:

$$\langle \varphi_i^3(1 - P_1)\varphi_j^3 \rangle = G_6^{1PI}(i, i, i\,|\,j, j, j) + 9G_{ij}G_4(i, i, j, j) + 6G_{ij}^3. \quad (3.45)$$

By the Lebowitz inequality, $G_4 \leq 0$. Hence, using the upper bound in Conjecture 3.2 [needed here only for $j_1 = j_2 = j_3$], we find

$$\langle \varphi_i^3(1 - P_1)\varphi_j^3 \rangle \leq 18 G_{ij}^3 \tag{3.46}$$

and hence

$$\Gamma(x) \geq - J\delta_{0x}^{NN} - \frac{\lambda_0^2}{2} G(x)^3 \qquad (x \neq 0). \tag{3.47}$$

Moreover, by the universal bounds proven in Appendix A (Proposition A.5),

$$\sum_x |x|^2 G(x)^3 \leq \text{const} \times \begin{Bmatrix} J^{-3}(1 + J\chi)^{4-d} & \text{if } d < 4 \\ J^{-3} \log(2+J\chi) & \text{if } d = 4 \\ J^{-3} & \text{if } d > 4 \end{Bmatrix}. \tag{3.48}$$

Thus, for the superrenormalizable case ($d < 4$ and g_0 fixed), we find, inserting (3.48) into (3.47) and (3.40) and using (3.35) and (2.52):

$$\frac{\xi_2^2}{J\chi} \leq \text{const} \times (1 + g_0^2). \tag{3.49}$$

This gives the desired bound on (3.37), and completes the argument of the preceding section.

Only slightly more argument [*30*] [*64*] is required to obtain a bound on the physical field-strength renormalization constant Z. By (3.47), we have

$$\sum_{\mathbf{x}} \Gamma(x_1, \mathbf{x}) \geq - \frac{\lambda_0^2}{2} G(x_1, \mathbf{0})^2 \sum_{\mathbf{x}} G(x_1, \mathbf{x}) \geq - O(e^{-3m|x_1|}) \tag{3.50}$$

as $|x_1| \to \infty$. It follows [*56*] [*64*] [*65*] that the spectral weight $d\nu_0(a)$ occurring in (2.55) is supported on $a \geq \cosh(3m) - 1$. (In other words, there is an upper mass gap extending to $3m$, and also the absence of CDD zeros below $3m$.) Thus, the integral appearing in (2.57) can be bounded:

$$\int_{\cosh(3m)-1} \frac{d\nu_0(a)}{(1 - \cosh m + a)^2} \leq \frac{9}{8} \int_{\cosh(3m)-1} \frac{d\nu_0(a)}{a^2}, \tag{3.51}$$

and hence, by (2.57) and (2.55),

$$Z^{-1} \leq \text{const} \times \left. \frac{\partial^2 \tilde{\Gamma}}{\partial p_1^2} \right|_{p=0} = - \text{const} \times \sum_x |x|^2 \Gamma(x). \tag{3.52}$$

Thus our previous argument proves also that

$$(ZJ)^{-1} \leq \text{const} \times (1 + g_0^2). \tag{3.53}$$

By (2.44), this also gives an upper bound on ξ/ξ_ϕ.

Summarizing the results of this chapter, we have proved the following:

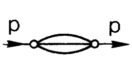
THEOREM 3.2. — Assume Conjecture 3.2, and consider lattice φ_d^4

models ($d < 4$) with g_0 fixed. Then $\xi_2^2/J\chi$, $(ZJ)^{-1}$ and ξ/ξ_ϕ are all bounded above as the critical point is approached; and if g_0 is sufficiently small (but nonzero), then the dimensionless renormalized coupling constant g is bounded away from zero as well. Thus, a continuum limit theory exists (by subsequences), satisfies all Osterwalder-Schrader axioms except perhaps Euclidean (rotation) invariance, and is non-Gaussian.

Remarks. — 1. For $d > 4$, we find [by (3.48)] that $\xi_2^2/J\chi$, $(ZJ)^{-1}$ and ξ/ξ_ϕ are all bounded provided that we take λ_0^{SM} bounded as we go to the critical point. (This extends [*30*, Theorem 5.1] and corrects an error in its proof; see Remark 2 following Proposition A.5. I should emphasize that the essential ideas of this section are taken almost verbatim from [*30*] [*64*].) Moreover, in this case the continuum limit theory is Gaussian; this is immediate consequence of (3.29). (This fact was anticipated in [*30*, Theorem 5.2], by an entirely different argument. For further discussion, see Appendix B.) Of course, it has recently been proven [*229-231*] that the continuum limit for φ_d^4 ($d > 4$) is Gaussian *no matter how* λ_0 *is varied* as we go to the critical point (see also Chapter 4); but it is possible for the limit to be achieved at different rates (i. e. with different critical exponents) for different modes of variation of λ_0. Moreover, one limit might be an ordinary free field (cf. [*30*, Theorem 5.2]) while another is a generalized free field. As will be seen in Section 4.2, λ_0^{SM} bounded is a natural borderline.

2. The same comments hold for $d = 4$, again by (3.48), provided that we take $\lambda_0^{SM} \leq O(1/\log \xi)$ as we go to the critical point. As will be seen in Section 4.2, this also is a natural borderline.

3. The heuristic basis of this section's argument is the following: For $d < 4$ (with g_0 fixed), the leading perturbative contribution to the field-strength renormalization, namely

$\frac{\partial^2}{\partial p^2}$ 

$$\Big|_{p=0} = \int d^d x \, | x |^2 G_0(x)^3 \qquad (3.54)$$

is ultraviolet convergent. The rigorous argument uses a correlation inequality to bound the exact field-strength renormalization by a quantity of the same form as (3.54), but with the exact propagator G replacing the free propagator G_0. We then use infrared bounds (and the Schrader—Messager—Miracle-Sole inequality) to bound this quantity by a constant times its free-field value [cf. (3.48)]. An exactly analogous (but simpler) argument [*87*] bounds the specific heat by a constant times its leading perturbative contribution

$$\Big|_{p=0} = \int d^d x G_0(x)^2 \,. \qquad (3.55)$$

4. The use of the G_6^{1PI} conjecture instead of the Γ_6 conjecture avoids some annoying extra terms [*30*, Proposition 3.2 and Theorems 5.1 and

6.2] which have the effect of shifting the critical dimension (speciously) from 4 to 5. The ease of the argument (3.45)-(3.46) shows again that the G_6^{1PI} conjecture is the « natural » one. The reason, of course, is that the field equations (3.14) and (3.43) give rise to correlation functions which are one-particle-irreducible in a *single* channel.

5. Paes-Leme [*65*] proves a $(3 - \varepsilon)m$ exponential decay rate for $\Gamma(x)$ [cf. (3.50)] *at high temperature.*

4. SPECULATIONS ON THE TRIVIALITY OF φ_d^4 ($d \geq 4$)

I would like to offer a theoretical prediction at the 5 % confidence level: within five years, there will be a rigorous construction of the solutions of $\lambda(\varphi^4)_4$ and of spin 1/2 quantum electrodynamics in four-dimensional space-time.

— Arthur Wightman (1977) [*88*].

4.1. Critical Review of Previous Work.

The existence or nonexistence of interacting quantum field theories in four-dimensional space-time has been the prime open question of axiomatic and constructive quantum field theory since their inception [*89*]. Despite much work and much controversy [*16*] [*90*], this remains an unsolved and apparently extremely difficult problem. In fact, recent renormalization-group ideas [*29*] [*91-94'*] have made clear that the behavior of the continuum limit in non-asymptotically-free field theories is inherently a strong-coupling problem, hence inaccessible by purely perturbative methods. As a result, the subject has remained in limbo since the pioneering work of the mid-1950's: although new insights [*91-96*] have *clarified* the problem, they have contributed (as yet) little toward *solving* it.

My purpose in this chapter is to make a modest contribution toward solving this problem, for the particular case of the φ^4 theory in space-time dimension $d \geq 4$. I argue in favor of the *nonexistence*: that is, I argue that the continuum limit of φ_d^4 lattice models ($d \geq 4$) is necessarily a (generalized) free field, irrespective of how the charge renormalization is performed. The essential physical idea goes back to Landau, Pomeranchuk and collaborators [*16-29*] a quarter-century ago; the key new ingredient is to use correlation inequalities to justify (in part) the Landau approximation. Although the arguments presented here are far from a complete rigorous proof, they do open up, in my opinion, a number of promising avenues for further investigation.

It is widely believed [*93*] [*94*] [*97*] (with, however, considerable dissent [*88*] [*98*]) that a nontrivial continuum φ_d^4 theory does not exist in dimension $d \geq 4$ (3). (Even the dissenters would generally concede the nonexistence for sufficiently high d – say, $d > 4$). The evidence in either direction is extremely sketchy, but it can be summarized in four categories: high-temperature expansions, random-walk ideas, partial summation of Feynman diagrams, and renormalization-group methods. (The last two are closely connected, as I shall explain below).

HIGH-TEMPERATURE EXPANSIONS [*99*].

This approach is theoretically unprejudiced and conceptually straightforward; on the other hand, its results are inherently inconclusive, inasmuch as they are based on numerical analyses with (as yet) uncontrolled errors. The basic idea is simple: the correlation functions of the φ_d^4 lattice model can be expanded in a power series in the nearest-neighbor coupling J, with a nonzero radius of convergence [*100-102*]; if this series converges up to the critical point J_c (or can be transformed to do so), then one can in principle extract a complete knowledge of the model as a function of the bare parameters B_0, λ_0 and J; in particular, one can survey all possible charge renormalizations and determine whether any of them lead to a nontrivial (i. e. non-Gaussian) continuum limit. Of course, the inherent difficulty is in extracting reliable numerical information from a *finite* number of terms of the infinite series. Although there do exist circumstances in which rigorous upper and/or lower bounds can be obtained from finite orders in a power series [*103*], none of these mathematical results is known (or believed) to apply to the case at hand (but see [*104*]).

The existing high-temperature analyses are of two kinds: those which survey the entire parameter space (B_0, λ_0, J) [*94*] [*105*] [*106*] (4), and those which survey only a restricted subset (usually the Ising model $\lambda_0 = \infty$) [*107-120″*]. The latter case, though sufficient for the purposes of the statistical mechanics of critical phenomena, in which λ_0 is fixed, is insufficient for the purposes of field theory, in which arbitrary charge renormalizations are allowed. If, however, the renormalized coupling g is monotonic in the bare coupling g_0 (at each fixed ξ) — as is generally believed [*12*] [*34*] [*86*] (with, however, some dissent for dimension $d \geq 3$ [*105*] [*106*]) — then it suffices [*12*] to investigate the continuum limit of the Ising model. This question can then be posed as follows: is the hyperscaling relation [*73*] [*107*] [*121*]

$$d\nu_\phi - 2\Delta_4 + \gamma = 0 \tag{4.1}$$

true (and unmodified by logarithms)? It is known rigorously [*72-74*]

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(3) Aizenman [*229*] [*230*] and Fröhlich [*231*] have recently proven essentially this result, for $d > 4$.

(4) Actually, one of the parameters, say B_0, is superfluous in view of the scaling (2.10).

that $dv_\phi - 2\Delta_4 + \gamma \geq 0$; the continuum limit of the Ising model is non-Gaussian if and only if the *equality* (4.1) holds (unmodified by logarithms). This is known to be true for $d=1$ [*122*] and $d=2$ [*123, 230*]; it is a two-decade-old open question for $d = 3$ [*124*] [*125*] [*107*] [*108*] [*111-115*] [*117-120″*] ([5]). It is generally believed that (4.1) holds for $d = 4$, but modified by logarithms [*34*] [*109*] [*110*] [*116*] and that (4.1) does *not* hold for $d > 4$. (These expectations are based, in part, on the renormalization group, discussed and criticized below. Also, Aizenman [*229*] [*230*] and Fröhlich [*231*] have quite recently *proven* this result for $d > 4$.) The latest high-temperature-series results for $d = 4$ [*109*] [*110*] [*116*] [*117*] are consistent with these beliefs, but do not rule out alternate possibilities [*119*]. A fair judgement, I believe, would be to describe these computational efforts as heroic but (as yet) inconclusive.

The theoretical agnosticism of the high-temperature-series method is not only its strength but also its weakness: whichever conclusion the numbers may lead to, one gains little physical insight into *why* it should be so. The other three methods have the opposite property: it is not obvious whether the arguments are to be believed, but they do in any case give a (more or less) clear picture of the *physical mechanism* by which φ_d^4 $(d \geq 4)$ is alleged to become free in the continuum limit.

RANDOM-WALK IDEAS.

Consider first, by way of analogy [*126*], a problem from nonrelativistic quantum mechanics: the motion of a particle in d-dimensional space under the influence of a repulsive delta-function potential. By analogy with quantum field theory, we consider the *regularized* Schrödinger operator

$$H_\varepsilon = -\Delta + \lambda_\varepsilon \delta_\varepsilon(x), \tag{4.2}$$

where δ_ε is an approximate delta function $\left[\delta_\varepsilon \geq 0,\ \delta_\varepsilon(x) = 0 \text{ if } |x| > \varepsilon,\right.$ and $\left.\int d^dx\,\delta_\varepsilon(x) = 1 \text{ with } \delta_\varepsilon \in C^\infty \text{ (say)}\right]$. Then ε^{-1} is the analogue of the ultraviolet cutoff Λ, and λ_ε is the « bare charge ». Now it can be shown [*127*; *33*, p. 72-73] that if $d \geq 2$, then the operator H_ε converges in strong resolvent sense to $H_0 = -\Delta$, as $\varepsilon \downarrow 0$, *irrespective of the choice of* λ_ε *as a function of* ε, provided only that $\lambda_\varepsilon \geq 0$. (If $d \geq 4$, the restriction $\lambda_\varepsilon \geq 0$ is unnecessary.) This result has a very simple interpretation in terms of the Feynman-Kac formula for the Euclidean propagator e^{-TH_ε}: the probability of a Brownian path hitting the ball of radius ε within the *finite* time interval $[0, T]$ *vanishes* as $\varepsilon \downarrow 0$, in dimension $d \geq 2$. (This follows, by countable additivity of the measure, from the fact [*33*, p. 83-84] that

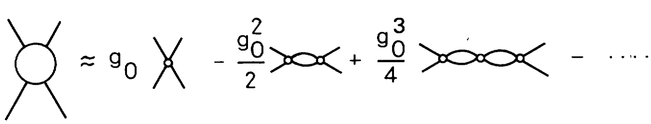
([5]) Of course, if hyperscaling fails for $d = 3$, then the monotonicity must also fail, since nontrivial continuum φ_3^4 models are known to exist !

a point is hit with probability zero.) Since $\lambda_\varepsilon \geq 0$ implies that the Feynman-Kac integrand is bounded (between 0 and 1), the contribution of this set of paths vanishes as $\varepsilon \downarrow 0$, and only the free-particle result remains. (This argument is a slight variant of the one given in [*33*].)

The analogous argument in quantum field theory is based on a representation of φ^4 theory due to Symanzik [*128*] [*129*], in terms of a gas of Brownian paths (see also [*130-131″*]). The interaction occurs whenever a Brownian path intersects either itself or another of the paths. But this occurs with probability zero, in dimension $d \geq 4$! [*129*] [*132*] (One should also ask about the probability that two paths come within ε of each other. This also vanishes as $\varepsilon \downarrow 0$ for $d > 4$; the case $d = 4$ is quite delicate.) Thus, by analogy with (4.2), one expects that the theory approaches a free field as the ultraviolet cutoff $\Lambda \to \infty$, *irrespective of the choice of* $\lambda_0(\Lambda)$ *as a function of* Λ, provided only that $\lambda_0 \geq 0$. Rigorous results in this direction are, as yet, meager; but see [*133-134*]. *Added Note:* Fröhlich *et al.* [*231*] [*232*] have recently used the random-walk ideas to derive rigorous correlation inequalities which imply, among other things, that the $\Lambda \to \infty$ limit is indeed a (generalized) free field, at least for dimension $d > 4$. The proof of Aizenman [*229*] [*230*] is also based on somewhat related ideas.

Partial Summation of Feynman Diagrams.

This is, in essence, the method of Landau, Pomeranchuk and collaborators [*16-23*] (but see also [*24-29*]), whose solution of approximate Schwinger-Dyson equations in the high-energy asymptotic regime is equivalent to the summation of the « leading logarithms » in the Feynman perturbation series. A very simple but instructive example is the bubble approximation to the connected 4-point function in φ_4^4 theory (with ultraviolet cutoff Λ):



(4.3)

The sum is just a geometric series; we find

$$g \approx \frac{g_0}{1 + (16\pi^2)^{-1} g_0 \log(\Lambda/m)} \tag{4.4}$$

for $\Lambda \gg m$. [Here we use the standard field-theoretic normalization convention, calling g what we previously called g_{STD} and m what we previously called $\tilde{m}_{\text{STD}}$; cf. (2.68) and (2.69).] Thus

$$0 \leq g \leq g_{\text{crit}}(\Lambda) \equiv \frac{16\pi^2}{\log(\Lambda/m)} \to 0 \tag{4.5}$$

as $\Lambda \to \infty$, *irrespective of the choice of* g_0 *as a function of* Λ, provided only that $g_0 \geq 0$. Thus, in this approximation, the continuum limit of a φ^4 theory is inevitably a (generalized) free field. This is the famous « zero-charge difficulty » of Landau [*16-25*] [*135*]. Of course, the real question is whether it is a property of the full φ^4 theory, or merely an artifact of the above (incredibly crude) approximation. In Section 4.2, I shall argue (but not prove !) that it is the former.

The physical mechanism underlying the result (4.4)-(4.5) is simple enough: *charge screening* [*18*] [*19*] [*136*]. It is most easily explained in the analogous example of quantum electrodynamics: a bare positive charge, for example, polarizes the vacuum (exactly as if the vacuum had a dielectric constant greater than unity), attracting the electrons of the virtual electron-positron pairs and repelling the positrons; thus the observed charge g is less than the bare charge g_0. As the ultraviolet cutoff Λ tends to infinity, the screening becomes complete, and g tends to zero. Moreover (and this is the crucial point), *it does no good to make* g_0 *increase with* Λ: for this has the side effect of increasing the efficiency of the screening, so that the increase in the denominator in (4.4) nullifies the gain in the numerator ! Of course, it is an open question whether this behaviour occurs also in the full, unapproximated theory.

It might be argued, however, that φ_4^4 has a perfectly good renormalized perturbation series. If this series is not asymptotic to a continuum φ_4^4 theory, then what *does* it mean ? Although I am unable to answer this question (but see [*137*]), it is worth demonstrating that the existence of a renormalized perturbation series has no bearing on the existence or nonexistence of the exact theory. Consider, for example, the bubble sum (4.3): renormalized perturbation theory would instruct us to fix $g > 0$ and to blindly choose $g_0(\Lambda)$, order-by-order, however necessary to maintain the selected g. But inspection of (4.4) reveals what we would, in our blindness, be doing: for Λ sufficiently large [so that $g > g_{\text{crit}}(\Lambda)$], we would be taking $g_0 < 0$, which is sure to lead to trouble [*138*] [*93*] [*135*]. (In the nonperturbative lattice theory (2.12), for example, this is clearly forbidden.) Sure enough, we get the trouble we deserve: the *renormalized* infinite-cutoff bubble sum (4.3) exhibits a tachyon pole (in contradiction with axiomatic requirements), the famous « Feldman-Landau ghost » [*139*] [*140*] [*24*] [*25*] [*135*] [*141*].

In any case, the success of the bubble sum in predicting the triviality of φ_4^4 is also its downfall: precisely *because* it vanishes as $\Lambda \to \infty$, it follows that *even a single one of the neglected diagrams is large compared to the sum* *of the ones considered.* This discouraging observation applies, of course, not only to the bubble sum, but to *any* incomplete partial sum, no matter how comprehensive, in which g is found to vanish as $\Lambda \to \infty$ (6). As a result, the Landau argument is tantalizing but ultimately inconclusive. We are left with the same old question, with even greater urgency: what can we say on an *exact*, nonperturbative basis about the φ_4^4 theory ?

Remarks. — 1. The exactly-soluble Lee model [*142-144*] [*80*] [*81*] exhibits precisely the behavior discussed above: the « zero-charge difficulty » and, in an otherwise unobjectionable renormalized perturbation series, the « ghost ». This is simply because, in the Lee model (which lacks anti-particles and crossing symmetry), the bubble sum is exact. Of course, this can hardly be taken as evidence that the same pathologies occur in realistic field theories such as the φ^4 theory. But it does show that such pathologies are not inconsistent with the existence of a renormalized perturbation series.

2. The same qualitative behavior is found in the summation of *all* « leading-logarithm » 4-point graphs (not just the bubble graphs). This is the so-called *parquet sum* [*20*] [*22*] [*145*] [*146*], which has also been applied to $(\bar{\psi}\psi)^2$ fermion models [*28*] [*147*], to the infrared behavior of φ^4 models [*148-152*], and to various solid-state problems [*153-156*]. The parquet approximation can be derived from the exact crossing-symmetric Bethe-Salpeter equations [*22*] [*146*] [*150*] [*157-160*] by approximating the 2-particle-irreducible (2PI) 4-point function as a point vertex [*146*] [*150*]. Of course, its use for studying the ultraviolet behavior of renormalizable (or nonrenormalizable) field theories is subject to the objections noted above (see also [*150*] and below).

3. The bubble sum (including crossed graphs so that we now get a geometric series in *each* channel) is also the leading term in the 1/N expansion for an N-component O(N)-symmetric φ^4 theory [*36*] [*135*] [*141*] [*161*]. Coleman, Jackiw and Politzer [*135*] argue that, although the Landau argument is suspect because it takes lowest-order perturbation theory too seriously in a regime where the renormalization-group-invariant coupling is in fact large (see below), the 1/N-expansion result is not open to any such easy criticism, since the location of the ghost pole is independent of the expansion parameter 1/N. However, this argument is insufficient: it is perfectly possible (and perhaps even likely [*131*]) that the remainder

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(6) Of course, this does not necessarily mean that partial resummations of the perturbation series are worthless. One could imagine, for example, rearranging the perturbation series into a number (presumably infinite) of infinite classes of diagrams, such that the *sum* of each class is well-behaved (or even vanishing) as $\Lambda \to \infty$. (This is essentially what occurs in renormalization-group-improved perturbation theory for *asymptotically-free* theories.) Of course, such a rearrangement of a highly divergent series is far from obviously valid, but neither is it manifest nonsense.

term in the 1/N expansion is *nonuniform* in the ultraviolet cutoff Λ, thereby invalidating the 1/N expansion for the study of the limit $\Lambda \to \infty$.

4. A similar attempt to sum infinite classes of divergent (but cutoff) graphs in the hope of finding a gentler behavior after summation was proposed about 15 years ago under the name of *peratization*. It has been applied to nonrenormalizable field theories [*162-168*] and to singular potentials in nonrelativistic quantum mechanics [*169*] [*170*], but not with much success [*170-174*].

5. Similar methods have been used, from a point of view opposite to that taken here, by Parisi [*175*].

Renormalization-Group Methods.

It was discovered long ago [*176-178*] [*29*] that the Landau *et al.* « zero-charge » and « ghost » results can also be derived by renormalization-group methods. The Landau formulae are, in fact, those found by lowest-order renormalization-group-improved perturbation theory [*141*] [*179*] —that is, by calculating the renormalization-group coefficient functions in lowest-order perturbation theory and then solving the renormalization-group equation *exactly*. (This is clearly equivalent to a partial summation of Feynman diagrams, although it may not always be clear which ones [*179*] [*180*].) This interpretation also makes clear the unreliability of the Landau result: for it is based on taking seriously the lowest-order perturbative result for the β function, i. e.

$$\beta(g) = \frac{3}{16\pi^2} g^2 \tag{4.6}$$

for φ_4^4 theory, even for *large g* where it is clearly nonsense. Assuming (4.6), one finds that the running coupling constant $\bar{g}(t)$, defined by $d\bar{g}/dt = \beta(\bar{g})$, is given by

$$\bar{g}(t) = \frac{g}{1 - (3/16\pi^2)gt}, \tag{4.7}$$

which reaches $+\infty$ at a *finite* t and then reappears at $-\infty$. [This is essentially the same as (4.4), up to a factor 3 arising from the fact that (4.6) incorporates contributions from all three channels. Crudely speaking, $\bar{g}(t)$ is analogous to the bare coupling $g_0(\Lambda)$ at cutoff $\Lambda \approx e^t m$.] The divergence of $\bar{g}(t)$ at finite t also induces a « ghost pole » in the various correlation functions. This disaster occurs for any theory in which

$$\int_\varepsilon^\infty \frac{dg}{\beta(g)} < \infty . \tag{4.8}$$

On the other hand, it could be avoided if β has a zero for some $g^* > 0$, or if β increases sufficiently slowly at infinity. Thus, the key question is the behavior of $\beta(g)$ at non-necessarily-small g: this is clearly a question which goes beyond perturbation theory.

Remarks. — 1. By « renormalization group » I always mean the field-theoretic renormalization group of Gell-Mann and Low, Callan, Symanzik, Weinberg, 'tHooft and others [*34*] [*91-93*] [*181*]. The « modern » renormalization group of Wilson [*94*] [*182*], which works within an infinite-dimensional space of Hamiltonians, is much deeper. For some crude results on φ_4^4 in this approach, see [*94*, Section 13]; for the $d = 4$ Ising model, see [*183*] [*184*].

2. The above discussion is vague because I cannot make much sense out of it. The field-theoretic renormalization group is concerned with how various continuum φ^4 theories transform into each other under dilations. But if the only continuum φ_4^4 theory is $g = 0$, what does the function $\beta(g)$ even mean ?

3. I have perhaps been too hasty in my repeated assertion that strong-coupling behavior cannot be computed from perturbation theory. In fact, resummation methods have been successfully applied to the anharmonic oscillator [*185-187*] and to φ_3^4 [*71*] [*188-190*] (but see [*119*]), allowing the quantitative (and quite accurate) computation of strong-coupling behavior from a manageable number of terms of the weak-coupling perturbation series. Khuri [*98*] has attempted an analogous Borel summation of the perturbation series for $\beta(g)$ in φ_4^4, using the first four terms of the series [*191*] [*192*]. However, Khuri's work is based on an *ad hoc* prescription for handling the renormalon singularities which seems difficult to justify; therefore his results cannot, in my opinion, be taken seriously (see also [*137*]).

4. The earlier work of Khuri [*193*] [*194*] is, on the other hand, quite interesting. It tends to show that if a nontrivial continuum φ_4^4 theory exists, then the correlation functions cannot be analytic in a sector in the complex g-plane; in particular, they cannot be the inverse Borel transform of any distribution [*195*] [*196*]. (This casts further doubt on the results of [*98*].) Essentially, the Landau ghost has reappeared, this time at slightly complex g [*197*].

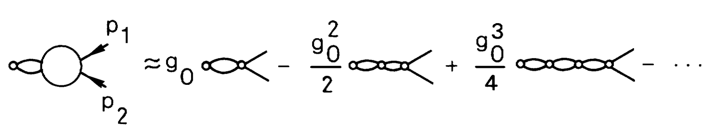
5. The renormalization group is often cited [*34*] [*107*] [*109*] [*110*] [*116*] as predicting that the $d = 4$ Ising model has mean-field critical exponents with logarithmic corrections. This argument implicitly assumes that the Ising model is within the « infrared domain of attraction » of the $g = 0$ fixed point —an assumption which may well be true, but is certainly an open question.

4.2. A Possible Destructive Approach to φ_4^4.

The Lee model is a very special one, considerably differing in several respects from physical interactions; and the validity of Pomeranchuk's proofs has been doubted. In my opinion such doubts are unfounded.
... It therefore seems to me inopportune to attempt an improvement in the rigour of Pomeranchuk's proofs, especially as the brevity of life does not allow us the luxury of spending time on problems which will lead to no new results.
— L. D. Landau (1959) [*198*].

In this section I shall give some heuristic arguments —and some ideas for a possible proof — that the continuum limit of φ_4^4 lattice theories ($d \geq 4$) is necessarily a (generalized) free field, *irrespective of how the charge renormalization is performed.* The arguments are motivated by the bubble sum (4.3)-(4.5); the idea is to use correlation inequalities (or integral-equation arguments) to show that the qualitative behavior found in the bubble sum is also a feature of the exact theory.

Let us first consider, for purposes of motivation, the bubble approximation to the connected 4-point function *with two arguments tied together:*



(4.9)

$$\approx \frac{g_0 F_\Lambda(p_1 + p_2)}{1 + \frac{1}{2} g_0 F_\Lambda(p_1 + p_2)} \tilde{G}(p_1)\tilde{G}(p_2) \tag{4.10}$$

where $F_\Lambda(p_1 + p_2)$ is the bubble integral with cutoff Λ and momentum $p_1 + p_2$ flowing through, i. e.

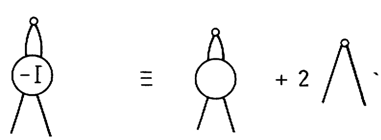
$$F_\Lambda(p) = \int^\Lambda \frac{d^4k}{(2\pi)^4} \tilde{G}(k)\tilde{G}(p - k) \tag{4.11}$$

$$\sim \log \frac{\Lambda^2}{p^2 + m^2} \qquad \text{(very crudely !)}. \tag{4.12}$$

Thus, as $\Lambda \to \infty$, $F_\Lambda(p)$ also approaches ∞. Provided that $g_0(\Lambda)$ is not chosen to vanish too fast as $\Lambda \to \infty$, we find that (4.10) approaches $\tilde{G}(p_1)\tilde{G}(p_2)$, which is just the Ising-like form (3.21) [up to a factor of 2 which should not be taken too seriously, in view of the complete mutilation of crossing symmetry involved in this approximation]. This behavior can also be expressed by saying that the fluctuations of the field φ^2 are much smaller than those of the field φ; in other words, the field φ is tending to become concentrated near two values, $-c$ and $+c$ (for some c). We thus conjecture the following physical picture: *ultraviolet divergences for $d \geq 4$ drive the theory toward Ising-like behavior for the field φ^2 (and more generally φ^n) as $\Lambda \to \infty$; and the theory becomes Gaussian in this limit.* This conjectured behavior is at least consistent with the belief that the Ising model itself becomes Gaussian in the continuum limit for $d \geq 4$ (recently *proven* by Aizenman [*229*] [*230*] and Fröhlich [*231*] for $d > 4$).

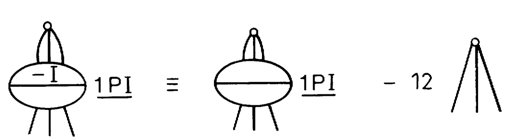
Remark. — If we take $g_0(\Lambda) \leq O(1/\log \Lambda)$ as $\Lambda \to \infty$, then (4.10) can avoid approaching the Ising-like form. Of course, this procedure is also guaranteed, by (3.29), to give us a Gaussian theory in the continuum limit. (See also Appendix B and Remark 2 of Section 3.3.) Thus, we get a Gaussian theory either way, but the rate of approach may be different; also, the limit might be an ordinary free field if $g_0(\Lambda) \leq O(1/\log \Lambda)$ but a generalized free field otherwise.

Motivated by the above conjecture, we rewrite the exact field equation (3.18)/(3.20) so as to exhibit explicitly the deviations from the Ising-like behavior (3.21) and (3.24); we then try to argue that these deviations vanish as $\Lambda \to \infty$. It is most convenient to use pictures rather than write out the equations. We define first the deviations from the Ising-like forms:



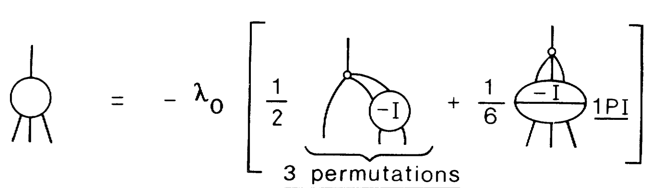
(4.13)

and



(4.14)

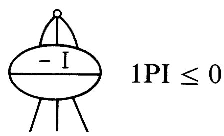
Substituting this into (3.20), we find



(4.15)

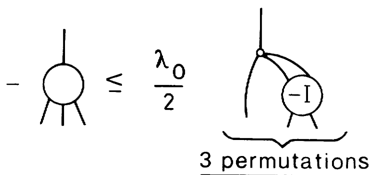
(Note that the tree-graph term has disappeared. This is as it had better be:

in the Ising limit $\lambda_0 \to \infty$, the left side approaches a finite limit, so the brackets had better vanish.) By Conjecture 3.2, we have



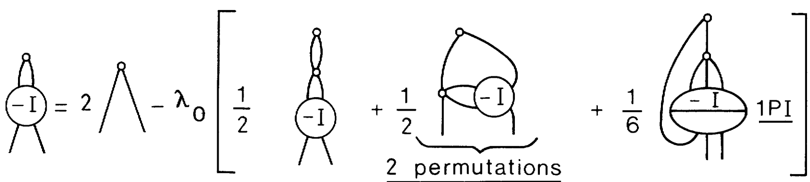
(4.16)

pointwise, so that



(4.17)

pointwise. Thus, to get an upper bound on $-\overline{u_4}$ (and hence on g), it suffices to get an upper bound on (4.13) evaluated at zero momentum. To get an equation for (4.13) in terms of itself, we tie together two arguments in (4.15):



(4.18)

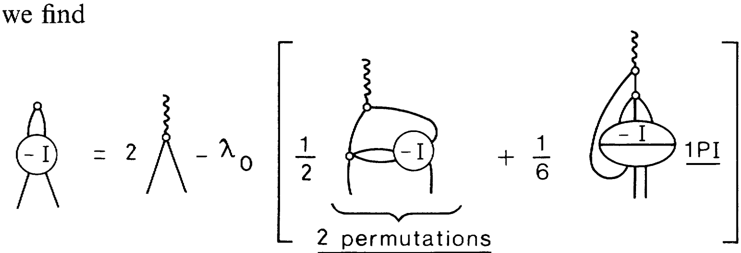
We can now bring the first term in the brackets to the other side and explicitly invert the resulting operator; defining

$$F(p) = \int \frac{d^d k}{(2\pi)^d} \tilde{G}(k)\tilde{G}(p-k) \tag{4.19}$$

(remember that this is an integral on the torus) and

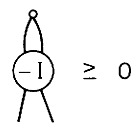
$$\text{〰〰} = \left[1 + \frac{\lambda_0}{2} F(p)\right]^{-1}, \tag{4.20}$$

we find



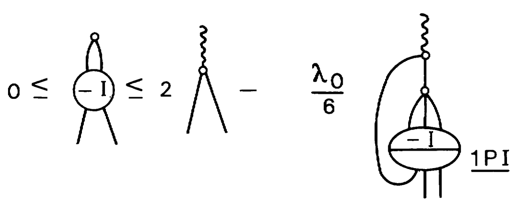
(4.21)

The first term on the right side of (4.21) is clearly the bubble sum (4.9); inserted into (4.17) it would give the bubble sum (4.3) in each of three channels. The question is: do the other two terms on the right side of (4.21) alter the qualitative behavior ? I have no definitive answer, but I can say this much: the *first* term in the brackets does not do so. This is because, by the correlation inequality (3.23), we have



(4.22)

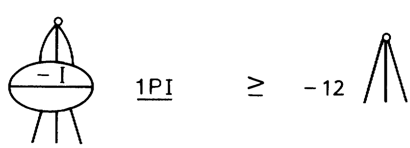
pointwise. Obviously the 2-point functions occurring in that term are also pointwise positive. The wiggly propagator (4.20) is not necessarily pointwise positive in position space, but it is positive when *summed* over position space, i. e. at zero momentum. We thus have



(4.23)

at zero momentum. In other words, the first term in the brackets in (4.21) would, if treated exactly, merely reinforce the vanishing of (4.13) found in the bubble approximation.

This is the end of the line: I do not know how to handle the 6-point function term occurring in (4.23). By (4.16) it has a sign which could hurt us. Of course, Conjecture 3.2 gives the lower bound

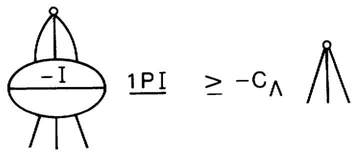


(4.24)

but this does us no good: inserted in (4.23) it produces an ultraviolet-divergent bubble which cancels the damping factor carried by the wiggly propagator; as a result, that term is O(1) rather than the hoped-for $O(1/\log \Lambda)$.

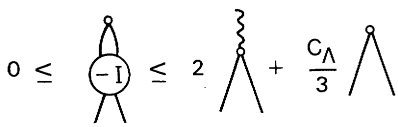
However, (4.23) *does* lend support to the physical picture conjectured above, for we have the following self-consistency result: *Assume* that the

6-point deviation from Ising vanishes in a suitable sense as the ultraviolet cutoff is removed, e. g.



(4.25)

pointwise, with $C_\Lambda \to 0$ as $\Lambda \to \infty$. Then the 4-point deviation from Ising also vanishes in this limit, in the sense that



(4.26)

at zero momentum, which vanishes as $\Lambda \to \infty$. [Here we assume, as before, that $\lambda_0(\Lambda)$ does not go to zero too fast as $\Lambda \to \infty$; to be precise, we assume that $\lambda_0 F(0) \to \infty$ as $\Lambda \to \infty$. Of course, if this is not the case, then the limit is Gaussian anyway, by (3.29).]

Remarks. — 1. Quite possibly the above argument can be extended, using the higher-point field equations, to show that if a suitable *8-point* deviation from Ising vanishes as $\Lambda \to \infty$, then so do the 6-point and 4-point ones, and so on. (However, one has fewer correlation inequalities available, at least at present, as one moves to higher-point functions.) Such a self-consistency argument would be useful from a heuristic point of view (if not from a rigorous one) by giving further confidence in the physical conjecture underlying the arguments of this section. Indeed, perhaps one can develop a formal perturbation expansion in powers of $1/\log \Lambda$ for the various correlation functions.

2. I have sloughed over an important point by tacitly assuming that the bubble integral (4.19) diverges as $\Lambda \to \infty$; since *interacting* propagators G occur, this is not entirely trivial. Let us return to the statistical-mechanical normalization convention, and evaluate the relevant quantities. Ignoring the 6-point function term in (4.23), and inserting (4.23) into (4.17), we find

$$0 \leq -\bar{u}_4 \leq 3\lambda_0 \chi^4 \left[1 + \frac{\lambda_0}{2} F(0) \right]^{-1},$$

or

$$0 \leq g \leq \frac{3\lambda_0 \chi^2}{\xi_2^d} \left[1 + \frac{\lambda_0}{2} F(0) \right]^{-1}. \tag{4.28}$$

To show that $g \to 0$ as $\xi \to \infty$, irrespective of the charge renormalization $\lambda_0(\xi)$, it suffices to show that

$$F(0)\xi^d/\chi^2 \to \infty \tag{4.29}$$

as $\xi \to \infty$. (I am sloughing over the distinction between ξ and ξ_ϕ.) That is, we need a *lower* bound on

$$F(0) = \int \frac{d^dk}{(2\pi)^d} \tilde{G}(k)^2 = \sum_x G(x)^2 . \tag{4.30}$$

I *conjecture* that

$$F(0) \geq \text{const} \times \begin{Bmatrix} \chi^2(1+\xi)^{-d} & \text{if } d < 4 \\ \chi^2(1+\xi)^{-4}\log(2+\xi) & \text{if } d = 4 \\ \chi^2(1+\xi)^{-4} & \text{if } d > 4 \end{Bmatrix}; \tag{4.31}$$

this would imply (4.29) for $d \geq 4$. But I have so far been unable to prove (4.31). The *weaker* inequality

$$F(0) \geq \text{const} \times \chi^2(1+\xi)^{-d} \tag{4.32}$$

for all d can be derived as follows: using the spectral representation (2.37) consecutively in all lattice directions, one finds

$$\begin{aligned} \tilde{G}(p) &\geq \tilde{G}(0) \prod_{i=1}^{d} \frac{\cosh m - 1}{\cosh m - \cos p_i} \\ &\geq \chi \prod_{i=1}^{d} \frac{m^2}{m^2 + p_i^2}; \end{aligned} \tag{4.33}$$

inserting this into (4.30), one obtains (4.32). One *expects* better than (4.33); for example if G had a Euclidean-invariant Källén-Lehmann representation, (4.33) would be replaced by

$$\tilde{G}(p) \geq \chi \frac{m^2}{m^2 + p^2}, \tag{4.34}$$

which does imply (4.31). Of course, a lattice theory cannot be exactly Euclidean-invariant, but one does expect approximate Euclidean invariance for theories near the critical point (i. e. with $\xi \gg 1$). Equivalently, a proof of (4.34)/(4.31) is connected with the Ornstein-Zernike decay of the 2-point function [*65*] [*199*] [*200*], i. e. to the expectation that

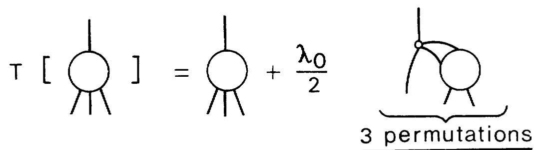
$$G(x) \leq \text{const} \times e^{-m|x|} |x|^{-b} \tag{4.35}$$

with $b = (d-1)/2$ rather than (the apparently also allowed) $b = 0$.

A partially alternate approach to proving the triviality of continuum φ_d^4 ($d \geq 4$) is based on the observation that the field equation (3.18)/(3.20) is a *linear* integral equation for G_4, with the tree graph and the 6-point-function term acting as an inhomogeneity. (The kernel of this integral equation depends, of course, on the interacting propagator G.) The bubble sum tells us that *if* this integral operator is mutilated so as to replace the three permutations in (3.20) by a single one (thereby wrecking crossing symmetry), then the solution G_4 becomes small in the continuum limit (provided we invoke Conjecture 3.2 to bound the inhomogeneity G_6^{1PI}). It would be of great interest, therefore, to investigate the properties of the unmutilated linear integral equation

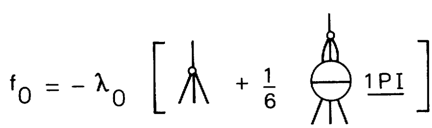
$$T[G_4] = f_0, \tag{4.36}$$

where



(4.37)

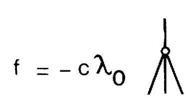
and



(4.38)

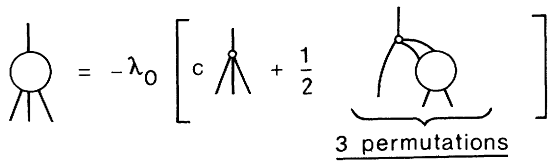
The conjecture is that T has no nullspace, and that $T^{-1}[f]$ is « small » for « reasonable » inhomogeneities f. A rigorous understanding of the properties of the linear operator T could well lead to a rigorous proof of the triviality of the continuum limit for $d \geq 4$ [modulo, as always, the correlation inequalities required to bound the inhomogeneity (4.38)].

As a warm-up to this problem, it might be interesting to study the integral equation $T[G_4] = f$ for a restricted class of inhomogeneities f, for example,



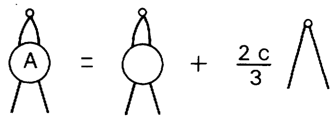
(4.39)

Here $c = 1$ corresponds to the naive (Gaussian) approximation $G_6^{1PI} = 0$, while $c = 3$ corresponds to the approximation giving G_6^{1PI} its Ising-like value (3.24). As explained earlier in this section, I believe that $c = 3$ is a good approximation for $d \geq 4$ and large Λ; but the point now is that *it does not matter what c is*, as long as it is bounded — this just alters the solution by an irrelevant constant factor. Let us study, therefore, the integral equation with the inhomogeneity (4.39), i. e.



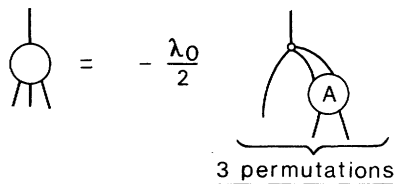
(4.40)

By analogy with (4.13), define



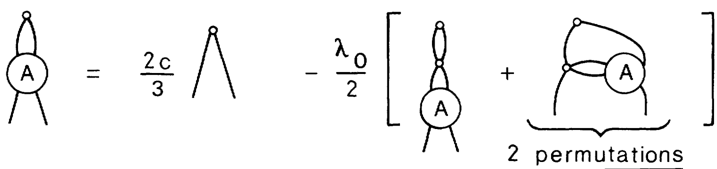
(4.41)

We then have



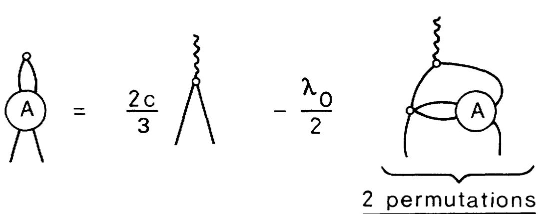
(4.42)

and



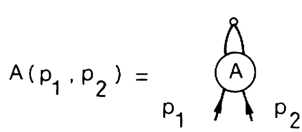
(4.43)

As before, we can bring the first term in the brackets to the other side; introducing the wiggly propagator (4.20), we find



(4.44)

Let us now define



(4.45)

and

$$B(p_1, p_2) = \frac{3}{2c} A(p_1, p_2)\tilde{G}(p_1)^{-1}\tilde{G}(p_2)^{-1} . \tag{4.46}$$

(This just amputates the legs on A, for convenience.) Then (4.43)/(4.44) can be written

$$\left[1 + \frac{\lambda_0}{2} F(p_1 + p_2)\right] B(p_1, p_2)$$
$$= 1 - \frac{\lambda_0}{2} \int \frac{d^dk}{(2\pi)^d} \tilde{G}(k)\tilde{G}(p_1 + p_2 + k)[B(p_1, k) + B(p_2, k)] , \tag{4.47}$$

where the bubble F is defined by (4.19), and is expected to behave crudely like (4.12) [in the field-theoretic normalization convention].

Unfortunately, I am unable to say much of anything about the properties of the linear integral equation (4.47). I *conjecture* that the solution is unique and behaves crudely for large Λ like the leading approximation

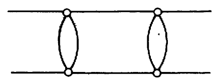
$$B(p_1, p_2) \approx \left[1 + \frac{\lambda_0}{2} F(p_1 + p_2)\right]^{-1} , \tag{4.48}$$

i.e. $B \sim O(1/\log \Lambda)$, but I have no particularly strong evidence for this belief. Perhaps an approximate solution in the regime $\Lambda \gg p \gg m$ can be found by assuming that $B(p_1, p_2)$ is a « slowly varying » function of p_1, p_2 [as (4.48) is]; see [*20*] [*22*] [*145*] [*148*] [*149*] [*152*] for a similar approximation in the parquet problem.

Remarks. — 1. In the first part of this section we were manipulating the *exact* field equation, interpreted as one true statement (among many) about the φ_d^4 theory; thus, we did not need to consider an *arbitrary* solution of the equations, but could restrict our study to those solutions satisfying other known exact properties, such as correlation inequalities. This crucial fact allowed the use of (4.22) to discard the crossed terms involving the 4-point function. Here, however, we are considering the *approximate* field equation (4.40); thus, we have no right to invoke correlation inequalities. (Moreover, since the magnitude of c is unknown, so is the sign of A.) Now I *suspect* that a solution to (4.47) necessarily satisfies $B(p_1, p_2) \geq 0$ [if so, it would immediately imply that (4.48) is an upper bound], but this must be *demonstrated* using only (4.47).

2. The approximate equation (4.40) is closely related to, but *not* identical to, the parquet approximation [*20*] [*22*] [*145*] [*146*]. Every graph obtained in the iteration of (4.40) is a parquet graph, but not all parquet graphs are obtained; in fact, the only graphs obtained are those in which two of the four external legs meet immediately at a vertex, and in which the graph obtained by the deletion of this vertex and its two attached external legs

still has this property. [This is immediate from inspection of (4.40).] For example, the parquet graph



(4.49)

and higher ladder graphs do *not* appear in the iterative solution to (4.40). Moreover, those graphs which do occur are given different weights than would be assigned in the parquet approximation (even if $c = 1$); this is because some of the graphs that would have been contributed by the G_6^{1PI} term are parquet graphs.

3. The approximate equation (4.40) is also considered, from a different point of view, by Bender *et al.* [*201*].

4.3. Another Possible Destructive Approach to φ_4^4.

In the preceding section I explained an approach to φ_4^4 that would, if correct, give an understanding of the *dynamical mechanism* by which φ_4^4 becomes trivial in the continuum limit (and would even allow calculation of the rate at which this occurs as $\Lambda \to \infty$). Here I should like to explain an alternate idea which might yield an essentially *axiomatic* proof that continuum φ_d^4 ($d \geq 4$) is necessarily a (generalized) free field. The disadvantage of this method is that it affords less physical insight; the advantage is that it is much more general, covering a wide range of possible ways in which one might try to construct a continuum φ_d^4 theory, including but not limited to lattice approximations.

The key observation is that the Griffiths/Lebowitz inequality (3.23), or more generally the Ginibre/Lebowitz inequality [*46*] [*74*]

$$- 2 \min \left[G_{k_1k_3}G_{k_2k_4}, \text{ permutations} \right] \leq G_4(k_1, k_2, k_3, k_4) \leq 0 \quad (4.50)$$

[see (2.109)], is *violated* in every order of renormalized perturbation theory for the continuum φ_4^4 theory, for any configuration of k_1, k_2, k_3, k_4 which brings a pair of arguments sufficiently close while keeping all others well separated [7]. This is because the bringing together of two arguments produces an ultraviolet divergence in G_4 — proportional to $\log(1/\varepsilon)$, where ε is the separation of the arguments — even though the bounds (3.23) and (4.50) remain perfectly finite. (The same is true for the tree graph in dimension $d > 4$; the divergence is proportional to ε^{4-d}. Higher graphs, being unrenormalizable for $d > 4$, can't really be considered.) Thus, the

[7] It is even allowed to have *two* close pairs, provided that these two pairs are kept separated from each other.

ultraviolet behavior of G_4 in any continuum φ_d^4 field theory ($d \geq 4$) must be considerably *softer* than that predicted by perturbation theory. On the other hand, as I shall argue (but not prove !), this is unlikely for an interacting field theory. The only way out is to have $G_4 = 0$, i. e. a Gaussian theory.

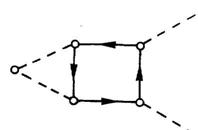
The belief that the exact ultraviolet behavior of G_4 must be at least as « hard » as that found in lowest-order perturbation theory comes from at least two sources:

1) Analogy with the 2-point function.

Positivity of the metric in the physical Hilbert space (equivalently, Osterwalder-Schrader positivity in Euclidean space) implies, via the Källén-Lehmann spectral representation, that the exact 2-point function is at least as singular in the ultraviolet region as is the free 2-point function. Of course, this is hardly much evidence that the same behavior occurs for the connected 4-point function, but it is suggestive. (Even the analogy is not precise: we wish to compare G_4 not to its free-field value (which is zero !), but to its value in *first*-order perturbation theory.)

2) Conformal-invariant skeleton theory.

If the Callan-Symanzik β function [*92*] [*93*] [*181*] has a zero at some g^*, then it is believed [*202*] [*203*] that the continuum φ_4^4 theory *at* $g = g^*$ will be asymptotically conformal-invariant at momenta $p \gg m$. Now conformal invariance (together with Osterwalder-Schrader positivity) puts considerable constraints on the correlation functions, and it may be possible to test whether these constraints are compatible with (3.23) and (4.50). Unfortunately, the conformal-invariant 4-point function retains considerable arbitrariness [*204-206*], so I have been unable to say anything concrete. However, in a conformal-invariant Yukawa theory (or theory of two interacting scalar fields), the 2-point and 3-point functions are completely determined, up to a multiplicative constant, by the dimensions of the fields [*207-209*]; and it is explicitly seen [*207*] that they are at least as singular in the ultraviolet as the ordinary perturbative forms. Moreover, in the conformal-invariant skeleton expansion [*207*] for the 4-point function, (3.23) is violated: the graph



(4.51)

(with conformal-invariant propagators and vertices) is logarithmically divergent for $d = 4$, irrespective of the dimensions of the fields. Again, this does not rule out the possibility that the exact 4-point function might be softer than each individual term in the skeleton expansion, but it is suggestive.

Remarks. — 1. The *amputated* 2-point and 3-point functions have *softer* ultraviolet behavior in a conformal-invariant theory than in ordinary perturbation theory [*207*]; it is crucial, therefore, that (3.23) and (4.50) refer to the *unamputated* 4-point function G_4. For this reason, an analogous argument based on the LSZ theorem on the vertex function [*79-84*] would not work (cf. the comments in [*80*]).

2. The correlation inequalities (3.23) and (4.50) are proven first, of course, in the lattice theory; they obviously carry over to the continuum limit. To be sure, other methods of taking the continuum limit (for example, those employing Pauli-Villars regularization) could cause these inequalities to be violated in the cutoff theory. But I think it is fair to say that any *continuum* scalar field theory violating (3.23) and (4.50), though it may be a perfectly good theory, ought not to be called a φ^4 *theory*.

3. Hidenaga Yamagishi has pointed out to me the possibility that φ_4^4 exists but that $g^* = \infty$, so that no conformal-invariant theory exists. Alternatively, one could have $g^* < \infty$ but things sufficiently singular so that no theory *at* g^* exists.

The problem can thus be stated: is (4.50) compatible with Osterwalder-Schrader positivity (or O-S positivity *and* conformal invariance [*209*] [*210*]) with $d \geq 4$ and $G_4 \not\equiv 0$?

Remark. — If one assumes also that the given continuum theory satisfies the Lee-Yang theorem (as a limit of lattice φ^4 theories would certainly do), then $G_4 \equiv 0$ implies that the theory is a (generalized) free field [*15*].

5. SOME OPEN QUESTIONS

This has been a long paper, with both many facts and many conjectures. Its purpose is to propose an alternate program for the construction of the φ_3^4 theory, and a possible program for the destruction of the φ_d^4 theory ($d \geq 4$). The first of these programs has been carried quite far; the second is still extremely speculative. Since my goal is to stimulate further research, I should like to close by stating as clearly as I can that which I do *not* know.

Only two points remain incomplete in the φ_3^4 program: the proof of Conjecture 3.2 and the proof of Euclidean invariance.

1) CONJECTURE 3.2.

This conjecture appears quite difficult to prove; the key stumbling block is the appearance of the inverse propagators Γ in the definition of G_6^{1PI}, which makes (3.25) quite unlike any correlation inequality yet proven. It is worth noting that Γ makes sense in an arbitrary finite lattice model; it is just the matrix inverse of the positive-definite matrix

$$G_{ij} = \langle \varphi_i \varphi_j \rangle . \quad (5.1)$$

As a first step towards understanding inequalities involving Γ, one might attempt:

1*a*) Prove (or disprove) that $\Gamma_{ij} \leq 0$ for $i \neq j$, for a suitable class of lattice models. This is known to be true for Gaussian ferromagnets (by a trivial calculation), for spin $-\frac{1}{2}$ Ising models of up to 3 sites (!) (by direct calculation or by using [*10*, Corollary A.3] combined with Griffiths' inequality

$$\langle \varphi_i \varphi_j \rangle = \langle \varphi_i \varphi_j \varphi_k^2 \rangle \geq \langle \varphi_i \varphi_k \rangle \langle \varphi_j \varphi_k \rangle), \quad (5.2)$$

and for Euclidean-invariant O-S-positive field theories (by the spectral representation [*34*]). Certain special cases are also known for reflection-positive lattice models, by the spectral representation (2.55). But the goal is to prove the inequality by purely algebraic or combinatoric means, *without* using the detailed geometric structure of the lattice. Already for 4×4 matrices the conditions for a matrix G to have an inverse with negative off-diagonal elements become quite complicated [*211-213*].

1*b*) Prove (or disprove) Conjecture 3.2 and (3.27) for the *one-site* φ^4 model (also called the $d = 0$ model or the « toy integral »).

1*c*) Study Conjecture 3.2 numerically, either by Schrödinger techniques for the case $d = 1$ [*86*], or by high-temperature expansions resummed with Padé approximants.

1*d*) Verify that Conjecture 3.2 is valid in the leading order of perturbation theory around the Ising model [*214-216*], i. e. that (4.16) holds in leading order in $1/\lambda_0$.

2) Euclidean invariance.

The disadvantage of the lattice as an ultraviolet cutoff is that it is far from obvious how Euclidean invariance is recovered in the continuum limit. Indeed, this is a detailed dynamical question [*94*, Section 12.3], and is presumably *not* true in general (e. g. compare [*217*] [*217'*] with [*218-218''*]). Of course, the φ_3^4 (and φ_2^4) theories constructed here *are* known to be Euclidean-invariant [*3-6*] [*10*] [*11*] [*236*], but the proof uses all the complicated machinery which our original motivation was to avoid. At present, I have no good idea how to prove the Euclidean invariance of the continuum limit within the present approach. Perhaps the method of Streater [*219*] [*220*] can be of some use. A first step might be to prove a Lorentz-invariant spectral condition for the 2-point function, by first proving a lattice analogue; the conjecture is that the « energy gap » $\omega(\boldsymbol{p})$ [*56*] [*65*] should be bounded below by something which approaches $|\boldsymbol{p}|$ in the continuum limit. Preliminary investigations of this issue have been made by the author and Arthur Wightman, using the diagonal spectral representation [*57*].

The program sketched here for the study of the φ_d^4 theory ($d \geq 4$) is at a much earlier stage. Among the key open questions are:

3) Figure out what to do with the G_6^{1PI} term in (4.23).

4) Prove (or disprove) the bound (4.31).

5) Study rigorously the integral equation (4.36)-(4.38). As a warm-up problem, study rigorously (4.40)/(4.47). Or even try to calculate, non-rigorously, the qualitative behavior of the solutions of (4.47).

6) Prove (or disprove) that (4.50) combined with Osterwalder-Schrader positivity (or O-S positivity *plus* conformal invariance) implies $G_4 \equiv 0$ for $d \geq 4$. Or try it first for $d \gg 4$.

Finally, here are some related interesting questions:

7) Prove (or disprove) that ξ_ϕ is a decreasing function of B_0, and an increasing function of J (cf. Remark 2 following Proposition 2.1).

8) Prove (or disprove) that g is an increasing function of g_0, *for fixed* ξ (the so-called Schrader monotonicity [*12*] [*86*] [*105*] [*106*] [*235*]). This is presumably a much harder problem than # 7, although of the same genre. It is worth noting that the analogous monotonicity can be proven in the φ_0^4 theory [*221*].

9) Can the methods of Aizenman [*229*] [*230*] or Fröhlich *et al.* [*231*] [*232*] be used to verify (or falsify), at least for $d > 4$, the physical picture conjectured in Section 4.2 ? Specifically, is it true that (4.13) and (4.14) approach zero in the continuum limit ?

10) What can be said about φ_4^4 using the methods of Aizenman [*229*] [*230*] or Fröhlich *et al.* [*231*] [*232*] ?

APPENDIX A

UNIVERSAL BOUNDS ON THE 2-POINT FUNCTION

The main purpose of this appendix is to derive universal upper bounds on expressions of the form

$$\sum_x |x|^\alpha G(x)^\beta \tag{A.1}$$

with $\alpha \geq 0$ and $\beta \geq 1$. Along the way, however, we shall derive some pointwise upper bounds on $G(x)$ [Lemmas A.1 through A.3] which are of some interest in their own right.

Henceforth we consider G to be the *truncated* 2-point function

$$G(x) = \langle \varphi_0 \varphi_x \rangle - \langle \varphi_0 \rangle \langle \varphi_x \rangle \tag{A.2}$$

in some translation-invariant, ergodic equilibrium state. The results of this appendix are then applicable to one-component models with arbitrary (not necessarily even) single-spin measure and ferromagnetic nearest-neighbor interaction, either above or below the critical temperature.

We shall use the following properties of G:

(A) $G(x) \geq 0$ (A.3)

(B) $\tilde{G}(p) \geq 0$ (A.4)

(C) $\tilde{G}(p) \leq \text{const}/Jp^2$ [with $J > 0$] (A.5)

(D) There exists a universal constant $c > 0$ such that $G(y) \geq G(x)$ whenever $|y| \leq c|x|$. (A.6)

(A) is a consequence of either the FKG inequality [*6*] [*49*] or the Percus inequality [*45*] (or of Griffiths' second inequality if the single-spin measure is even). (B) follows from the positive definiteness of G, by Bochner's theorem. (C) is the Fröhlich-Simon-Spencer infrared bound [*58*] [*55*] [*59*], which is a consequence of reflection positivity. (D) is the most subtle of the four properties; it is a consequence of the Schrader — Messager — Miracle-Sole inequalities (2.29) and (2.30), which imply that

$$G(y) \geq G(x) \qquad \text{whenever} \qquad |y|_1 \leq |x|_\infty . \tag{A.7}$$

In particular, this occurs whenever $|y| \leq d^{-1}|x|$; so (D) holds with $c = d^{-1}$.

Remarks. — 1. Strictly speaking, (B) says that $\tilde{G}$ is a positive *measure*. By ergodicity, G vanishes at infinity at least in the mean-square sense, so Wiener's theorem [*222*, Theorem XI.114] implies that $\tilde{G}$ has no pure point part. The assertion of (C) is then that $\tilde{G}$ is in fact absolutely continuous with respect to Lebesgue measure, with Radon-Nikodym derivative bounded (a. e.) by const/Jp^2. The derivation of this property is slightly subtle when $d \leq 2$, because the bound is non-integrable. However, it is no loss for our purposes to assume that $\chi < \infty$ in this case (i. e. $G \in l^1$, hence $\tilde{G} \in L^\infty$); then no difficulties arise. (Actually, the bound is true without this assumption, as Jean Bricmont has explained to me.)

2. (D) holds equally well for non-even single-spin measures, because the Schrader — Messager — Miracle-Sole inequality has been proven also in « Percus form » [*54*, Theorem 2; *223*].

3. Jean Bricmont has remarked to me that (A)-(D) hold also for the plane rotator; see [*224*] for the Schrader — Messager — Miracle-Sole inequalities in this case, and [*31'*] for an application of Lemma A.3.

4. At times we shall need only the following weakened version of property (D):

(D') There exists a universal constant $c' > 0$

$$\text{such that } \# \{ y \in \mathbb{Z}^d : G(y) \geq G(x) \} \geq c'(|x| + 1)^d . \quad (A.8)$$

It is easy to see that (D) implies (D').

Define now the « susceptibility »

$$\chi = \tilde{G}(0) = \sum_x G(x) . \quad (A.9)$$

Note that $G(x) \geq 0$ implies that

$$|\tilde{G}(p)| \leq \chi \qquad \text{for all } p . \quad (A.10)$$

We then have the following pointwise upper bounds on $G(x)$:

LEMMA A.1. — Assume (A) and (D'). Then

$$0 \leq G(x) \leq \chi / c'(|x| + 1)^d . \quad (A.11)$$

Proof. — $\chi = \sum_y G(y) \geq c'(|x| + 1)^d G(x)$ by (D'). ■

Remark. — A similar argument shows that

$$0 \leq G(x) \leq \text{const} \times \chi \xi_\phi^\phi / (|x| + 1)^{d+\phi} \quad (A.12)$$

for $x \neq 0$.

LEMMA A.2. — Assume (A) and (C). Then

$$0 \leq G(0) \leq \text{const} \times \begin{Bmatrix} J^{-1}(J\chi)^{(2-d)/2} & \text{if } d < 2 \\ J^{-1} \log (1 + J\chi) & \text{if } d = 2 \\ J^{-1} & \text{if } d > 2 \end{Bmatrix} \quad (A.13)$$

If, moreover, either (B) or (D) holds, then $0 \leq G(x) \leq G(0)$ and so $G(x)$ is also bounded by (A.13).

Proof. — By (A) [cf. (A.10)] and (C), we have

$$G(0) = (2\pi)^{-d} \int d^d p \tilde{G}(p) \leq (2\pi)^{-d} \int d^d p \min [\chi, \text{const}/Jp^2] . \quad (A.14)$$

From this we easily deduce (A.13). $G(x) \leq G(0)$ is an easy consequence of either (B) or (D). ■

LEMMA A.3. — Assume (C) and (D) and $d > 2$. Then

$$G(x) \leq \text{const}/J(|x| + 1)^{d-2} . \quad (A.15)$$

Proof. — Let $\chi_L(x) = 1$ if $|x|_\infty \leq L$, 0 otherwise. Then

$$\tilde{\chi}_L(p) = \prod_{j=1}^{d} \frac{\sin\left(L + \frac{1}{2}\right)p_j}{\sin (p_j/2)} . \quad (A.16)$$

Now

$$(\chi_L * G * \chi_L)(0) = (2\pi)^{-d}\int d^dp\tilde{G}(p)\tilde{\chi}_L(p)^2 \leq \text{const} \times J^{-1}\int d^dp\tilde{G}_0(p)\tilde{\chi}_L(p)^2$$
$$= \text{const} \times J^{-1}(\chi_L * G_0 * \chi_L)(0), \quad (A.17)$$

where $\tilde{G}_0(p) = \left[2\sum_{i=1}^{d}(1 - \cos p_i)\right]^{-1}$ is the free massless lattice field (defined only for $d > 2$). Now $G_0(x) \sim |x|^{-(d-2)}$ for large $|x|$, so that G_0 is in weak $L^{d/(d-2)}$. Hence, by the generalized Young inequality (or equivalently, the Hardy — Littlewood — Sobolev inequality) [*225*, p. 30-32],

$$(\chi_L * G * \chi_L)(0) = \sum_{\substack{|x|_\infty \leq L \\ |y|_\infty \leq L}} G(x - y) \geq (2L + 1)^{2d} \min_{|y|_\infty \leq 2L} G(y), \quad (A.19)$$

so that

$$\min_{|y|_\infty \leq 2L} G(y) \leq \text{const} \times J^{-1}(2L + 1)^{-(d-2)}. \quad (A.20)$$

But if we take $L = [(c/2d^{1/2})|x|]$, we have $|y| \leq d^{1/2}|y|_\infty \leq c|x|$ wherever $|y|_\infty \leq 2L$, so that property (D) implies that

$$G(x) \leq \min_{|y|_\infty \leq 2L} G(y) \leq \text{const} \times J^{-1}(|x| + 1)^{-(d-2)}. \quad \blacksquare \quad (A.21)$$

Remarks. — 1. My proof of Lemma A.3 is based on ideas of Bricmont *et al.* [*31*]. They obtain the much weaker conclusion (but good enough for their purposes)

$$G(x) \leq \text{const} \times \begin{cases} J^{-1}(|x| + 1)^{-1} \log(|x| + 2) & \text{if } d = 3 \\ J^{-1}(|x| + 1)^{-1} & \text{if } d \geq 4 \end{cases} \quad (A.22)$$

under (essentially) the much weaker hypothesis (2.29) [which is a consequence of reflection positivity] instead of (D).

2. For $d = 3$ it suffices to assume (B) and (D') in place of (D). To see this, note that (B) and (C) together imply that $\tilde{G}$ is in weak $L^{3/2}$; so by the weak Hausdorff-Young inequality [*225*, p. 31-32], G is in weak L^3, i. e.

$$\#\{ y \in \mathbb{Z}^d : |G(y)| > c_1\varepsilon/J \} < \varepsilon^{-3} \quad (A.23)$$

for a suitable universal constant c_1. But take $\varepsilon = c'^{-1/3}(|x| + 1)^{-1}$. Then $G(x) > c_1\varepsilon/J$ together with (D') would contradict (A.23). Hence $G(x) \leq c_1\varepsilon/J$, i. e. (A.15) holds. I do not know whether a similar proof can be constructed for $d \geq 4$; it would have to be more subtle, since the weak Hausdorff-Young inequality applies only for $1 < p < 2$, hence $2 < d < 4$.

3. Fröhlich, Simon and Spencer [*58*] note that, alas, the p-space infrared bound $\tilde{G}(p) \leq J^{-1}\tilde{G}_0(p)$ does *not* imply its x-space analogue $G(x) \leq J^{-1}G_0(x)$. It is pleasant to know, therefore, that this x-space bound does hold *up to a universal multiplicative constant*, provided that (D) is valid.

We are now prepared to derive an upper bound on (A.1). We first prove a special case by a direct p-space method [which does not require hypothesis (D)]; we then prove the general case by an x-space method using Lemmas A.1-A.3. To state the bound, define first the critical dimension

$$d_c = (\alpha + 2\beta)/(\beta - 1) > 2; \quad (A.24)$$

d_c is the dimension at which (A.1) barely diverges if $G = G_0$, the free massless lattice field.

We can easily handle in p-space the case $\alpha = 0$, β integer:

Proposition A.4. — Assume (A), (B) and (C), and let $\beta \geq 1$ be an integer. Then

$$\sum_x G(x)^\beta \leq \text{const} \times \begin{cases} J^{-\beta}(J\chi)^{(\beta-1)(d_c-d)/2} & \text{if } d < d_c \quad (A.25a) \\ J^{-\beta}\log(1+J\chi) & \text{if } d = d_c \quad (A.25b) \\ J^{-\beta} & \text{if } d > d_c \quad (A.25c) \end{cases}$$

with $d_c = 2\beta/(\beta - 1)$.

Proof. — Let $\beta = n$; we then compute in p-space

$$\sum_x G(x)^n = \text{const} \times (\underbrace{\tilde{G} * \ldots * \tilde{G}}_{n \text{ factors}})(0) \tag{A.26}$$

$$\leq \text{const} \times \| \tilde{G} \|^n_{n/(n-1)} \tag{A.27}$$

by Young's inequality. (For $n = 1$ it is trivial, for $n = 2$ it is the Plancherel theorem.) We now use (A), (B) and (C):

$$\| \tilde{G} \|_{n/(n-1)} = \left(\int d^dp \, | \tilde{G}(p) |^{n/(n-1)} \right)^{(n-1)/n} \leq \left(\int d^dp \min [\chi, \text{const}/Jp^2]^{n/(n-1)} \right)^{(n-1)/n} \tag{A.28}$$

This is easily seen to imply (A.25*a*) [if $d < d_c$] or (A.25*c*) [if $d > d_c$]. If $d = d_c$, on the other hand, (A.28) does *not* imply (A.25*b*) but rather the weaker bound with log replaced by $\log^{n-1}$ (note that here $n \geq 2$). Even if we return to (A.26) and use instead the *generalized* Young inequality [*225*, p. 31-32], which allows all but two of the factors $\tilde{G}$ to be estimated by their *weak* $L^{n/(n-1)}$ norms, we still get only $\log^{2(n-1)/n}$. To get the correct behavior, we insert into (A.26) the bound

$$0 \leq \tilde{G}(p) \leq \text{const} \times J^{-1}\tilde{G}_a(p), \tag{A.29}$$

where

$$\tilde{G}_a(p) = \left[2a + 2\sum_{i=1}^{d} (1 - \cos p_i) \right]^{-1} \tag{A.30}$$

is the free massive lattice field and $a = (2J\chi)^{-1}$. We then return to x-space to compute $\sum_x G_a(x)^n$; since G_a satisfies hypothesis (D) [it is, after all, the 2-point function of a nearest-neighbor Gaussian ferromagnet], this can be estimated by the method of Proposition A.5 below [cf. (A.34)]. The result is (A.25*b*). ■

Remarks. — 1. There ought to be a straightforward p-space argument yielding (A.25*b*); I have simply been too stupid to see it.

2. The case $\beta = 2$ of Proposition A.4 has been used to derive a rigorous upper bound on the specific heat [*87*].

3. Hölder's inequality allows interpolation to nonintegral β, but the optimal result (A.25) is obtained only in certain cases. See, however, Proposition A.5.

4. The p-space method used above can also be extended to cover certain cases with $\alpha = 2k \leq \beta = n$ (k and n integers); this requires, however, the new infrared bound [*57*]

$$| \partial\tilde{G}/\partial p_i | \leq \text{const} \times | p |^{-1}\tilde{G}(p). \tag{A.31}$$

The proof of (A.31) is somewhat subtle: it requires the use of the spectral representation (2.37) combined with a new spectral representation [*57*] based on *diagonal* reflection positivity.

Proposition A.5. — Assume (A), (B), (C) and (D) [or for $d < 4$, assume (A), (B), (C) and (D')]. If $0 \le \alpha \le (\beta - 1)d$, then

$$\sum_x |x|^\alpha G(x)^\beta \le \text{const} \times \begin{cases} J^{-\beta}(1 + J\chi)^{(\beta-1)(d_c-d)/2} & \text{if } d < d_c \quad (A.32a) \\ J^{-\beta}\log(2 + J\chi) & \text{if } d = d_c \quad (A.32b) \\ J^{-\beta} & \text{if } d > d_c \quad (A.32c) \end{cases}$$

Proof. — Consider first the case $d > 2$. Then (A.11) and (A.15) hold (for $d = 3$ this relies on Remark 2 following Lemma A.3 if (D') is assumed). For $d > d_c$ we write simply

$$\sum_x |x|^\alpha G(x)^\beta \le \text{const} \times \sum_x |x|^\alpha (1/J(|x| + 1)^{d-2})^\beta . \tag{A.33}$$

By definition of d_c, the sum converges; hence (A.32c) holds. For $d \le d_c$ we split up the region of summation:

$$\begin{aligned}\sum_x |x|^\alpha G(x)^\beta &\le \text{const} \times \Bigg[\sum_{|x| \le (J\chi)^{1/2}} |x|^\alpha (1/J(|x| + 1)^{d-2})^\beta \\ &+ \Big(\sup_{|x| > (J\chi)^{1/2}} |x|^\alpha (\chi/(|x| + 1)^d)^{\beta - 1}\Big)\Big(\sum_{|x| > (J\chi)^{1/2}} G(x)\Big)\Bigg] \\ &\le \text{const} \times \Bigg[\sum_{|x| \le (J\chi)^{1/2}} |x|^\alpha (1/J(|x| + 1)^{d-2})^\beta + \chi \sup_{|x| > (J\chi)^{1/2}} |x|^\alpha (\chi/(|x| + 1)^d)^{\beta-1} \Bigg]. \end{aligned} \tag{A.34}$$

The second term is finite if $\alpha \le (\beta - 1)d$, in which case it is bounded by (A.32a). The first term is bounded by (A.32a) if $d < d_c$, or (A.32b) if $d = d_c$. This completes the proof for $d > 2$.

For $d \le 2$ (which is always in the case $d < d_c$), the result is already proven for $\alpha = 0$ and β integer, by Proposition A.4. The case $\alpha = 0$ and general $\beta \ge 1$ is obtained by a simple application of Hölder's inequality to interpolate between integer values of β. Finally, for general $\alpha \le (\beta - 1)d$, we write

$$\begin{aligned}\sum_x |x|^\alpha G(x)^\beta &\le \Big(\sum_x G(x)^{\beta - \alpha/d}\Big)\big(\sup_x |x|^\alpha G(x)^{\alpha/d}\big) \\ &\le \text{const} \times \Big(\sum_x G(x)^{\beta - \alpha/d}\Big)\big(\sup_x |x|^\alpha (\chi/(|x| + 1)^d)^{\alpha/d}\big) \\ &\le \text{const} \times \big(J^{-(\beta - \alpha/d)}(J\chi)^{\beta - \alpha/d - (\beta - \alpha/d - 1)d/2}\big)\chi^{\alpha/d} \\ &= \text{const} \times J^{-\beta}(J\chi)^{(\beta-1)(d_c - d)/2} , \end{aligned} \tag{A.35}$$

where on the second line we used (A.11) and on the third line we used the $\alpha = 0$ case of the proposition (with β replaced by $\beta - \alpha/d$). ■

Remarks. — 1. The argument (A.34) works also for $d < 2$ (i. e. $d = 1$!) by using (A.13) in place of (A.15) in the first term of (A.34). However, this method is apparently insufficient for $d = 2$: it fails by logarithms to give the desired result.

2. Glimm and Jaffe [*30*, Theorem 5.1] assert the case $\alpha = 2$, $\beta = 3$, $d > d_c = 4$ of Proposition A.5, but their proof contains a gap. In essence, they have assumed that $\tilde{G}(p)$ behaves at worst like the free massless lattice field in all relevant respects (in particular, that (A.31) holds), but this does *not* follow solely from the spectral representation they use [our (2.37)].

3. The above methods can also be used (for what it's worth) to study the moments of *convolutions* of powers of G (as occur, for example, in [*30*, Theorems 5.1 and 6.2]).

4. Cases with $\alpha > (\beta - 1)d$ can also be studied, by using (A.12) in place of (A.11), for some $\phi > \alpha/(\beta - 1) - d$. Of course, the resulting bound then depends on ξ_ϕ as well as on J and χ; by (2.52) this is a loss.

APPENDIX B

SOME INEQUALITIES FOR CRITICAL EXPONENTS

Two types of bounds, which we shall call *universal* and *non-universal*, have been used in the rigorous study of critical phenomena and constructive quantum field theory. They correspond to different ways of approaching the critical point:

1. Universal bounds.

These are bounds which hold, with universal constants, for wide classes of models, independently of specific features such as coupling constants. For example, the infrared bound (2.32) holds for all nearest-neighbor Ising models, irrespective of the single-spin measure. (In fact, all of the bounds proven in Section 2.2 and Appendix A are universal bounds.) This type of bound is appropriate to the study of constructive quantum field theory, which allows an arbitrary manner of approach to the critical point (e. g. arbitrary charge renormalization).

2. Non-universal bounds.

These are bounds in which the constants are not universal, but depend on various parameters of the model. For example, consider a statistical-mechanical model in which one fixes the single-spin measure [e. g. fixes λ_0 and B_0 in (2.3)] and increases the nearest-neighbor coupling J toward its critical value J_c. Now, for $J \leq J_c$ we have

$$G(x\,;J) \leq G(x\,;J_c) \leq C(1 + |x|)^{-(d-2+\eta)} \tag{B.1}$$

by Griffiths' inequality and (one) definition of the critical exponent η. However, the constant C depends on the single-spin measure. Thus, non-universal bounds are appropriate only for the study of certain restricted ways of approaching the critical point (e. g. increasing J with λ_0 and B_0 fixed, or decreasing B_0 with λ_0 and J fixed). These are, however, precisely the limits of interest in the statistical-mechanical theory of critical phenomena.

As a first example, let us consider the specific heat C_H in an Ising or φ^4 model. It was shown in [87], using the Griffiths and Lebowitz inequalities, that

$$0 \leq C_H \leq \frac{J^2}{2} \sum_{\substack{x,i_1,i_2 \\ |i_1|=|i_2|=1}} G(x)G(x + i_1 + i_2) \leq 2d^2J^2 \sum_x G(x)^2, \tag{B.2}$$

where the last step uses the Schwarz inequality. (We consider only the case $J < J_c$ for simplicity; then the magnetization M is zero.) By rewriting (B.2) in momentum space and using the infrared bound (2.32) [with $c = 0$], it was shown in [87] that

$$0 \leq C_H \leq \text{const} \times \begin{Bmatrix} (1 + J\chi)^{2-\frac{d}{2}} & \text{if } d < 4 \\ \log(2 + J\chi) & \text{if } d = 4 \\ 1 & \text{if } d > 4 \end{Bmatrix}, \tag{B.3}$$

with a constant depending *only* on d. (The same result can be obtained directly in position space, using Proposition A.5.) This is a universal bound on the specific heat; as noted in [87], it implies the critical-exponent inequality

$$\alpha \leq \max\left[0, \left(2 - \frac{d}{2}\right)\gamma\right]. \tag{B.4}$$

(Critical exponents are defined in [32] [124] [226].) However, if *all* one wants is a critical-exponent inequality, one can use a non-universal bound and do better than (B.4). For, by Lemma A.1 and inequality (B.1), we have

$$0 \leq G(x) \leq \min\left[C(1 + |x|)^{-(d-2+\eta)}, c\chi(1 + |x|)^{-d}\right] \tag{B.5}$$

for all $J < J_c$. Since $n \geq 0$ [58], this improves the bound (A.15) used in the proof of Proposition A.5, at the price of a non-universal constant C. Imitating (A.33) and (A.34), it is easy to show that

$$\sum_x G(x)^2 \leq \text{const} \times \begin{cases} (1 + \chi)^{\frac{4-d-2\eta}{2-\eta}} & \text{if } d < 4 - 2\eta \\ \log(2 + \chi) & \text{if } d = 4 - 2\eta \\ 1 & \text{if } d > 4 - 2\eta \end{cases} \tag{B.6}$$

where the constant is now *non-universal*. This together with (B.2) implies the improved critical-exponent inequality

$$\alpha \leq \max\left[0, \left(2 - \frac{d}{2-\eta}\right)\gamma\right]. \tag{B.7}$$

It is left to the reader to plug in the numbers for the $d = 2$ and $d = 3$ Ising models [190] to see how worthless the change from (B.4) to (B.7) really is, from any but a conceptual point of view.

Remarks. — 1. The idea of using (B.1) to derive critical-exponent inequalities is due to Fisher [32], who used the method to derive the critical-exponent inequality $\gamma \leq (2 - \eta)\nu_\phi$. This improves the inequality $\gamma \leq 2\nu_\phi$ obtainable from the universal bound (2.52), just as (B.7) improves (B.4).

2. A heuristic « proof » of an inequality weaker than (B.7) [but incomparable with (B.4)] is given by Glimm and Jaffe [85]. They claim $\alpha \leq (4 - d - 2\eta)\nu$; this follows from (B.7) and Fisher's [32] inequality $\gamma \leq (2 - \eta)\nu$.

3. An analogous argument can be carried through for the critical-isotherm exponents; the analogue of (B.1) is

$$G(x; H) \leq G(x; H = 0) \leq C(1 + |x|)^{-(d-2+\eta)} \tag{B.8}$$

for $H \geq 0$, which is a consequence of the GHS inequality. (Here G is the *truncated* 2-point function.) Following [87] and the above argument, one finds

$$\alpha_c \leq \max\left[0, \left(2 - \frac{d}{2-\eta}\right)(\delta - 1), \delta - 3\right]. \tag{B.9}$$

However, the argument does *not* work for the low-temperature exponents: the analogue of (B.1) has not been proven, and is quite likely false. (Cf. Fisher's [32] inability to prove $\gamma' \leq (2 - \eta)\nu'$.)

4. A careful examination of the above proof shows that the full strength of (B.1) was not used; it suffices to take a weaker (and more usual [32]) definition of η, namely

$$\sum_{|x| \leq R} G(x; J_c) \leq O(R^{2-\eta}) \tag{B.10}$$

as $R \to \infty$.

As a second example, let us consider the field-strength renormalization constant Z. As shown in Section 3.3, we have, assuming Conjecture 3.2,

$$0 \leq Z^{-1} \leq \text{const} \times \left[J + \lambda_0^2 \sum_x |x|^2 G(x)^3 \right]. \tag{B.11}$$

Thus, by Proposition A.5, we have the universal bound

$$0 \leq Z^{-1} \leq \text{const} \times J + \text{const} \times \lambda_0^2 J^{-3} \times \begin{Bmatrix} (1 + J\chi)^{4-d} & \text{if } d < 4 \\ \log (2 + J\chi) & \text{if } d = 4 \\ 1 & \text{if } d > 4 \end{Bmatrix}. \tag{B.12}$$

Now fix λ_0 and increase J toward its critical value J_c (this is the statistical-mechanical situation, as opposed to the superrenormalizable field-theoretic situation considered in Section 3.3). Then we derive immediately the critical-exponent inequality

$$0 \leq \zeta \leq \max\,[0, (4 - d)\gamma] \tag{B.13}$$

closely analogous to (B.4). [Here ζ is the critical exponent of Glimm and Jaffe [*30*] [*68*], defined by $Z \sim (J_c - J)^\zeta$, and not to be confused with the traditional exponent ζ [*32*]. $\zeta \geq 0$ follows from (2.39).] However, this can be improved by using the non-universal bound (B.5) when estimating (B.11). Again imitating (A.33) and (A.34), it is easy to show that

$$\sum_x |x|^2 G(x)^3 \leq \text{const} \times \begin{Bmatrix} (1 + \chi)^{\frac{8-2d-3\eta}{2-\eta}} & \text{if } d < 4 - \frac{3}{2}\eta \\ \log (2 + \chi) & \text{if } d = 4 - \frac{3}{2}\eta \\ 1 & \text{if } d > 4 - \frac{3}{2}\eta \end{Bmatrix} \tag{B.14}$$

where the constant is now non-universal. Thus

$$0 \leq \zeta \leq \max \left[0, \left(4 - d - \frac{(d-1)\eta}{2-\eta} \right) \gamma \right], \tag{B.15}$$

which improves (B.13) since $\eta \geq 0$. Moreover, (2.39)/(2.40) imply that

$$2\nu - \zeta \leq \gamma. \tag{B.16}$$

Combined with Fisher's [*32*] inequality

$$\gamma \leq (2 - \eta)\nu \tag{B.17}$$

[which is a simple consequence of (B.1)], we find

$$0 \leq \eta \leq \zeta/\nu. \tag{B.18}$$

Combining this with (B.15) and (B.17), we deduce that

$$0 \leq \eta \leq \max \left[0, \left(2 - \frac{d}{2} \right) \right]. \tag{B.19}$$

(B.15) and (B.19) are further results [along with (B.4)/(B.7)] asserting that critical exponents take their mean-field values for dimension $d \geq 4$ (possibly modified by logarithms for $d = 4$).

Remarks. — 1. (B.18) and (B.19) were already proven by Glimm and Jaffe [*30*] under an *ad hoc* hypothesis on the 2-point function. The key observation made here is that this hypothesis is unnecessary, and can be replaced by (B.1). (Actually, Glimm and Jaffe obtained somewhat weaker results due to their use of Γ_6 instead of G_6^{1PI}.)

2. Unfortunately the above argument does not apply to the Ising model ($\lambda_0 = \infty$), although the conclusion is presumably still true.

Finally, let us derive some critical-exponent inequalities from the bound (3.29) and its improvement (4.17) [both of which rely on Conjecture 3.2]. Indeed, (3.29) yields immediately the bound

$$0 \leq -\overline{u_4} \leq 3\lambda_0\chi^4 . \tag{B.20}$$

Now consider the statistical-mechanical situation in which λ_0 is fixed. Then we derive immediately the critical-exponent inequality

$$\Delta_4 \leq \frac{3}{2}\gamma . \tag{B.21}$$

[Here Δ_4 is the « gap exponent » [*124*] [*73*] [*61*] defined by $-\overline{u_4} \sim (J_c - J)^{-\gamma-2\Delta_4}$.] Using the further inequalities $1 \leq \gamma \leq (2-\eta)\nu_\phi \leq 2\nu_\phi$ [*32*] [*227*] [*228*], we find

$$d\nu_\phi - 2\Delta_4 + \gamma \geq d\nu_\phi - 2\gamma \geq (d-4)\nu_\phi \geq (d-4)/2 ; \tag{B.22}$$

thus, the hyperscaling relation (4.1) is violated for φ^4 models in dimension $d > 4$. Equivalently, the dimensionless renormalized coupling constant g vanishes as the critical point is approached, for $d > 4$; the continuum limit is a (generalized) free field.

Remarks. — 1. Unfortunately the above argument does not apply to the Ising model ($\lambda_0 = \infty$). But Aizenman [*229*] [*230*] has recently shown, by a beautiful argument, the universal bound

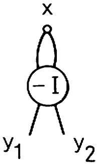
$$0 \leq -\overline{u_4} \leq 2\chi^4 \tag{B.23}$$

for Ising models (and also a generalization for φ^4 models); thus (B.21) and (B.22) follow. A similar bound has also been proven by Fröhlich [*231*].

2. The vanishing of g described in (B.22) ff. should not be confused with that conjectured in Chapter 4. We are considering here the statistical-mechanical situation in which λ_0 is held fixed as the critical point is approached; by (2.13) this means, for $d > 4$, that $\lambda_0^{FT} \to 0$. Thus it is hardly surprising that $g \to 0$ for *this* rather silly choice of charge renormalization. The tree-graph contribution goes to zero; and the import of (3.29) is that the ultraviolet divergences do *not* cause an amplification which destroys this vanishing in the exact theory. In Chapter 4 we argued for a much stronger conjecture, namely that for $d \geq 4$ one has $g \to 0$ *irrespective of the choice of charge renormalization.*

3. (3.29) and (2.52) imply in fact that $g \to 0$ whenever $\lambda_0^{FT} \to 0$, not just in the particular case λ_0 = fixed. However, this only implies that the continuum limit is a *generalized* free field; to be an ordinary free field, λ_0^{FT} might have to go to zero sufficiently fast, for example, $\lambda_0^{FT} \leq O(\xi^{4-d})$ so that λ_0^{SM} is bounded ($d > 4$); see e. g. [*30*, Theorem 5.2].

A slight improvement can be obtained by using (4.17) instead of (3.29) [equivalently, by not discarding the middle term in the brackets in (3.18)/(3.20)]. For note that



$$= \quad < \varphi_x^2 ; \varphi_{y_1} \varphi_{y_2} > \tag{B.24}$$

Now by a fluctuation-dissipation relation [*61*, Appendix],

$$\sum_{x,y_1} \langle \varphi_x^2 ; \varphi_{y_1}\varphi_{y_2} \rangle = -2\partial\chi/\partial B_0 . \tag{B.25}$$

Hence

$$0 \leq -\overline{u_4} \leq -3\lambda_0\chi^2\partial\chi/\partial B_0 . \tag{B.26}$$

So if we consider a slightly modified way of going to the critical point, in which we fix $\lambda_0 \geq 0$ and $J > 0$ and decrease B_0 towards its critical value B_{0c}, we find that

$$\Delta_4 \leq \gamma + \frac{1}{2}; \tag{B.27}$$

in view of $\gamma \geq 1$, this is a slight improvement of (B.21). [Of course, for $d \geq 4$, we expect $\gamma = 1$.] Here the critical exponents are defined in the modified way

$$-\overline{u_4} \sim (B_0 - B_{0_c})^{-\gamma-2\Delta_4} \tag{B.28}$$

$$-\partial\chi/\partial B_0 \sim (B_0 - B_{0_c})^{-\gamma-1} \qquad [\text{hence } \chi \sim (B_0 - B_{0_c})^{-\gamma}] . \tag{B.29}$$

Presumably (B.27) is also valid for the ordinary way of going to the critical point, although I do not know how to prove it.

Note Added (April 1982).

In the year that has elapsed since this paper was submitted for publication, considerable progress has been made on the subjects treated here. In addition, some points made in this paper merit clarification.

1. The continuum limit is defined in this paper by convergence of the Schwinger distributions in $\mathscr{S}'(\mathbb{R}^{nd})$. This is natural from the point of view of Euclidean field theory. However, only the Schwinger distributions at *noncoinciding arguments* enter into the Osterwalder-Schrader [*13*] reconstruction of the Minkowski-space quantum field theory. Thus, a more general construction would consider convergence not in $\mathscr{S}'(\mathbb{R}^{nd})$ but in $\mathscr{S}'_0(\mathbb{R}^{nd})$, the space of distributions at noncoincident points. This would allow the construction, from lattice approximations, of quantum field theories more singular than those constructed here, and of quantum field theories not satisfying Nelson-Symanzik positivity. It is worth noting, however, that convergence in $\mathscr{S}'(\mathbb{R}^{nd})$ is natural from the point of view of the *extended* Osterwalder-Schrader axioms of [*237*] [*238*].

2. For the same reason, the condition $g \neq 0$, which ensures that the Euclidean theory is non-Gaussian, does *not* necessarily imply that the reconstructed Minkowski-space quantum field theory is something other than a generalized free field; for this it is necessary to verify that $G_4 \neq 0$ for some *noncoinciding* arguments.

3. The construction given in Section 2.3, which enforces a very particular set of mass and field-strength normalization conditions, is *one* way of taking the continuum limit; it is not by any means the only one. Most generally, one should allow *any* sequence of lattice φ^4_d models with lattice spacings $a_m \to 0$, and seek to determine all possible continuum limits.

I thank Jürg Fröhlich for bringing these three items to my attention.

4. In Section 4.1 I stated that the method of high-temperature expansions is « theoretically unprejudiced »; this is, in fact, only true in the idealized case in which the full infinite series is available. In practice, one must try to extract numerical estimates of the critical exponents from a *finite* number of terms of the series; this *always* requires assumptions on the critical behavior, e. g. power-law behavior either with or without confluent singularities. Such assumptions may well be inspired by theoretical considerations, e. g. the renormalization-group predictions of the existence and nature of the confluent singularities. Moreover, the choice of assumption has a profound influence on the numerical results obtained [*119*].

5. Using the « mean-field bound » [*239*] $G(0) \geq \text{const}/J$ (for ξ not near zero), I can prove the « infrared lower bound » $G(p) \geq G(\pi, \ldots, \pi) \geq \text{const}/J$ for $d > 3$. (A similar inequality, modified by logarithms, holds for $d = 3$.) It follows that $F(0) \geq \text{const}/J^2$, which is *stronger* than the conjecture (4.31) for $d > 4$. However, the crucial case $d = 4$ is still open.

6. Brydges, Fröhlich and the author [*233*] have succeeded in proving strengthened forms of the correlation inequalities (3.29) and (3.30), *without* using Conjecture 3.2 or any other unproved result. The proof employs the random-walk methods of [*232*]. We have then used these inequalities to give a simple proof [*234*] of the existence and nontriviality of the continuum limit for φ_3^4 (with the conventional mass renormalization), proving all Osterwalder-Schrader axioms (including a mass gap) except rotation invariance.

7. I now suspect that φ_4^4 is more difficult and subtle than is implied by the bubble-graph-motivated considerations of Section 4.2. In any case, the question is wide open !

Note Added in Proof: The proof given here of Proposition 2.1 is insufficient; a correct proof can be given along the lines of Proposition 5.1 of [*234*].

REFERENCES

[*1*] J. GLIMM and A. JAFFE, Positivity of the φ_3^4 Hamiltonian. *Fortsch. Physik*, t. **21**, 1973, p. 327-376.

[*2*] J. FELDMAN, The $\lambda\varphi_3^4$ theory in a finite volume. *Commun. Math. Phys.*, t. **37**, 1974, p. 93-120.

[*3*] J. S. FELDMAN and K. OSTERWALDER, The Wightman axioms and the mass gap for weakly coupled $(\varphi^4)_3$ quantum field theories. *Ann. Phys.*, t. **97**, 1976, p. 80-135.

[*4*] J. MAGNEN and R. SÉNÉOR, The infinite volume limit of the φ_3^4 model. *Ann. Inst. Henri Poincaré*, t. **A 24**, 1976, p. 95-159.

[*5*] J. MAGNEN and R. SÉNÉOR, Phase space cell expansion and Borel summability for the Euclidean φ_3^4 theory. *Commun. Math. Phys.*, t. **56**, 1977, p. 237-276.

[*5'*] G. GALLAVOTTI, Some aspects of the renormalization problems in statistical mechanics and field theory. *Mem. Accad. Lincei*, t. **15**, 1978, p. 23-59.

[*5''*] G. GALLAVOTTI, On the ultraviolet stability in statistical mechanics and field theory. *Ann. Mat. Pura Appl.*, t. **120**, 1979, p. 1-23.

[*5'''*] G. BENFATTO, M. CASSANDRO, G. GALLAVOTTI, F. NICOLÒ, E. OLIVIERI, E. PRESUTTI and E. SCACCIATELLI, Some probabilistic techniques in field theory. *Commun. Math. Phys.*, t. **59**, 1978, p. 143-166.

[*5''''*] G. BENFATTO, M. CASSANDRO, G. GALLAVOTTI, F. NICOLÒ, E. OLIVIERI, E. PRESUTTI and E. SCACCIATELLI, Ultraviolet stability in Euclidean scalar field theories. *Commun. Math. Phys.*, t. **71**, 1980, p. 95-130.

[*6*] B. SIMON, *The* $P(\varphi)_2$ *Euclidean (quantum) field theory*. Princeton, N. J., Princeton University Press, 1974.

[*7*] J. ROSEN, Mass renormalization for the $\lambda\varphi^4$ Euclidean lattice field theory. *Adv. Appl. Math.*, t. **1**, 1980, p. 37-49.

[*8*] J. GLIMM and A. JAFFE, Remark on the existence of φ_4^4. *Phys. Rev. Lett.*, t. **33**, 1974, p. 440-442.

[*9*] G. A. BAKER, Jr., Self-interacting, boson, quantum field theory and the thermodynamic limit in *d* dimensions. *J. Math. Phys.*, t. **16**, 1975, p. 1324-1346.

[*10*] F. GUERRA, L. ROSEN and B. SIMON. The $P(\varphi)_2$ Euclidean quantum field theory as classical statistical mechanics. *Ann. Math.*, t. **101**, 1975, p. 111-259.

[*11*] Y. M. PARK, Convergence of lattice approximations and infinite volume limit in the $(\lambda\varphi^4\text{-}\sigma\varphi^2\text{-}\mu\varphi)_3$ field theory. *J. Math. Phys.*, t. **18**, p. 354-366, 1977.

[12] R. SCHRADER, A possible constructive approach to φ_4^4. I. *Commun. Math. Phys.*, t. **49**, 1976, p. 131-153, II. *Ann. Inst. Henri Poincaré*, t. **A 26**, 1977, p. 295-301, III. *Commun. Math. Phys.*, t. **50**, 1976, p. 97-102.

[13] K. OSTERWALDER and R. SCHRADER, Axioms for Euclidean Green's functions. I. *Commun. Math. Phys.*, t. **31**, 1973, p. 83-112. II. *Commun. Math. Phys.*, t. **42**, 1975, p. 281-305.

[14] J. GLIMM and A. JAFFE, Functional integral methods in quantum field theory. In: *New developments in quantum field theory and statistical mechanics* (1976 Cargèse lectures). M. Lévy and P. Mitter, eds., New York and London, Plenum Press, 1977.

[15] C. M. NEWMAN, Inequalities for Ising models and field theories which obey the Lee-Yang theorem. *Commun. Math. Phys.*, t. **41**, 1975, p. 1-9.

[16] L. D. LANDAU, *Collected papers of L. D. Landau* (D. ter Haar, ed.), New York-London-Paris: Gordon and Breach, 1965.

[17] L. LANDAU, On the quantum theory of fields. In: *Niels Bohr and the development of physics* (W. Pauli, ed.), London, Pergamon Press, 1955. Reprinted in [16].

[18] L. LANDAU and I. POMERANCHUK, On point interactions in quantum electrodynamics. *Dokl. Akad. Nauk S. S. S. R.*, t. **102**, 1955, p. 489-492. English translation in [16].

[19] I. POMERANČUK, Vanishing of the renormalized charge in electrodynamics and in meson theory. *Nuovo Cim.*, t. **3**, 1956, p. 1186-1203.

[20] I. Ya. POMERANCHUK, V. V. SUDAKOV and K. A. TER-MARTIROSYAN, Vanishing of renormalized charges in field theories with point interaction. *Phys. Rev.*, t. **103**, 1956, p. 784-802.

[21] A. D. GALANIN, On the possibility of formulating a meson theory with several fields. *Zh. Eksp. Teor. Fiz.*, t. **32**, 1957, p. 552-558 [*Soviet Phys., J. E. T. P.*, t. **5**, 1957, p. 460-464].

[22] I. T. DIATLOV, V. V. SUDAKOV and K. A. TER-MARTIROSYAN, Asymptotic meson-meson scattering theory. *Zh. Eksp. Teor. Fiz.*, t. **32**, 1957, p. 767-780 [*Soviet Phys., J. E. T. P.*, t. **5**, 1957, p. 631-642].

[23] A. A. ABRIKOSOV, A. D. GALANIN, L. P. GORKOV, L. D. LANDAU, I. Ya. POMERANCHUK and K. A. TER-MARTIROSYAN, Possibility of formulation of a theory of strongly interacting fermions. *Phys. Rev.*, t. **111**, 1958, p. 321-328. Reprinted in [16].

[24] E. S. FRADKIN, The asymptote of Green's function in quantum electrodynamics. *Zh. Eksp. Teor. Fiz.*, t. **28**, 1955, p. 750-752 [*Soviet Phys., J. E. T. P.*, t. **1**, 1955, p. 604-606].

[25] E. S. FRADKIN, The problem of the asymptote of the Green function in the theory of mesons with pseudoscalar coupling. *Zh. Eksp. Teor. Fiz.*, t. **29**, 1955, p. 377-379 [*Soviet Phys., J. E. T. P.*, t. **2**, 1956, p. 340-342].

[26] D. A. KIRZHNITS, Contribution to field theory involving a cut-off factor. *Zh. Eksp. Teor. Fiz.*, t. **32**, 1957, p. 534-541 [*Soviet Phys., J. E. T. P.*, t. **5**, 1957, p. 445-451].

[27] S. KAMEFUCHI, A comment on Landau's method of integration in quantum electrodynamics. *Mat. Fys. Medd. Dan. Vid. Selsk.*, t. **31**, 1957, no. 6.

[28] A. A. ANSEL'M, Asymptotic theory of a one-dimensional four-fermion interaction. *Zh. Eksp. Teor. Fiz.*, t. **35**, 1958, p. 1522-1531 [*Soviet Phys., J. E. T. P.*, t. **8**, 1959, p. 1065-1071].

[29] N. N. BOGOLIUBOV and D. V. SHIRKOV, *Introduction to the theory of quantized fields.* New York, Interscience, 1959, Chapter VIII, especially p. 528-529.

[30] J. GLIMM and A. JAFFE, Particles and scaling for lattice fields and Ising models. *Commun. Math. Phys.*, t. **51**, 1976, p. 1-13.

[31] J. BRICMONT, J.-R. FONTAINE, J. L. LEBOWITZ and T. SPENCER, Lattice systems with a continuous symmetry. II. Decay of correlations. *Commun. Math. Phys.*, t. **78**, 1981, p. 363-371.

[31′] J. BRICMONT, J.-R. FONTAINE, J. L. LEBOWITZ, E. H. LIEB and T. SPENCER, Lattice systems with a continuous symmetry. III. Low temperature asymptotic expansion for the plane rotator model. *Commun. Math. Phys.*, t. **78**, 1981, p. 545-566.

[32] M. E. FISHER, Rigorous inequalities for critical-point correlation exponents. *Phys. Rev.*, t. **180**, 1969, p. 594-600.

[33] B. SIMON, *Functional integration and quantum physics.* New York-San Francisco-London, Academic Press, 1979.

[34] E. BRÉZIN, J. C. LEGUILLOU and J. ZINN-JUSTIN, Field theoretical approach to critical phenomena. In: *Phase transitions and critical phenomena*, t. **6**. C. Domb and M. S. Green, eds., London-New York-San Francisco, Academic Press, 1976.

[35] K. SYMANZIK, Regularized quantum field theory. In: *New developments in quantum field theory and statistical mechanics* (1976 Cargèse Lectures). M. Lévy and P. Mitter, eds. New York and London, Plenum Press, 1977.

[36] K. G. WILSON, Quantum field-theory models in less than 4 dimensions. *Phys. Rev.*, t. **D 7**, 1973, p. 2911-2926.

[37] R. B. GRIFFITHS, Phase transitions. In: *Statistical mechanics and quantum field theory* (1970 Les Houches lectures). C. DeWitt and R. Stora, eds. New York-London-Paris, Gordon and Breach, 1971.

[38] G. S. SYLVESTER, *Continuous-spin Ising ferromagnets.* M. I. T. dissertation (mathematics), 1976.

[39] R. L. DOBRUSHIN, The description of a random field by means of conditional probabilities and conditions of its regularity. *Teor. Veroya. Prim.*, t. **13**, 1968, p. 201-229 [*Theor. Prob. Appl.*, t. **13**, 1968, p. 197-224].

[40] R. L. DOBRUSHIN, Gibbsian random fields for lattice systems with pairwise interactions. *Funkts. Anal. Prilozh.*, t. **2**, no. 4, 1968, p. 31-43 [*Funct. Anal. Appl.*, t. **2**, 1968, p. 292-301].

[41] O. E. LANFORD III and D. RUELLE, Observables at infinity and states with short range correlations in statistical mechanics. *Commun. Math. Phys.*, t. **13**, 1969, p. 194-215.

[42] C. PRESTON, *Random fields.* Lecture notes in mathematics # 534, Berlin-Heidelberg-New York, Springer-Verlag, 1976.

[43] M. CASSANDRO, E. OLIVIERI, A. PELLEGRINOTTI and E. PRESUTTI, Existence and uniqueness of DLR measures for unbounded spin systems. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, t. **41**, 1978, p. 313-334.

[44] J. L. LEBOWITZ and E. PRESUTTI, Statistical mechanics of systems of unbounded spins. *Commun. Math. Phys.*, t. **50**, 1976, p. 195-218 ; t. **78** (E), 1980, p. 151.

[45] G. S. SYLVESTER, Inequalities for continuous-spin Ising ferromagnets. *J. Stat. Phys.*, t. **15**, 1976, p. 327-341.

[46] C. M. NEWMAN, Gaussian correlation inequalities for ferromagnets. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, t. **33**, 1975, p. 75-93.

[47] G. S. SYLVESTER, Representations and inequalities for Ising model Ursell functions. *Commun. Math. Phys.*, t. **42**, 1975, p. 209-220.

[48] J. BRICMONT, The Gaussian inequality for multicomponent rotators. *J. Stat. Phys.*, t. **17**, 1977, p. 289-300.

[49] G. A. BATTLE and L. ROSEN, The FKG inequality for the Yukawa_2 quantum field theory. *J. Stat. Phys.*, t. **22**, 1980, p. 123-192.

[50] C. M. NEWMAN, Moment inequalities for ferromagnetic Gibbs distributions. *J. Math. Phys.*, t. **16**, 1975, p. 1956-1959.

[51] R. SCHRADER, New correlation inequalities for the Ising model and $P(\phi)$ theories. *Phys. Rev.*, t. **B 15**, 1977, p. 2798-2803.

[52] A. MESSAGER and S. MIRACLE-SOLE, Correlation functions and boundary conditions in the Ising ferromagnet. *J. Stat. Phys.*, t. **17**, 1977, p. 245-262.

[53] G. C. HEGERFELDT, Correlation inequalities for Ising ferromagnets with symmetries. *Commun. Math. Phys.*, t. **57**, 1977, p. 259-266.

[54] D. SZÀSZ, Correlation inequalities for non-purely-ferromagnetic systems. *J. Stat. Phys.*, t. **19**, 1978, p. 453-459.

[55] J. FRÖHLICH, R. ISRAEL, E. H. LIEB and B. SIMON, Phase transitions and reflection positivity. I. General theory and long range lattice models. *Commun. Math. Phys.*, t. **62**, 1978, p. 1-34.

[56] R. S. SCHOR, The particle structure of ν-dimensional Ising models at low temperatures. *Commun. Math. Phys.*, t. **59**, 1978, p. 213-233.

[57] A. D. SOKAL, *Representations and inequalities for the 2-point function in classical lattice systems.* Unpublished manuscript, 1980.

[58] J. FRÖHLICH, B. SIMON and T. SPENCER, Infrared bounds, phase transitions and continuous symmetry breaking. *Commun. Math. Phys.*, t. **50**, 1976, p. 79-85.

[59] B. SIMON, New rigorous existence theorems for phase transitions in model systems. In: *Statistical Physics*, « *Statphys 13* » (1977 I. U. P. A. P. conference, Haifa), part 1, D. Cabib *et al.*, eds. *Annals of the Israel Physical Society*, t. **2**, 1978.

[60] B. SIMON, Correlation inequalities and the mass gap in $P(\phi)_2$. I. Domination by the two point function. *Commun. Math. Phys.*, t. **31**, 1973, p. 127-136.

[61] A. D. SOKAL, More inequalities for critical exponents. *J. Stat. Phys.*, t. **25**, 1981, p. 25-50.

[62] B. SIMON, Decay of correlations in ferromagnets. *Phys. Rev. Lett.*, t. **44**, 1980, p. 547-549.

[63] B. SIMON, Correlation inequalities and the decay of correlations in ferromagnets. *Commun. Math. Phys.*, t. **77**, 1980, p. 111-126.

[64] J. GLIMM and A. JAFFE, Three-particle structure of ϕ^4 interactions and the scaling limit. *Phys. Rev.*, t. **D 11**, 1975, p. 2816-2827.

[65] P. J. PAES-LEME, Ornstein-Zernike and analyticity properties for classical lattice spin systems. *Ann. Phys.*, t. **115**, 1978, p. 367-387.

[66] O. A. MCBRYAN and J. ROSEN, Existence of the critical point in ϕ^4 field theory. *Commun. Math. Phys.*, t. **51**, 1976, p. 97-105.

[67] E. H. LIEB, A refinement of Simon's correlation inequality. *Commun. Math. Phys.*, t. **77**, 1980, p. 127-135.

[68] J. GLIMM and A. JAFFE, Critical exponents and elementary partices. *Commun. Math. Phys.*, t. **52**, 1977, p. 203-209.

[69] A. D. SOKAL, Unpublished, 1980.

[70] J. L. LEBOWITZ and A. MARTIN-LÖF, On the uniqueness of the equilibrium state for Ising spin systems. *Commun. Math. Phys.*, t. **25**, 1972, p. 276-282.

[71] G. A. BAKER Jr., B. G. NICKEL, M. S. GREEN and D. I. MEIRON, Ising-model critical indices in three dimensions from the Callan-Symanzik equation. *Phys. Rev. Lett.*, t. **36**, 1976, p. 1351-1354.

[72] J. GLIMM and A. JAFFE, Absolute bounds on vertices and couplings. *Ann. Inst. Henri Poincaré*, t. **A 22**, 1975, p. 97-107.

[73] R. SCHRADER, New rigorous inequality for critical exponents in the Ising model. *Phys. Rev.*, t. **B 14**, 1976, p. 172-173.

[74] A. D. SOKAL, *Improved rigorous upper bound for the renormalized 4-point coupling.* In preparation.

[75] D. G. BOULWARE and L. S. BROWN, Tree graphs and classical fields. *Phys. Rev.*, t. **172**, 1968, p. 1628-1631.

[76] R. W. JOHNSON, Finite integral equations for Green's functions for $:\phi^4:$ coupling. *J. Math. Phys.*, t. **11**, 1970, p. 2161-2165.

[77] M. SUZUKI, Generalized exact formula for the correlations of the Ising model and other classical systems. *Phys. Lett.*, t. **19**, 1965, p. 267-268.

[78] F. SCHWABL, Generating functional and Green's function hierarchy of the Ising model. *Ann. Phys.*, t. **54**, 1969, p. 1-16.
[79] H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN, Zur Vertexfunktion in quantisierten Feldtheorien. *Nuovo Cim.*, t. **2**, 1955, p. 425-432.
[80] G. BARTON, *Introduction to advanced field theory*. New York, Interscience, 1963, Chapters 10-12.
[81] K. W. FORD, Problem of ghost states in field theories. *Phys. Rev.*, t. **105**, 1957, p. 320-327.
[82] E. FERRARI and G. JONA-LASINIO, Remarks on the appearance of ghost states in relativistic field theories. *Nuovo Cim.*, t. **16**, 1960, p. 867-885.
[83] K. SYMANZIK, On the many-particle structure of Green's functions in quantum field theory. *J. Math. Phys.*, t. **1**, 1960, p. 249-273, Section 5 and Appendix B.
[84] L. E. EVANS, High energy theorem in quantum electrodynamics. *Nucl. Phys.*, t. **17**, 1960, p. 163-168.
[85] J. GLIMM and A. JAFFE, Critical exponents and renormalization in the ϕ^4 scaling limit. In: *Quantum dynamics, models and mathematics* (1975 Bielefeld symposium), L. Streit, ed. Wien-New York, Springer-Verlag, 1976. *Acta Physica Austriaca Suppl.*, t. **XVI**, 1976, p. 147-166.
[86] D. MARCHESIN, *The scaling limit of the ϕ^2 field in the anharmonic oscillator: existence and numerical studies*. NYU preprint, 1978.
[87] A. D. SOKAL, A rigorous inequality for the specific heat of an Ising or ϕ^4 ferromagnet. *Phys. Lett.*, t. **71 A**, 1979, p. 451-453.
[88] A. S. WIGHTMAN, Should we believe in quantum field theory? In: *The whys of subnuclear physics* (1977 Erice lectures). A. Zichichi, ed. New York, Plenum Press, 1979.
[89] R. F. STREATER and A. S. WIGHTMAN, *PCT, spin and statistics, and all that*. New York-Amsterdam, Benjamin, 1964. Second edition, 1978.
[90] G. KÄLLÉN, Review of consistency problems in quantum electrodynamics. In: *Quantum electrodynamics*. P. Urban, ed., Wien-New York, Springer-Verlag, 1965. *Acta Physica Austriaca Suppl.*, t. **II**, 1965, p. 133-161.
[91] K. G. WILSON, Renormalization group and strong interactions. *Phys. Rev.*, t. **D 3**, 1971, p. 1818-1846.
[92] S. COLEMAN, Dilatations. In: *Properties of the fundamental interactions* (1971 Erice lectures), A. Zichichi, ed., Bologna, Editrice Compositori, 1973.
[93] D. J. GROSS, Applications of the renormalization group to high-energy physics. In: *Methods in field theory* (1975 Les Houches lectures), R. Balian and J. Zinn-Justin, eds., Amsterdam-New York, North-Holland, 1976.
[94] K. G. WILSON and J. KOGUT, The renormalization group and the ε expansion. *Phys. Rep.*, t. **12**, 1974, p. 75-200.
[94′] R. D. PISARSKI, Fixed-point structure of $(\varphi^6)_3$ at large N. *Phys. Rev. Lett.*, t. **48**, 1982, p. 574-576.
[95] S. L. ADLER, Short-distance behavior of quantum electrodynamics and an eigenvalue condition for α. *Phys. Rev.*, t. **D 5**, 1972, p. 3021-3047; t. **D 7**, 1973, (E), p. 1948.
[96] K. JOHNSON and M. BAKER, Some speculations on high-energy quantum electrodynamics. *Phys. Rev.*, t. **D 8**, 1973, p. 1110-1122.
[97] B. M. MCCOY and T. T. WU, Non-perturbative quantum field theory. *Scientia Sinica*, t. **22**, 1979, p. 1021-1032.
[98] N. N. KHURI, Zeros of the Gell-Mann—Low function and Borel summations in renormalizable theories. *Phys. Lett.*, t. **82 B**, 1979, p. 83-88.
[99] C. DOMB and M. S. GREEN, *Phase transitions and critical phenomena*. Vol. 3, Series expansions for lattice models. London-New York, Academic Press, 1974.
[100] H. KUNZ, Analyticity and clustering properties of unbounded spin systems. *Commun. Math. Phys.*, t. **59**, 1978, p. 53-69.

[*101*] G. S. SYLVESTER, Weakly coupled Gibbs measures. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, t. **50**, 1979, p. 97-118.
[*102*] J. GLIMM, A. JAFFE and T. SPENCER, The particle structure of the weakly coupled $P(\phi)_2$ model and other applications of high temperature expansions. Part II: The cluster expansion. In: *Constructive quantum field theory* (1973 Erice lectures), G. Velo and A. Wightman, eds., Berlin-Heidelberg-New York, Springer-Verlag, 1973.
[*103*] G. A. BAKER, Jr., *Essentials of Padé approximants*. New York, Academic Press, 1975.
[*104*] J. D. BESSIS, J. M. DROUFFE and P. MOUSSA, Positivity contraints for the Ising ferromagnetic model. *J. Phys.*, t. **A 9**, 1976, p. 2105-2124.
[*105*] G. A. BAKER, Jr. and J. M. KINCAID, Continuous-spin Ising model and $\lambda: \varphi^4:_d$ field theory. *Phys. Rev. Lett.*, t. **42**, 1979, p. 1431-1434; t. **44** (E), 1980, p. 434.
[*106*] G. A. BAKER, Jr. and J. M. KINCAID, The continuous-spin Ising model, $g_0 : \phi^4 :_d$ field theory, and the renormalization group. *J. Stat. Phys.*, t. **24**, 1981, p. 469-528.
[*107*] G. A. BAKER, Jr., Analysis of hyperscaling in the Ising model by the high-temperature series method. *Phys. Rev.*, t. **B 15**, 1977, p. 1552-1559.
[*108*] D. S. GAUNT and M. F. SYKES, The critical exponent γ for the three-dimensional Ising model. *J. Phys.*, t. **A 12**, 1979, L 25-L 28.
[*109*] S. MCKENZIE, M. F. SYKES and D. S. GAUNT, The critical isotherm of the four-dimensional Ising model. *J. Phys.*, t. **A 12**, 1979, p. 743-746.
[*110*] D. S. GAUNT, M. F. SYKES and S. MCKENZIE, Susceptibility and fourth-field derivative of the spin-1/2 Ising model for $T > T_c$ and $d = 4$. *J. Phys.*, t. **A 12**, 1979, p. 871-877.
[*111*] S. MCKENZIE, The exponent γ for the spin-1/2 Ising model on the face-centered cubic lattice. *J. Phys.*, t. **A 12**, 1979, L 185-L 188.
[*112*] J. OITMAA and J. HO-TING-HUN, The critical exponent γ for the three-dimensional Ising model. *J. Phys.*, t. **A 12**, 1979, L 281-L 282.
[*113*] B. G. NICKEL and B. SHARPE, On hyperscaling in the Ising model in three dimensions. *J. Phys.*, t. **A 12**, 1979, p. 1819-1834.
[*114*] J. J. REHR, Confluent singularities and hyperscaling in the spin-1/2 Ising model. *J. Phys.*, t. **A 12**, 1979, L 179-L 183.
[*115*] J. ZINN-JUSTIN, Analysis of Ising model critical exponents from high temperature series expansion. *J. de Physique*, t. **40**, 1979, p. 969-975.
[*116*] S. MCKENZIE and D. S. GAUNT, A test of hyperscaling for the spin-1/2 Ising model in four dimensions. *J. Phys.*, t. **A 13**, 1980, p. 1015-1021.
[*117*] D. BESSIS, P. MOUSSA and G. TURCHETTI, Subdominant critical indices for the ferromagnetic susceptibility of the spin-1/2 Ising model. *J. Phys.*, t. **A 13**, 1980, p. 2763-2773.
[*118*] B. NICKEL, Hyperscaling and universality in 3 dimensions. *Physica*, t. **106 A**, 1981, p. 48-58.
[*119*] B. G. NICKEL, The problem of confluent singularities. In: *Phase Transitions* (1980 Cargèse lectures), M. Lévy, J.-C. LeGuillou and J. Zinn-Justin, eds., New York and London, Plenum Press, 1982.
[*120*] R. ROSKIES, Hyperscaling in the Ising model on the simple cubic lattice. *Phys. Rev.*, t. **B 23**, 1981, p. 6037-6042.
[*120'*] J. ZINN-JUSTIN, Analysis of high temperature series of the spin S Ising model on the body-centred cubic lattice. *J. de Physique*, t. **42**, 1981, p. 783-792.
[*120''*] R. Z. ZOSKIES, Reconciliation of high temperature series and renormalization group results by suppressing confluent singularities. *Phys. Rev.*, t. **B 24**, 1981, p. 5305-5317.
[*121*] M. E. FISHER, General scaling theory for critical points. In: *Collective properties of physical systems* (24th Nobel symposium). B. Lundqvist and S. Lundqvist, eds., Uppsala, Almqvist & Wiksell, 1974.

[122] D. ISAACSON, The critical behavior of ϕ_1^4. *Commun. Math. Phys.*, t. **53**, 1977, p. 257-275.
[123] B. M. McCOY, C. A. TRACY and T. T. WU, Two-dimensional Ising model as an exactly soluble relativistic quantum field theory: explicit formulas for n-point functions. *Phys. Rev. Lett.*, t. **38**, 1977, p. 793-796.
[124] M. E. FISHER, The theory of equilibrium critical phenomena. *Reports Prog. Phys.*, t. **30**, 1967, p. 615-730.
[125] W. J. CAMP, D. M. SAUL, J. P. VANDYKE and M. WORTIS, Series analysis of corrections to scaling for the spin-pair correlations of the spin-s Ising model: confluent singularities, universality and hyperscaling. *Phys. Rev.*, t. **B 14**, 1976, p. 3990-4001.
[126] T. SPENCER, Private communication.
[127] C. FRIEDMAN, Perturbations of the Schrödinger equation by potentials with small support. *J. Funct. Anal.*, t. **10**, 1972, p. 346-360.
[128] K. SYMANZIK, Euclidean quantum field theory. I. Equations for a scalar model. *J. Math. Phys.*, t. **7**, 1966, p. 510-525.
[129] K. SYMANZIK, Euclidean quantum field theory. In: *Local quantum theory* (1968 Varenna lectures), R. Jost, ed., New York-London, Academic Press, 1969.
[130] D. BRYDGES and P. FEDERBUSH, A lower bound for the mass of a random Gaussian lattice. *Commun. Math. Phys.*, t. **62**, 1978, p. 79-82.
[131] A. J. KUPIAINEN, On the $1/n$ expansion. *Commun. Math. Phys.*, t. **73**, 1980, p. 273-294.
[131′] R. L. WOLPERT, Wiener path intersections and local time. *J. Funct. Anal.*, t. **30**, 1978, p. 329-340.
[131″] R. L. WOLPERT, Local time and a particle picture for Euclidean field theory. *J. Funct. Anal.*, t. **30**, 1978, p. 341-357.
[132] A. DVORETZKY, P. ERDÖS and S. KAKUTANI, Double points of paths of Brownian motion in n-space. *Acta Sci. Math.* (Szeged), t. **12 B**, 1950, p. 75-81.
[133] M. J. WESTWATER, On Edwards' model for long polymer chains. *Commun. Math. Phys.*, t. **72**, 1980, p. 131-174.
[133′] J. WESTWATER, On Edwards' model for polymer chains. III. Borel summability. *Commun. Math. Phys.*, t. **84**, 1982, p. 459-470.
[134] G. F. LAWLER, A self-avoiding random walk. *Duke Math. J.*, t. **47**, 1980, p. 655-693.
[135] S. COLEMAN, R. JACKIW and H. D. POLITZER, Spontaneous symmetry breaking in the O(N) model for large N. *Phys. Rev.*, t. **D 10**, 1974, p. 2491-2499.
[136] V. B. BERESTETSKIĬ, Zero-charge and asymptotic freedom. *Usp. Fiz. Nauk*, t. **120**, 1976, p. 439-454 [*Soviet Phys., Uspekhi*, t. **19**, 1976, p. 934-943].
[137] P. OLESEN, On vacuum instability in quantum field theory. *Phys. Lett.*, t. **73 B**, 1978, p. 327-329.
[138] S. COLEMAN, Unpublished. Cited in [93].
[139] G. FELDMAN, Modified propagators in field theory (with application to the anomalous magnetic moment of the nucleon). *Proc. Roy. Soc.* (London), t. **A 223**, 1954, p. 112-129.
[140] L. D. LANDAU, A. A. ABRIKOSOV and I. M. KHALATNIKOV, An asymptotic expression for the photon Green function in quantum electrodynamics. *Dokl. Akad. Nauk S. S. S. R.*, t. **95**, 1954, p. 1177-1180. English translation in [16].
[141] D. J. GROSS and A. NEVEU, Dynamical symmetry breaking in asymptotically free field theories. *Phys. Rev.*, t. **D 10**, 1974, p. 3235-3253, Appendix.
[142] T. D. LEE, Some special examples in renormalizable field theory. *Phys. Rev.*, t. **95**, 1954, p. 1329-1334.
[143] G. KÄLLÉN and W. PAULI, On the mathematical structure of T. D. Lee's model of a renormalizable field theory. *Dan. Mat. Fys. Medd.*, t. **30**, no. 7, 1955.
[144] S. S. SCHWEBER, *An introduction to relativistic quantum field theory*. Evanston, Illinois and Elmsford, New York, Row, Peterson and Co., 1961, Section 12*b*.
[145] A. A. ANSEL'M, Solution of equation for the meson-meson scattering amplitude

in the asymptotic region. *Zh. Eksp. Teor. Fiz.*, t. **38**, 1960, p. 1887-1890 [*Soviet Phys., J. E. T. P.*, t. **11**, 1960, p. 1356-1358].

[*146*] T. T. Wu, Perturbation theory of pion-pion interaction. II. Two-pion approximation. *Phys. Rev.*, t. **126**, 1962, p. 2219-2226.

[*147*] A. B. Zamolodchikov, Yu. M. Makeenko and K. A. Ter-Martirosyan, A theory of direct four-fermion interactions. *Zh. Eksp. Teor. Fiz.*, t. **71**, 1976, p. 24-45 [*Soviet Phys., J. E. T. P.*, t. **44**, 1976, p. 11-22].

[*148*] A. I. Larkin and D. E. Khmel'nitksiĭ, Phase transition in uniaxial ferroelectrics. *Zh. Eksp. Teor. Fiz.*, t. **56**, 1969, p. 2087-2098 [*Soviet Phys., J. E. T. P.*, t. **29**, 1969, p. 1123-1128].

[*149*] A. M. Polyakov, Properties of long and short range correlations in the critical region. *Zh. Eksp. Teor. Fiz.*, t. **57**, 1969, p. 271-283 [*Soviet Phys., J. E. T. P.*, t. **30**, 1970, p. 151-157].

[*150*] K. Symanzik, Infrared singularities and small-distance-behaviour analysis. *Commun. Math. Phys.*, t. **34**, 1973, p. 7-36.

[*151*] T. Tsuneto and E. Abrahams, Skeleton graph expansion for critical exponents. *Phys. Rev. Lett.*, t. **30**, 1973, p. 217-220.

[*152*] S. L. Ginzburg, Vertex functions and Green functions in the (4-ε)-dimensional theory of phase transitions. *Zh. Eksp. Teor. Fiz.*, t. **66**, 1974, p. 647-654 [*Soviet Phys., J. E. T. P.*, t. **39**, 1974, p. 312-315].

[*153*] A. A. Abrikosov, Electron scattering on magnetic impurities in metals and anomalous resistivity effects. *Physics*, t. **2**, 1965, p. 5-37.

[*154*] Yu. A. Bychkov, L. P. Gor'kov and I. E. Dzyaloshinskiĭ, Possibility of superconductivity type phenomena in a one-dimensional system. *Zh. Eksp. Teor. Fiz.*, t. **50**, 1960, p. 738-758 [*Soviet Phys., J. E. T. P.*, t. **23**, 1966, p. 489-501].

[*155*] B. Roulet, J. Gavoret and P. Nozières, Singularities in the X-ray absorption and emission of metals. I. First-order parquet calculation. *Phys. Rev.*, t. **178**, 1969, p. 1072-1083.

[*156*] P. Nozières, J. Gavoret and B. Roulet, Singularities in the X-ray absorption and emission of metals. II. Self-consistent treatment of divergences. *Phys. Rev.*, t. **178**, 1969, p. 1084-1096.

[*157*] E. C. Poggio, On the renormalization method: a different outlook, and the energy-momentum tensor in the $g\phi^4$ theory. *Ann. Phys.*, t. **81**, 1973, p. 481-518.

[*158*] R. W. Haymaker and R. Blankenbecler, Variational principles for crossing-symmetric off-shell equations. *Phys. Rev.*, t. **171**, 1968, p. 1581-1587.

[*159*] R. J. Yaes, Nonassociativity of the operators in the crossing-symmetric Bethe-Salpeter equations. *Phys. Rev.*, t. **D 2**, 1970, p. 2457-2463.

[*160*] W. Becker and D. Grosser, Crossing and unitarity in the context of the Bethe-Salpeter equation. *Nuovo Cim.*, t. **10 A**, 1972, p. 343-361.

[*161*] R. G. Root, Effective potential for the O(N) model to order 1/N. *Phys. Rev.*, t. **D 10**, 1974, p. 3322-3334.

[*162*] G. Feinberg and A. Pais, A field theory of weak interactions. I. *Phys. Rev.*, t. **131**, 1963, p. 2724-2761. II. *Phys. Rev.*, t. **133**, 1964, B 477-B 486.

[*163*] B. A. Arbuzov and A. T. Filippov, Vertex function in nonrenormalizable field theory. *Nuovo Cim.*, t. **38**, 1965, p. 796-806.

[*164*] B. A. Arbuzov and A. T. Filippov, Iteration method in nonrenormalizable field theory. *Zh. Eksp. Teor. Fiz.*, t. **49**, 1965, p. 990-999 [*Soviet Phys., J. E. T. P.*, t. **32**, 1966, p. 688-693].

[*165*] W. Güttinger, R. Penzl and E. Pfaffelhuber, Peratization of unrenormalizable field theories. *Ann. Phys.*, t. **33**, 1965, p. 246-271.

[*166*] T. D. Lee, Analysis of divergences in a neutral-spin-1-meson theory with parity-nonconserving interactions. *Nuovo Cim.*, t. **59 A**, 1969, p. 579-598.

[167] M. K. VOLKOV, Methods of quantum field theory with rapidly increasing spectral functions. *Fortsch. Physik*, t. **19**, 1971, p. 757-782.

[168] L. NITTI, M. F. PELLICORO and M. VILLANI, Higher-order corrections in nonrenormalizable quantum field theories. *Lett. Nuovo Cim.*, t. **3**, 1972, p. 541-547.

[169] N. N. KHURI and A. PAIS, Singular potentials and peratization. I. *Rev. Mod. Phys.*, t. **36**, 1964, p. 590-595.

[170] W. M. FRANK, D. J. LAND and R. M. SPECTOR, Singular potentials. *Rev. Mod. Phys.*, t. **43**, 1971, p. 36-98.

[171] T. T. WU, Theory of Toeplitz determinants and the spin correlations of the two-dimensional Ising model. I. *Phys. Rev.*, t. **149**, 1966, p. 380-401, Section 8(H).

[172] A. A. SLAVNOV and A. E. SHABAD, Elimination of unphysical singularities in the Feinberg-Pais theory of weak interactions. *Yad. Fiz.*, t. **1**, 1965, p. 721-728 [*Sov. J. Nucl. Phys.*, t. **1**, 1965, p. 514-519].

[173] M. B. HALPERN, Nonrenormalizable field theories. *Ann. Phys.*, t. **39**, 1966, p. 351-375.

[174] V.Sh. GOGOKHIYA and A. T. FILIPPOV, Approximate methods for solving the Schrödinger equation with a singular potential. *Yad. Fiz.*, t. **15**, 1972, p. 1294-1306 [*Sov. J. Nucl. Phys.*, t. **15**, 1972, p. 714-719].

[175] G. PARISI, The theory of non-renormalizable interactions. The large N expansion. *Nucl. Phys.*, t. **B 100**, 1975, p. 368-388.

[176] J. C. TAYLOR, The form of the divergencies in quantum electrodynamics. *Proc. Roy. Soc.* (London), t. **234 A**, 1956, p. 296-300.

[177] J. C. TAYLOR, Renormalization in meson theories. *Proc. Camb. Philos. Soc.*, t. **52**, 1956, p. 534-537.

[178] K. A. TER-MARTIROSIAN, Charge renormalization for an arbitrary, not necessarily small, value of e_0. *Zh. Eksp. Teor. Fiz.*, t. **31**, 1956, p. 157-159 [*Soviet Phys., J. E. T. P.*, t. **4**, 1957, p. 443-444].

[179] H. D. POLITZER, Asymptotic freedom: An approach to the strong interactions. *Phys. Rep.*, t. **14**, 1974, p. 129-180.

[180] E. S. FRADKIN and O. K. KALASHNIKOV, Renormalized set of equations for the Green functions in the Yang-Mills field theory and its asymptotic solution. *J. Phys.*, t. **A 9**, 1976, p. 159-168.

[181] J. C. COLLINS and A. J. MACFARLANE, New methods for the renormalization group. *Phys. Rev.*, t. **D 10**, 1974, p. 1201-1212.

[182] K. G. WILSON, Renormalization group and critical phenomena. I. Renormalization group and the Kadanoff scaling picture. *Phys. Rev.*, t. **B 4**, 1971, p.3174-3183. II. Phase-space cell analysis of critical behavior. *Phys. Rev.*, t. **B 4**, 1971, p. 3184-3205.

[183] L. P. KADANOFF, A. HOUGHTON and M. C. YALABIK, Variational approximations for renormalization group transformations. *J. Stat. Phys.*, t. **14**, 1976, p. 171-203; t. **15** (E), 1976, p. 263.

[184] H. W. J. BLÖTE and R. H. SWENDSEN, Critical behavior of the four-dimensional Ising model. *Phys. Rev.*, t. **B 22**, 1980, p. 4481-4483.

[185] B. SIMON, Coupling constant analyticity for the anharmonic oscillator. *Ann. Phys.*, t. **58**, 1970, p. 76-136.

[186] S. GRAFFI, V. GRECCHI and B. SIMON, Borel summability: application to the anharmonic oscillator. *Phys. Lett.*, t. **32 B**, 1970, p. 631-634.

[187] G. PARISI, The perturbative expansion and the infinite coupling limit. *Phys. Lett.*, t. **69 B**, 1977, p. 329-331.

[188] G. A. BAKER, Jr., B. G. NICKEL and D. I. MEIRON, Critical indices from perturbation analysis of the Callan-Symanzik equation. *Phys. Rev.*, t. **B 17**, 1978, p. 1365-1374.

[189] J. C. LEGUILLOU and J. ZINN-JUSTIN, Critical exponents for the *n*-vector model in three dimensions from field theory. *Phys. Rev. Lett.*, t. **39**, 1977, p. 95-98.

[190] J. C. LEGUILLOU and J. ZINN-JUSTIN, Critical exponents from field theory. *Phys. Rev.*, t. **B 21**, 1980, p. 3976-3998.

[191] A. A. VLADIMIROV, Methods of calculating many-loop diagrams and renormalization-group analysis of the ϕ^4 theory. *Teor. Mat. Fiz.*, t. **36**, 1978, p. 271-278 [*Theor. Math. Phys.*, t. **36**, 1978, p. 732-737].

[192] F. M. DITTES, Yu. A. KUBYSHIN and O. V. TARASOV, Four-loop approximation in the ϕ^4 model. *Teor. Mat. Fiz.*, t. **37**, 1978, p. 66-73 [*Theor. Math. Phys.*, t. **37**, 1978, p. 879-884].

[193] N. N. KHURI, Solutions of the Callan-Symanzik equation in a complex neighborhood of zero coupling. *Phys. Rev.*, t. **D 12**, 1975, p. 2298-2310.

[194] N. N. KHURI, Borel summability and the renormalization group. *Phys. Rev.*, t. **D 16**, 1977, p. 1754-1761.

[195] A. D. SOKAL, An improvement of Watson's theorem on Borel summability. *J. Math. Phys.*, t. **21**, 1980, p. 261-263.

[196] J. ZINN-JUSTIN, Asymptotic estimate of perturbation theory at large orders. In: *Hadron structure and lepton-hadron interactions* (1977 Cargèse lectures). M. Lévy *et al.*, eds., New York-London, Plenum Press, 1979.

[197] G. PARISI, The physical basis of asymptotic estimates in perturbation theory. In: *Hadron structure and lepton-hadron interactions* (1977 Cargèse lectures). M. Lévy *et al.*, eds., New York-London, Plenum Press, 1979.

[198] L. D. LANDAU, Fundamental problems. In: *Theoretical physics in the twentieth century, a memorial volume to Wolfgang Pauli*, M. Fierz and V. F. Weisskopf, eds., New York, Interscience, 1960. Reprinted in [16].

[199] D. B. ABRAHAM and H. KUNZ, Ornstein-Zernike theory of classical fluids at low density. *Phys. Rev. Lett.*, t. **39**, 1977, p. 1011-1014.

[200] G. A. BAKER, Jr. and D. BESSIS, *Asymptotic decay of correlations in the spin-1/2* Ising model. *J. Math. Phys.*, t. **22**, 1981, p. 1264-1266.

[201] C. M. BENDER, G. S. GURALNIK, R. W. KEENER and K. OLAUSSEN, Numerical study of truncated Green's-function equations. *Phys. Rev.*, t. **D 14**, 1976, p. 2590-2595.

[202] B. SCHROER, A necessary and sufficient condition for the softness of the trace of the energy-momentum tensor. *Lett. Nuovo Cim.*, t. **2**, 1971, p. 867-872.

[203] J. C. COLLINS, Renormalization of the energy-momentum tensor in ϕ^4 theory. *Phys. Rev.*, t. **D 14**, 1976, p. 1965-1976.

[204] E. C. POGGIO, Short-distance behavior of a $g\phi^4$ theory, and conformal-invariant four-point functions. *Phys. Rev.*, t. **D 8**, 1973, p. 2431-2446.

[205] V. K. DOBREV, G. MACK, V. B. PETKOVA, S. G. PETROVA and I. T. TODOROV, *Harmonic analysis on the* n-*dimensional Lorentz group and its application to conformal quantum field theory* (Lecture notes in physics # 63), Berlin-Heidelberg-New York, Springer-Verlag, 1977.

[206] B. GEYER, V. A. MATVEEV, D. ROBASCHIK and E. WIECZOREK, Analytical properties of conformal invariant four point functions. *Rep. Math. Phys.*, t. **10**, 1976, p. 203-211.

[207] G. MACK and I. T. TODOROV, Conformal-invariant Green functions without ultraviolet divergences. *Phys. Rev.*, t. **D 8**, 1973, p. 1764-1787.

[208] E. S. FRADKIN and M. Ya. PALCHIK, Conformally invariant solution of quantum field theory equations (1). *Nucl. Phys.*, t. **B 99**, 1975, p. 317-345.

[209] V. K. DOBREV, V. B. PETKOVA, S. G. PETROVA and I. T. TODOROV, Dynamical derivation of vacuum operator-product expansion in Euclidean conformal quantum field theory. *Phys. Rev.*, t. **D 13**, 1976, p. 887-912.

[210] G. MACK, Osterwalder-Schrader positivity in conformal invariant quantum field theory. In: *Trends in elementary particle theory* (Lecture notes in physics # 37). H. Rollnik and K. Dietz, eds., Berlin-Heidelberg-New York, Springer-Verlag, 1975.

[211] R. A. WILLOUGHBY, The inverse M-matrix problem. *Lin. Alg. Appl.*, t. **18**, 1977, p. 75-94.

[212] T. L. MARKHAM, Nonnegative matrices whose inverses are M-matrices. *Proc. Amer. Math. Soc.*, t. **36**, 1972, p. 326-330.

[213] B. B. MURPHY, On the inverses of M-matrices. *Proc. Amer. Math. Soc.*, t. **62**, 1977, p. 196-198.

[214] G. CAGINALP, The ϕ^4 lattice field theory as an asymptotic expansion about the Ising limit. *Ann. Phys.*, t. **124**, 1980, p. 189-207.

[215] G. CAGINALP, Thermodynamic properties of the ϕ^4 lattice field theory near the Ising limit. *Ann. Phys.*, t. **126**, 1980, p. 500-511.

[216] F. CONSTANTINESCU, Strong-coupling expansion of the continuous-spin Ising model. *Phys. Rev. Lett.*, t. **43**, 1979, p. 1632-1635.

[217] L. H. KARSTEN and J. SMIT, The vacuum polarization with SLAC lattice fermions. *Phys. Lett.*, t. **85 B**, 1979, p. 100-102.

[217′] J. M. RABIN, *Long-range interactions in lattice field theory*. SLAC-Report 240, June 1981.

[218] H. S. SHARATCHANDRA, Continuum limit of lattice gauge theories in the context of renormalized perturbation theory. *Phys. Rev.*, t. **D 18**, 1978, p. 2042-2055.

[218′] N. KIMURA and A. UKAWA, Energy-momentum dispersion of glueballs and the restoration of Lorentz invariance in lattice gauge theories. *Nucl. Phys.*, t. **B205** [FS5], 1982, p. 637-647.

[218″] C. B. LANG and C. REBBI, Potential and restoration of rotational symmetry in SU(2) lattice gauge theory. *Phys. Lett.*, t. **115 B**, 1982, p. 137-142.

[219] R. F. STREATER, Connection between the spectrum condition and the Lorentz invariance of $P(\phi)_2$. *Commun. Math. Phys.*, t. **26**, 1972, p. 109-120.

[220] E. P. HEIFETS and E. P. OSIPOV, The energy-momentum spectrum in the $P(\phi)_2$ quantum field theory. *Commun. Math. Phys.*, t. **56**, 1977, p. 161-172.

[221] A. D. SOKAL, Unpublished, 1980.

[222] M. REED and B. SIMON, *Methods of modern mathematical physics. III. Scattering theory*, New York-San Francisco-London, Academic Press, 1979.

[223] A. D. SOKAL, Unpublished, 1980.

[224] L. MONROE and P. A. PEARCE, Correlation inequalities for vector spin models. *J. Stat. Phys.*, t. **21**, 1979, p. 615-633.

[225] M. REED and B. SIMON, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, New York-San Francisco-London, Academic Press, 1975.

[226] H. E. STANLEY, *Introduction to phase transitions and critical phenomena*, New York, Oxford University Press, 1971.

[227] J. GLIMM and A. JAFFE, ϕ_2^4 quantum field model in the single-phase region: differentiability of the mass and bounds on critical exponents. *Phys. Rev.*, t. **D 10**, 1974, p. 536-539.

[228] G. A. BAKER, Jr., Critical exponent inequalities and the continuity of the inverse range of correlation. *Phys. Rev. Lett.*, t. **34**, 1975, p. 268-270.

[229] M. AIZENMAN, Proof of the triviality of φ_d^4 field theory and some mean-field features of Ising models for $d > 4$. *Phys. Rev. Lett.*, t. **47**, 1981, p. 1-4, 886 (E).

[230] M. AIZENMAN, Geometric analysis of φ^4 fields and Ising models (parts I and II). *Commun. Math. Phys.*, t. **86**, 1982, p. 1-48.

[231] J. FRÖHLICH, On the triviality of $\lambda\varphi_d^4$ theories and the approach to the critical point in $d \geq 4$ dimensions. *Nucl. Phys.*, t. **B 200** [FS4], 1982, p. 281-296.

[232] D. BRYDGES, J. FRÖHLICH and T. SPENCER, The random walk representation of classical spin systems and correlation inequalities. *Commun. Math. Phys.*, t. **83**, 1982, p. 123-150.

[*233*] D. C. BRYDGES, J. FRÖHLICH and A. D. SOKAL, *The random-walk representation of classical spin systems and correlation inequalities. II. The skeleton inequalities.* Courant Institute preprint.

[*234*] D. C. BRYDGES, J. FRÖHLICH and A. D. SOKAL, *A new proof of the existence and nontriviality of the continuum φ_2^4 and φ_3^4 quantum field theories.* Courant Institute preprint.

[*235*] R. SCHRADER and E. TRÄNKLE, Analytic and numerical evidence from quantum field theory for the hyperscaling relation $d\nu = 2\Delta - \gamma$ in the $d = 3$ Ising model. *J. Stat. Phys.*, t. **25**, 1981, p. 269-290.

[*236*] K. OSTERWALDER, Unpublished, cited in [*235*, section 1].

[*237*] W. DRIESSLER and J. FRÖHLICH, The reconstruction of local observable algebras from the Euclidean Green's functions of relativistic quantum field theory. *Ann. Inst. Henri Poincaré*, t. **A 27**, 1977, p. 221-236.

[*238*] J.-P. ECKMANN and H. EPSTEIN, Time-ordered products and Schwinger functions. *Commun. Math. Phys.*, t. **64**, 1979, p. 95-130.

[*239*] A. D. SOKAL, Mean-field bounds and correlation inequalities. *J. Stat. Phys.*, t. **28**, 1982, p. 431-439.

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