ANNALES DE L'I. H. P., SECTION A

MAURO FRANCAVIGLIA
DEMETER KRUPKA

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Annales de l'I. H. P., section A, tome 37, nº 3 (1982), p. 295-315
<http://www.numdam.org/item?id=AIHPA_1982__37_3_295_0>

Ann. Inst. Henri Poincaré, Section A :
Vol. XXXVII, n° 3, 1982, p. 295-315. *Physique théorique.*

The Hamiltonian formalism in higher order variational problems

by

Mauro FRANCAVIGLIA (*) and Demeter KRUPKA
Department of Mathematics, Faculty of Science,
J. E. Purkyně University
Janáčkovo nám. 2a, 66295 BRNO (Czechoslovakia)

ABSTRACT. — In the framework of the theory of Lagrangian structures of any order r, the meaning of a natural intrinsic « regularity condition » for the Lagrangian forms is investigated. This condition is used to characterize so called Hamiltonian extremals and to define an invertible r-th order Legendre transformation. Hamilton equations of order r are explicitly calculated in terms of new « phase space » variables, thus providing a completely equivalent counterpart to Euler-Lagrange equations (for regular Lagrangians).

RÉSUMÉ. — Dans le cadre de la théorie des structures Lagrangiennes d'ordre r quelconque, on étudie la signification d'une « condition de régularité » naturelle et intrinsèque pour les formes Lagrangiennes. Cette condition est utilisée pour caractériser ce qu'on appelle les extrémales Hamiltoniennes et pour définir une transformation de Legendre inversible d'ordre r. On établit explicitement les équations de Hamilton d'ordre r en termes de nouvelles variables d'espace de phase, obtenant ainsi un système complètement équivalent aux équations d'Euler-Lagrange (pour des Lagrangiens réguliers).

(*) *Permanent address:* Istituto di Fisica Matematica « J.-L. Lagrange », Universitá di Torino, Via C. Alberto 10, 10123 Torino (Italy).

Annales de l'Institut Henri Poincaré-Section A-Vol. XXXVII, 0020-2339 1982/295/$ 5,00/

0. INTRODUCTION

As it is well known, classical mechanical systems with finitely many degrees of freedom can be alternatively described from either a Lagrangian viewpoint or a Hamiltonian viewpoint. The relations between these two possible formulations are fairly well established. In particular, it is known that a Lagrangian system admits an *equivalent* Hamiltonian counterpart if the Lagrangian function is regular (in a standard sense), and, as a consequence, the classical Legendre transformation is an invertible mapping from « velocity space » to « phase space ».

It is also well known that the Lagrangian formulations of classical mechanics arise from so-called « (classical) variational principles », whereby the equations of motion are derived by requiring the stationarity of « action functionals ».

Lagrangian methods and variational principles are in fact very common in all domains of physics and they can be considered as the most frequently used approaches to the formulation of physical field theories. While in classical mechanics only « configurations » and « velocities » are involved in the Lagrangian, in a generic field theory the Lagrangian function (which is supposed to contain all informations pertaining to the physical theory itself) is allowed to depend on higher order derivatives of the fields. This is the source of a widespread interest in « higher order » variational principles and their rigorous mathematical foundations.

Thanks to the contributions of several authors it has been reached a relatively complete geometrical description of a rigorous framework for higher order variational principles, especially by means of the calculus of variations on fiber bundles and the use of jet structures (see e. g. [*1*]-[*10*] and references cited therein). We shall here rely on the approach developed by one of us in [*7*]-[*9*]. It is based on assigning a « configuration bundle » (Y, X, π) together with an r-th order Lagrangian λ, which is a differential form on the r-th order jet prolongation of this bundle. Taking in particular $X = R$, $Y = M \times R$ and $r = 1$, we can obtain the Lagrangian formulation of classical mechanics, where M is the « configuration space » of the mechanical system considered. So called *higher order mechanics* is obtained by taking X and Y as before and letting $r > 1$.

In the geometrical framework above, the corresponding Lagrangian formalism is very well established at all orders r (see e. g. [*9*] [*6*]), containing classical Lagrangian mechanics as a particular case. A completely satisfactory Hamiltonian formalism has been formulated for 1-st order variational principles, containing classical Hamiltonian mechanics as a particular case (see e. g. [*1*] [*3*] [*11*] [*12*]). For the first order case, in fact, it has been shown that if λ satisfies an appropriate natural regularity condition, there exists an equivalent Hamiltonian formalism (of order 1), which may be obtained using an appropriate 1-*st order Legendre transformation* which maps the original bundle j^1Y to the so called (1st order) *Legendre bundle*, which plays the role of « phase space ».

A natural question then arises: does there exist an appropriate generalization of the Hamilton formalism to higher order cases ? And if so, under which conditions on the Lagrangian ?

As far as we know, no fully satisfactory answer to this question has been given up to now. We remark that partial steps towards constructing higher order Hamiltonian formalism have been taken by several authors, both in the framework of higher order mechanics (e. g. [*13*] [*14*]) and in field theory (e. g. [*15*]).

In this paper we shall consider the question from the point of view of the theory of *Lagrangian structures*, according to [*8*]. We shall show that a suitable natural intrinsic regularity condition can be imposed to the Lagrangian λ of arbitrary order r, so that an appropriate equivalent Hamiltonian counterpart exists for the corresponding Lagrangian theory, and we shall derive the explicit (local) Hamiltonian equations.

In Section 1 the basic notations are introduced. Section 2 contains a discussion about the r-th order Poincaré-Cartan form, which is a fundamental notion of the theory. In Section 3 the notion of *Hamiltonian extremals* is introduced (cf. [*16*]) and it is shown that Hamiltonian extremals are in one-to-one correspondence with critical sections of λ, provided λ satisfies the aforementioned regularity condition. Section 4 contains a technical lemma which under the same regularity condition defines (locally) a natural r-th order Legendre transformation, which may be interpreted as a change of local coordinates in the underlying jet bundle. This enables us, in Section 5, to derive a set of r-th order equations, that we call *Hamiltonian equations of order* r, which are equivalent to the $2r$-th order Euler-Lagrange equations of λ. These Hamiltonian equations reduce to the well known ones if we take $r = 1$. Finally, in Section 6 we consider some elementary examples within the framework of second order mechanics.

1. PRELIMINARIES AND NOTATIONS

Here, we shall fix the notations which shall be used throughout this paper and we shall recall some basic definitions from the theory of variational problems on jet bundles. We shall essentially follow some previous papers by one of us [*7*]-[*9*].

Throughout this paper all manifolds, all objects defined over them and all mappings between manifolds, are implicitely assumed to be smooth (in the C^∞-sense).

Let X be a paracompact, Hausdorff, n-dimensional (real) manifold. We shall assume for simplicity that X is connected, orientable and oriented.

(These restrictions are unessential. See [9] for the details of the theory when X is not orientable and not necessarily connected). We denote by (Y, X, π) any *fibered manifold* over X (here Y is the *total space*, X is the *base space*, $\pi : Y \to X$ is a surjective submersion). We set

$$m = \dim Y - \dim X = \dim Y - n .$$

We denote by (j^rY, X, π_r) the r-th jet prolongation of Y and by $(j^rY, j^lY, \pi_{r,l})$ the natural fibered structure of j^rY onto j^lY (where $0 \leq l \leq r-1$ and $Y \equiv j^0Y$).

If (V, ψ) is a fibered chart of Y, with local fibered coordinates (x^i, y^σ), $1 \leq i \leq n$, $1 \leq \sigma \leq m$, then the associated fibered chart of j^rY is denoted by (V_r, ψ_r). Here $V_r \equiv (\pi_{r,0})^{-1}(V) \subset j^rY$ and ψ_r defines local fibered coordinates $(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \ldots, y^\sigma_{j_1\ldots j_r})$, where $1 \leq i \leq n$, $1 \leq \sigma \leq m$, $1 \leq j_1 \leq j_2 \leq \ldots \leq j_r \leq n$. We stress that only non-decreasing sequences of integers appear.

In order to allow also arbitrary sequences of integers, which shall sometimes simplify the calculations in local coordinates, we introduce the following conventions. Let s be any integer, $1 \leq s \leq r$. Let $(i_1, \ldots, i_s)$ be an arbitrary s-tuple of integers $1 \leq i_k \leq n$ $(k = 1, \ldots, s)$. There exists one and only one non-decreasing s-tuple of integers $1 \leq j_1 \leq j_2 \leq \ldots \leq j_s \leq n$ which is a permutation of the given s-tuple $(i_1, \ldots, i_s)$. We call this non-decreasing s-tuple the *ordered form* of $(i_1, \ldots, i_s)$ and we denote it by $0(i_1, \ldots, i_s)$. Accordingly we formally define $y^\sigma_{i_1\ldots i_s}$ for an arbitrary s-tuple $(i_1, \ldots, i_s)$ by setting:

$$y^\sigma_{i_1\ldots i_s} \equiv y^\sigma_{0(i_1,\ldots,i_s)} . \tag{1.1}$$

Let now $a^{i_1\ldots i_s}$ be any collection of functions on V_r, which are totally symmetric in all superscripts. Whenever such a collection appears in some formula, the whole collection $\{ a^{i_1\ldots i_s} \}$ will be replaced by its distinguished member $a^{0(i_1,\ldots,i_s)}$. The same convention will be applied to any collection $b_{i_1\ldots i_s}$ of functions on V_r which are totally symmetric in all subscripts. Accordingly, this convention will be applied as follows to summations. If $a^{i_1\ldots i_s i_{s+1}\ldots i_r}$ (respectively $b_{i_1\ldots i_s}$) are expressions which are totally symmetric in their superscripts (resp. in their subscripts) then we shall denote by

$$a^{i_1\ldots i_s i_{s+1}\ldots i_r} b_{i_1\ldots i_s}$$

the expression obtained by extending the summation only to ordered non-decreasing sequences; namely, we set:

$$a^{i_1\ldots i_s i_{s+1}\ldots i_r} b_{i_1\ldots i_s} \equiv \sum_{0(i_1,\ldots,i_s)} a^{0(i_1,\ldots,i_s),i_{s+1},\ldots,i_r} b_{0(i_1,\ldots,i_s)} \equiv \sum_{1 \leq i_1 \leq \ldots \leq i_s \leq n} a^{i_1\ldots i_s i_{s+1}\ldots i_r} b_{i_1\ldots i_s} .$$

We shall use the following standard notations. By $i_{\Xi}\eta$ we denote the *inner product* of a vectorfield Ξ and a differential form η; $d_i\psi$ denotes the *formal derivative* (or *total derivative*) with respect to x^i of a function $\psi : V_r \to R$; if $\gamma : X \to Y$ is a section of π, its r-*th jet prolongation* is denoted by $j^r\gamma$ (we recall that $j^r\gamma$ is a section of π_r); similarly, $j^r\Xi$ denotes the r-*th jet prolongation* of a π-projectable vector field Ξ on Y (we recall that $j^r\Xi$ is a π_r-projectable vectorfield on j^rY).

The following (local) differential forms on j^rY, which we define by their coordinate expressions on V_r, will be used throughout:

$$\omega_0 \equiv dx^1 \wedge \dots \wedge dx^n ;$$
$$\omega_j \equiv (-1)^{j-1} dx^1 \wedge \dots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \dots \wedge dx^n , \quad (1 \le j \le n);$$
$$\omega^\sigma \equiv dy^\sigma - y^\sigma_i dx^i , \quad (1 \le \sigma \le m);$$
$$\omega^\sigma_{j_1 \dots j_l} = dy^\sigma_{j_1 \dots j_l} - y^\sigma_{ij_1 \dots j_l} dx^i , \quad (1 \le l \le n-1).$$

For further details see e. g. [*8*].

Let (Z, X, τ) be any fibered manifold. Recall that a vector-field $\Sigma : Z \to TZ$ is τ-*vertical* iff $T\tau \circ \Sigma = 0$ (this means, roughly speaking, that Σ is tangent to the fibres of Z). A differential p-form $\rho \in \Omega^p(Z)$ is τ-*horizontal* iff $\rho(\Sigma_1, \dots, \Sigma_p)$ vanishes whenever at least one of the vectorfields $\Sigma_1, \dots, \Sigma_p$ is τ-vertical.

Consider now (j^sY, X, π_s), where s is any integer. A p-form $\rho \in \Omega^p(j^sY)$, with $p \le n$, is called a *contact form* iff the following holds:

$$(1.2) \qquad (j^s\gamma)^*\rho = 0$$

for all sections $\gamma : X \to Y$ of π. Let then η be any p-form $n \in \Omega^p(j^sY)$. There exists one and only one π_{s+1}-horizontal p-form $h(\eta) \in \Omega^p(j^{s+1}Y)$ such that the following holds:

$$(1.3) \qquad (j^{s+1}\gamma)^*\, h(\eta) = (j^s\gamma)^*\, \eta$$

for all sections γ of π (see [*7*], p. 23). The form $h(\eta)$ will be called the *horizontal part* of η. We set:

$$(1.4) \qquad p(\eta) = (\pi_{s+1,s})^*\, \eta - h(\eta).$$

The form $p(\eta)$ is contact and we call it the *contact part* of η. We remark that the two operations h and p may be suitably extended to the differential forms of any degree $p \ge n + 1$ (see [*8*]) in such a way that the decomposition $(\pi_{s+1;s})^*\, \eta = p(\eta) + h(\eta)$ still holds.

We recall that from the operational viewpoint the action of h on p-form on j^sY ($p \le n$) is completely described by the following local coordinate relations:

$$(1.5) \qquad \begin{aligned} h(f) &= f \circ \pi_{s+1,s} ; \\ h(dx^i) &= dx^i ; \\ h(dy^\sigma) &= y^\sigma_k dx^k ; \\ h(dy^\sigma_{j_1 \dots j_l}) &= y^\sigma_{ij_1 j_2 \dots j_l} dx^i , \qquad (1 \le l \le s) ; \end{aligned}$$

where f is any function on V_s.

We finally recall the following definitions. A *Lagrangian of order r* for Y is a π_r-horizontal n-form $\lambda \in \Omega^n(J^r Y)$ (here $n = \dim x$). The quadruple (Y, X, π ; λ) will be referred to as a *Lagrangian structure of order r*. Lagrangian structures (of order r) are the natural framework for formulating and investigating variational problems (of order r).

2. THE GENERALIZED POINCARÉ-CARTAN FORM

In the calculus of variations on jet bundles a special role is played by the so-called *generalized Poincaré-Cartan form*, which, as we shall see later, is a fundamental concept for the transition from the Lagrangian formulation to the corresponding Hamiltonian formulation of a variational problem. The Poincaré-Cartan form for the order $r = 1$ was known since a long time (see e. g. [*3*] [*4*]). A generalization to the second order was first proposed in [*7*]; finally the generalized Poincaré-Cartan form for an arbitrary order r was defined an investigated in [*17*].

Let (Y, X, π ; λ) be a Lagrangian structure of order r. The following is a refinement of a theorem which was given in [*7*] (see also [*9*]).

THEOREM 1. — *There exists a n-form* $\Theta_\lambda \in \Omega^n(j^{2r-1}Y)$ *such that the following conditions hold:*

i) *for each* π_{2r-1}*-vertical vectorfield* Ξ *on* $j^{2r-1}Y$ *the* $(n-1)$*-form* $i_\Xi \Theta_\lambda$ *is* π_{2r-1}*-horizontal;*

ii) $h(\Theta_\lambda) = \lambda$;

iii) *for each* π_{2r-1}*-vertical* $\pi_{2r-1,0}$ *projectable vectorfield* Ξ *on* $j^{2r-1}Y$ *the n-form* $h(i_\Xi d\Theta_\lambda)$ *depends only on the* $\pi_{2r-1,0}$*-projection of the vectorfield* Ξ;

iv) *for each isomorphism* $\alpha : Y \to Y$ *of the fibered manifold* (Y, X, π) *the condition* $\Theta_{j_r\alpha * \lambda} = j^{2r-1}\alpha * \Theta_\lambda$ *holds.*

The n-form $\Theta_\lambda \in \Omega^n(j^{2r-1}Y)$ is called the *generalized Poincaré-Cartan form* associated with λ. In local fibered coordinates the restriction of Θ_λ to the fibered chart V_{2r-1} has the following explicit expression :

$$\Theta_\lambda = L\omega_0 + \left(\sum_{k=0}^{r-1} f_\sigma^{ij_1 \dots j_k} \omega_{j_1 \dots j_k}^\sigma \right) \wedge \omega_i . \tag{2.1}$$

Here we have set (locally):

$$\lambda = L\omega_0 , \tag{2.2}$$

where $L = L(x^i, y^\sigma, y^\sigma_{j_1}, \dots, y^\sigma_{j_1 \dots j_r})$ is a function from V_r to R. Moreover:

$$(2.3) \qquad f_\sigma^{j_1 \dots j_r} \equiv \frac{\partial L}{\partial y^\sigma_{j_1 \dots j_r}}$$

$$f_\sigma^{j_1 \dots j_s} \equiv \frac{\partial L}{\partial y^\sigma_{j_1 \dots j_s}} - d_k f_\sigma^{k j_1 \dots j_s}, \qquad (s = 1, \dots, r-1);$$

or more explicitly:

$$(2.4) \quad f_\sigma^{j_1 \dots j_s} \equiv \frac{\partial L}{\partial y^\sigma_{j_1 \dots j_s}} - d_{i_1} \frac{\partial L}{\partial y^\sigma_{i_1 j_1 \dots j_s}} + \dots + (-1)^{r-s} d_{i_1} \dots d_{i_{r-s}} \frac{\partial L}{\partial y^\sigma_{i_1 \dots j_s}}$$

(for $1 \leq s \leq r-1$).

We remark that from the structure of the coefficients $f_\sigma^{j_1 \dots j_s}$ we immediately infer that only the first one $f_\sigma^{j_1}$ depends on all coordinates $x^i, y^\sigma, y^\sigma_{j_1}, \dots, y^\sigma_{j_1, \dots j_{2r-1}}$. More precisely, $f_\sigma^{i_1 \dots i_s}$ is a function of the coordinates $(x^i, y^\sigma, y^\sigma_{j_1}, \dots, y^\sigma_{j_1 \dots j_{2r-s}})$ in $V_{2r-s} \equiv (\pi_{2r-s,0})^{-1}(V)$, for $1 \leq s \leq r$.

As it is well known the *Euler-Lagrange expressions* of the r-th order Lagrangian λ are the following local functions $\varepsilon_\sigma(L) : V_{2r} \to R$ defined for all local fibered charts $V_{2r} = (\pi_{2r,0})^{-1}(V)$ of $j^{2r}Y$:

$$(2.5) \qquad \varepsilon_\sigma(L) \equiv \frac{\partial L}{\partial y^\sigma} - d_{i_1} \frac{\partial L}{\partial y_{i_1}} + \dots + (-1)^r d_{i_1} \dots d_{i_r} \frac{\partial L}{\partial y_{i_1 \dots i_r}}.$$

The *Euler-Lagrange form associated with* λ is the (global) $(n+1)$-form $\varepsilon_\lambda \in \Omega^{n+1}(j^{2r}Y)$ defined by the local coordinate expressions:

$$(2.6) \qquad \varepsilon_\lambda |_{V_{2r}} \equiv \varepsilon_\sigma(L) \omega^\sigma \wedge \omega_0 .$$

It is immediately seen that ε_λ is $\pi_{2r,0}$-horizontal.

An important property of the generalized Poincaré-Cartan form Θ_λ is that the horizontal part of the $(n+1)$-form $d\Theta_\lambda$ coincides with the Euler-Lagrange form ε_λ itself; in formulae:

$$(2.7) \qquad h(d\Theta) = \varepsilon_\lambda .$$

More precisely we have the following lemma.

LEMMA 1. — *Let λ be a Lagrangian of order r for* Y. *Then the following decomposition holds:*

$$(2.8) \qquad d\Theta_\lambda = \varepsilon_\lambda + F_\lambda ,$$

where F_λ is a contact form on $j^{2r-1}Y$, locally defined by:

$$(2.9) \qquad F_\lambda |_{V_r} \equiv \sum_{0}^{r-1}{}_k \sum_{0}^{2r-1}{}_s \frac{\partial f_\sigma^{i j_1 \dots j_k}}{\partial y^\nu_{l_1 \dots l_s}} \omega^\nu_{l_1 \dots l_s} \wedge \omega^\sigma_{j_1 \dots j_k} \wedge \omega_i .$$

Proof. — By direct calculation from (2.1), (2.3) and (2.5) (see e. g. [9] [17] for details).

3. HAMILTONIAN EXTREMALS AND REGULARITY CONDITIONS

Let us suppose a Lagrangian structure of order r (Y, X, π ; λ) is given. We recall that the *action of* λ on a domain $\Omega \subseteq X$ is the mapping λ_Ω defined by:

$$\lambda_\Omega : \gamma \to \int_\Omega (j^r\gamma)^* \lambda, \tag{3.1}$$

where $\gamma : \Omega \to \pi^{-1}(\Omega) \subseteq Y$ is any local section of π. According to the standard definitions, the *extremals* (or *critical sections*) of λ are those sections which make all mappings λ_Ω stationary (in the suitably defined well known sense). We denote by Γ_λ the set of all critical sections of λ.

All sections $\gamma \in \Gamma_\lambda$ satisfy *Euler-Lagrange equations:*

$$\varepsilon_\lambda \circ j^{2r}\gamma = 0 \tag{3.2}$$

which are in fact equivalent to the well known local equations

$$\frac{\partial L}{\partial y^\sigma} - d_{i_1} \frac{\partial L}{\partial y^\sigma_{i_1}} + \ldots + (-1)^r d_{i_1} d_{i_2} \ldots d_{i_r} \frac{\partial L}{\partial y^\sigma_{i_1 \ldots i_r}} = 0 \tag{3.2'}$$

along the section γ itself. It is easy to show that equations (3.2) may be re-written as follows:

$$(j^{2r}\gamma)^* (i_{j^{2r}\xi}\varepsilon_\lambda) = 0, \tag{3.3}$$

for all π-vertical vectorfields ξ on Y. Using (2.8) and recalling that F_λ is contact, conditions (3.3) are immediately transformed into the following:

$$(j^{2r-1}\gamma)^* (i_{j^{2r-1}\xi} d\Theta_\lambda) = 0. \tag{3.4}$$

Equations (3.4) suggest to us the following definition:

DEFINITION 3.1. — *We say that a section* $\delta : X \to j^{2r-1}$ *is a Hamiltonian extremal of* λ *if it satisfies the following condition:*

$$\delta^* (i_\Xi d\Theta_\lambda) = 0 \tag{3.5}$$

for all π_{2r-1}-vertical $\pi_{2r-1,0}$-*projectable vectorfields* Ξ *on* $j^{2r-1}Y$.

We note that Hamiltonian extremals may alternatively be defined by means of some anholonomic jets [*18*] (added in proof).

The set of Hamiltonian extremals will be denoted by H_λ. If a section γ of π is a critical section of λ, then we can prove that its jet prolongation $j^{2r-1}\gamma$ is a Hamiltonian extremal. We then have a mapping $J_\lambda : \Gamma_\lambda \to H_\lambda$ given by:

$$J_\lambda(\gamma) = j^{2r-1}\gamma. \tag{3.6}$$

We are interested in investigating conditions on the Lagrangian λ under which the mapping J_λ is a bijection, i. e. conditions ensuring that each Hamiltonian extremal $\delta \in H_\lambda$ be the $(2r-1)$-jet prolongation of a critical section $\gamma \in \Gamma_\lambda$.

To this purpose we give first the following definition:

DEFINITION 3.2. — *We say that λ is regular at a point $j^r_x\gamma \in j^rY$ if the following condition holds: There exists a fibered chart V on Y, with coordinates (x^i, y^σ), such that $\pi_{r,0}(j^r_x\gamma) \in V$ and:*

$$\det \left\| \frac{\partial^r L}{\partial y^\sigma_{i_1 \ldots i_r} \quad \partial y^\nu_{i_1 \ldots i_r}} \right\| \neq 0 \tag{3.7}$$

at the point $j^r_x\gamma$, where $L : V \to R$ is defined by the chart representation (2.2) *of λ. We say that λ is regular in an open subset $W \subset j^rY$ if it is regular at each point of* W.

In (3.7) the rows (resp. columns) of the matrix are labelled by the multi-indices ${}^\sigma_{i_1 \ldots i_r}$ (resp. ${}^\nu_{i_1 \ldots i_r}$). We refer to (3.7) as the *regularity condition* (for λ). We remark that the regularity condition (3.7) does not depend on the choice of a fibered coordinate system around the point considered. It is therefore an intrinsic property of the Lagrangian λ itself.

Let us now assume that λ is regular. We shall calculate the explicit coordinate expression of $\delta^* (i_\Xi d\Theta_\lambda)$, where Ξ is any π_{2r-1}-vertical $\pi_{2r-1,0}$-projectable vectorfield in $j^{2r-1}Y$, i. e.:

$$\Xi = \sum_{k=0}^{2r-1} \Xi^\sigma_{i_1 \ldots i_k} \frac{\partial}{\partial y^\sigma_{i_1 \ldots i_k}} . \tag{3.8}$$

By a straightforward calculation we obtain:

$$i_\Xi \varepsilon_\lambda = (\varepsilon_\sigma(L)\Xi^\sigma)\omega_0 ; \tag{3.9}$$

$$\begin{aligned} i_\Xi F_\lambda = & \sum_{k=0}^{r-1} \sum_{l=0}^{r-1} \frac{\partial f^{ij_1 \ldots j_k}_\sigma}{\partial y^\nu_{i_1 \ldots i_l}} (\Xi^\nu_{i_1 \ldots i_l} \omega^\sigma_{j_1 \ldots j_k} - \Xi^\sigma_{j_1 \ldots j_k} \omega^\nu_{i_1 \ldots i_l}) \wedge \omega_i \\ & + \sum_{k=0}^{r-1} \sum_{l=r}^{2r-1} \frac{\partial f^{ij_1 \ldots j_k}_\sigma}{\partial y^\nu_{i_1 \ldots i_l}} \cdot (\Xi^\nu_{i_1 \ldots i_l} \omega^\sigma_{j_1 \ldots j_k} - \Xi^\sigma_{j_1 \ldots j_k} \omega^\nu_{i_1 \ldots i_l}) \wedge \omega_i . \end{aligned} \tag{3.10}$$

Assume now that δ is a Hamiltonian extremal of λ. By the arbitrariness of Ξ and the linearity of $\delta^* (i_\Xi d\Theta_\lambda)$ in Ξ, we should exploit (3.5) by requiring that *all* the coefficients at the various components of Ξ vanish identically. However, it turns out that it is enough to restrict our attention to only few of these coefficients. In fact, using (2.8) together with (3.9) and (3.10) we obtain a number of conditions, among which we find the following ones:

$$(3.11)\qquad \sum_{0}^{r-1}{}_k \frac{\partial f_\sigma^{ij_1\ldots j_k}}{\partial y^\nu_{i_1\ldots i_l}} \delta^* \omega^\nu_{j_1\ldots j_k} = 0, \qquad r \leq l \leq 2r-1;$$

$$(3.12)\qquad \sum_{r}^{2r-1}{}_l \frac{\partial f_\sigma^{ij_1\ldots j_k}}{\partial y^\nu_{i_1\ldots i_l}} \delta^* \omega^\nu_{i_1\ldots i_l} = 0, \qquad 0 \leq k \leq r-1.$$

These two sets of conditions imply, via a simple argument involving the regularity condition (3.2), that the following holds:

$$(3.13)\quad \delta^* \omega^\sigma = 0, \quad \ldots, \quad \delta^* \omega^\sigma_{i_1\ldots i_k} = 0 \quad (k = 1, \ldots, 2r-1).$$

Relations (3.13) imply in turn that there exists a section $\gamma : X \to Y$ such that:

$$(3.14)\qquad \delta = j^{2r-1}\gamma$$

At this point we see that it is not necessary to consider explicitly the further coefficients arising from (3.5). In fact, replacing (3.14) into the original equation (3.5) we see at once that γ is a critical section of λ.

Therefore, we have proved the following theorem:

THEOREM 2. — *Under the hypothesis that λ is a regular Lagrangian of order r in* Y, *i. e. under condition* (3.2), *the following two conditions are equivalent:*

i) $\delta : X \to j^{2r-1}Y$ *is a Hamiltonian extremal* (i. e. $\delta \in H_\lambda$);

ii) *there exists a critical section $\gamma \in \Gamma_\lambda$ such that $\delta = j^{2r-1}\gamma$.*

In other words we have proved the following: *the mapping $I_\lambda : \Gamma_\lambda \to H_\lambda$ is a bijection if λ is regular.* Thus we see that for regular Lagrangians the search for critical sections is equivalent to the search for Hamiltonian extremals.

We conclude by remarking that theorem 2 allows to replace condition (3.5) for Hamiltonian extremals with an equivalent condition, which shall be however most suited for deriving the Hamiltonian equations for λ. In fact, if λ is regular and $\delta \in H_\lambda$ is a Hamiltonian extremal, then it is $\delta = j^{2r-1}\gamma$ for some $\gamma \in \Gamma_\lambda$. Therefore, applying (1.3) to (3.5) we obtain the equivalent condition:

$$(3.15)\qquad (j^{2r}\gamma)^* h(i_\Xi d\Theta_\lambda) = 0;$$

or more simply:

$$(3.16)\qquad h(i_\Xi d\Theta_\lambda) = 0$$

(along the jet prolongation $j^{2r}\gamma$). This result should not surprise, because equation (3.15) is in fact equivalent to Euler-Lagrange equations (3.3).

Nevertheless, investigating condition (3.16) is simpler than investigating directly (3.5), because of the condition *iii*) of theorem 1.

We might give a further interpretation of our last condition (3.16). Let us first introduce a further notion, which will be helpful. We say that a section σ of π_s is an *extension* of a section δ of π_l (where $l \leq s - 1$) if the following holds:

$$\pi_{s,l} \circ \sigma = \delta \,. \tag{3.17}$$

Then we have the following immediate result: *a section δ of π_{2r-1} is a Hamiltonian extremal of λ if and only if all its extensions σ to $j^{2r}Y$ satisfy the condition*

$$\delta^* \, \pi_{2r,2r-1}{}^* \, i_{\Xi} d\Theta_\lambda = 0. \tag{3.18}$$

(It is in fact $\sigma^* \, \pi_{2r,2r-1}{}^* \, i_{\Xi} d\Theta_\lambda = (\pi_{2r,2r-1} \circ \sigma)^* \, (i_{\Xi} d\Theta_\lambda) = \delta^* \, (i_{\Xi} d\Theta_\lambda)$, by virtue of (3.17) itself). It is easy to see that (3.18) reduces to (3.16) if we take (2.8) into account and we remark that $(\pi_{2r,2r-1} \circ \sigma)^* \, i_{\Xi} F_\lambda$ vanishes by virtue of the steps done when proving theorem 2.

4. THE LEGENDRE TRANSFORMATION OF ORDER r

In this section we shall prove that under the regularity condition (3) a natural change of coordinates in j^rY may be defined, together with its natural prolongation to $j^{r-1}(j^rY)$. This change of local coordinates in $j^{r-1}(j^rY)$ will be suited for the search of Hamiltonian extremals (for λ), as we shall see in the next section.

Let then (Y, X, π ; λ) be a Lagrangian structure of order r. The following holds:

LEMMA 2. — *Let us suppose that λ is regular on a fibered chart $V_r = (\pi_{r,0})^{-1}(V) \subseteq j^rY$ in the sense of* (3.7).

Then define:

$$p_\sigma^{j_1 \ldots j_r}(x^i, y^\sigma, y_{j_1}^\sigma, \ldots, y_{j_1 \ldots j_r}^\sigma) = \frac{\partial L}{\partial y_{j_1 \ldots j_r}^\sigma} \equiv f_\sigma^{j_1 \ldots j_r} \tag{4.1}$$

and:

$$p_{\sigma, i_1 \ldots i_l}^{j_1 \ldots j_r} = d_{i_l} d_{i_{l-1}} \ldots d_{i_1} p_\sigma^{j_1 \ldots j_r} \tag{4.2}$$

for $1 \leq l \leq r-1$. The mappings ψ: $V_r \to V_r$ and $\hat{\psi} : (V_r)_{r-1} \to (V_r)_{r-1}$ respectively defined by:

$$\psi : (x^i, y^\sigma, y_{i_1}^\sigma, \ldots, y_{j_1 \ldots j_r}^\sigma) \to (x^i, y^\sigma, y_{j_1 \ldots j_{r-1}}^\sigma, p_\sigma^{j_1 \ldots j_r}) \tag{4.3}$$

and

$$(4.4)\quad \hat{\psi} : (x^i, \ldots, y^\sigma_{j_1\ldots j_{2r-1}}) \to (x^i, y^\sigma, \ldots, y^\sigma_{j_1\ldots j_{r-1}}, p^{j_1\ldots j_r}_\sigma, p^{j_1\ldots j_r}_{\sigma,i_1}, \ldots, p^{j_1\ldots j_r}_{\sigma,i_1\ldots i_{r-1}})$$

are diffeomorphisms, i. e. they induce a local change of coordinates in j^rY (*resp. in* $j^{r-1}(j^r\mathrm{Y})$).

Proof. — *i*) The mapping ψ is a diffeomorphism of V_r if and only if the jacobian of $p^{j_1\ldots j_r}_\sigma$ with respect to the given coordinates $y^\nu_{i_1\ldots i_r}$ is nonvanishing. But (4.1) gives us immediately:

$$(4.5)\qquad \frac{\partial p^{j_1\ldots j_r}_\sigma}{\partial y^\nu_{i_1\ldots i_r}} = \frac{\partial^2 \mathrm{L}}{\partial y^\sigma_{j_1\ldots j_r}\partial y^\nu_{i_1\ldots i_r}}.$$

Thus we see that ψ is a diffeomorphism if and only if (3.7) holds.

ii) We now turn to consider $\hat{\psi} : (V_r)_{r-1} \to (V_r)_{r-1}$. We shall prove that $\hat{\psi}$ is a diffeomorphism of $(V_r)_{r-1}$ by showing that the regularity condition (3.7) assures the invertibility of the functions $p^{j_1\ldots j_r}_{\sigma,i_1\ldots i_l}$ with respect to their higher order variables $y^\sigma_{j_1\ldots j_r,i_1\ldots i_l}$, for all integers $l (1 \leq l \leq r-1)$. In order to prove this we shall more precisely show that $p^{i_1\ldots i_r}_{\sigma,i_1\ldots i_l}$ is an invertible affine function of the highest order variables $y^\nu_{j_1\ldots j_r,i_1\ldots i_l}$, having $\frac{\partial^2 \mathrm{L}}{\partial y^\sigma_{j_1\ldots j_r}\partial y^\nu_{k_1\ldots k_r}}$ as coefficient matrix, namely that it has the form:

$$(4.6)\qquad p^{j_1\ldots j_r}_{\sigma,i_1\ldots i_l} = \frac{\partial^2 \mathrm{L}}{\partial y^\sigma_{j_1\ldots j_r}\partial y^\nu_{k_1\ldots k_r}} y^\nu_{k_1\ldots k_r,i_1\ldots i_l} + \Phi^{j_1\ldots j_r}_{\sigma,i_1\ldots i_l},$$

where the functions $\Phi^{j_1\ldots j_r}_{\sigma,i_1\ldots i_l}$ do not depend on $y^\nu_{k_1\ldots k_r,i_1\ldots i_l}$. (Invertibility follows of course from the regularity condition (3.7) on λ).

a) Take $l = 1$. Then we have by definition:

$$(4.7)\qquad p^{j_1\ldots j_r}_{\sigma,i_1} \equiv d_{i_1} p^{j_1\ldots j_r}_\sigma = \frac{\partial \mathrm{L}}{\partial y^\sigma_{j_1\ldots j_r}\partial y_{k_1\ldots k_r}} y^\nu_{k_1\ldots k_r,i_1} + \Phi^{j_1\ldots j_r}_{\sigma,i_1},$$

where $\Phi^{j_1\ldots j_r}_{\sigma,i_1}$ are suitable functions on V_r (i. e. they depend on the coordinates x^i, y^σ, $y_{i_1\ldots i_r}$ only). This proves our claim for $l = 1$.

b) Suppose our claim is correct for $l = s$ $(1 \leq s < r-1)$. We shall prove that it is also valid for $l = s + 1$. We have in fact, by the induction hypothesis:

$$p^{j_1\ldots j_r}_{\sigma,i_1\ldots i_s i_{s+1}} \equiv d_{i_{s+1}}(p^{j_1\ldots j_r}_{\sigma,i_1\ldots i_s}) = d_{i_{s+1}}\left[\frac{\partial^2 \mathrm{L}}{\partial y^\sigma_{j_1\ldots j_r}\partial y^\nu_{k_1\ldots k_r}} y^\nu_{k_1\ldots k_r,i_1\ldots i_s} + \Phi^{j_1\ldots j_r}_{\sigma,i_1\ldots i_s}\right],$$

where $\Phi^{j_1\ldots j_r}_{\sigma,i_1\ldots i_s}$ depend only on x^i, y^σ, $y^\sigma_{i_1}, \ldots, y^\sigma_{j_1\ldots j_r,k_1\ldots k_{s-1}}$. Calculating the derivative we find thence:

$$(4.8)\qquad p^{j_1\ldots j_r}_{\sigma,i_1\ldots i_{s+1}} = \frac{\partial^2 \mathrm{L}}{\partial y^\sigma_{j_1\ldots j_r}\partial y^\nu_{k_1\ldots k_r}} y^\nu_{k_1\ldots k_r,i_1\ldots i_s i_{s+1}} + \Phi^{j_1\ldots j_r}_{\sigma,i_1\ldots i_{s+1}},$$

where:

$$(4.9) \qquad \Phi^{j_1 \dots j_r}_{\sigma, i_1 \dots i_{s-1}} = d_{i_{s+1}} \Phi^{j_1 \dots j_r}_{\sigma, i_1 \dots i_s} + d_{i_{s+1}} \frac{\partial^2 L}{\partial y^\sigma_{j_1 \dots j_r} \partial y^\nu_{k_1 \dots k_r}} y^\nu_{k_1 \dots k_r, i_1 \dots i_s} .$$

It is clear that $\Phi^{j_1 \dots j_r}_{\sigma, i_1 \dots i_{s+1}}$ does not depend on the higher order coordinates $y^\nu_{k_1 \dots k_r, i_1 \dots i_{s+1}}$. This ends our proof.

The (local) transformation of coordinates $\hat{\psi}$ in $j^{r-1}(j^r Y)$ will be called *generalized Legendre transformation* (or *Legendre transformation of order r*). If the regularity condition (3.7) holds everywhere in $j^{r-1}(j^r Y)$, then the local mappings $\hat{\psi} : (V_r)_{r-1} \to (V_r)_{r-1}$ patch together to define a global mapping, establishing a diffeomorphism between the jet bundle $j^{r-1}(j^r Y)$ and a suitably defined *Legendre bundle of order r.* (Condition (3.7), as we remarked in section 3, has a global intrinsic meaning, even if it is explicitly given in a local form, see [*3*] [*11*] for a definition of the Legendre bundle of order 1). This problem, which is related to the definition of « phase space » for λ will be dealt with elsewhere.

5. HAMILTON EQUATIONS

We are now in position to derive the Hamiltonian equations for the Lagrangian structure $(Y, X, \pi; \lambda)$, under the assumption of regularity of λ. We shall in fact derive a set of explicit equations for the Hamiltonian extremals δ defined in section 3; according to Theorem 2 these differential equations for δ are equivalent to the Euler-Lagrange equations for $\gamma \in \Gamma_\lambda$ because of the regularity condition (3.7). These new equations will be called the *Hamiltonian equations* for λ (or, more generically, the *r-th order Hamiltonian equations*). Our procedure will consist in exploiting condition (3.9) in the new local coordinates defined by the regular change of variables $\hat{\psi} : (V_r)_{r-1} \to (V_r)_{r-1}$, which is admissible by virtue of lemma 2.

Since the mapping ψ defined by equation (4.3) is invertible, we may express $y^\nu_{i_1 \dots i_r}$ as a function of the new local coordinates $(x^i, y^\sigma, y^\sigma_{j_1, \dots j_{r-1}}, p^{j_1 \dots j_r}_\sigma)$; namely, we have

$$(5.1) \qquad y^\nu_{i_1 \dots i_r} = y^\nu_{i_1 \dots i_r}(x^i, y^\sigma, y^\sigma_{i_1}, \dots, y^\sigma_{i_1 \dots i_{r-1}}, p^{i_1 \dots i_r}_\sigma) .$$

Let us define $H : V_r \to R$ by setting

$$(5.2) \quad H(x^i, y^\sigma, \dots, p^{j_1 \dots j_r}_\sigma) = - L(x^i, y^\nu, \dots, y^\nu_{i_1 \dots i_{r-2}}, y^\nu_{j_1 \dots j_{r-1}}, y^\nu_{j_1 \dots j_r}(x^\nu, y^\sigma, \dots, p^{j_1 \dots j_r}_\sigma)) + p^{j_1 \dots j_r}_\sigma y^\sigma_{j_1 \dots j_r}(x^\nu, y^\sigma, \dots, p^{j_1 \dots j_r}_\sigma) ;$$

shortly we shall write:

$$(5.2') \qquad H = - L + p^{j_1 \dots j_r}_\sigma y_{j_1 \dots j_r} .$$

We call H the *Hamiltonian function* (of λ) in V_r. Let us now re-express the Poincaré-Cartan form Θ_λ in terms of H and of the new coordinates on V_r defined by (4.4). After some calculations we find:

$$(5.3)\quad \Theta_\lambda = -\mathrm{H}\omega_0 - \sum_{k=1}^{r-1} y^\sigma_{j_1\dots j_k} f_\sigma^{j_1\dots j_k}\omega_0 + \sum_{k=0}^{r-1} f_\sigma^{ij_1\dots j_k} dy^\sigma_{j_1\dots j_k} \wedge \omega_i\,,$$

(where the functions $f_\sigma^{j_1\dots j_s}$ are considered as functions of the new variables in $(V_r)_{r-1}$).

Let us now consider a π_{2r-1}-vertical projectable vectorfield Ξ, having the following local expression in the new coordinates:

$$(5.4)\quad \Xi = \Xi^\sigma \frac{\partial}{\partial y^\sigma} + \sum_{k=1}^{r-1} \Xi^\sigma_{j_1\dots j_k} \frac{\partial}{\partial y^\sigma_{j_1\dots j_k}} + \Xi_\sigma^{j_1\dots j_r} \frac{\partial}{\partial p_\sigma^{j_1\dots j_r}} + \sum_{l=1}^{r-1} \Xi^{j_1\dots j_r}_{\sigma,i_1\dots i_l} \frac{\partial}{\partial p^{j_1\dots j_r}_{\sigma,i_1\dots i_l}}\,.$$

We now turn to calculate explicitly the horizontal $(n+1)$-form $h(i_\Xi d\Theta_\lambda)$, with the aid of equation (5.3). A simple calculation gives us the following:

$$(5.5)\quad i_\Xi(d\mathrm{H}\wedge\omega_0) = \left(\frac{\partial \mathrm{H}}{\partial y^\sigma}\Xi^\sigma + \sum_{k=1}^{r-1} \frac{\partial \mathrm{H}}{\partial y_{j_1\dots j_k}} \Xi^\sigma_{j_1\dots j_k} + \frac{\partial \mathrm{H}}{\partial p_\sigma^{j_1\dots j_r}} \Xi_\sigma^{j_1\dots j_r}\right).\omega_0$$

where we have taken into account that H does not depend on the variables $p^{j_1\dots j_r}_{\sigma,i_1\dots i_l}$. Moreover we have:

$$(5.6)\quad i_\Xi d\left(\sum_{k=1}^{r-1} y^\sigma_{j_1\dots j_k} f_\sigma^{j_1\dots j_k}\right)\omega_0 = \sum_{k=1}^{r-1} \left((i_\Xi df_\sigma^{j_1\dots j_k}) y^\sigma_{j_1\dots j_k} + f^{j_1\dots j_k}\Xi^\sigma_{j_1\dots j_k}\right).\omega_0\,.$$

Finally, an easy calculation gives:

$$(5.7)\quad i_\Xi d\left(\left(\sum_{k=0}^{r-1} f_\sigma^{ij_1\dots j_k} dy^\sigma_{j_1\dots j_k}\right)\wedge\omega_i\right) = \sum_{k=0}^{r-1} \left((i_\Xi df_\sigma^{ij_1\dots j_k}) dy^\sigma_{j_1\dots j_k} - \Xi^\sigma_{j_1\dots j_k} df_\sigma^{ij_1\dots j_k}\right)\wedge\omega_i\,.$$

We now collect our results and calculate $h(i_\Xi d\Theta_\lambda)$ by using relations (5.3), (5.5), (5.6). To this purpose relations (1.5) will be helpful. In fact, we see immediately that when applying the operation h to the left hand sides of both equations (5.6) and (5.7) several identical terms of the form $(i_\Xi df_\sigma^{ij_1\dots j_k} y^\sigma_{ij_1\dots j_k})\omega_0$ are generated. As a consequence, in the final result only the term:

$$h((i_\Xi df_\sigma^{ij_1\dots j_{r-1}})dy^\sigma_{j_1\dots j_{r-1}} \wedge \omega_i) \equiv \Xi_\sigma^{j_1\dots j_r} y^\sigma_{j_1\dots j_r},$$

(which comes from (5.7)) will survive. After some standard calculations we obtain the following result:

$$h(i_\Xi d\Theta_\lambda) = \mathrm{G}.\omega_0, \tag{5.8}$$

where:

$$\mathrm{G} = -\left(\frac{\partial \mathrm{H}}{\partial y^\sigma}\Xi^\sigma + \sum_1^{r-1} \frac{\partial \mathrm{H}}{\partial y^\sigma_{j_1\dots j_k}} \Xi^\sigma_{j_1\dots j_k} + \frac{\partial \mathrm{H}}{\partial p_\sigma^{j_1\dots j_r}} \Xi_\sigma^{j_1\dots j_r}\right) - \sum_{1}^{r-1}{}_k f_\sigma^{j_1\dots j_k}\Xi^\sigma_{j_1\dots j_k} + \Xi_\sigma^{j_1\dots j_r} y^\sigma_{j_1\dots j_r} - \sum_1^{r-1}{}_k \Xi^\sigma_{j_1\dots j_k} d_i f_\sigma^{ij_1\dots j_k}. \tag{5.9}$$

Let us now require that condition (3.5) is satisfied for all π_{2r-1}-vertical projectable vectorfields Ξ. Taking (5.9) into account and collecting properly the terms, from the arbitrariness of the components Ξ^σ, $\Xi^\sigma_{j_1\dots j_k}$ and $\Xi_\sigma^{j_1\dots j_r}$ we derive the following system of differential equations:

$$\begin{aligned} &\frac{\partial \mathrm{H}}{\partial y^\sigma} + d_i f_\sigma^i = 0, \\ &\frac{\partial \mathrm{H}}{\partial y^\sigma_{j_1\dots j_k}} + f_\sigma^{j_1\dots j_k} + d_i f_\sigma^{ij_1\dots j_k} = 0, \qquad (k = 1, \dots, r-1); \\ &\frac{\partial \mathrm{H}}{\partial p_\sigma^{j_1\dots j_r}} - y^\sigma_{j_1\dots j_r} = 0. \end{aligned} \tag{5.10}$$

Equations (5.10) are satisfied along the Hamiltonian extremals $\delta \in \mathrm{H}_\lambda$. They can in fact be considered as a system of differential equations in the unknown variables $y^\sigma(x^i), \dots, y^\sigma_{j_1\dots j_{r-1}}(x_i), p_\sigma^{j_1\dots j_r}(x^i)$. We remark that also the variables $p_{\sigma,i_1\dots i_l}^{j_1\dots j_r}$ appear implicitly in equations (5.10), through the functions $f_\sigma^{j_1\dots j_k}$. However, this does not play a significant role. We recall, in fact, that lemma 2 assures us that the variables $p_{\sigma,i_1\dots i_l}^{j_1\dots j_r}$ may be easily re-expressed in terms of the original variables $y^\sigma_{j_1\dots j_r i_1\dots i_l}$. But theorem 2 tells us that δ is the jet prolongation of a section $\gamma : \mathrm{X} \to \mathrm{Y}$. Therefore the higher order derivatives $y^\nu_{j_1\dots j_r i_1\dots i_l}$ may be formally eliminated from equations (5.10), providing us with an unambiguous system of differential equations. See also later for some further remarks and examples.

Equations (5.10), which are equivalent to Lagrange equations (3.2′), will be called the *Hamiltonian equations* for λ. We remark that the system (5.10) is a system of differential equations of order r, while Euler-

Lagrange equations are of order $2r$. We therefore see that, as we should have expected, the Hamiltonian equations are, *at least in principle*, easier than the original Lagrange equations.

We can state our results in the form of the following theorem, which was in fact our principal aim.

THEOREM 3. — *Let* (Y, X, π) *be a fibered manifold and* λ *a regular Lagrangian of order* r *for* Y. *Then the critical sections* (*of* λ), *satisfying the Euler-Lagrange equations* (3.2′), *may be equivalently characterized by the Hamiltonian equations* (5.10).

Remarks. — *i*) We immediately see that our equations (5.10) are a genuine generalization of classical Hamiltonian equations of order 1. If in fact we set $r = 1$, the intermediate equations $(5.10)_{\text{II}}$ are empty and our system reduces to the well known system of canonical equations:

$$(5.11)\qquad \frac{\partial \mathrm{H}}{\partial y^\sigma} + d_i p^i_\sigma = 0\,, \qquad \frac{\partial \mathrm{H}}{\partial p^i_\sigma} - d_i y^\sigma = 0.$$

ii) We remark that the intermediate Hamiltonian equations $(5.10)_{\text{II}}$ are in fact equivalent to the following relations between the Lagrangian function L and the Hamiltonian function H in V_r:

$$(5.12)\qquad \frac{\partial \mathrm{H}}{\partial y^\sigma_{j_1\dots j_k}} = -\frac{\partial \mathrm{L}}{\partial y^\sigma_{j_1\dots j_k}}\,, \qquad (k = 1, \dots, r-1)\,.$$

(These relations generalize to all orders r the analogous well known relations for $r = 1$).

iii) We finally remark that our choice (5.2) for the Hamiltonian H seems to be the most convenient and the most natural one. Other choices for H might be proposed (and have in fact been proposed in the literature); for example, one could take:

$$\mathrm{H}^* = -\mathrm{L} + \sum_{1}^{r}{}_{k}\, y^\sigma_{j_1\dots j_k} f^{j_1\dots j_k}_\sigma$$

or any other « intermediate » choice in which some « lower order momenta » $f^{j_1\dots j_l}_\sigma$ $(l \neq r)$ undergo a sort of Legendre transformation.

Nevertheless, any such different choice would not produce different results. It would rather produce, however, a more complicated Hamiltonian theory. In fact any such new Hamiltonian would depend on some of the « higher order coordinates » $p^{j_1\dots j_r}_{\sigma,i_1\dots i_l}$, which do not appear instead in (5.2). As a consequence, the corresponding « Hamiltonian equations » would be more complicated than equations (5.10), even though perfectly equivalent to them (and to Euler-Lagrange equations, too !).

6. HIGHER ORDER MECHANICS. EXAMPLES

In this section we shall re-write equations (5.10) for the case of higher order mechanics and we shall give some elementary examples.

The case of r-th *order mechanics* is obtained by taking $X = R$ (the real line), $Y = M \times R$, where the m-dimensional manifold M is the *configuration space* of a mechanical system. We have then only one base coordinate x^i, which is usually denoted by the letter t. Accordingly, the local coordinates in $j^r Y$ will be denoted as follows:

$$(t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma, \ldots, \overset{(r)}{q^\sigma}),$$

or also:

$$(t, q^\sigma, \overset{(1)}{q^\sigma}, \overset{(2)}{q^\sigma}, \ldots, \overset{(r)}{q^\sigma}),$$

where the classical mechanical notation q^σ is used for the coordinates in configuration space M.

In these notations the Hamiltonian equations (5.10) reduce to:

$$(6.1)\qquad \begin{aligned} &\frac{\partial H}{\partial q^\sigma} + \frac{df_\sigma^{(1)}}{dt} = 0\,; \\ &\frac{\partial H}{\partial \overset{(k)}{q^\sigma}} + f_\sigma^{(k)} + \frac{df_\sigma^{(k+1)}}{dt} = 0\,, \qquad (k = 1, \ldots, r-1)\,; \\ &\frac{\partial H}{\partial p_\sigma} - \frac{d^r q^\sigma}{dt^r} = 0\,; \end{aligned}$$

where p_σ and $f_\sigma^{(k)} (k = 1, \ldots, r)$ are defined by:

$$(6.2)\qquad p_\sigma \equiv f_\sigma^{(r)} = \frac{\partial L}{\partial \overset{(r)}{q^\sigma}}\,;$$

$$(6.3)\qquad f_\sigma^{(s)} = \frac{\partial L}{\partial \overset{(s)}{q^\sigma}} - \frac{d}{dt} f_\sigma^{(s+1)} \qquad (s = 1, \ldots, r-1)\,.$$

Here p_σ are assumed as independent variables in place of $\overset{(r)}{q_\sigma}$, provided of course the regularity condition:

$$(6.4)\qquad \det \left\| \frac{\partial^2 L}{\partial \overset{(r)}{q^\sigma} \partial \overset{(r)}{q^\nu}} \right\| \neq 0$$

holds.

In particular, we shall be for simplicity interested in the case of *second* *order mechanics* ($r = 2$). In this case the previous relations take the following form:

$$p_\sigma \equiv \frac{\partial L}{\partial \ddot{q}^\sigma}, \tag{6.5}$$

$$f_\sigma^{(1)} \equiv \varphi_\sigma = \frac{\partial L}{\partial \dot{q}^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}^\sigma}, \tag{6.6}$$

and the Hamiltonian equations are:

$$\begin{aligned} &\frac{\partial H}{\partial q^\sigma} + \frac{d\varphi_\sigma}{dt} = 0, \\ &\frac{\partial H}{\partial \dot{q}^\sigma} + \varphi_\sigma + \frac{dp_\sigma}{dt} = 0, \\ &\frac{\partial H}{\partial p_\sigma} - \frac{d^2 q^\sigma}{dt^2} = 0. \end{aligned} \tag{6.7}$$

Here, p_σ are taken as independent variables (in place of $\ddot{q}^\sigma$) provided the regularity condition:

$$\det \left\| \frac{\partial^2 L}{\partial \ddot{q}^\sigma \partial \ddot{q}^\nu} \right\| \neq 0 \tag{6.8}$$

holds; φ_σ are instead functions of $(t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma, \dddot{q}^\sigma)$ which can be re-expressed as functions of $(t, q^\sigma, \dot{q}^\sigma, p_\sigma, \dot{p}_\sigma)$ $\left(\text{where } \dot{p}_\sigma \equiv \frac{d}{dt} p^\sigma\right)$, because of the relevant form of lemma 2.

We are now ready to discuss some very elementary 2nd order examples, just to show how Hamilton equations may work in place of the corresponding Lagrange equations.

a) Let us take $m = 1$ (*one dimensional mechanical systems*) and $r = 2$. Let the Lagrangian be defined by:

$$L(t, q, \dot{q}, \ddot{q}) = \frac{1}{2}(\ddot{q})^2 + U(t, q, \dot{q}), \tag{6.9}$$

where $U : R^3 \to R$ is a C^∞ function (*generalized potential*). Our Lagrangian function L is regular. We have:

$$p = \ddot{q}, \qquad \varphi = U'_{\dot{q}} - \frac{d\ddot{q}}{dt}, \tag{6.10}$$

where $U'_{\dot{q}}$ denotes $\frac{\partial U}{\partial \dot{q}}$. Analogous notations will be used for the other partial derivatives of U. Euler-Lagrange equations are:

$$\frac{\partial U}{\partial q} - \frac{d}{dt}\frac{\partial U}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0, \tag{6.11}$$

with $\dot{q} = \frac{dq}{dt}$ and $\ddot{q} = \frac{d\dot{q}}{dt}$. Thence we find easily:

$$(6.11') \qquad \frac{d^4q}{dt^4} - (U''_{\dot{q}\dot{q}})\frac{d^2q}{dt^2} - (U''_{\dot{q}q})\frac{dq}{dt} - U''_{\dot{q}t} + U'_q = 0 .$$

Let us now calculate Hamilton equations, in terms of the new variables $(t, q, \dot{q}, p)$. Relation $(6.10)_{\mathrm{I}}$ inverts to give $\ddot{q}(p) = p$; therefore $H(t, q, \dot{q}, p)$ is given by:

$$(6.12) \qquad H = \frac{1}{2}p^2 - U(t, q, \dot{q}),$$

while $\varphi(t, q, \dot{q}, p, \dot{p})$ is given by:

$$(6.13) \qquad \varphi = U'_{\dot{q}}(t, q, \dot{q}) - \dot{p} .$$

Applying equation (6.7) we find then:

$$(6.14) \qquad \begin{aligned} & - U'_q + \frac{d}{dt}(U'_{\dot{q}} - \dot{p}) = 0 , \\ & - U'_{\dot{q}} + (U'_{\dot{q}} - \dot{p}) + \frac{dp}{dt} = 0 , \\ & p - \frac{d^2q}{dt^2} = 0 . \end{aligned}$$

Equivalently:

$$(6.14') \qquad \begin{aligned} & - U'_q + (U''_{\dot{q}\dot{q}})\frac{d^2q}{dt^2} + (U''_{\dot{q}q})\frac{dq}{dt} + U''_{\dot{q}t} - \frac{d\dot{p}}{dt} = 0 . \\ & \dot{p} = \frac{dp}{dt}, \\ & p = \frac{d^2q}{dt^2} . \end{aligned}$$

It is immediate to see that the 2nd-order equations (6.14′) are in fact equivalent to the 4th-order equations (6.11′).

b) Let us generalize the example *a*) to the case of many degrees of freedom m and to Lagrangians of the « dynamical type »:

$$(6.15) \qquad L(t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma) = \frac{1}{2} a_{\sigma\rho}(t, q^\lambda, \dot{q}^\lambda)\ddot{q}^\sigma\ddot{q}^\rho + U(t, q^\lambda, \dot{q}^\lambda),$$

where $\| a_{\sigma\rho}(t, q, \dot{q}) \|$ is a symmetric $m \times m$ matrix.

If L is regular the matrix $\| a_{\sigma\rho} \|$ is regular and we shall denote by $\| a^{\sigma\rho}(t, q^\lambda, \dot{q}^\lambda) \|$ its inverse. We have soon the Euler-Lagrange equations:

$$(6.16) \qquad \frac{1}{2}\frac{\partial a_{\lambda\rho}}{\partial q^\sigma}\ddot{q}^\lambda\ddot{q}^\rho + \frac{\partial U}{\partial q^\sigma} - \frac{d}{dt}\left(\frac{1}{2}\frac{\partial a_{\lambda\rho}}{\partial \dot{q}^\sigma}\ddot{q}^\lambda\ddot{q}^\rho + \frac{\partial U}{\partial \dot{q}^\sigma}\right) + \frac{d}{dt}\left(\frac{d}{dt}(a_{\sigma\rho}\ddot{q}^\rho)\right) = 0 ,$$

which might be re-written as follows:

$$a_{\sigma\rho}\frac{d^4q^\rho}{dt^4} + F_\sigma(t, q^\lambda, \dot{q}^\lambda, \ddot{q}^\lambda, \dddot{q}^\lambda) = 0 \qquad (6.16')$$

for a suitable function F_σ. The explicit form of F_σ is interesting for further investigations; however, it is not relevant to our present purpose and it shall therefore be omitted.

We now turn to Hamilton equations. We have:

$$p_\sigma(t, q^\lambda, \dot{q}^\lambda, \ddot{q}^\lambda) = a_{\sigma\rho}(t, q^\lambda, \dot{q}^\lambda)\ddot{q}^\rho, \qquad (6.17)$$

$$\varphi_\sigma(t, q^\lambda, \dot{q}^\lambda, \ddot{q}^\lambda, \dddot{q}^\lambda) = \frac{1}{2}\frac{\partial a_{\lambda\rho}}{\partial \dot{q}^\sigma}\ddot{q}^\lambda\ddot{q}^\rho + \frac{\partial U}{\partial \dot{q}^\sigma} - \frac{d}{dt}(a_{\sigma\rho}\ddot{q}^\rho). \qquad (6.18)$$

Inverting (6.17) and re-expressing φ_σ in terms of $(t, q^\lambda, \dot{q}^\lambda, p_\lambda, \dot{p}_\lambda)$ we find:

$$\ddot{q}^\sigma(t, q^\lambda, \dot{q}^\lambda, p_\lambda) = a^{\rho\sigma}(t, q^\lambda, \dot{q}^\lambda)p_\rho, \qquad (6.17')$$

$$\begin{aligned} \varphi_\sigma(t, q^\lambda, \dot{q}^\lambda, p_\lambda, \dot{p}_\lambda) &= \frac{1}{2}\frac{\partial a_{\lambda\rho}}{\partial \dot{q}^\sigma}a^{\lambda\varepsilon}a^{\rho\mu}p_\varepsilon p_\mu + \frac{\partial U}{\partial \dot{q}^\sigma} - \dot{p}_\sigma \\ &= -\frac{1}{2}\frac{\partial a^{\mu\varepsilon}}{\partial \dot{q}^\sigma}p_\mu p_\varepsilon + \frac{\partial U}{\partial \dot{q}^\sigma} - \dot{p}_\sigma. \end{aligned} \qquad (6.18')$$

Replacing (6.17′) into Hamilton equations (6.7) for the Hamiltonian H;

$$H = \frac{1}{2}a^{\sigma\rho}(t, q^\lambda, \dot{q}^\lambda)p_\sigma p_\rho - U(t, q^\lambda, \dot{q}^\lambda) \qquad (6.19)$$

we find the following system:

$$\begin{aligned} &\frac{1}{2}\frac{\partial a^{\lambda\rho}}{\partial q^\sigma}p_\lambda p_\rho - \frac{\partial U}{\partial q^\sigma} + \frac{d}{dt}\left(-\frac{1}{2}\frac{\partial a^{\mu\varepsilon}}{\partial \dot{q}^\sigma}p_\varepsilon p_\mu + \frac{\partial U}{\partial \dot{q}^\sigma} - \dot{p}_\sigma\right) = 0, \\ &-\dot{p}_\sigma + \frac{dp_\sigma}{dt} = 0, \\ &a^{\sigma\rho}p_\rho - \frac{d^2q^\sigma}{dt^2} = 0 \end{aligned} \qquad (6.20)$$

which is of course equivalent to the Euler-Lagrange equations (6.16).

ACKNOWLEDGMENTS

This research has been partially supported by Italian CNR-GNFM. One of the authors (D. K.) acknowledges the hospitality of Prof. D. Galletto and his collaborators at the Institute of Mathematical Physics « J.-L. Lagrange » of Torino, where joint research was begun. The other author (M. F.) acknowledges the hospitality of the Department of Mathematics of Brno University, where this paper was done.

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(Manuscrit reçu le 19 novembre 1981)