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Section A :
Physique théorique.

Generic properties of classical n-body systems, in one dimension, and crystal theory

by

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ABSTRACT. — We investigate generic properties of classical n-body systems interacting through two-body potentials. First we prove that for almost all choice of the potential (in the Baire sense), the corresponding n-body energies are simultaneously Morse functions on the configuration spaces. Then this result is used to prove the stability of a certain symmetry property of equilibriums, namely, the existence of a center of symmetry, in the sense that it is preserved under small but arbitrary variations of the potential. Finally, a large class of realistic interactions is proved to give rise to such symmetric equilibriums for any number of particles.

RÉSUMÉ. — On étudie les propriétés génériques des systèmes classiques à n corps interagissant par des potentiels à deux corps. On montre d'abord que pour presque tout potentiel (au sens de Baire), les énergies à n corps sont simultanément des fonctions de Morse sur les espaces de configuration. On utilise ensuite ce résultat pour prouver la stabilité d'une certaine propriété des positions d'équilibre, à savoir l'existence d'un centre de symétrie, dans le sens qu'elle est préservée sous des variations petites mais arbitraires du potentiel. Enfin, on montre qu'une large classe d'interactions réalistes donne lieu à de tels équilibres symétriques pour un nombre quelconque de particules.

I. INTRODUCTION

The classical theory of crystals, which is still uncompleted, has been mainly approached from the following point of view: one chooses a realistic interaction potential and one studies the structure of the minimal energy configurations for the corresponding n-body problem [3]-[8].

However, unless the potentials are completely specified by the physical theory, as in the Coulomb case, one usually takes more or less phenomenological expressions for them. In such a situation, the relevance of the conclusions depends on their stability with respect to physically allowed variation of these potentials.

Concerning the classical theory of crystal, since various elements yield the same lattice structure, it seems that only qualitative properties of the potential are involved in its ability to produce this symmetry.

Thus, we are led to consider the problem from an opposite point of view: namely, to stress on the study of these qualitative properties of potentials, using the language and methods of functional analysis.

In this work, we prove that, for « almost all » two-body interaction, the corresponding n-body potential energies are simultaneously Morse functions: for any n, the equilibrium configurations are non degenerate critical points of the energies, i. e., the corresponding Hessians are of maximal rank. Such a property implies in particular the non existence of soft phonons and allows the study of the trajectory of an equilibrium under an arbitrary variation of the potential.

Using this result, we show that a certain symmetry property, namely the existence of a center of symmetry, is stable with respect to variations of the potential: if a given equilibrium presents this symmetry, then any neighbouring interaction gives rise to a perturbed equilibrium with the same symmetry. In this way is achieved the first step towards the proof of the generic existence of cristal structures.

Finally, we consider a class of hard-core potentials which are attractive at large distances and satisfy a certain convexity property. We prove that for any potential in this class, there exists for any n a unique equilibrium, and that this equilibrium presents a center of symmetry. Moreover, this class is an open subset of the space of potentials with respect to the Whitney topology.

II. NOTATIONS AND TOPOLOGIES OF THE POTENTIALS

Let us consider a translation and reflection invariant two body interaction, described by a potential $\phi \in C^{\infty}(]a, \infty[)$, where $a \geqq 0$ is the diameter of a possible hard core.

The configuration space of $n + 1$ particles ($n \geqq 1$) is:

$$Q^{(n+1)} = \{ q \in \mathbb{R}^{n+1} ; q_i + a < q_{i+1}, i = 1, \ldots, n \} .$$

The translation invariance is taken into account by reducing the configuration space to:

$$X^{(n)} = \{ x \in \mathbb{R}^n ; x_i > a, i = 1, \ldots, n \} .$$

where $x_i = q_{i+1} - q_i$. $Q^{(n+1)}$ and $X^{(n)}$ are open sub-manifolds of $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$ respectively.

The potentiel energy for the $n + 1$ particles interacting via the given ϕ, belongs to $C^\infty (X^{(n)})$ and is given by:

$$\phi^{(n)}(x) = \sum_{I}^{(n)} \phi(x_I) \tag{1}$$

where $x_I = \sum_{i \in I} x_i$ and where the summation $\sum_{I}^{(n)}$ involves all the intervals I in the set $\{ 1, \ldots, n \}$.

The Hamiltonian flow on the cotangent bundle $T^*Q^{(n+1)}$ is complete only if $\lim_{x \to a+} \phi(x) = + \infty$.

However, this restriction will not be assumed in the following, since it has no effect on the existence and the local properties of equilibriums lying in $Q^{(n+1)}$, all the distances between particles beeing strictly larger than the hard core diameter.

A configuration described by $x \in X^{(n)}$ corresponds to an equilibrium if and only if all the partial derivatives of $\phi^{(n)}$ vanish at x; i. e. if and only if x is a critical point for $\phi^{(n)}$.

Let $\{ \varepsilon_1, \ldots, \varepsilon_n \}$ be the canonical dual basis of $\mathbb{R}^n$. It follows from (1) that the differential $d\phi^{(n)}$ can be written:

$$d\phi^{(n)}(x) = \sum_{I}^{(n)} \phi'(x_I)\varepsilon_I \tag{2}$$

where:

$$\varepsilon_I = \sum_{i \in I} \varepsilon_i$$

Let $T_xX^{(n)} \simeq \mathbb{R}^n$ be the tangent space to $X^{(n)}$ at x. If $\xi \in T_xX^{(n)}$, then:

$$d\phi^{(n)}(x) . \xi = \sum_{I}^{(n)} \phi'(x_I)\xi_I$$

where:

$$\xi_I = \varepsilon_I . \xi = \sum_{i \in I} \xi_i$$

If $x \in X^{(n)}$ corresponds to an equilibrium for $\phi^{(n)}$, the stability properties of this equilibrium are described by the Hessian of $\phi^{(n)}$ at x:

$$H_x\phi^{(n)} = \sum_{I}^{(n)} \phi''(x_I)\varepsilon_I \otimes \varepsilon_I \tag{3}$$

In fact, $\dfrac{\partial^2\phi^{(n)}}{\partial x_i \partial x_j}(x) = \sum_{i,j \in I}^{(n)} \phi''(x_I)$, and for $\xi, \zeta \in T_xX^{(n)}$:

$$H_x\phi^{(n)}(\xi) = \sum_{I}^{(n)} \phi''(x_I)\xi_I . \varepsilon_I \tag{3'}$$

$$H_x\phi^{(n)}(\xi, \zeta) = \sum_{I}^{(n)} \phi''(x_I)\xi_I\zeta_I \tag{3''}$$

A critical point x for $\phi^{(n)}$ is called non degenerate if and only if the rank of $H_x\phi^{(n)}$ is maximal. If D is any subset of $X^{(n)}$, $\phi^{(n)}$ is said to be a Morse function on D iff each critical point of $\phi^{(n)}$ lying in D is non degenerate.

The expression (3) defines a coordinate-free symmetric bilinear form only if $d\phi^{(n)}(x) = 0$. However, we shall consider such a formula in the following at regular points of $\phi^{(n)}$.

Our purpose is to investigate the perturbation of an equilibrium, corresponding to a variation of the two body interaction. Now, it is clear that such a variation steps in only by the values it takes in the neighbourhoods of the distances actually realized in the given configuration. Thus, it appears that the set of all configurations splits into classes, which are described by means of equalities of distances between particles. So we are led to set the following definition:

For any $x \in X^{(n)}$, we define the stratum $S(x)$ as the subset of $X^{(n)}$ given by:

$$S(x) = \{ y \in X^{(n)}, x_I = x_J \Leftrightarrow y_I = y_J \} \tag{4}$$

Thus we get a partition of the configuration space $X^{(n)}$, and one easily checks the following properties:

1) The closure of the strata are intersections of $X^{(n)}$ with vector subspaces of $\mathbb{R}^n$.

2) The equations for their tangent spaces are

$$T_x S(x) = \{ \xi \in \mathbb{R}^n, x_I = x_J \Rightarrow \xi_I = \xi_J \}$$

which is the vector subspace spanned by $S(x)$.

3) The following boundary property holds:

$$S \cap \overline{S}' \neq \varnothing \Rightarrow S \subset \overline{S}' - S' \quad \text{and} \quad T_xS \subset T_x\overline{S}' \quad \text{for} \quad x \in S \cap \overline{S}'$$

4) There exists a largest and a smallest stratum, defined by:

— for the largest: $x_I \neq x_J \quad \forall I \neq J$.

— for the smallest (periodic): $x_1 = x_2 \ldots = x_n$.

On the other hand, the relations defining a stratum $S(x)$ imply restrictions on the differential of the energy $\phi^{(n)}$, and eventually on its Hessian, which can be written in the general form:

$$d\phi^{(n)}(x) = \sum_{I}^{(n)} b_I \varepsilon_I \tag{5}$$

$$H_x\phi^{(n)}(x) = \sum_{I}^{(n)} c_I \varepsilon_I \otimes \varepsilon_I \tag{6}$$

where the coefficients $b = \{ b_I \}$ et $c = \{ c_I \}$ $\left(I \text{ running over the } \frac{n(n+1)}{2} \right.$ non empty intervals of $\{ 1, \ldots, n \}$), are elements of the following subspace $B(x)$ of $\mathbb{R}^{\frac{n(n+1)}{2}}$:

$$B(x) = \{ b \in \mathbb{R}^{\frac{n(n+1)}{2}} ; x_I = x_J \Rightarrow b_I = b_J \} \tag{7}$$

In the sequel, it will be useful to consider the canonical mapping from $\mathbb{R}^n$ into $\mathbb{R}^{\frac{n(n+1)}{2}}$ defined by:

$$x \to i(x) = \{ x_I \}$$

One can easily check that, for any stratum S, and for any $x \in S$, one has:

$$i(S) \subset i(T_xS) \subset B,$$

where B is the subspace of $\mathbb{R}^{\frac{n(n+1)}{2}}$ associated with S by the definition (7).

It will be equally useful to introduce the homomorphism δ from $\mathbb{R}^{\frac{n(n+1)}{2}}$ into the dual $\mathbb{R}^{n*}$ of $\mathbb{R}^n$, defined by:

$$b \to \delta(b) = \sum_{I}^{(n)} b_I \varepsilon_I$$

the kernel of which has the dimension $\frac{n(n-1)}{2}$.

Topologies on the potential space $C^\infty(]a, \infty[)$.

The existence, and symmetry properties of the equilibrium configurations, depend only on the two body interaction potential ϕ.

Whenever ϕ is not completely specified by physical requirements, the stability of these properties with respect to admissible variation of the potential must be questionned. Then, one must give a precise definition of what is meant by « small variation » of ϕ, in other words, one must define a topology on the potential space.

Let us briefly recall the definitions and fundamental properties of the weak and strong topologies on $C^\infty(X)$, where X is a paracompact manifold ([*1*]):

— Weak (or compact-open) topology: A sequence $\{ \phi_n \}_{n \geqq 0}$ converges to ϕ if and only if, for any compact set K of X, the restrictions of ϕ_n to K converge to the restriction of ϕ, together with the derivatives of all orders, the convergence being uniform on K.

— Strong (or Whitney) topology: A sequence $\{ \phi_n \}_{n \geqq 0}$ converges to ϕ if and only if there exists a compact set K such that, on one hand, the restrictions of ϕ_n to K converge to the restriction of ϕ as in the weak topology case and, on the other hand, for n large enough, all ϕ_n are equal to ϕ out of K.

These two topologies are equal if X is compact, otherwise the Whitney topology is strictly stronger than the weak topology.

Equipped with either topology, $C^\infty(X)$ is a Baire space, i. e. residual sets (countable intersections of dense open sets) are dense.

We now consider these topologies from the point of view of the stability of critical points and of the Morse property in $C^\infty(X)$. Recall that $\phi \in C^\infty(X)$ is a Morse function if all of the critical points of ϕ are nondegenerate, i. e. the corresponding Hessians are of maximal rank.

Let $\phi \in C^\infty(X)$, and x be a nondegenerate critical point of ϕ:

$$d\phi(x) = 0, \qquad rk\mathrm{H}_x\phi = \dim X.$$

Then we have:

1) Let U be any open set in X such that $x \in U$. Then there exists a weak (and consequently strong) open neighbourhood $V(\phi)$ such that $\forall \psi \in V(\phi)$, ψ admits a non-degenerate critical point y in U.

This property corresponds to the stability of non degenerate critical points.

2) If $\phi \in C^\infty(X)$ is a Morse function, there exists a neighbourhood $V(\phi)$ in the strong topology containing only Morse functions. If X is non compact, there does not exist any weak neighbourhood with this property.

3) The set of Morse functions is dense in $C^\infty(X)$ for the strong (and consequently, for the weak) topology. However, although in the strong topology the set of Morse functions is open, one can see that, in the weak topology, this set is of empty interior, if X is non compact.

From the point of view of the physicist, two potentials should be considered close to each other if the physical quantities of the corresponding system are close to each other. More specially, to a « small » variation of the potentiel, must be associated a « small » variation of the equilibriums. Thus we see that weak and strong topologies are relevant to situations respectively corresponding to local and global properties of ϕ.

Morse functions and transversality.

We recall without demonstration (cf. [1] [2]), the equivalence between the Morse property for a function $\phi \in C^\infty(X)$ and a certain transversality property for the one-jet of ϕ: $j^1\phi \in C^\infty(X, J^1(X, \mathbb{R}))$, where

$$J^1(X, \mathbb{R}) = \{ (x, a, \theta), x \in X, a \in \mathbb{R}, \theta \in T_x^*X \}$$

and $j^1\phi(x) = (x, \phi(x), d\phi(x))$.

Let U be any subset in X. One can prove (cf. [1]), that the two following properties are equivalent:

— $\phi \in C^\infty(X)$ is a Morse function on U,

— $j^1\phi \in C^\infty(X, J^1(X, \mathbb{R}))$ is transverse on U to the submanifold $N = \{ (x, a, 0), x \in X, a \in \mathbb{R} \}$ of $J^1(X, \mathbb{R})$, i. e. $d\phi(x) = 0$ implies:

$$T_x j^1\phi(T_xX) + T_{j^1\phi(x)}N = T_{j^1\phi(x)}J^1(X, \mathbb{R}).$$

III. GENERICITY OF THE MORSE PROPERTY

In this section, we prove that the set $\mathscr{M}$ of potentials ϕ such that, for all $n \geqq 1$, $\phi^{(n)}$ is a Morse function on $X^{(n)}$, is a residual set in $C^\infty(]a, +\infty[)$ for both weak and strong topologies.

This result is achieved in two steps:

1) Let n be fixed, S a stratum of $X^{(n)}$ and let U be a relatively compact open set in S. Then the set $\mathscr{M}(S, \overline{U})$ of ϕ such that the restriction $\phi^{(n)}_{|S}$ of $\phi^{(n)}$ to S is a Morse function on $\overline{U}$, is a dense open set in $C^\infty(]a, \infty[)$ for both weak and strong topologies.

However, it is clear that this property does not imply that $\phi^{(n)}$ is a Morse function on $\overline{U}$, for the Hessian can be degenerate in directions transverse to the stratum.

2) Under the same assumptions, the set $\mathscr{M}(X^{(n)}, \overline{U})$ of ϕ such that $\phi^{(n)}$ is a Morse function on $\overline{U}$ is a dense open set of $C^\infty(]a, \infty[)$, for both weak and strong topologies.

Then the claimed residuality follows from countable intersection of $\mathscr{M}(X^{(n)}, \overline{U})$, relatively to n, to the strata of X^n, and to countable coverings of each stratum by relatively compact open sets.

Coverings of the strata

Let S be any stratum of $X^{(n)}$. Now we define a covering of S by relatively compact open sets in the following way:

Let $x \in S$ and $i(x) = \{ x_I \} \in \mathbb{R}^{\frac{n(n+1)}{2}}$. Then there exist finite collections

$\{ U_I \}$, $\{ V_I \}$ and $\{ W_I \}$ of relatively compact open sets in $]a, +\infty[$, labelled by the set of intervals I in $\{ 1, \ldots, n \}$, such that:

$$\left.\begin{array}{ll} 1) & x_I \in U_I,\ \overline{U}_I \subset V_I, \text{ and } \overline{V}_I \subset W_I \\ 2) & x_I = x_J \Rightarrow U_I = U_J,\ V_I = V_J \text{ and } W_I = W_J \\ 3) & x_I \neq x_J \Rightarrow W_I \cap W_J = \phi \end{array}\right\} \quad (10)$$

The intersections (in S) of the pull-backs of $\{ U_I \}$, $\{ V_I \}$, $\{ W_I \}$, by means of the mappings $x \to x_I$, are relatively compact open sets U, V, W, such that $x \in U$, $\overline{U} \subset V$, $\overline{V} \subset W$.

Carrying out this construction for all x, one can, since S is paracompact, take out a countable covering of the stratum by a collection $\{ U_\alpha \}$ corresponding to a certain sequence $\{ x_\alpha \}$. With this sequence are also associated the collections $\{ V_\alpha \}$ and $\{ W_\alpha \}$, in such a way that, for all α, one has: $x_\alpha \in U_\alpha$, $\overline{U}_\alpha \subset V_\alpha$ and $\overline{V}_\alpha \subset W_\alpha$.

Density of $\mathcal{M}(S, V_\alpha)$ with respect to the strong topology.

For any stratum S of $X^{(n)}$ and for any point x_α of the sequence $\{ x_\alpha \}$, let $\mathcal{M}(S, V_\alpha)$ be the set of ϕ such that the restriction $\phi^{(n)}_{|S}$ is a Morse function in V_α.

We prove that $\mathcal{M}(S, V_\alpha)$ is dense in $C^\infty(]a, \infty[)$ with respect to the strong topology, and consequently also with respect to the weak topology. We use a method which consists in constructing locally affine perturbations ε in such a way that $(\phi + \varepsilon)^{(n)}_{|S}$ is a Morse function on V_α.

Locally affine perturbation.

Let us write $\{ U_I \}$, $\{ V_I \}$, $\{ W_I \}$ the open sets of $]a, \infty[$ associated to x_α. For each pair $\{ V_I, W_I \}$, one can take an Urysohn function ρ_I on $]a, +\infty[$ such that:

$$\rho_{I|V_I} = 1, \ \rho_{I|W_I^c} = 0,$$

where W_I^c is the complement of W_I.

Let B the subspace of $\mathbb{R}^{\frac{n(n+1)}{2}}$ corresponding to S and defined by (7). The above construction allows us to associate to any $b \in B$ the variation $\varepsilon_b \in C^\infty(]a, \infty[)$ defined by:

$$\varepsilon_b(t) = \sum_I^{(n)} \frac{\rho_I(t)}{m_I} b_I(t - x_{\alpha,I})$$

where the summation $\sum_I^{(n)}$ runs as before, over all non empty intervals of $\{1, \ldots, n\}$, and where m_I is the number of J such that $x_{\alpha,I} = x_{\alpha,J}$. This definition is justified by the following property of $\varepsilon_b^{(n)}$:

For $x \in V_\alpha$:

$$\varepsilon_b^{(n)}(x) = \sum_I^{(n)} b_I(x_I - x_{\alpha,I}) \tag{12}$$

$$d\varepsilon_b^{(n)}(x) = \sum_I^{(n)} b_I\varepsilon_I = \delta(b) \tag{13}$$

In other words, $\varepsilon_b^{(n)}$ is an affine function on V_α. Moreover, one can check that since the compact support of ε_b is independant of b, the mapping $b \to \varepsilon_b$ from B to $C^\infty(]a, \infty[)$ is continuous for the strong topology, i. e. ε_b tends to 0 when b tends to 0.

LEMMA 1. — For $\phi \in C^\infty(]a, \infty[)$, and for almost all $b \in B$, $(\phi + \varepsilon_b)_{|S}^{(n)}$ is a Morse function en V_α.

Proof. — Let $F : V_\alpha \times B \to J^1(S, \mathbb{R})$ be the mapping defined by:

$$F(x, b) = j^1(\phi + \varepsilon_b)_{|S}^{(n)}(x)$$

so that if one identifies $J^1(S, \mathbb{R})$ with:

$$\{ (x, a, \theta), x \in S, a \in \mathbb{R}, \theta \in T_x^*S \}$$
$$F(x, b) = (x, (\phi + \varepsilon_b)^{(n)}(x), d(\phi + \varepsilon_b)_{|S}^{(n)}(x))$$

Considering (13), we have for all $x \in V_\alpha$:

$$d(\phi + \varepsilon_b)_{|S}^{(n)}(x) = (d\phi^{(n)}(x) + \delta(b))_{|T_xS}$$

Let N be the closed submanifold of $J^1(S, \mathbb{R})$ given by:

$$N = \{ (x, a, 0), x \in S, a \in \mathbb{R} \}$$

It follows from a general theorem [*1*] that, if F is transverse to N, then for almost all $b \in B$, the partial mappings: $x \to F(x, b)$ are also transverse to N. But one can immediately check that such a property for a certain b is equivalent to the property: $(\phi + \varepsilon^b)_{|S}^{(n)}$ is a Morse function on V_α. Thus it is sufficient to prove the transversality of F. The point is then to verify that, if $F(x, b) \in N$, i. e. $d(\phi + \varepsilon_b)_{|S}^{(n)}(x) = 0$, we have:

$$T_{x,b}F(T_xS, B) + T_{F(x,b)}N = T_{F(x,b)}J^1(S, \mathbb{R})$$

But, with the identification $T_{F(x,b)}J^1(S, \mathbb{R}) = T_xS \times \mathbb{R} \times T_x^*S$, we have:

$$T_{x,b}F(\xi, \beta) = (\xi, d(\phi + \varepsilon_b)^{(n)}(x).\xi + \delta(\beta).(x - x_\alpha), H_x\phi^{(n)}(\xi) + \delta(\beta))$$

and

$$T_{F(x,b)}N = \{ (\xi, \alpha, 0), \xi \in T_xS, \alpha \in \mathbb{R} \} = T_xS \times \mathbb{R} \times \{ 0 \}$$

So, it is sufficient, for F to be transverse to N, that $\delta(B)$ should be « large enough », more precisely that:

$$\delta(B)_{|T_xS} = T_x^*S .$$

Now if ξ belongs to the kernel of $\delta(B)$ in T_xS, take $\beta = i(\xi)$. Then $\delta(\beta).\xi = \sum_I^{(n)} \xi_I^2 = 0$ implies $\xi = 0$, which completes the demonstration.

Q. E. D.

Openness of $\mathscr{M}(S, \overline{U}_\alpha)$ with respect to the weak topology.

Let $\mathscr{M}(S, \overline{U}_\alpha)$ be the set of ϕ such that the restriction $\phi_{|S}^{(n)}$ is a Morse function on $\overline{U}_\alpha$, and let j_S be the mapping $\phi \in C^\infty(]a, +\infty[) \to \phi_{|S}^{(n)} \in C^\infty(S)$, which is continuous in the weak topology, as one can easily check. Then $\mathscr{M}(S, \overline{U}_\alpha)$ is the pull-back with respect to j_S of $M(S, \overline{U}_\alpha)$, the subset of $C^\infty(S)$ of the functions which are Morse on $\overline{U}_\alpha$. It follows from a general theorem ([*2*]) that this set is a (dense) open set. Thus we can conclude that $\mathscr{M}(S, \overline{U}_\alpha)$ is a weak (and consequently strong) open set in $C^\infty(]a, +\infty[)$.

PROPOSITION 1. — For any n, any stratum S in $X^{(n)}$ and any open set U_α in the covering of S, the set $\mathscr{M}(S, \overline{U}_\alpha)$ of ϕ such that $\phi_{|S}^{(n)}$ is a Morse function on $\overline{U}_\alpha$, is a weak and strong dense open set in $C^\infty(]a, +\infty[)$.

Proof. — Since $\overline{U}_\alpha \subset V_\alpha$, we have $\mathscr{M}(S, V_\alpha) \subset \mathscr{M}(S, \overline{U}_\alpha)$. Lemma 1 implies that $\mathscr{M}(S, \overline{U}_\alpha)$ is weakly and strongly dense. The openness follows from the previous remarks. Q. E. D.

Density of $\mathscr{M}(X^{(n)}, \overline{U}_\alpha)$.

Let S be any stratum in $X^{(n)}$ and U_α any open set in its covering. We prove that the set $\mathscr{M}(X^{(n)}, \overline{U}_\alpha)$ of ϕ such that $\phi^{(n)}$ is a Morse function on $\overline{U}_\alpha$, is dense in $C^\infty(]a, +\infty[)$ with respect to the strong (and consequently to the weak) topology.

Let $\phi_0 \in C^\infty(]a, +\infty[)$ and $\mathscr{V}(\phi_0)$ be any strong neighbourhood of ϕ_0. It follows from proposition 1 that there exists $\phi \in \mathscr{V}(\Phi_0)$ such that $\phi_{|S}^{(n)}$ is a Morse function on $\overline{U}_\alpha$.

We now prove that a local quadratic perturbation is sufficient to remove the possible partial degeneracy of the critical points in $\overline{U}_\alpha$, in directions transverse to S.

Since $\phi_{|S}^{(n)}$ has a finite number, say k, of critical points in $\overline{U}_\alpha$, there exists a neighbourhood $\mathscr{W}(\phi)$ in $\mathscr{M}(S, \overline{U}_\alpha) \cap \mathscr{V}(\phi_0)$ in which we can vary ϕ without increasing the number of critical points in $\overline{U}_\alpha$.

Local quadratic perturbations.

To any $x \in \overline{U}_\alpha$, we associate the following fonction $\eta \in C^\infty(]a, +\infty[)$:

$$\eta(t) = \frac{1}{2} \sum_{I}^{(n)} \rho_I(t)(t - x_I)^2 \frac{1}{m_I} \tag{14}$$

One easily checks that, for $y \in V_\alpha$

$$\eta^{(n)}(y) = \frac{1}{2} \sum_{I}^{(n)} (y_I - x_I)^2$$

$$d\eta^{(n)}(y) = \sum_{I}^{(n)} (y_I - x_I)\varepsilon_I$$

$$H_x\eta^{(n)} = \sum_{I}^{(n)} \varepsilon_I \otimes \varepsilon_I .$$

In other words, $\eta^{(n)}$ is a quadratic function in V_α, centered at x.

LEMMA 2. — Let $\phi_0 \in C^\infty(]a, \infty[)$. Then there exists a function $\psi \in \mathscr{M}(X^{(n)}, \overline{U}_\alpha)$ in any strong neighbourhood $\mathscr{V}(\phi_0)$.

Proof. — Let $\phi \in \mathscr{M}(S, \overline{U}_\alpha) \cap \mathscr{V}(\phi_0)$ and $\mathscr{W}(\phi)$ as above. If x is a critical point for $\phi^{(n)}$, then it is a non degenerate critical point for the restriction $\phi^{(n)}_{|S}$. Consider the potentials $\phi + \lambda\eta$, where $\lambda \in \mathbb{R}$ and η is defined by (14) with the given $x \in \overline{U}_\alpha$.

For λ small enough, $\phi + \lambda\eta$ is in $\mathscr{W}(\phi)$. On the other hand, one can check that, since $d\eta^{(n)}(x) = 0$, x remains a critical point of $(\phi + \lambda\eta)^{(n)}$. The Hessian of $(\phi + \lambda\eta)^{(n)}$ at x is:

$$H_x(\phi + \lambda\eta)^{(n)} = H_x\phi^{(n)} + \lambda \sum_{I}^{(n)} \varepsilon_I \otimes \varepsilon_I$$

As the rank of $H_x\eta^{(n)}$ is n, the rank of $H_x(\phi + \lambda\eta)^n$ is maximal for all λ, except for at most n values solutions of $\det H_x(\phi + \lambda\eta)^{(n)} = 0$.

Thus the possible degeneracy of the critical point x is removable in the neighbourhood $\mathscr{W}(\phi)$.

Moreover, one needs at most k variations of this kind to get a function $\psi \in \mathscr{W}(\phi)$ which, on one hand, belongs to $\mathscr{V}(\phi_0)$ and, on the other hand, gives a potential energy $\psi^{(n)}$ which is a Morse function in $\overline{U}_\alpha$. This completes the proof. Q. E. D.

Openness of $\mathcal{M}(X^{(n)}, \overline{U}_\alpha)$ with respect to the weak topology.

Now we know that the set $M(X^{(n)}, \overline{U}_\alpha)$ of functions in $C^\infty(X^{(n)})$ which are Morse functions on the compact set $\overline{U}_\alpha$, is a (dense) open set for the weak topology.

The mapping $\phi \in C^\alpha(]a, +\infty[) \to \phi^{(n)} \in C^\infty(X^{(n)})$ being continuous for the weak topology, the corresponding pull back $\mathcal{M}(X^{(n)}, \overline{U}_\alpha)$ of $M(X^{(n)}, \overline{U}_\alpha)$ is thus a weak (and consequently, strong) open set of $C^\infty(]a, \infty[)$.

PROPOSITION 2. — For any n, any stratum S of $X^{(n)}$, and any U_α in its covering, the set $\mathcal{M}(X^{(n)}, \overline{U}_\alpha)$ of ϕ such that $\phi^{(n)}$ is a Morse function on $\overline{U}_\alpha$ is a dense open set of $C^\infty(]a, \infty[)$ with respect to both weak and strong topologies.

Proof. — The density follows from lemma 2 and openness from the previous remarks. Q. E. D.

COROLLARY 1. — For n fixed, we have a finite number of strata S in $X^{(n)}$, and each of them is covered by a countable collection of open sets U_α. The set $\mathcal{M}(X^{(n)})$ of ϕ such that $\phi^{(n)}$ is a Morse function on $X^{(n)}$ is thus a countable intersection of the dense open sets $\mathcal{M}(X^{(n)}, \overline{U}_\alpha)$, i. e. is a residual set. Since $C^\infty(]a, \infty[)$ is a Baire space for the weak and strong topologies, $\mathcal{M}(X^n)$ is dense for both topologies.

Finally, the set $\mathcal{M} = \bigcap_{n \geq 1} \mathcal{M}(X^{(n)})$ of ϕ such that, for all n, $\phi^{(n)}$ is a Morse function, is also residual, and thus dense in $C^\infty(]a, \infty[)$ for both topologies.

Consequences. — *a)* The set of potentials bounded from below is a strongly open set in $C^\infty(]a, \infty[)$. The intersection of $\mathcal{M}$ with this open set is dense in it, which means that, generically, such a potential admits minimal energy configurations which are non degenerate critical points.

b) Consider for all $b > a$ the set $\mathcal{U}_b$ of the potentials $\phi \in C^\infty(]a, \infty[)$ such that:

1) $\lim_{t \to a} \phi(t) = +\infty$: hard core

2) $t \geq b \Rightarrow \phi'(t) > 0$: attraction at large distances (15)

One checks easily that $\mathcal{U}_b$ is strongly open. Then, for all n, $\mathcal{M}(X^n) \cap \mathcal{U}_b$ is strongly open.

In fact, if $\phi \in \mathcal{U}_b$, the critical points of $\phi^{(n)}$ all lie in the compact set:

$$K_b^{(n)} = \{ x \in X^{(n)}, x_i \leq b \quad i = 1, \dots, n \}.$$

Proposition 2 implies that the set $\mathcal{M}(X^{(n)}, K_b^{(n)})$ of ϕ such that $\phi^{(n)}$ is a Morse function on $K_b^{(n)}$ is strongly open.

One easily checks that $\mathcal{M}(X^{(n)}, K_b^{(n)}) \cap \mathcal{U}_b$ is contained in $\mathcal{M}(X^{(n)}) \cap \mathcal{U}_b$.

The opposite inclusion being obvious, the equality of these two sets implies the strong openness of $\mathcal{M}(X^{(n)}) \cap \mathcal{U}_b$. It follows that $\mathcal{M}(X^n) \cap \mathcal{U}$ is strongly open for any n, where $\mathcal{U} = \bigcup_{b>a} \mathcal{U}_b$.

COROLLARY 2. — As a last remark, we prove the existence of a strong open set contained in $\mathcal{M} = \bigcap_{n \geqq 1} \mathcal{M}(X^{(n)})$, which is only a priori a residual set. Let $b > a$ and $\phi \in \mathcal{U}_b$ (defined by (15)). For any n, the critical points of $\phi^{(n)}$ lie in the compact set $K_b^{(n)}$ defined above. When b is small enough, we give a sufficient and open condition (16) for the Hessian of $\phi^{(n)}$ to be positive definite. Thus $\phi^{(n)}$ has only one critical point in $X^{(n)}$, which is its minimum. Let us write:

$$H_x\phi^{(n)} = \sum_{i,j} h_{ij}\varepsilon_i \otimes \varepsilon_j$$

$$h_{ij} = \sum_{i,j\in I}^{(n)} \phi''(x_I)$$

For $H_x\phi^{(n)}$ to be positive definite, it is sufficient that for all $i = 1, 2, \ldots, n$:

$$h_{ii} > \sum_{j \neq i} | h_{ij} | .$$

But, when $i \leqq j$:

$$h_{ij} = \sum_{\substack{1 \leqq p \leqq i \\ j \leqq q \leqq n}} \phi''(x_{[p,q]})$$

If $\phi \in \mathcal{U}_b$, the critical points of $\phi^{(n)}$ satisfy: $a < x_i \leqq b$, thus

$$(q - p + 1)a < x_{[p,q]} \leqq (q - p + 1)b .$$

For $r \in \mathbb{N}^*$, define $\alpha_r = \sup_{r.a \leqq t \leqq r.b} | \phi''(t) |$,

Then if $i < j$:

$$| h_{ij} | \leqq \sum_{\substack{1 \leqq p \leqq i \\ j \leqq q \leqq n}} \alpha_{q-p+1} \leqq \sum_{r \geqq |j-i|+1} (r - 1)\alpha_r$$

which yields for any i:

$$\sum_{j \neq i} | h_{ij} | < 2\sum_{s \geqq 1} \sum_{r \geqq s+1} (r - 1)\alpha_r = 2\sum_{r \geqq 2} (r - 1)^2\alpha_r$$

and:

$$h_{ii} \geqq \phi''(x_i) - \sum_{r \geqq 2} r\alpha_r$$

Thus we get the following positivity condition for the Hessian, in $K_b^{(n)}$ and any n:

$$\operatorname*{Inf}_{a<t\leqq b} \phi''(t) > \sum_{r\geqq 2} [r + 2(r-1)^2] \operatorname*{Sup}_{r.a\leq t\leq r.b} |\phi''(t)| \tag{16}$$

This condition is clearly open for the strong topology, and certainly non empty if $b < 2a$.

Let $\mathscr{V}_b$ be the subset of $\phi \in \mathscr{U}_b$ that satisfy the condition (16), and $\mathscr{V} = \bigcup_{b>a} \mathscr{V}_b$.

Then $\mathscr{V}$ is a non empty strongly open set in $C^\infty(]a, \infty[)$, contained in all the $\mathscr{M}(X^{(n)})$, and thus contained in $\mathscr{M}$. Notice that $\mathscr{V}$ corresponds to realistic potentials, since it consists of hard core potentials attractive at large distances, and satisfying a certain convexity condition.

IV. PERTURBATION OF EQUILIBRIUM CONFIGURATIONS AND STABILITY OF THE STRATA

In the following, we shall be concerned with perturbations of non degenerate equilibrium configurations, corresponding to small enough, but arbitrary variations of the two body interaction.

A main point in our conclusion is the existence, for any n, of a stratum $\mathscr{S}^{(n)}$ of $X^{(n)}$ which is stable with respect to variations of the potential, in the following sense. If $x \in \mathscr{S}^{(n)}$ is a non degenerate critical point of $\phi^{(n)}$, there exists a weak neighbourhood $\mathscr{V}$ of ϕ such that, for all ψ in $\mathscr{V}$, $\psi^{(n)}$ has a non degenerate critical point, close to x, which also lies in $\mathscr{S}^{(n)}$.

This property is not obvious, since the dimension of $\mathscr{S}^{(n)}$ is $\left[\frac{n+1}{2}\right]$, integer part of $\frac{n+1}{2}$.

Trajectory of critical points.

Let $\phi \in C^\infty(]a, \infty[)$, and assume that, for a given n, $x_0 \in X^{(n)}$ is a non degenerate critical point of $\phi^{(n)}$.

Then there exist two relatively compact open sets, U and V in $X^{(n)}$ such that:

$$\begin{array}{ll} 1) & x_0 \in \mathrm{U} \\ 2) & \overline{\mathrm{U}} \subset \mathrm{V} \\ 3) & d\phi^{(n)} \neq 0 \quad \text{on} \quad \overline{\mathrm{V}} - \{ x_0 \} \end{array} \tag{17}$$

Now, proposition 1 implies the existence of a weak neighbourhood $\mathscr{W}$ of 0 in $C^\infty(]a, \infty[)$ such that for any $\psi \in \mathscr{W}$ and $\lambda \in [0, 1]$, $(\phi + \lambda\psi)^{(n)}$ is a Morse function on $\overline{\mathrm{V}}$ with a unique critical point in U.

Then we have the following lemma:

LEMMA 3. — Let x_0 be a non degenerate critical point of $\phi^{(n)}$ and let U, V satisfy the conditions (17). There exists a neighbourhood $\mathscr{W}$ of 0 in $C^\infty(]a, \infty[)$, such that $\forall \varepsilon \in \mathscr{W}$, the trajectory: $\lambda \in [0, 1] \to x_\lambda \in \mathrm{U}$ of the critical point of $(\phi + \lambda\varepsilon)^{(n)}$ is C^∞ and satisfy the following equation:

$$\mathrm{H}_{x_\lambda}(\phi + \lambda\varepsilon)^{(n)}(\xi_\lambda) + d\varepsilon^{(n)}(x_\lambda) = 0 \tag{19}$$

where:

$$\xi_\lambda = \frac{dx_\lambda}{d\lambda} \in \mathrm{T}_{x_\lambda}\mathrm{X}^{(n)} \tag{20}$$

Proof. — Let $\mathscr{W}$ be as above. For any fixed $\varepsilon \in \mathscr{W}$, let x_λ be the critical point of $(\phi + \lambda\varepsilon)^{(n)}$ in U. We have:

$$d(\phi + \lambda\varepsilon)^{(n)}(x_\lambda) = 0 .$$

The derivation of this relation yields the equation of the trajectory:

$$\mathrm{H}_{x_\lambda}(\phi + \lambda\varepsilon)^{(n)}\left(\frac{dx_\lambda}{d\lambda}\right) + d\varepsilon^{(n)}(x_\lambda) = 0$$

which is equivalent to (19) and (20).

The regularity of the trajectory follows directly from that of ϕ and ε.

Q. E. D.

If $\mathrm{B}(x_\lambda)$ is the subspace of $\mathbb{R}^{\frac{n(n+1)}{2}}$ associated by formula (7) to the stratum $\mathrm{S}(x_\lambda)$, then $d\varepsilon^{(n)}(x_\lambda) \in \delta(\mathrm{B}(x_\lambda))$ and:

$$\xi_\lambda \in \{ \mathrm{H}_{x_\lambda}(\phi + d\varepsilon)^{(n)} \}^{-1}\delta(\mathrm{B}(x_\lambda))$$

We are thus led to give the following general definition. For x any non degenerate critical point of $\psi^{(n)}$, let:

$$\mathrm{E}_{x,\psi} = \{ \mathrm{H}_x\psi^{(n)} \}^{-1}\delta(\mathrm{B}(x)) \tag{21}$$

It is the subspace of $\mathrm{T}_x\mathrm{X}^{(n)}$ spanned by the tangents at x to the trajectories of the critical point corresponding to any variation of the potential ψ.

Infinitesimal and local stabilities of strata.

We can now give the following infinitesimal stability condition.

DEFINITION. — Let S be a stratum of $X^{(n)}$ and $x \in S$. Then the stratum S is stable at x if, for any potential ψ such that x is a non degenerate critical point of $\psi^{(n)}$, the trajectories of the critical point corresponding (using the method of lemma 3) to any variation of the potential, are all tangent to S at x.

This condition is clearly equivalent to $E_{x,\psi} \subset T_xS$, for x any non degenerate critical point of $\psi^{(n)}$.

LEMMA 4. — A stratum S of $X^{(n)}$ is stable if and only if $\dim S = \dim \delta(B)$, where B is the subspace of $\mathbb{R}^{\frac{n(n+1)}{2}}$ associated with S. The stratum S is then stable at any point.

Proof. — Let us first prove that $T_xS \subset E_{x,\psi}$.
Formula (3′) gives:

$$H_x\psi^{(n)}(\xi) = \sum_{I}^{(n)} \psi''(x_I)\xi_I\varepsilon_I$$

For $\xi \in T_xS$, one checks that the family $\{ \psi''(x_I).\xi_I \}$ is contained in the subspace B of $\mathbb{R}^{\frac{n(n+1)}{2}}$ associated with S by formula (7). Then there exists $b \in B$ such that, $\forall I$, $\psi''(x_I)\xi_I = b_I$ and thus $H_x\psi^{(n)}(\xi) = \delta(b)$.
In other words:

$$\xi = \{ H_x\psi^{(n)} \}^{-1}\delta(b) \in E_{x,\psi} .$$

The stability condition $E_{x,\psi} \subset T_xS$ is thus equivalent to the property: $T_xS = E_{x,\psi}$ when x is a non degenerate critical point of $\psi^{(n)}$.
Since the Hessian is non degenerate, it follows that $\dim S = \dim \delta(B)$.
Conversely, if $\dim S = \dim \delta(B)$, the inclusion $T_xS \subset E_{x,\psi}$ implies $T_xS = E_{x,\psi}$ and yields the stability of S at x.
Finally, since the condition $\dim S = \dim \delta(B)$ is independant of x and ψ, the stability at one point of S is equivalent to the stability everywhere in S.
Conversely, if $T_xS \neq E_{x,\psi}$ for a certain pair (x, ψ), then $\dim S < \dim \delta(B)$ and this inequality extends in the same way to the whole stratum.
Q. E. D.

We now prove that the infinitesimal stability is equivalent to the local stability defined as follows.

DEFINITION. — A stratum S of $X^{(n)}$ is locally stable if, for any $x \in S$ and all ϕ such that x is a non degenerate critical point of $\phi^{(n)}$, there exists a neighbourhood $\mathscr{W}$ of 0 in $C^\infty(]a, \infty[)$ satisfying the condition:
$\forall \varepsilon \in \mathscr{W}$, $(\phi + \varepsilon)^{(n)}$ has a non degenerate critical point in S close to x.

Let us notice that, if S is not infinitesimally stable, then dim S < dim $\delta(B)$. Under these conditions, lemma 3 implies that there exists some trajectories of critical points going out of the stratum. Thus S is not locally stable. Conversely, we have the following result.

LEMMA 5. — If S is a stratum of $X^{(n)}$ such that dim S = dim $\delta(B)$, then S is locally stable.

Proof. — Let x_0 be a non degenerate critical point of $\phi^{(n)}$, belonging to a stratum S. Lemma 3 gives an open neighbourhood U of x_0 in S and an open neighbourhood $\mathscr{W}$ of 0 in $C^\infty(]a, +\infty[)$, such that for all $\varepsilon \in \mathscr{W}$, there exists a trajectory $\lambda \in [0, 1] \rightarrow x_\lambda \in U$ of critical points of $(\phi + \lambda\varepsilon)^{(n)}$.

Let us prove that if dim S = dim $\delta(B)$, then this trajectory is contained in S.

Consider the one-parameter vector field on U defined by:

$$(\lambda, x) \in [0, 1] \times U \rightarrow \xi_\lambda(x) = \{ H_x(\phi + \lambda\varepsilon)^{(n)} \}^{-1} d\varepsilon^{(n)}(x) \qquad (22)$$

Equations (19) and (20) show that the trajectory is an integral curve of the vector field (22). On the other hand, it is clear that $S \cap U$ is an integral manifold since $\forall x \in S$, $\lambda \in [0, 1]$, $\xi_\lambda(x) \in T_xS$. The smoothness of the Hessian implies uniqueness of the solution of the system (19)-(20), for given initial conditions. It follows that, for $x_0 \in S$, the trajectory of critical points is contained in $S \cap U$. More specially, the critical point of $(\phi + \varepsilon)^n$ belongs to S. Q. E. D.

Remark. — Stability of $\overline{S}$.

Let S be a stratum of $X^{(n)}$. The closure $\overline{S}$ in $X^{(n)}$ is a submanifold without boundary and the topological boundary $\overline{S} - S$ is a union of strata with lower dimensions.

Assume that S is stable, i. e. dim S = dim $\delta(B)$ and that $x \in S_1 \subset \overline{S} - S$ is a non degenerate critical point of $\phi^{(n)}$. If B_1 is the subspace of $\mathbb{R}^{\frac{n(n+1)}{2}}$ associated to the stratum S_1, then $B_1 \subset B$ and:

$$E_{x,\phi} = \{ H_x\phi^{(n)} \}^{-1}\delta(B_1) \subset \{ H_x\phi^{(n)} \}^{-1}\delta(B) = T_x\overline{S}$$

The demonstration of lemma 5 fits immediately, and implies the stability of $\overline{S}$.

In other words, if x is a critical point that belongs to an unstable stratum S_1 in the boundary of a stable stratum S, then some variations of the two body potential will shift x out of S_1, but any of them will leave it in $\overline{S}$.

In the following, we shall consider three particular strata, and we shall prove that one of them, namely the symmetric stratum, is the lowest dimensional stable stratum.

The open stratum.

For any n, there exists a unique stratum of maximal dimension, the open stratum of equation:

$$S_0^{(n)} = \{ x \in X^{(n)}, I \neq J \Rightarrow x_I \neq x_J \} \tag{23}$$

This stratum is open an dense in $X^{(n)}$ and the space $B^{(n)}$ corresponding is $\mathbb{R}^{\frac{n(n+1)}{2}}$. Thus $\phi(B_0^{(n)}) = \mathbb{R}^{n*}$ and $\dim S_0^{(n)} = \dim \delta(B_0^{(n)})$.

Consequently this stratum is stable, but its stability follows merely from its openness.

The « periodic » stratum.

This stratum, which is the lowest dimensional stratum in $X^{(n)}$, is given by

$$S_p^{(n)} = \{ x \in X^{(n)} : x_1 = x_2 \ldots = x_n \} \tag{24}$$

It is diffeomorphic to $]a, +\infty[$, and of dimension 1. The corresponding subspace $B_p^{(n)}$ is given by:

$$B_p^{(n)} = \{ b \in \mathbb{R}^{\frac{n(n+1)}{2}} : |I| = |J| \Rightarrow b_I = b_J \} \tag{24'}$$

and is isomorphic to $\mathbb{R}^n$ by the mapping: $b \in B_p^{(n)} \rightarrow (b^{(1)}, \ldots, b^{(n)}) \in \mathbb{R}^n$ where $b_I = b^{(|I|)}$.

Using elementary linear algebra, one can check that $\dim \delta(B_p^{(n)}) = \left[\frac{n+1}{2}\right]$, integer part of $\frac{n+1}{2}$.

Thus $\dim S_p^{(n)} < \dim \delta(B_p^{(n)})$ for $n \geqq 2$, and the « periodic » stratum is not stable.

The symmetric stratum.

Consider the stratum:

$$\mathcal{S}^{(n)} = \{ x \in X^{(n)}, x_i = x_j \Leftrightarrow i = n - j \text{ or } i = j \} \tag{25}$$

The closure of $\mathcal{S}^{(n)}$ in $X^{(n)}$ is:

$$\overline{\mathcal{S}^{(n)}} = \{ x \in X^{(n)} : x_i = x_{n-i}, i = 1, 2, \ldots, n-1 \}$$

One checks that $\dim \mathcal{S}^{(n)} = \left[\frac{n+1}{2}\right]$ and that the corresponding space $\mathcal{B}^{(n)}$ is defined by:

$$\mathcal{B}^{(n)} = \{ b \in \mathbb{R}^{\frac{n(n+1)}{2}} : I = (n+1) - J \Rightarrow b_I = b_J \} \tag{25'}$$

where $I = (n + 1) - J$ means that I and J are two intervals of $\{ 1, \ldots, n \}$, symmetric with respect to $\frac{n+1}{2}$. It follows that

$$\dim \mathcal{B}^{(n)} = \frac{n(n+1)}{4} + \frac{1}{2}\left[\frac{n+1}{2}\right].$$

If $b \in \mathscr{B}^{(n)}$, $\delta(b) = \sum_{i=1}^{n} c_i \varepsilon_i$, with $c_i = c_{n-i}$.

Consequently, $\dim \delta(\mathscr{B}^{(n)}) \leqq \left[\frac{n+1}{2}\right]$. On the other hand, we know that $\dim \mathscr{S}^{(n)} \leqq \dim \delta(\mathrm{B}^{(n)})$. It follows that $\dim \mathscr{S}^{(n)} = \dim \delta(\mathscr{B}^{(n)}) = \left[\frac{n+1}{2}\right]$.

In other words, the symmetric stratum is stable, and thus its closure as well.

We conclude this part by the proof that $\mathscr{S}^{(n)}$ is the lowest dimensional stable stratum of $\mathrm{X}^{(n)}$, in the following sense. If S is another stable stratum, then $\dim \mathrm{S} > \dim \mathscr{S}^{(n)}$ and $\mathscr{S}^{(n)} \subset \overline{\mathrm{S}} - \mathrm{S}$.

Let S be a stable stratum of $\mathrm{X}^{(n)}$. Then $\mathrm{S}_p^{(n)} \subset \overline{\mathrm{S}}$, where $\mathrm{S}_p^{(n)}$ is the « periodic » stratum.

If $x \in \mathrm{S}_p^{n)}$ is a non degenerate critical point of $\phi^{(n)}$, one must have $\mathrm{E}_{x,\phi} \subset \mathrm{T}_x\overline{\mathrm{S}}$. On the other hand, $\mathrm{E}_{x,\phi} = \{ \mathrm{H}_x \phi^{(n)} \}^{-1} \delta(\mathrm{B}_p^{(n)})$ is of the same dimension as $\delta(\mathrm{B}_p^{(n)})$, i. e. $\left[\frac{n+1}{2}\right]$, and since $\overline{\mathscr{S}^{(n)}}$ is stable, $\mathrm{E}_{x,\phi}$ is contained in $\mathrm{T}_x\overline{\mathscr{S}^{(n)}}$. It follows that $\mathrm{E}_{x,\phi} = \mathrm{T}_x\overline{\mathscr{S}^{(n)}} \subset \mathrm{T}_x\overline{\mathrm{S}}$.

Since $\overline{\mathscr{S}^{(n)}}$ and $\overline{\mathrm{S}}$ are both affine submanifolds of $\mathrm{X}^{(n)}$, either $\overline{\mathscr{S}^{(n)}} = \overline{\mathrm{S}}$, or, the stratification implies $\mathscr{S}^{(n)} \subset \overline{\mathrm{S}} - \mathrm{S}$ and $\dim \mathrm{S} > \dim \mathscr{S}^{(n)}$.

The physical meaning of this result is the following. If $x \in \mathscr{S}^{(n)}$ is a non degenerate equilibrium configuration for a given potential ϕ, then there exists a weak neighbourhood of ϕ in which any potential yields a perturbed equilibrium configuration in $\mathscr{S}^{(n)}$. Moreover, the breakdown of this symmetry can occure only through the degeneracy of the Hessian.

V. EXISTENCE OF SYMMETRIC EQUILIBRIUM CONFIGURATIONS

We saw that the symmetric strata $\mathscr{S}^{(n)}$ are stable. The interest of this property would be however restricted if the existence of equilibrium configurations in such strata was exceptional. Actually, nothing of this kind occurs, and more precisely, the potentials which are attractive at large distance generate interactions such that, for any n, there exists at least one equilibrium in $\overline{\mathscr{S}^{(n)}}$. This result follows mainly from a property of $\mathscr{S}^{(n)}$, and more generally of stable strata.

LEMMA 6. — Let $\phi \in \mathrm{C}^\infty(]a, \infty[)$ and let S be a stable stratum in $\mathrm{X}^{(n)}$, n arbitrary. Then any critical point of the restriction $\phi_{|\overline{\mathrm{S}}}^{(n)}$ is a critical point of $\phi^{(n)}$. If $x \in \overline{\mathrm{S}}$ is a degenerate critical point for $\phi_{|\overline{\mathrm{S}}}^{(n)}$, then x is also degenerate for $\phi^{(n)}$.

Proof. — Let $x \in \overline{S}$ be such that $d\phi^{(n)}_{|\overline{S}}(x) = 0$, i. e. $\forall \xi \in T_x\overline{S}$, $d\phi^{(n)}(x).\xi = 0$. If B is the subspace of $\mathbb{R}^{\frac{n(n+1)}{2}}$ corresponding to S, the stability implies $\delta(B) = \delta \circ i(T_x\overline{S})$.

In other words, there exists $\zeta \in T_x\overline{S}$ such that

$$d\phi^{(n)}(x) = \sum_I^{(n)} \phi'(x_I)\varepsilon_I = \sum_I^{(n)} \zeta_I \varepsilon_I$$

Thus we have for all $\xi \in T_x\overline{S}$:

$$d\phi^{(n)}(x).\xi = \sum_I^{(n)} \xi_I . \zeta_I = 0 .$$

Since the bilinear form $(\xi, \zeta) \to \sum_I^{(n)} \xi_I\zeta_I$ is positive and non degenerate, it follows that $\zeta = 0$ and $d\phi^{(n)}(x) = 0$.

If $x \in \overline{S}$ is a degenerate critical point of $\phi^{(n)}_{|\overline{S}}$, there exists $\zeta \in T_x\overline{S}$ such that:

$$\forall \xi \in T_x\overline{S}, \quad H_x\phi^{(n)}(\xi, \zeta) = 0$$

The linear form $H_x\phi^{(n)}(\xi) = \sum_I^{(n)} \phi''(x_I)\zeta_I\varepsilon_I$, which belongs to $\delta(B)$, is thus zero on $T_x\overline{S}$. As in the previous case, we can conclude that it is zero everywhere, i. e. $H_x\phi^{(n)}(\xi) = 0$, so that x is a degenerate critical point for $\phi^{(n)}$. Q. E. D.

Now, consider a potential ϕ belonging to $\mathcal{U}_b$ (defined by (15)). For any n, the critical points of $\phi^{(n)}$ lie in the compact set

$$K_b^{(n)} = \{ x \in X^{(n)}, x_i \leqq b, i = 1 \dots n \} .$$

Observe then that $\phi^{(n)}_{|\overline{\mathcal{S}^{(n)}}}$ has certainly a relative minimum in $K_b^{(n)}$.

In fact, the minimum of $\phi^{(n)}$ on the boundary of $\overline{\mathcal{S}^{(n)}} \cap K_b^{(n)}$ is reached at a point such that $x_i > a\ \forall i$, and $x_j = x_{n-j} = b$ for at least one j.

Define $\xi \in T_x\overline{\mathcal{S}^{(n)}}$ by $\xi_j = \xi_{n-j} = 1$ and $\xi_i = 0$ for all $i \neq j$ and $n - j$.

Then $d\phi^{(n)}(x).\xi = \sum_I^{(n)} \phi'(x_I)\xi_I > 0$, since $j \in I$ or $n - j \in I$ implies $x_I \geqq b$ and $\phi'(x_I) > 0$.

Since ξ points out of $\overline{\mathcal{S}^{(n)}} \cap K_b^{(n)}$ in $\overline{\mathcal{S}^{(n)}}$, it follows that $\phi^{(n)} \mid \overline{\mathcal{S}^{(n)}}$ reaches a relative minimum in $\overline{\mathcal{S}^{(n)}} \cap K_b^{(n)}$.

Lemma 6 asserts that this critical point for the restriction to $\overline{\mathcal{S}^{(n)}}$ is a critical point for $\phi^{(n)}$. Thus we get:

PROPOSITION 3. — Let $\mathcal{U} = \bigcup_{b>a} \mathcal{U}_b$ of $C^\infty(]a, \infty[)$ defined from (15) be the set of hard core potentials which are attractive at large distances. Then $\mathcal{U}$ contains a residual set of potentials ϕ such that, for any n, $\phi^{(n)}$ has a non degenerate critical point in the symmetric stratum $\overline{\mathcal{S}^{(n)}}$.

Consider the open set $\mathcal{V} = \bigcup_{b>a} \mathcal{V}_b$, where $\mathcal{V}_b$ is the subset of $\phi \in \mathcal{U}_b$ satisfying the convexity condition (16). Then, for all $\phi \in \mathcal{V}$ and any n, the only critical point of $\phi^{(n)}$ is non degenerate and belongs to $\overline{\mathcal{S}^{(n)}}$.

In other words, proposition 3 exhibits an open set $\mathcal{V}$ of realistic potentials for which the ground states for any number of particles are symmetric.

Proof. — The result follows from lemma 6, a subsequent remark and from the consequences of proposition 2. Q. E. D.

VI. CONCLUSION

We have thus proved that, in one dimension, and for a large class of realistic two body potentials, there exists for any number of particles, an equilibrium configuration which is symmetric with respect to a center. But in one dimension, there is only one non trivial symmetry transformation, namely the inversion, and thus only one Bravais lattice, the periodic lattice. Now, if we could extend our results to the case of infinite configurations, the translationnal invariance suggests that the periodic stratum should be stable in an appropriate sense.

In a recent work [*4*], G. C. Hamrick and C. Radin investigate the perturbations of an infinite periodic equilibrium configurations, and prove that an arbitrary small perturbation yields a non periodic equilibrium configuration.

In fact, this result is not contradictory with ours, since the situation they study is precisely non generic: the initial interaction is of finite range, and so the n-points energies are not Morse functions.

Moreover, the « perturbation » does not tend to zero, with the coupling constant, even for the weak topology.

In several dimensions, one may hope that the configurations space is stratified with respect to the symmetry groups of the configurations, and that the crystallographic point groups correspond directly to stable strata. However there exist other finite subgroups of the rotations, namely the dihedral groups and the icosahedron group.

Actually one can define a stratification corresponding to the symmetry groups of the configurations. It is unclear wether this stratification is stable or wether some symmetry groups play a special part in the stability and genericity properties. These questions will be studied in a subsequent paper.

REFERENCES

[1] M. GOLUBITSKY, V. GUILLEMIN, *Stable mappings and their singularities* (Springer).
[2] M. HIRSH, *Differential topology* (Springer).
[3] C. S. GARDNER and C. RADIN, *J. Stat. Phys.*, t. **20**, 1979, p. 719.
[4] G. C. HAMRICK and C. RADIN, *J. Stat. Phys.*, t. **21**, 1979, p. 601.
[5] R. C. HEITMANN and C. RADIN, *J. Stat. Phys.*, t. **22**, 1980, p. 281.
[6] W. J. VENTEVOGEL, *Physica*, t. **92 A**, 1978, p. 343.
[7] W. J. VENTEVOGEL and B. R. A. NIJBOER, *Physica*, t. **98 A**, 1979, p. 274.
[8] W. J. VENTEVOGEL and B. R. A. NIJBOER, *Physica*, t. **99 A**, 1979, p. 569.

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