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J. DIMOCK

BERNARD S. KAY

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# Classical wave operators and asymptotic quantum field operators on curved space-times

by

**J. DIMOCK (\*)**

Department of Mathematics SUNY at Buffalo,  
Buffalo, N. Y. 14214, U. S. A.

and

**Bernard S. KAY (\*\*)**

Institute for Theoretical Physics, University of Bern,  
Sidlerstrasse 5, CH-3012 Bern, Switzerland

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**ABSTRACT.** — We consider a class of Lorentzian metrics on  $\mathbb{R}^4$  which are stationary at time-like infinity and Minkowskian either at space-like or time-like infinity. For these metrics, we construct wave operators for the classical Klein-Gordon equation. For certain transient cases we also construct inverse wave operators. Given the classical wave operators, we show how to construct in and out field operators for the quantum Klein-Gordon equation and a particle interpretation for this theory. Finally, we give several results of a general nature on the construction of classical and quantum scattering operators paying special attention to the stationary case. The methods and results of the paper could be generalized to other external field problems (not just gravitational).

**RÉSUMÉ.** — Nous considérons une classe de métriques lorentziennes sur  $\mathbb{R}^4$  stationnaires à l'infini temporel et minkowskiennes à l'infini spatial ou à l'infini temporel. Dans ces métriques, nous construisons des

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opérateurs d'onde pour l'équation de Klein-Gordon classique. Pour certains cas de champs gravitationnels éphémères, nous construisons aussi les opérateurs d'onde inverses. Les opérateurs d'onde classiques étant donnés, nous expliquons comment construire des opérateurs de champs *in* et *out* pour l'équation de Klein-Gordon quantique et obtenons une interprétation en termes de particules pour cette théorie. Finalement, nous démontrons plusieurs résultats de nature plus générale concernant la construction d'opérateurs de diffusion en développant particulièrement le cas stationnaire. On pourrait généraliser les méthodes et résultats de cet article à d'autres problèmes de champs externes (non seulement gravitationnels).

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## I. INTRODUCTION

The theory of linear quantum fields on a curved space-time is now reasonably well understood on a mathematical level [10] [13] [14] [15]. As for the physical interpretation it is now clear that for generic space-times the concepts of vacuum and particle are inappropriate. If however the field operator approaches a free field operator in the distant past and/or future then a particle interpretation is possible in the appropriate asymptotic sense. We are concerned with studying how this happens, and further with the construction of a scattering operator to relate the notions of in-particle and out-particle.

In particular we study the covariant Klein-Gordon equation

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2)u = 0 \quad m > 0$$

(It will be supposed throughout that the mass is not zero.)

We have isolated conditions on the space-time which give one control over the classical scattering theory for this equation. This is just the input needed for studying asymptotic behavior of the corresponding quantum fields.

Our first restriction is that the manifold is taken to be  $\mathbb{R}^4$ . This is mainly for simplicity and we expect that our methods could be adapted to certain other manifolds as well.

For the Lorentzian metric we have several conditions.

A. The coefficients  $g^{\mu\nu}$  are bounded  $C^\infty$  functions on  $\mathbb{R}^4$ . There exist positive constants  $C_1, C_2$  such that  $g^{00} \geq C_1$  and  $-g^{ij}\xi_i\xi_j \geq C_2|\xi|^2$

B. 
$$\int \|\partial_0 g_{\mu\nu}(t, \cdot)\|_\infty dt < \infty$$

C. For some  $2 \leq p \leq \infty$ , there are constants  $C, \varepsilon > 0$  such that if  $\eta_{\mu\nu}$  is the Minkowski metric

$$\|g^{\mu\nu}(t, \cdot) - \eta^{\mu\nu}\|_p + \|\partial_k(g^{\mu\nu}(t, \cdot) - \eta^{\mu\nu})\|_p \leq C |t|^{3/p-1-\varepsilon},$$

$$|t| \geq 1, \quad k = 1, 2, 3.$$

The significance of these conditions is roughly as follows (details in Chapter II). Condition A implies that the space-time is globally hyperbolic with surfaces  $x^0 = \text{constant}$  as Cauchy surfaces. As a consequence one has existence and uniqueness for the global Cauchy problem. Condition A also bounds the growth of solutions by growth in energy. Condition B says that the metric is asymptotically stationary, and is needed so that the energy of solutions is bounded in time. Condition C says that the metric becomes Minkowskian at space-like infinity ( $2 \leq p < \infty$ ), and/or at time-like infinity ( $3/(1 + \varepsilon) < p \leq \infty$ ). Stronger decay to  $\eta^{\mu\nu}$  in space-like directions (smaller  $p$ ) allows weaker decay or even growth in time-like directions. Note that for stationary metrics ( $t$ -independent) it is sufficient that  $\|g^{\mu\nu} - \eta^{\mu\nu}\|_p$  and  $\|\partial_k(g^{\mu\nu} - \eta^{\mu\nu})\|_p$  are finite for some  $2 \leq p < 3$ . Thus metrics for which  $g^{\mu\nu}(\vec{x}) - \eta^{\mu\nu}$  and its spatial derivatives are  $O(|\vec{x}|^{-1-\delta})$  for some  $\delta > 0$  are permitted.

Under conditions ABC we show the existence of classical wave operators  $\Omega^\pm$  in the free energy norm (Chapter III). These give the free asymptotic behavior of solutions, and an immediate consequence is the existence of free asymptotic fields for the quantum problem and a corresponding particle interpretation (Chapter V). If in addition  $\text{Ran } \Omega^- = \text{Ran } \Omega^+$  one can define a classical scattering operator. This ensures the existence of a scattering automorphism on the field algebra for the quantum case.

With further assumptions we develop more detailed results. For a class of transient metrics we prove classical asymptotic completeness, that is  $\Omega^\pm$  are onto (Chapter IV). For stationary metrics we show that various definitions of a vacuum are equivalent, and that the scattering automorphism (when it exists) is implemented by a unitary scattering operator (Chapter VI).

This completes the survey of our results. A preliminary announcement and further exposition can be found in [17]. Related results for the classical problem have recently been announced by Paneitz and Segal [22], who have a somewhat different approach to the quantum problem. Furlani [11] has some results for scattering on non-globally hyperbolic manifolds. We also mention the work of Cotta-Ramusino, Krüger, and Schrader [7] who study scattering for the Schrödinger equation on curved 3-space.

Our results have implications for other external field problems. For example, consider a charged scalar field in an external electromagnetic field as described by the equation

$$((\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu) + m^2)u = 0$$

Then one should be able to formulate conditions analogous to our ABC and prove similar theorems. Many of these results would be new. For the existing literature on this problem see [8] [11] [19] [23] [24] [26].

Finally we remark that in any quantum problem on curved space-time it is desirable to formulate things in a representation and coordinate independent way, as much as possible. There is a general algebraic framework accommodating these ideas [10], which however we do not employ in this paper. We do maintain representation independence and only pick special representations when the problem selects them for us. Coordinate independence is not emphasized and would only be realized by stating our conditions ABC in the form « there is a global coordinate system such that... ». It would be nice to have a more intrinsic (manifestly covariant) set of conditions.

## II. PRELIMINARIES

### A. Notation.

Our metric has signature  $(+, -, -, -)$ . Tensors are assumed to be  $C^\infty$  (for convenience) and components of tensors are always taken with respect to the standard coordinate system on  $\mathbb{R}^3$  or  $\mathbb{R}^4$ . We use Roman letters  $(i, j, k, \dots)$  for indices going from 1 to 3, Greek letters  $(\mu, \nu, \dots)$  for indices going from 0 to 3, and employ the summation convention. For any tensor  $\mathbf{t}$  (or object with indices) we define a norm  $|\mathbf{t}|$  to be the square root of the sum of the squares of the components. Thus if  $\mathbf{t} = t^\mu{}_\nu \partial/\partial x^\mu \otimes dx^\nu$  we have  $|\mathbf{t}| = \left( \sum_{\mu, \nu} |t^\mu{}_\nu|^2 \right)^{1/2}$  (some authors write  $|t^\mu{}_\nu|$  for  $|\mathbf{t}|$ ). If  $\mathbf{t}$  is a tensor field we define the  $L_p$ -norm  $\|\mathbf{t}\|_p = \left( \int |\mathbf{t}|^p \right)^{1/p}$  and  $\|\mathbf{t}\|_\infty = \sup |\mathbf{t}|$ .

Throughout  $C$  denotes an arbitrary constant which may vary from line to line.

### B. The metric.

We begin by introducing some general notation for our metric  $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$ . We define lapse and shift functions by

$$\alpha = (g^{00})^{-1/2} \quad \beta^j = -g^{j0}(g^{00})^{-1} \quad (2.1)$$

We let  $\gamma_{ij} = -g_{ij}$  be the positive definite 3-metric induced on the constant time hypersurfaces. One then has

$$g_{00} = \alpha^2 - \beta^i \beta_i \quad g_{0i} = -\beta_i \quad (2.2)$$

where  $\beta_i = \gamma_{ij}\beta^j$ . One can show that the inverse 3-metric  $\gamma^{ij}$  is related to the inverse 4-metric by

$$g^{ij} = -\gamma^{ij} + \beta^i\beta^j/\alpha^2 \tag{2.3}$$

Furthermore the determinants  $|g| = |\det \{g_{\mu\nu}\}|$  and  $\gamma = \det \{\gamma_{ij}\}$  are related by

$$|g|^{1/2} = \alpha\gamma^{1/2} \tag{2.4}$$

LEMMA II.1. — Under assumption A:

- a)  $k_1|\xi|^2 \leq \gamma^{ij}\xi_i\xi_j \leq k_2|\xi|^2$  for some constants  $k_1, k_2$
- b)  $k_2^{-1}|\mathbf{v}|^2 \leq \gamma_{ij}v^iv^j \leq k_1^{-1}|\mathbf{v}|^2$
- c)  $\beta_j, \beta^j$ , and  $g_{\mu\nu}$  are bounded
- d)  $\alpha, g_{00}, \gamma, |g|$  are bounded above and below (i. e. away from zero).

*Proof.* — a) follows directly from assumption A and (2.3). If  $\Gamma$  is the matrix  $\{\gamma_{ij}\}$ , then (a) says  $k_1|\xi|^2 \leq |\Gamma^{-1/2}\xi|^2 \leq k_2|\xi|^2$  and if we set  $\xi = \Gamma^{1/2}\mathbf{v}$  we get (b). Now  $\beta^i$  is bounded and by (b)  $\gamma_{ij}$  is bounded so  $\beta_i = \gamma_{ij}\beta^j$  is bounded. The boundedness of  $g_{\mu\nu}$  follows which completes (c). By (2.3) again we have for  $\hat{\beta} = \beta_i dx^i$

$$\beta_i\beta^i - (\beta_i\beta^i)^2/\alpha^2 \geq C_2|\hat{\beta}|^2 \geq C_2k_2^{-1}(\beta_i\beta^i)$$

Thus if  $\hat{\beta} \neq 0$  we have  $g_{00} = \alpha^2 - \beta_i\beta^i$  bounded below, and this is also true if  $\hat{\beta} = 0$ . Finally  $\gamma$  is bounded above since the  $\gamma_{ij}$  are, and since  $\gamma^{-1} = \det(\Gamma^{-1})$  is also bounded,  $\gamma$  is bounded below. Similarly for  $|g|$  (or use (2.4)) so (d) is proved.

LEMMA II.2. — Under assumption A, every light cone is contained in a fixed cone, i. e. there is a constant M such that if  $g_{\mu\nu}v^\mu v^\nu \geq 0$  then

$$M^2(v^0)^2 - \sum_j (v^j)^2 \geq 0.$$

*Proof.* —

$$\begin{aligned} g_{\mu\nu}v^\mu v^\nu &= (\alpha^2 - \beta_i\beta^i)(v^0)^2 - 2\beta_iv^iv^0 - \gamma_{ij}v^iv^j \\ &= \alpha^2(v^0)^2 - \gamma_{ij}(v^i - v^0\beta^i)(v^j - v^0\beta^j) \\ &\leq \alpha^2(v^0)^2 - k_2^{-1}|\tilde{\mathbf{v}} - v^0\boldsymbol{\beta}|^2 \end{aligned}$$

where  $\tilde{\mathbf{v}} = v^i\partial/\partial x^i$ . Thus if  $g_{\mu\nu}v^\mu v^\nu \geq 0$  we have  $|\tilde{\mathbf{v}} - v^0\boldsymbol{\beta}| \leq C|v^0|$  and hence  $|\tilde{\mathbf{v}}| \leq M|v^0|$ .

### C. The Klein-Gordon Equation.

Now we are ready to discuss the Klein-Gordon equation which can also be written as

$$(\square_g + m^2)u = (|g|^{-1/2}\partial_\mu g^{\mu\nu}|g|^{1/2}\partial_\nu + m^2)u = 0 \tag{2.5}$$

Associated with this equation is the form

$$\sigma(u_1, u_2) = \int_{x^0=t} (u_1 \partial_\mu u_2 - u_2 \partial_\mu u_1) n^\mu \gamma^{1/2} d^3x \quad (2.6)$$

where  $n_\mu = \delta_\mu^0 \alpha$  is the unit normal to the constant time hypersurfaces and  $n^\mu = g^{\mu 0} \alpha$ . If  $u_1, u_2$  are solutions then  $\sigma(u_1, u_2)$  is independent of  $t$ , as may be seen by using the divergence theorem.

We pose the Cauchy problem in terms of the variable.

$$p = \gamma^{1/2} n^\mu (\partial_\mu u) = |g|^{1/2} g^{0\mu} (\partial_\mu u)$$

which is the canonical momentum in a Hamiltonian formalism. The existence and uniqueness theorem is:

**THEOREM II.3.** — Under Assumption A, given  $f, p \in C_0^\infty(\mathbb{R}^3)$  there is a unique  $C^\infty$  solution  $u$  of the Klein-Gordon equation such that  $f = u(t_0, \cdot)$  and  $p = (\gamma^{1/2} n^\mu (\partial_\mu u))(t_0, \cdot)$ . The solution has compact support on every other constant time hypersurface.

This result is straightforward to prove using the standard method of energy estimates (e. g. [12]). We do not go into details of the proof, but will need to derive detailed energy estimates for other purposes. Alternatively one can note that Lemma II.2 implies that the metric is globally hyperbolic and that the surfaces  $x^0 = \text{constant}$  are Cauchy surfaces. Then a general theorem of Leray [5] [10] [18] gives existence and uniqueness for the Cauchy problem.

#### D. Energy Estimates.

The energy-momentum tensor for our problem is given by

$$T_{\mu\nu} = (\partial_\mu u)(\partial_\nu u) - \frac{1}{2} g_{\mu\nu} (g^{\alpha\beta} (\partial_\alpha u)(\partial_\beta u) - m^2 u^2) \quad (2.7)$$

Let  $X$  be the vector field  $\partial/\partial x^0$  so that  $X^\mu = \delta_\mu^0$ . We define the energy  $E(t) = E_t(u)$  by

$$\begin{aligned} E(t) &= \int_{x^0=t} T_{\mu\nu} X^\mu n^\nu \gamma^{1/2} d^3x \\ &= \int_{x^0=t} T_0^0 |g|^{1/2} d^3x \end{aligned} \quad (2.8)$$

The energy density  $T_0^0$  can be written as

$$T_0^0 = \frac{1}{2} (g^{00} (\partial_0 u)^2 - g^{ij} (\partial_i u)(\partial_j u) + m^2 u^2)$$

and is positive.

The following two theorems bound growth of energy (and related functions) in time. Quite similar bounds have been obtained by Choquet-Bruhat, Christodoulou, and Francaviglia [6]. Note that under assumption B, the next theorem gives that energy is bounded in time.

**THEOREM II.4.** — Let  $u$  be a solution of  $(\square_g + m^2)u = 0$  as in Theorem II.3. Then the energy  $E(t)$  satisfies for some  $C$

$$E(t) \leq E(0) \exp \left( C \int_0^t \|(\partial_0 \mathbf{g})(s, \cdot)\|_\infty ds \right)$$

*Proof.* — By the divergence theorem

$$E(t) - E(s) = \int_{s < x^0 < t} \nabla_\mu (\mathbf{T}^{\mu\nu} X_\nu) |g|^{1/2} d^4x$$

Thus  $E(t)$  is differentiable and

$$E'(t) = \int_{x^0=t} \nabla_\mu (\mathbf{T}^{\mu\nu} X_\nu) |g|^{1/2} d^3x$$

The derivative is evaluated using  $\nabla_\mu \mathbf{T}^{\mu\nu} = 0$  and

$$\nabla_{(\mu} X_{\nu)} = \frac{1}{2} (\mathcal{L}_{\mathbf{X}} \mathbf{g})_{\mu\nu} = \frac{1}{2} \partial_0 g_{\mu\nu}$$

and so

$$E'(t) = \frac{1}{2} \int_{x^0=t} \mathbf{T}^{\mu\nu} (\partial_0 g_{\mu\nu}) |g|^{1/2} d^3x$$

We now estimate  $|\mathbf{T}^{\mu\nu} \partial_0 g_{\mu\nu}| \leq |\mathbf{T}| |\partial_0 \mathbf{g}|$  and note that by Lemma II.1 we have  $|\mathbf{T}^{\mu\nu}| \leq C \mathbf{T}_0^0$  and hence  $|\mathbf{T}| \leq C \mathbf{T}_0^0$ . Thus we have

$$E'(t) \leq C \|(\partial_0 \mathbf{g})(t, \cdot)\|_\infty E(t)$$

If we put  $f(t) = C \|(\partial_0 \mathbf{g})(t, \cdot)\|_\infty$ . Then  $E' \leq fE$ . Then

$$F(t) = \exp \left( - \int_0^t f(t') dt' \right) E(t) \text{ satisfies}$$

$$F' = (E' - fE) \exp \left( - \int_0^t f dt \right) \leq 0$$

Therefore  $F(t) \leq F(0)$  which is our result.

The next result is a generalization of this theorem which we shall need in Chapter IV. It concerns a collection of scalar fields  $u_A$  which satisfy a coupled set of Klein-Gordon equations. (Later,  $u_A$  will be a derivative  $\partial_\mu u$  of a solution of the Klein-Gordon equation).

**THEOREM II.5.** — Given  $M_{AB}, N_{AB} \in C^\infty(\mathbb{R}^4)$ , let  $u_A$  solve :

$$(\square_g + m^2)u_A - M_{AB}^{\mu} \partial_\mu u_B - N_{AB} u_B = 0$$



let  $E_A(t) = E_t(u_A)$  be the energy (2.8) and let  $E(t) = \sum_A E_A(t)$ . Then for some constant  $C$

$$E(t) \leq E(0) \exp \left( C \int_0^t [\|(\partial_0 g)(s, \cdot)\|_\infty + \|\mathbf{M}(s, \cdot)\|_\infty + \|\mathbf{N}(s, \cdot)\|_\infty] ds \right)$$

*Proof.* — We analyze  $E_A$  as in the previous theorem, but now we replace  $\nabla_\mu T^{\mu\nu} = 0$  by

$$\nabla_\mu (T^{\mu\nu}(u_A)) = (\partial^\nu u_A)(\square_g + m^2)u_A$$

This yields

$$E'_A(t) = \int_{x^0=t} [(\partial_0 u_A)(M_{AB}^\mu \partial_\mu u_B + N_{AB} u_B) + T^{\mu\nu} \subseteq (u_A) \partial_0 g_{\mu\nu}] |g|^{1/2} d^3x$$

We write

$$|(\partial_0 u_A) M_{AB}^\mu (\partial_\mu u_B)| \leq \|\mathbf{M}\| |\partial u|^2$$

and use Lemma II.1 to obtain

$$|\partial u|^2 = \sum_{\mu, B} |\partial_\mu u_B|^2 \leq C \sum_B T_0^0(u_B)$$

The other terms are treated similarly and we have

$$E'_A(t) \leq C(\|\mathbf{M}(s, \cdot)\|_\infty + \|\mathbf{N}(s, \cdot)\|_\infty + \|(\partial_0 g)(s, \cdot)\|_\infty) E(t)$$

Then  $E'(t)$  satisfies the same type of bound and this gives the result.

### E. The Hamiltonian Formalism.

Scattering is conveniently discussed in a first order formalism which we now develop (cf. [15]). For any solution  $u$  of the Klein-Gordon equation as in Theorem II.3 we define  $F(t) \in C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$  by

$$F(t) = (f(t), p(t)) = (u(t, \cdot), (|g|^{1/2} g^{0\mu} \partial_\mu u)(t, \cdot))$$

Then we have that

$$\frac{d}{dt} F(t) = -H(t)F(t) \tag{2.9}$$

where

$$H(t) = \begin{bmatrix} -\beta^i \partial_i & -\alpha \gamma^{-1/2} \\ -\partial_i (\alpha \gamma^{1/2} \gamma^{ij} \partial_j \cdot) + \alpha \gamma^{1/2} m^2 & -\partial_i (\beta^i \cdot) \end{bmatrix}$$

Conversely any solution of this equation has a first entry which satisfies the Klein-Gordon equation. Existence and uniqueness of solutions for (3.1) follows from Theorem II.3

For any  $F$  let  $F(t)$  be the solution of (2.9) such  $F(s) = F$ . The time evo-

lution operator sending  $F = F(s)$  to  $F(t)$  is denoted  $\mathcal{T}(t, s)$ . Then  $\mathcal{T}(t, s)$  maps  $C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$  to itself, is linear, and satisfies

$$\begin{aligned} \mathcal{T}(t, t) &= I \\ \frac{d}{dt} \mathcal{T}(t, s)F &= -H(t)\mathcal{T}(t, s)F \quad (\text{pointwise}) \end{aligned}$$

Furthermore by the uniqueness of solutions

$$\begin{aligned} \mathcal{T}(t, s)^{-1} &= \mathcal{T}(s, t) \\ \mathcal{T}(t, s)\mathcal{T}(s, u) &= \mathcal{T}(t, u) \end{aligned} \tag{2.10}$$

A special case of the above is the Minkowski metric for which the Hamiltonian is

$$H_0 = \begin{bmatrix} 0 & -I \\ -\Delta + m^2 & 0 \end{bmatrix}$$

The evolution operator in this case is denoted  $\mathcal{T}_0(t, s)$  or  $\mathcal{T}_0(t - s)$ .

We now define the bilinear form  $\sigma$  on  $C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$  by defining for  $F_1 = (f_1, p_1)$  and  $F_2 = (f_2, p_2)$

$$\sigma(F_1, F_2) = \int (f_1 p_2 - p_1 f_2) d^3x$$

This is the translation of our earlier definition (2.6) into the first order formalism and so

$$\sigma(\mathcal{T}(t, s)F_1, \mathcal{T}(t, s)F_2) = \sigma(F_1, F_2)$$

Note that  $\sigma$  is a linear symplectic form (i. e. skew-symmetric and weakly non-degenerate). Operators such as  $\mathcal{T}(t, s)$  which leave  $\sigma$  invariant are called symplectic.

We want to extend our dynamics to a Hilbert space setting. We define our Hilbert space  $\mathcal{A}$  to be the completion of  $C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$  in the free energy norm  $\|F\|$  defined for  $F = (f, p)$  by

$$\|F\|^2 = \frac{1}{2} \int (p^2 + \delta^{ij}(\partial_i f)(\partial_j f) + m^2 f^2) d^3x$$

Alternatively  $\mathcal{A}$  is the Sobolev space  $H^1(\mathbb{R}^3) \oplus L_2(\mathbb{R}^3)$  with this norm. Note that the associated inner product satisfies  $(F, G) = \sigma(F, H_0 G)$ .

We want to compare this norm with the energy of solutions as defined previously. This energy can be written  $E(t) = \|F(t)\|_t^2$  where

$$\begin{aligned} \|F\|_t^2 &= \int_{x^0=t} \frac{1}{2} (\gamma^{-1} p^2 + \gamma^{ij}(\partial_i f)(\partial_j f) + m^2 f^2) |g|^{1/2} d^3x \\ &+ \int_{x^0=t} p(\beta^k \partial_k f) d^3x \end{aligned}$$

(The associated inner product satisfies  $(F, G)_t = \sigma(F, H(t)G)$  here as well [15]). The next lemma says that these norms are uniformly equivalent.

LEMMA II. 6. — Under condition A there are positive constants  $K_1, K_2$  such that

$$K_1 \| F \|_t \leq \| F \| \leq K_2 \| F \|_t$$

*Proof.* — The first inequality follows easily by Lemma 2.1. For the second we note that by condition A,  $\gamma^{ij}\xi_i\xi_j - (\beta^i\xi_i)^2\alpha^{-2} \geq 2C|\xi|^2$  for some C. We choose  $a > 1$  so that  $-(a^2 - 1)(\beta^i\xi_i)^2\alpha^{-2} \geq -C|\xi|^2$ . Adding these inequalities gives

$$\gamma^{ij}\xi_i\xi_j - a^2(\beta^i\xi_i)^2\alpha^{-2} \geq C|\xi|^2 \tag{2.11}$$

Now we write

$$\begin{aligned} p\beta^k\partial_k f &= |g|^{1/2}(a^{-1}\gamma^{-1/2}p)(a\alpha^{-1}\beta^k\partial_k f) \\ &\geq -\frac{1}{2}|g|^{1/2}(a^{-2}\gamma^{-1}p^2 + a^2\alpha^{-2}(\beta^k\partial_k f)^2) \end{aligned}$$

which combined with (2.11) gives

$$\begin{aligned} \| F \|_t^2 &\geq \frac{1}{2} \int ((1 - a^{-2})\gamma^{-1}p^2 + C\delta^{ij}(\partial_i f)(\partial_j f) + m^2 f^2) |g|^{1/2} d^3x \\ &\geq K_1^2 \| F \|^2 \quad \blacksquare \end{aligned}$$

As a consequence of this lemma and Theorem II.4 we have for

$$\begin{aligned} F &\in C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3) \\ \| \mathcal{T}(t, 0)F \| &\leq C_1 \| F \| \exp\left(\frac{C_2}{2} \int_0^t \| (\partial_0 g)(s, \cdot) \|_\infty ds\right) \end{aligned} \tag{2.12}$$

Then  $\mathcal{T}(t, 0)$  extends to a bounded operator on  $\mathcal{A}$  with the same bound. Similarly we extend  $\mathcal{T}(t, s)$  to all of  $\mathcal{A}$ . The relations (2.10) still hold. Furthermore  $\sigma$  is a continuous form on  $\mathcal{A} \times \mathcal{A}$  and the extended  $\mathcal{T}(t, s)$  is still symplectic. Finally (2.12) gives us uniform boundedness of  $\mathcal{T}(t, 0)$ :

THEOREM II. 7. — Under conditions A, B,  $\| \mathcal{T}(t, 0) \|$  is bounded in time.

### III. CLASSICAL SCATTERING

We now pose the scattering problem. The first question is to find data with given free asymptotic behavior. That is for either  $t \rightarrow \pm \infty$ , if we are given  $F \in \mathcal{A}$  we want to find  $G \in \mathcal{A}$  such that

$$\lim_{t \rightarrow \pm \infty} \| \mathcal{T}(t, 0)G - \mathcal{T}_0(t, 0)F \| = 0 \tag{3.1}$$

Assuming conditions A, B so that  $\mathcal{T}(t, 0)$  is bounded this is solved by showing that the wave operators

$$\Omega^\pm F = \lim_{t \rightarrow \pm\infty} \mathcal{T}(0, t)\mathcal{T}_0(t, 0)F$$

exist, and taking  $G = \Omega^\pm F$ .

The other half of the scattering problem is to find free asymptotic behavior for given data. That is given  $G \in \mathcal{A}$ , find  $F \in \mathcal{A}$  so that (3.1) holds. This is solved by showing that

$$\tilde{\Omega}^\pm G = \lim_{t \rightarrow \pm\infty} \mathcal{T}_0(0, t)\mathcal{T}(t, 0)G$$

exists and taking  $F = \tilde{\Omega}^\pm G$ .

We define the domains  $\text{Dom } \Omega^\pm, \text{Dom } \tilde{\Omega}^\pm$  to be those vectors for which the limits exist. These are subspaces of  $\mathcal{A}$ .

**PROPOSITION III. 1.** — Under assumptions A, B

- a)  $\text{Dom } \Omega^\pm, \text{Dom } \tilde{\Omega}^\pm$  are closed
- b)  $\text{Ran } \Omega^\pm = \text{Dom } \tilde{\Omega}^\pm, \text{Ran } \tilde{\Omega}^\pm = \text{Dom } \Omega^\pm$
- c)  $\Omega^\pm \tilde{\Omega}^\pm = I$  on  $\text{Dom } \tilde{\Omega}^\pm, \tilde{\Omega}^\pm \Omega^\pm = I$  on  $\text{Dom } \Omega^\pm$
- d)  $\Omega^\pm, \tilde{\Omega}^\pm$  are symplectic.

*Proof.* — a) follows easily using the fact that  $\|\mathcal{T}(t, 0)\|, \|\mathcal{T}_0(t, 0)\|$  are bounded. For (b.) let  $\Omega(t) = \mathcal{T}(0, t)\mathcal{T}_0(t, 0)$  and  $\tilde{\Omega}(t) = \mathcal{T}_0(0, t)\mathcal{T}(t, 0)$ . If  $G \in \text{D}(\tilde{\Omega}^\pm)$  then we have

$$\begin{aligned} \|\Omega(t)\tilde{\Omega}^\pm G - G\| &= \|\Omega(t)(\tilde{\Omega}^\pm - \tilde{\Omega}(t))G\| \\ &\leq C \|(\tilde{\Omega}^\pm - \tilde{\Omega}(t))G\| \quad (\text{by Theorem II. 7}) \\ &\rightarrow 0 \quad \text{as } t \rightarrow \pm\infty \end{aligned}$$

Thus  $\text{Ran } \tilde{\Omega}^\pm \subset \text{Dom } \Omega^\pm$  and  $\Omega^\pm \tilde{\Omega}^\pm = I$  on  $\text{Dom } (\tilde{\Omega}^\pm)$  and hence  $\text{Dom } \tilde{\Omega}^\pm \subset \text{Ran } \Omega^\pm$ . The same statements hold with  $\Omega^\pm, \tilde{\Omega}^\pm$  interchanged. This completes the proof of (b.), (c.), and (d.) follows by the continuity of  $\sigma$ . ■

The next theorem, our main result, shows that  $\text{Dom } \Omega^\pm = \mathcal{A}$ . Statements about  $\text{Dom } \tilde{\Omega}^\pm = \text{Ran } \Omega^\pm$  are more difficult and we only consider a special case in the next chapter. For now we note that if  $\text{Ran } \Omega^+ = \text{Ran } \Omega^-$  (weak asymptotic completeness) then we may define the classical scattering operator

$$S = \tilde{\Omega}^+ \Omega^-$$

which relates asymptotic behavior in the past and future. The operator S is bounded and symplectic, as is its inverse

$$S^{-1} = \tilde{\Omega}^- \Omega^+$$

**THEOREM III. 2.** — Under assumptions A, B, C the limits

$$\Omega^\pm F = \lim_{t \rightarrow \pm\infty} \mathcal{T}(0, t)\mathcal{T}_0(t, 0)F$$

exist for all  $F \in \mathcal{A}$ .

*Proof.* — We employ the usual Cook formalism for scattering theory (e. g. [23]). Since  $\text{Dom } \Omega^\pm$  is closed it is sufficient to prove convergence on a dense domain and we choose  $C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$ . For  $F, G$  in this domain we compute for  $\Omega(t) = \mathcal{T}(0, t)\mathcal{T}_0(t, 0)$

$$\frac{d}{dt} \sigma(G, \Omega(t)F) = \sigma(G, \mathcal{T}(0, t)(H(t) - H_0)\mathcal{T}_0(t, 0)F)$$

(Here we use that  $\mathcal{T}(t, 0)$  is symplectic and  $H(t)$  is skew-symplectic. The differentiation under the integral sign is easily justified.) If we now integrate we obtain the identity

$$(\Omega(t) - \Omega(t'))F = \int_{t'}^t \mathcal{T}(0, s)(H(s) - H_0)\mathcal{T}_0(s, 0)F ds$$

This holds in the weak symplectic sense, i. e. applying  $\sigma(G, \cdot)$  (under the integral) gives an identity. But since  $(G, \cdot) = \sigma(G, H_0 \cdot)$  we also have that the identity holds weakly for  $G$  in a dense set and hence weakly in  $\mathcal{A}$ . The identity thus holds as stated with the integral interpreted weakly.

To show that  $\Omega(t)F$  has a strong limit as  $t \rightarrow \pm \infty$  it therefore suffices to show that

$$\int_{-\infty}^{\infty} \| (H(t) - H_0)\mathcal{T}_0(t, 0)F \| dt < \infty \tag{3.2}$$

where we again use that  $\| \mathcal{T}(0, t) \|$  is bounded.

Writing this out with  $(f(t), p(t)) = \mathcal{T}_0(t, 0)F$  we can dominate (3.2) by a sum of terms of the form  $\int \| \chi(t, \cdot) \|_2 dt$  where

$$\begin{aligned} \chi = & \beta^i \partial_i f, & \partial_k(\beta^i \partial_i f), & (1 - \alpha\gamma^{-1/2})p, \\ & \partial_k(1 - \alpha\gamma^{-1/2})p, & m^2(1 - \alpha\gamma^{1/2})f, \\ & \partial_i(\alpha\gamma^{1/2}\gamma^{ij} - \delta^{ij})\partial_j f, & \partial_i(\beta^i p). \end{aligned}$$

Expanding out the derivatives we may dominate by a sum of terms of the form

$$\int \| Z(t, \cdot)u_0(t, \cdot) \|_2 dt \tag{3.3}$$

where

$$Z = \beta^i, \quad \partial_k \beta^i, \quad (1 - \alpha\gamma^{-1/2}), \quad \partial_k(1 - \alpha\gamma^{-1/2}), \\ (1 - \alpha\gamma^{1/2}), \quad \alpha\gamma^{1/2}\gamma^{ij} - \delta^{ij}, \quad \partial_i(\alpha\gamma^{1/2}\gamma^{ij})$$

and where

$$u_0 = f, p, \partial_k f, \partial_k p, \partial_i \partial_j f$$

Note that all the  $u_0$  are derivatives of  $f$  and hence are solutions of the free Klein-Gordon equation.

To estimate (3.3) we choose  $p$  from condition C and have by Hölder's inequality

$$\|Z(t, \cdot)u_0(t, \cdot)\|_2 \leq \|Z(t, \cdot)\|_p \|u_0(t, \cdot)\|_q, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}$$

By the following lemmas we have

$$\|Z(t, \cdot)\|_p \leq C |t|^{3/p-1-\varepsilon}, \quad |t| \geq | \tag{3.4}$$

Furthermore the free solutions  $u_0$  satisfy

$$\|u_0(t, \cdot)\|_q \leq C |t|^{-3/2+3/q}, \quad |t| \geq | \tag{3.5}$$

For  $q = \infty$  this result is well known [1] [17] [20] and the result for  $2 \leq q < \infty$  follows since supports grow like  $0(t^3)$ . Combining the above estimates gives

$$\|Z(t, \cdot)u_0(t, \cdot)\|_2 \leq C |t|^{-1-\varepsilon}$$

and hence convergence of the integral.

**LEMMA III.3.** — Under assumption C the following functions  $Z$  satisfy (3.4):  $\alpha - 1$ ,  $\beta^i$ ,  $\gamma^{ij} - \delta^{ij}$ ,  $\gamma^{-1} - 1$  as well as  $\partial_k \alpha$ ,  $\partial_k \beta^i$ ,  $\partial_k \gamma^{ij}$ ,  $\partial_k \gamma^{-1}$ .

*Proof.* — C says that  $g^{\mu\nu} - \eta^{\mu\nu}$  satisfies (3.4). Then

$$|\alpha - 1| \leq C |\alpha^2 - 1| \leq C' |g^{00} - 1|$$

does also. The same estimate for  $\beta^j$  and  $\gamma^{ij} - \delta^{ij}$  follows easily. We also have  $|\gamma^{-1} - 1| \leq C \max_{ij} |\gamma^{ij} - \delta^{ij}|$  which gives (3.4) again.

C also says that  $\partial_k g^{\mu\nu}$  satisfies (3.4) and this gives the second set of estimates. For  $\partial_k \gamma^{-1}$  we use the identity  $\partial \gamma^{-1} / \partial \gamma^{ij} = \gamma^{-1} \gamma_{ij}$  so that

$$\partial_k \gamma^{-1} = (\partial_k \gamma^{ij}) \gamma^{-1} \gamma_{ij}$$

**LEMMA III.4.** — Under assumption C the following functions  $Z$  satisfy (3.4):

$$(\alpha \gamma^{1/2} - 1), \quad (1 - \alpha \gamma^{-1/2}), \quad \partial_k (1 - \alpha \gamma^{-1/2}), \quad \partial_i (\alpha \gamma^{1/2} \gamma^{ij}), \quad (\alpha \gamma^{1/2} \gamma^{ij} - \delta^{ij})$$

*Proof.* — These follow by straightforward manipulation of the results of the previous lemma.

#### IV. TRANSIENT GRAVITATIONAL FIELDS

In this section we specialize to a class of transient space-times and show that  $\tilde{\Omega}^\pm$  also exists on all of  $\mathcal{A}$ . Then we have asymptotic completeness:  $\text{Ran } \Omega^\pm = \mathcal{A}$ , i. e. all states are scattering states.

This class of space-times is obtained by strengthening condition B to

control more derivatives of the metric, and by restricting condition C to  $p = \infty$ . Explicitly we adjoin to A the conditions

$$\begin{aligned} \text{B') } i) \quad & \int \|(\partial_\alpha g^{\mu\nu})(t, \cdot)\|_\infty dt < \infty \quad \forall \alpha, \mu, \nu \\ ii) \quad & \int \|(\partial_\alpha \Gamma^\lambda)(t, \cdot)\|_\infty dt < \infty \quad \forall \alpha, \lambda \end{aligned}$$

where 
$$\Gamma^\lambda = g^{\mu\nu} \Gamma^\lambda_{\mu\nu}$$

C') There are positive constants C,  $\varepsilon$  such that for  $|t| \geq 1$

$$\begin{aligned} i) \quad & \|g^{\mu\nu}(t, \cdot) - \eta^{\mu\nu}\|_\infty \leq C |t|^{-1-\varepsilon} \\ ii) \quad & \|\partial_k(g^{\mu\nu}(t, \cdot) - \eta^{\mu\nu})\|_\infty \leq C |t|^{-1-\varepsilon} \end{aligned}$$

Condition C' picks out metrics which, although not necessarily decaying at spacelike infinity become Minkowskian at timelike infinity, hence the name transient. Note that since  $\partial_\alpha g_{\mu\nu} = -(\partial_\alpha g^{\rho\sigma})g_{\rho\mu}g_{\sigma\nu}$  conditions A and B' imply that  $\int \|(\partial_\alpha g_{\mu\nu})(\cdot, t)\|_\infty dt < \infty$  and hence that condition B holds. (One can also show that conditions B and C' (ii) together imply B' (i)).

THEOREM IV.1. — Under assumptions A, B', C' the limits

$$\tilde{\Omega}^\pm F = \lim_{t \rightarrow \pm\infty} \mathcal{T}_0(0, t) \mathcal{T}(t, 0) F$$

exist for all  $F \in \mathcal{A}$ .

*Proof.* — We proceed as in the proof of Theorem III.2. Since  $\mathcal{T}_0(t, 0)$  is norm preserving, it is sufficient to show

$$\int \| (H(t) - H_0) \mathcal{T}(t, 0) F \| dt < \infty$$

If we let  $(f(t), p(t)) = \mathcal{T}(t, 0) F$  then this integral can be bounded by a sum of terms of the form

$$\int \| Z(t, \cdot) v(t, \cdot) \|_2 dt$$

where Z is exactly as in Theorem III.2 and  $v$  is one of  $f, p, \partial_i f, \partial_i p, \partial_i \partial_j f$ . Now we estimate

$$\| Z(t, \cdot) v(t, \cdot) \|_2 \leq \| Z(t, \cdot) \|_\infty \| v(t, \cdot) \|_2$$

By condition C' we have as before that  $\| Z(t, \cdot) \|_\infty \leq C |t|^{-1-\varepsilon}$ , so it suffices to show that  $\| v(t, \cdot) \|_2$  is bounded to complete the proof. The functions  $v$  are derivatives of solutions of the full Klein-Gordon equation. Since  $\| \mathcal{T}(t, 0) F \|$  is bounded we have that  $\| f \|_2, \| \partial_k f \|_2$  and  $\| p \|_2$  are bounded. In the next lemma we show that second derivatives of solutions

have bounded  $L_2$  norms. This gives  $\|\partial_i \partial_j f\|_2$  bounded and also  $\|\partial_k p\|_2$  since

$$\begin{aligned} \|\partial_k p\|_2 &= \|\partial_k(|g|^{1/2} g^{0\nu} \partial_\nu f)\|_2 \\ &\leq \| |g|^{1/2} g^{0\nu} \|_\infty \|\partial_k \partial_\nu f\|_2 + \|\partial_k(|g|^{1/2} g^{0\nu})\|_\infty \|\partial_\nu f\|_2 \end{aligned}$$

and we may bound the  $L_\infty$  norms using Lemmas II.1, III.3.

**LEMMA IV.2.** — Under assumptions A, B' if  $(\square_g + m^2)u = 0$  then  $\|(\partial_\mu \partial_\nu u)(t, \cdot)\|_2$  is bounded in  $t$ .

*Proof.* — Applying  $\partial_\alpha$  to

$$(g^{\mu\nu} \partial_\mu \partial_\nu - \Gamma^\lambda \partial_\lambda + m^2)u = 0$$

one obtains

$$(\square_g + m^2) \partial_\alpha u + (\partial_\alpha g^{\mu\nu}) \partial_\mu \partial_\nu u - (\partial_\alpha \Gamma^\lambda) (\partial_\lambda u) = 0$$

We regard this as a system of equations for scalar fields  $\partial_\alpha u$ . By condition B' and Theorem II.5 we conclude that the energy  $E_t(\partial_\alpha u)$  given by (2.8) is bounded in time. By condition A we have for any  $w$ ,  $\|(\partial_\beta w)(\cdot, t)\|_2^2 \leq CE_t(w)$ . Thus  $\|\partial_\beta(\partial_\alpha u)\|_2^2 \leq CE_t(\partial_\alpha u)$  and so is bounded as required.

## V. QUANTUM SCATTERING

### A. Algebraic Formulation.

We now study the quantum scattering problem. Our formalism is influenced by the work of I. Segal (e. g. [25]). The starting point is a representation  $W$  of the CCR over the real Hilbert space  $\mathcal{A}$  with symplectic form  $\sigma$ . That is we have a function  $W$  from  $F \in \mathcal{A}$  to unitary operators  $W(F)$  on some complex Hilbert space such that

$$W(F_1)W(F_2) = e^{-i\sigma(F_1, F_2)/2} W(F_1 + F_2)$$

We assume  $t \rightarrow W(tF)$  is strongly continuous and then  $W(F) = e^{i\sigma(\Phi, F)}$  for some self-adjoint operator  $\sigma(\Phi, F)$  which satisfies

$$[\sigma(\Phi, F_1), \sigma(\Phi, F_2)] = i\sigma(F_1, F_2)$$

on a suitable domain. Defining  $\varphi(f) = \sigma(\Phi, (0, f))$  and  $\pi(f) = -\sigma(\Phi, (f, 0))$  we have  $\sigma(\Phi, F) = \varphi(p) - \pi(f)$  when  $F = (f, p)$ . These satisfy the usual  $[\varphi(f_1), \pi(f_2)] = i(f_1, f_2)$ .

We also consider the  $C^*$ -algebra  $\mathfrak{A}$  generated by the operators  $W(F)$  of some representation. Any two pairs  $(W_1, \mathfrak{A}_1)$  and  $(W_2, \mathfrak{A}_2)$  are equivalent in the sense that there is a unique isomorphism  $i: \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$  such that  $i[W_1(F)] = W_2(F)$  [4] [28].

The operator  $W(F)$  corresponds to the field operator at time zero.



The dynamics are specified by defining a field at time  $t$  by

$$W_t(F) = W(\mathcal{T}(0, t)F) \quad F \in \mathcal{A}$$

Since  $\mathcal{T}(0, t)$  is symplectic,  $W_t(F)$  defines a representation of the CCR with generator  $\sigma(\Phi_t, F) = \sigma(\Phi, \mathcal{T}(0, t)F)$ . Thus formally  $\Phi_t = \mathcal{T}(t, 0)\Phi$  just as for the classical problem. Since  $\mathcal{T}(0, t)$  is invertible,  $W_t(F)$  generates the same algebra as  $W(F)$  and by the general equivalence result we have a time evolution automorphism  $\alpha(t, 0)$  on  $\mathfrak{A}$  such that  $\alpha(t, 0)W(F) = W_t(F)$ .

Assuming the existence of  $\Omega^\pm$  as in Theorem III.2 we define asymptotic fields by

$$W_{\text{out/in}}(F) = W(\Omega^\pm F) \quad F \in \mathcal{A}$$

These are given a free time evolution:

$$W_{\text{out/in},t}(F) = W_{\text{out/in}}(\mathcal{T}_0(0, t)F)$$

If  $W$  is strongly continuous in  $\mathcal{A}$ , these give the asymptotic behavior of  $W_t$  in the sense that for any vector  $\psi$  in the representation space

$$\begin{aligned} \| [W_t(\mathcal{T}_0(t, 0)F) - W_{\text{out},t}(\mathcal{T}_0(t, 0)F)]\psi \| &= \| [W_t(\mathcal{T}_0(t, 0)F) - W_{\text{out}}(F)]\psi \| \\ &= \| W((\mathcal{T}(0, t)\mathcal{T}_0(t, 0) - \Omega^+)F)\psi \| \\ &\rightarrow 0 \quad \text{as } t \rightarrow \infty \end{aligned}$$

For such representations we may also characterize  $W_{\text{out/in}}(F)$  by

$$W_{\text{out/in}}(F) = s\text{-}\lim_{t \rightarrow \pm\infty} \alpha(t, 0)\alpha_0(0, t)W(F)$$

The maps  $W_{\text{out/in}}$  give new representations of the CCR since the  $\Omega^\pm$  are symplectic. On the other hand the elements  $W_{\text{out/in}}(F)$ , which have generators  $\sigma(\Phi_{\text{out/in}}, F) = \sigma(\Phi, \Omega^\pm F)$ , generate subalgebras  $\mathfrak{A}_{\text{out/in}}$  of the original algebra  $\mathfrak{A}$ . Elements of  $\mathfrak{A}_{\text{in/out}}$  roughly correspond to observables which can be measured at spatial infinity in the distant past or future. By the general equivalence result there is a scattering isomorphism  $\mathcal{J}: \mathfrak{A}_{\text{in}} \rightarrow \mathfrak{A}_{\text{out}}$  such that  $\mathcal{J}[W_{\text{in}}(F)] = W_{\text{out}}(F)$ . If we have  $\text{Ran } \Omega^+ = \text{Ran } \Omega^-$  then  $\mathfrak{A}_{\text{in}} = \mathfrak{A}_{\text{out}}$  and  $\mathcal{J}$  is an automorphism on this algebra. We also have that  $W_{\text{out}}(F) = W_{\text{in}}(S^{-1}F)$ , and so  $\sigma(\Phi_{\text{out}}, F) = \sigma(\Phi_{\text{in}}, S^{-1}F)$ , or formally  $\Phi_{\text{out}} = S\Phi_{\text{in}}$ .

Note that all of the above constructions are representation independent. They provide an algebraic description of scattering phenomena. To make contact with physics we must next say something about states and their interpretation.

### B. States.

1) We review some well-known material about states on the CCR. Given any representation  $(W, \mathfrak{A})$  and a vector  $\psi$  which is cyclic for the  $W(F)$

we define a state  $\omega(\cdot) = (\psi, [\cdot]\psi)$  on  $\mathfrak{A}$ . Then  $\mu(F) = \omega(W(F))$  satisfies

- a)  $\mu(0) = 1$
- b)  $t \rightarrow \mu(tF)$  is continuous
- c) for any  $c_1, \dots, c_n \in \mathbb{C}, F_1, \dots, F_n \in \mathcal{A}$

$$\sum_{ij} \mu(F_i - F_j) e^{-i/2\sigma(F_i, F_j)} \bar{c}_i c_j \geq 0$$

Conversely given a generating function  $\mu(F)$  satisfying these conditions there is a representation  $(W, \mathfrak{A})$  with cyclic vector  $\psi$  giving rise to it, and the representation is unique up to unitary equivalence.

2) One may define the free vacuum  $\omega_0$  on the CCR by the generating functional

$$\omega_0(W(F)) = e^{-1/2\|K_0 F\|^2}$$

where  $K_0: \mathcal{A} \rightarrow L_2(\mathbb{R}^3, \mathbb{C})$  is defined by

$$K_0 F = \frac{1}{\sqrt{2}} (B_0^{1/2} f + iB_0^{-1/2} p)$$

$$B_0 = (-\Delta + m^2)^{1/2}$$

Concretely one may begin with Fock space over  $L_2(\mathbb{R}^3, \mathbb{C})$  with vacuum  $\Omega_0 = (1, 0, 0, \dots)$  and define  $\sigma(\Phi, F)$  to be the self adjoint extension of

$$\sigma(\Phi, F) = -i(a^*(K_0 F) - (a^*(K_0 F))^*)$$

where  $a^*$  is the usual creation operator. Then  $\omega_0 = (\Omega_0, [\cdot]\Omega_0)$  and  $W_0(F) = e^{i\sigma(\Phi, F)}$  give the above generating functional.  $W_0$  is the usual free representation of the CCR's.

As is well-known  $\omega_0$  is stationary under the free time evolution  $\alpha_0(t)$  and in fact under an associated representation of the full Poincaré group by algebra automorphisms. Furthermore  $\alpha_0(t)$  is implemented by a unitary group with positive energy. Hence  $\omega_0$  is interpreted as a no particle state for this dynamics. Similarly vectors  $\psi_h = (0, h, 0, 0, \dots)$  in Fock space give rise to states  $\omega_{0,h} = (\psi_h, [\cdot]\psi_h)$  on  $\mathfrak{A}$  which transform irreducibly under these Poincaré transformations and have positive energy. The  $\omega_{0,h}$  are then interpreted as one-particle states. Higher particle states are treated similarly.

3) We are now ready to study states on the CCR and classify them with respect to their behavior under the full dynamics. This is difficult to do directly and instead we classify states by their behavior on the asymptotic form for  $W_t(F)$  which is the free field  $W_{in/out,t}(F)$ . Thus a state  $\omega_{in}$  on  $\mathfrak{A}$  is interpreted as containing no particles in the distant past if

$$\omega_{in}(W_{in}(F)) = \omega_0(W_0(F)),$$

a state  $\omega_{in,h}$  is interpreted as having a single asymptotic particle in the distant

past if  $\omega_{in,h}(W_{in}(F)) = \omega_{0,h}(W_0(F))$ , and so forth. These identities will also hold at any time  $t$  (by  $W_{0,t}(F) = W_0(\mathcal{T}_0(0, t)F)$  and  $W_{in,t}(F) = W_{in}(\mathcal{T}_0(0, t)F)$ ).

Note that the definitions only determine these states on  $\mathfrak{A}_{in/out}$  (where existence is trivial). In general there will be many extensions to the full algebra. Different extensions presumably correspond to different occupations of bound states.

### C. The Quantum Scattering Operator.

We now suppose that  $\text{Ran } \Omega^+ = \text{Ran } \Omega^-$  so  $S$  exists. We restrict attention to the algebra  $\mathfrak{A}_{in} = \mathfrak{A}_{out}$  which is generated by  $W_{in}(F)$  or  $W_{out}(F) = W_{in}(S^{-1}F)$ . Suppose further we choose the free representation for  $W_{in}(F)$  (which has the physical interpretation). We ask whether the scattering automorphism  $\mathcal{S}: W_{in}(F) \rightarrow W_{out}(F)$  is unitarily implemented. Is there a unitary operator  $\mathcal{S}$  so that

$$\mathcal{S}^{-1}W_{in}(F)\mathcal{S} = W_{out}(F) ?$$

If so  $\mathcal{S}$  gives detailed information about scattering.

To find sufficient conditions for implementability we refer to the free representation of the CCR over  $L_2(\mathbb{R}^3, \mathbb{C})$  regarded as a real symplectic space with symplectic form  $2 \text{Im}(\dots)$ . As is well-known this consists of unitary operators  $\widehat{W}(\psi)$ ,  $\psi \in L_2(\mathbb{R}^3, \mathbb{C})$ , (e. g. on Fock space) such that

$$\widehat{W}(\psi_1)\widehat{W}(\psi_2) = e^{-i\text{Im}(\psi_1, \psi_2)}\widehat{W}(\psi_1 + \psi_2)$$

and a cyclic vector  $\widehat{\Omega}$  such that  $\widehat{\omega} = (\widehat{\Omega}, [\cdot] \widehat{\Omega})$  satisfies

$$\widehat{\omega}(\widehat{W}(\psi)) = e^{-1/2\|\psi\|^2}$$

Note that  $\widehat{W}(K_0 \cdot)$  is a representation over  $(\mathcal{A}, \sigma)$  since  $K_0$  is symplectic:  $2 \text{Im}(K_0 F_1, K_0 F_2) = \sigma(F_1, F_2)$ . Furthermore a pair  $(\widehat{W}(K_0 \cdot), \widehat{\omega})$  is a realization of the  $(W, \omega_0)$  introduced earlier, and hence of the  $(W_{in}, \omega_{in})$  we are now considering.

Next consider the operators

$$\Sigma = K_0 S K_0^{-1} \quad \Sigma^{-1} = K_0 S^{-1} K_0^{-1}$$

which are real linear and symplectic from the dense domain  $K_0 \mathcal{A} \subset L_2(\mathbb{R}^3, \mathbb{C})$  onto itself. Suppose that  $\Sigma, \Sigma^{-1}$  extend to bounded (symplectic) operators on  $L_2(\mathbb{R}^3, \mathbb{C})$ . If there is an  $\mathcal{S}$  such that  $\mathcal{S}\widehat{W}(\psi)\mathcal{S}^{-1} = \widehat{W}(\Sigma\psi)$  then  $W_{in}(F) = \widehat{W}(K_0 F)$  satisfies  $\mathcal{S}W_{in}(F)\mathcal{S}^{-1} = W_{in}(SF)$  as required.

Thus when  $\Sigma$  and  $\Sigma^{-1}$  are bounded, it suffices to study the implementability of  $\widehat{W}(\psi) \rightarrow \widehat{W}(\Sigma\psi)$ . If  $\Sigma$  is unitary it is well-known that this transformation is implementable (by the second quantization of  $\Sigma$ ). More generally there is Shale's Theorem [27] which says that the transformation is implementable if and only if  $\Sigma^+ \Sigma - I$  is Hilbert-Schmidt, where  $\Sigma^+$  is the real adjoint of  $\Sigma$  defined with respect to  $\text{Re}(\dots)$ . (We remark that

it is enough that  $\Sigma^+\Sigma - I$  on  $K_0\mathcal{A}$  extends to a Hilbert-Schmidt operator. For then it follows that  $\Sigma$  extends to a bounded operator (since

$$\|\Sigma\psi\|^2 = (\psi, (\Sigma^+\Sigma - I)\psi) + \|\psi\|^2)$$

and that  $\Sigma^{-1}$  extends to a bounded operator (since  $\Sigma^+ \supset -i\Sigma^{-1}i$ )).

For the transient case discussed in § IV we may define  $S$  and  $\Sigma$ , but unitary implementability is difficult. If however  $g$  is strictly Minkowskian off a compact set the unitary implementability has been proved [9] [29].

### VI. STATIONARY SPACE-TIMES

We continue to assume conditions A, B, C and suppose in addition that  $g_{\mu\nu}$  is actually independent of  $t$  (so B is trivial). We call such a space-time stationary. For the classical dynamics this means that  $H$  is independent of  $t$  and that time evolution has the form  $\mathcal{T}(t, s) = \mathcal{T}(t - s)$  where  $\mathcal{T}(t)$  satisfies  $\mathcal{T}(t)\mathcal{T}(s) = \mathcal{T}(t + s)$ . It is then straightforward to obtain the intertwining relations

$$\begin{aligned} \mathcal{T}(t)\Omega^\pm F &= \Omega^\pm \mathcal{T}_0(t)F & F \in \mathcal{A} \\ \mathcal{T}_0(t)\tilde{\Omega}^\pm F &= \tilde{\Omega}^\pm \mathcal{T}(t)F & F \in \text{Ran } \Omega^\pm \end{aligned} \tag{6.1}$$

We do not study the classical problem further, although there are many interesting questions.

To discuss the quantum case we recall the concept of a « one-particle structure » for  $(\mathcal{A}, \sigma, \mathcal{T}(t))$  [14] [16]. This is a triple  $(K, \mathcal{H}, e^{-iBt})$  consisting of a complex Hilbert space  $\mathcal{H}$  regarded as a real symplectic space (with symplectic form  $2 \text{Im}(\cdot, \cdot)$ ), a symplectic operator  $K: \mathcal{A} \rightarrow \mathcal{H}$  onto a dense domain in  $\mathcal{H}$ , and a unitary group  $e^{-iBt}$  on  $\mathcal{H}$  with strictly positive generator  $B$ , such that

$$K\mathcal{T}(t) = e^{-iBt}K \tag{6.2}$$

An example we have already met is the one-particle structure

$$(K_0, L_2(\mathbb{R}^3, \mathbb{C}), e^{-iB_0t}) \quad \text{for} \quad (\mathcal{A}, \sigma, \mathcal{T}_0(t)).$$

Kay has shown that one-particle structures exist for a class of stationary space-times including our own [14]. Furthermore it is known that one-particle structures are unique up to unitary equivalence [16] [30], a fact we make use of below.

Once one has a one-particle structure one may define a vacuum state (the clothed vacuum) on the algebra  $\mathfrak{A}$  by the generating functional

$$\omega(W(F)) = e^{-1/2\|KF\|^2}$$

This  $\omega$  is invariant under time evolution,  $\omega \circ \alpha_t = \omega$ , and in the represen-

tation defined by  $\omega$ ,  $\alpha_t$  is unitarily implemented with positive energy.

We have two results for these stationary space-times.

**THEOREM VI.1.** —  $\omega | \mathfrak{A}_{\text{in/out}} = \omega_{\text{in/out}}$ .

*Proof.* — By the intertwining relations (6.1), (6.2), we have

$$(\mathbf{K}\Omega^\pm)\mathcal{T}_0(t) = e^{-i\mathbf{B}t}(\mathbf{K}\Omega^\pm)$$

This suggests that  $\mathbf{K}\Omega^\pm$  can define a one-particle structure for the free dynamics. In fact define  $\mathcal{H}_\pm \subset \mathcal{H}$  by

$$\mathcal{H}_\pm = \overline{\text{Ran } \mathbf{K}\Omega^\pm}$$

Then  $e^{-i\mathbf{B}t}$  leaves  $\mathcal{H}_\pm$  invariant.  $\mathcal{H}_\pm$  is *a priori* only real linear. However let  $\mathbf{P}_\pm$  be the projection onto  $\mathcal{H}_\pm$ . Since  $[\mathbf{P}_\pm, e^{-i\mathbf{B}t}] = 0$  we may conclude by a theorem of Weinless ([30], Theorem 1.2) that  $\mathbf{P}_\pm$  is complex linear. Hence  $\mathcal{H}_\pm$  is complex linear. Thus  $(\mathbf{K}\Omega^\pm, \mathcal{H}_\pm, e^{-i\mathbf{B}t})$  is a one-particle structure for  $(\mathcal{A}, \mathcal{T}_0(t))$  and since  $(\mathbf{K}_0, L_2, e^{-i\mathbf{B}_0t})$  is also, we have by the uniqueness that there are unitary operators  $\mathbf{U}_\pm : L_2(\mathbb{R}^3, \mathbb{C}) \rightarrow \mathcal{H}_\pm$  such that

$$\mathbf{K}\Omega^\pm = \mathbf{U}_\pm \mathbf{K}_0$$

Now we may compute

$$\begin{aligned} \omega(\mathbf{W}_{\text{out/in}}(\mathbf{F})) &= \omega(\mathbf{W}(\Omega^\pm \mathbf{F})) \\ &= \exp\left(-\frac{1}{2} \|\mathbf{K}\Omega^\pm \mathbf{F}\|^2\right) \\ &= \exp\left(-\frac{1}{2} \|\mathbf{K}_0 \mathbf{F}\|^2\right) \end{aligned}$$

and hence  $\omega$  agrees with  $\omega_{\text{out,in}}$ .

**THEOREM VI.2.** — For a stationary space-time with  $\text{Ran } \Omega^+ = \text{Ran } \Omega^-$ , the scattering automorphism is unitarily implementable.

*Proof.* — From the intertwining relations (6.1), we have

$$[\mathcal{T}_0(t), \mathbf{S}] = 0$$

It follows that both  $(\mathbf{K}_0\mathbf{S}, L_2, e^{-i\mathbf{B}_0t})$  and  $(\mathbf{K}_0, L_2, e^{-i\mathbf{B}_0t})$  are one-particle structures for  $(\mathcal{A}, \mathcal{T}_0(t))$ . By uniqueness we have  $\mathbf{K}_0\mathbf{S} = \hat{\Sigma}\mathbf{K}_0$  for some unitary  $\hat{\Sigma}$  on  $L_2(\mathbb{R}^3, \mathbb{C})$ . In other words  $\Sigma = \mathbf{K}_0\mathbf{S}\mathbf{K}_0^{-1}$  extends to a unitary. As pointed out in § V.3 this gives the implementability.

*Remarks.* — Both these theorems should be valid for general external field problems. To our knowledge they have not previously been proved for bosons. For Dirac fields in an external potential results of this type may be found in [2] [3] [21].

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