

# ANNALES DE L'I. H. P., SECTION A

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*Annales de l'I. H. P., section A*, tome 37, n° 2 (1982), p. 115-127

[http://www.numdam.org/item?id=AIHPA\\_1982\\_\\_37\\_2\\_115\\_0](http://www.numdam.org/item?id=AIHPA_1982__37_2_115_0)

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## Zero mass, 2 dimensional, real time Sine Gordon model without u. v. cut-offs

by

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**ABSTRACT.** — We prove the existence of a limit for matrix elements of the time evolution operator in the interaction picture, for a zero mass two dimensional Sine Gordon model in a space time box when the ultra-violet cut-off is removed, and for  $\alpha^2 < 2\pi$ .

**RÉSUMÉ.** — Nous prouvons l'existence d'une limite pour les éléments de matrice de l'opérateur d'évolution temporel dans le schéma d'interaction, pour le modèle de Sine Gordon de masse nulle à deux dimensions dans une boîte d'espace-temps quand la régularisation ultraviolette est supprimée, et ceci pour  $\alpha^2 < 2\pi$ .

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### § 1. INTRODUCTION

In a previous paper, hereafter referred as [1], we have shown that a generalized Poisson process [2] can be associated to some relativistic (we mean real time) models. We mentioned that it applies to trigonometric

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This work was completed with the support of the Norwegian Research Council for Science and the Humanities under the project Matematisk Seminar.

interactions [3], and in particular to the Sine-Gordon model [4], which is defined by the interaction (see section 2 for notations):

$$H_I = \int_{\Lambda} dx : \cos(\alpha\phi_{\kappa}(x)) :_{\mu},$$

$\kappa$  is an ultraviolet cut-off and  $\Lambda$  a space cut-off.

This model has some broader generality than it could appear since it is equivalent to the massive Thirring model [5], plus Yukawa interaction (see eg. [6] [7]).

Trigonometric interactions have been studied extensively in the Euclidean region (see e. g. [8] to [12]), showing that at least for  $\alpha^2 < 4\pi$  the two dimensional Euclidean model exists [13] [14] in the limit  $\kappa \rightarrow 0$ ,  $\Lambda \nearrow \mathbb{R}$ .

Then the Osterwalder-Schrader reconstruction theorem (see eg. [15] [17]) allows to assert the existence of a relativistic model.

In this paper, it is this detour that we want to avoid by using Poisson measures instead of Gaussian measures, which can be no longer used since of the characteristics of the relativistic propagators.

Number of estimates as well as the general strategy of [13] and [14] can be adapted in the zero mass case, where we use essentially the existence of the grand partition function of a neutral Coulomb gas in two dimensions [16].

The lack of comparison between the relativistic free propagators for different masses does not allow to treat the  $m \neq 0$  model, in contrast to the Euclidean case and also make necessary a further estimate that is not needed for the Euclidean model. This reduces our existence proof to  $\alpha^2 < 2\pi$ .

In the second section we briefly describe the notations and the main results of [1]. We also mention the definition of the Poisson measure through the characteristic functional of the Poisson process, naturally associated with trigonometric interaction models.

The third section deals with the limit  $m = 0$ , whereas in the fourth section we derive the main result of this paper, namely the existence of the limit:

$$\lim_{\kappa \rightarrow 0} (\psi_{fg} e^{iH_0 t} e^{-i(H_0 + H_I^{\kappa})t} \psi_{fg})$$

where  $H_0$  is the free relativistic Hamiltonian and  $\psi_{fg}$  are coherent states.

## § 2. THE SINE GORDON MODEL WITH CUT-OFF IN TWO DIMENSIONAL SPACE TIME

In this section we define the notations and recall the results of [1] which are needed in the sequel.

$\phi$  is a scalar neutral relativistic Bose field of mass  $m$  in two space time dimensions.

$\Lambda$  is a finite box in  $\mathbb{R}$  viz.  $\Lambda = \{x; |x| < \Lambda/2\}$ .

The sine Gordon model is defined by the interaction:

$$H_I = \lambda \int_{\Lambda} dx : \cos (\alpha \phi_x(x)) :_{\mu}, \tag{2.1}$$

where

$$\phi_x(x) = \int_{\mathbb{R}} dy \phi(y, 0) \chi_x(x - y), \tag{2.2}$$

is the zero time field with ultraviolet cut-off, viz.  $\chi_x$  is the function:

$$\chi_x(x) = \frac{1}{\varkappa} \chi\left(\frac{1}{\varkappa} x\right), \tag{2.3}$$

where  $\chi$  is a function of  $D(\mathbb{R})$  which is:

$$\text{symmetric positive} \tag{2.4}$$

$$\chi(x) = 0, \quad \text{if } |x| \geq 1. \tag{2.5}$$

$$\int_{\mathbb{R}} \chi(x) dx = 1. \tag{2.6}$$

Moreover, the Wick order in (2.1) is taken with respect to a mass  $\mu$ , which is strictly positive (see eg. [17]):

$$: \exp (i \alpha \phi_x(x)) :_{\mu} = \exp (i \alpha \phi_x(x)) \exp \left( \frac{\alpha^2}{4} \int_{\mathbb{R}} dk \frac{|\tilde{\chi}_x(k)|^2}{\sqrt{k^2 + \mu^2}} \right), \tag{2.7}$$

where  $\tilde{\chi}_x$  is the Fourier transform of  $\chi_x$ :

$$\tilde{\chi}_x(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} \chi_x(x) dx.$$

We notice that [17] [7]:

$$\lambda \int_{\Lambda} dx : \cos (\alpha \phi_x(x)) :_{\mu} = \lambda(m, \mu, \varkappa) \int_{\Lambda} dx : \cos (\alpha \phi_x(x)) :_m, \tag{2.8}$$

when  $m > 0$ , with:

$$\lambda(m, \mu, \varkappa) = \lambda \exp \left( \frac{\alpha^2}{4} \int_{\mathbb{R}} dk |\chi_x(k)|^2 \left( \frac{1}{\sqrt{k^2 + \mu^2}} - \frac{1}{\sqrt{k^2 + m^2}} \right) \right). \tag{2.9}$$

Finally  $H_0$  is the relativistic free Hamiltonian corresponding to the mass  $m$ .

With these notations one has the following, [2]:

**PROPOSITION (2.10).** — The application:

$$f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}) \rightarrow \mathbf{C}(f) \\ = \exp \left\{ |\lambda| \left( \int_0^t dt \int_{\Lambda} dx \left\{ \cos \left( \alpha \int_{\mathbb{R}} d\xi d\xi' \Delta_{\mathbb{R}}^{(m)}(x - \xi + \xi'; t) \chi_x(\xi) f(\xi) \right) - 1 \right\} \right) \right\}$$

is a normalized, continuous, positive type function.

$\Delta_{\mathbf{R}}^{(m)}(x, t) = \frac{1}{2\pi} \int \frac{dke^{ikx}}{\sqrt{k^2 + m^2}} \Theta(t) \sin(t\sqrt{k^2 + m^2})$  is the usual retarded function.

As a corolary, by Minlos theorem (see eg. [19] [17]), there exists a positive measure  $p_{\Lambda|\lambda}$  on  $\mathcal{S}'_{\mathbf{R}}(\mathbf{R})$  such that:

$$C(f) = \int_{\mathcal{S}'_{\mathbf{R}}(\mathbf{R})} dp_{\Lambda|\lambda}(\varphi) \exp(i\varphi(f)). \tag{2.11}$$

Moreover:

$$C(f) = E[\exp(i\Phi_{t=0}(f))], \tag{2.12}$$

where  $\Phi$  is a generalized Poisson process.

Actually, in [1] we have proved that the generalized Poisson process

can be realized as follows:  $\Omega = \bigsqcup_n \Omega_n$ ,

$$\Omega_0 = \{ \omega_0 \}, \tag{2.13}$$

$$\Omega_n = \{ \omega = (n, t_1 \dots t_n, \varepsilon_1 \dots \varepsilon_n, x_1 \dots x_n) \}, \quad n > 0, \tag{2.14}$$

where  $0 < t_1 < t_2 \dots < t_n < T$ ,  $\varepsilon_i = \pm 1$  and  $x_i \in \Lambda$ .

Let  $V_{a_i \eta_i \mathcal{B}_i}^{(n)} = \{ (n, t_i, \varepsilon_i, x_i, t_i \in a_i, \varepsilon_i = \eta_i, x_i \in \mathcal{B}_i) \}$ , where the  $a_i$ 's are disjoint ordered Lebesgue measurable subsets of  $[0, T]$ , and the  $\mathcal{B}_i$ 's Lebesgue measurable subsets of  $\Lambda$ . They generate a Borel  $\sigma$ -algebra  $\mathcal{F}$ . Then:

$$P_{\Lambda|\lambda}(V_{a_i \eta_i \mathcal{B}_i}^{(n)}) = \frac{|\lambda|^n}{2^n} \prod_{i=1}^n |a_i| |\mathcal{B}_i|, \tag{2.15}$$

where the  $|a_i|$  and  $|\mathcal{B}_i|$  are the Lebesgue measure of  $a_i$  and  $\mathcal{B}_i$  respectively, is a bounded positive measure on  $\Omega$ , [19] [1].

The generalized process  $\Phi$  can be written, [1]:

$$\Phi(x, t)(\omega = (n, t_i, \varepsilon_i, x_i)) = \alpha \sum_{i=1}^n \varepsilon_i \int_{\mathbf{R}} d\xi \Delta_{\mathbf{R}}^{(m)}(x - x_i - \xi; t_i - t) \chi_x(\xi), \tag{2.16}$$

and one has:

$$E[\exp(i\Phi(f))] = e^{-|\lambda|\Delta T} \int_{\Omega} P_{\Lambda|\lambda}(d\omega) \{ e^{i\Phi(f)} \}(\omega). \tag{2.17}$$

In [1] it was proved that the expectation value of the operator  $\exp(iH_0 T) \exp(-i(H_0 + H_1)T)$  in between coherent states:

$$\Psi_{fg} = \exp(i(\pi(f) - \phi(g)))\Omega_F, \quad f, g \in \mathcal{S}'_{\mathbf{R}}(\mathbf{R}), \tag{2.18}$$

where  $\pi$  is the conjugate momentum of  $\phi$  and  $\Omega_F$  the vacuum Fock state, is given by:

$$(\Psi_{fg} | \exp(iH_0 T) \exp(-i(H_0 + H_1)T) \Psi_{fg}) = \int_{\Omega} P_{\Lambda|\lambda}(d\omega) F_{xm}(\omega), \tag{2.19}$$

where  $F_{xm}$  is an integrable function on  $\Omega$ , with the following structure:

$$F_{xm}(\omega) = F^{(1)}(\omega)F_{fg}^{(2)}(\omega)F_{xm}^{(3)}(\omega), \tag{2.20}$$

where:

$$F^{(1)}(\omega = (n, t_i, \varepsilon_i, x_i)) = \left( \frac{-i\lambda}{|\lambda|} \right)^n, \tag{2.21}$$

$$F_{fg}^{(2)}(\omega = (n, t_i, \varepsilon_i, x_i)) = \exp \left( i\alpha \sum_{i=1}^n \varepsilon_i \int_{\mathbb{R}} \int_{\mathbb{R}} d\xi d\zeta \chi_{\kappa}(\xi) \Delta_{\mathbb{R}}^{(m)}(x_i - \xi - \zeta; t_i) g(\zeta) \right) \tag{2.22}$$

$$\exp \left( -i\alpha \sum_{i=1}^n \varepsilon_i \int_{\mathbb{R}} \int_{\mathbb{R}} d\xi d\zeta \chi_{\kappa}(\xi) \partial_0 \Delta_{\mathbb{R}}^{(m)}(x_i - \xi - \zeta; t_i) f(\zeta) \right),$$

and:

$$F_{xm}^{(3)}(\omega = (n, t_i, \varepsilon_i, x_i)) = \exp \left( \frac{n\alpha^2}{4} \int_{\mathbb{R}} |\tilde{\chi}_{\kappa}(k)|^2 \left( \frac{1}{\sqrt{k^2 + \mu^2}} - \frac{1}{\sqrt{k^2 + m^2}} \right) dk \right) \tag{2.23}$$

$$\exp \left( -\frac{\alpha^2}{4} \sum_{i \neq j}^n \varepsilon_i \varepsilon_j \int_{\mathbb{R}} \int_{\mathbb{R}} d\xi d\zeta \chi_{\kappa}(x_i - \xi) \Delta_{\mathbb{F}}^{(m)}(\xi - \zeta, t_i - t_j) \chi_{\kappa}(x_j - \zeta) \right),$$

where  $\Delta_{\mathbb{F}}^{(m)}$  is the Feynman propagator:

$$\Delta_{\mathbb{F}}^{(m)}(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} \frac{e^{-i|t|\sqrt{k^2 + m^2}}}{\sqrt{k^2 + m^2}} dk. \tag{2.24}$$

$F^{(1)}$  does not depend on the cut-off, and  $F^{(2)}$  is of modulus one.  $F^{(3)}$  depends explicitly on the cut-off and becomes singular if  $\kappa$  tends to zero. In the next section we shall control its singularity.

### § 3. THE LIMIT $m = 0$

Our goal is to prove that at least for small  $\alpha$ , we can deal with the singularity which appears in  $\Delta_{\mathbb{F}}^{(m)}$ . This will be done only in the case  $m = 0$  and in this section we shall prove that the limit  $m = 0$  exists in (2.19).

PROPOSITION (3.1). — One has the following limit:

$$\lim_{m \rightarrow 0} \int_{\Omega} P_{\Lambda|\lambda|}(d\omega) F^{(1)}(\omega) F_{xm}^{(3)}(\omega) = \int_{\Omega} P_{\Lambda|\lambda'}(d\omega) F^{(1)}(\omega) F_{\kappa}^{(3)}(\omega),$$

where:

$$\lambda' = \lambda \left( \frac{2}{\mu} e^{-\gamma} \right)^{\frac{\alpha^2}{4\pi}}, \tag{3.2}$$

$\gamma$  is the Euler's constant, and:

$$\begin{aligned}
 &F_x^{(3)}(\omega = (n, t_i, \varepsilon_i, x_i)) \\
 &= \exp\left(\frac{n\alpha^2}{4} \int_{\mathbb{R}} dk (|\tilde{\chi}_x(k)|^2 - \frac{1}{2\pi}) \left(\frac{1}{\sqrt{k^2 + \mu^2}} - \frac{1}{|k|}\right)\right) \\
 &\quad \exp\left(\frac{\alpha^2}{4\pi} \sum_{i \neq j} \varepsilon_i \varepsilon_j \int_{\mathbb{R}} \int_{\mathbb{R}} d\xi d\zeta \chi_x(x_i - \xi) \ln((\xi - \zeta)^2 - (t_i - t_j)^2)^{1/2} \chi_x(x_j - \zeta)\right), \\
 &\qquad\qquad\qquad \text{if } \sum_{i=1}^n \varepsilon_i = 0, \quad = 0 \text{ otherwise} \quad (3.3)
 \end{aligned}$$

with the usual determination of  $\ln(z)$ .

*Proof.* — First we prove the pointwise convergence of  $F_{xm}^{(3)}$  to  $F_x^{(3)}$ . Indeed if we remark that (see [20] for notations):

$$\begin{aligned}
 \Delta_F^{(m)}(x, t) &= \frac{1}{\pi} K_0(im\sqrt{t^2 - x^2}), \quad 0 < |x| < |t| \\
 &= \frac{1}{\pi} K_0(m\sqrt{x^2 - t^2}), \quad 0 < |t| < |x|, \quad (3.4)
 \end{aligned}$$

where:

$$K_0(z) = -\ln\left(\frac{cz}{2}\right) I_0(z) + 2 \sum_{n \geq 1} \frac{1}{n} I_{2n}(z), \quad (3.5)$$

where  $\ln(c) = \gamma$ , the Euler's constant, then:

$$\begin{aligned}
 F_{xm}^{(3)}(\omega = (n, t_i, \varepsilon_i, x_i)) &= \left(\frac{m}{\mu}\right)^{n\alpha^2/4\pi} \left(\frac{mc}{2}\right)^{\alpha^2/4\pi \left(\left(\sum_i \varepsilon_i\right)^2 - n\right)} \\
 &\quad \exp\left(\frac{n\alpha^2}{4} \int_{\mathbb{R}} dk (|\tilde{\chi}_x(k)|^2 - \frac{1}{2\pi}) \left(\frac{1}{\sqrt{k^2 + \mu^2}} - \frac{1}{\sqrt{k^2 + m^2}}\right)\right) \\
 &\quad \exp\left(\frac{\alpha^2}{4\pi} \ln\left(\frac{mc}{2}\right) \sum_{i \neq j} \varepsilon_i \varepsilon_j \int_{\mathbb{R}} \int_{\mathbb{R}} d\xi d\zeta \chi_x(x_i - \xi) A_0(m, \xi, \zeta, t_i, t_j) \chi_x(x_j - \zeta)\right) \quad (3.6) \\
 &\quad \exp\left(-\frac{\alpha^2}{4\pi} \sum_{i \neq j} \varepsilon_i \varepsilon_j \int_{\mathbb{R}} \int_{\mathbb{R}} d\xi d\zeta \chi_x(x_i - \xi) \sum_{k=1}^{\infty} A_{2k}(m, \xi, \zeta, t_i, t_j) \chi_x(x_j - \zeta)\right) \\
 &\quad \exp\left(\frac{\alpha^2}{4\pi} \sum_{i \neq j} \varepsilon_i \varepsilon_j \int_{\mathbb{R}} \int_{\mathbb{R}} d\xi d\zeta \chi_x(x_i - \xi) \ln(\sqrt{(\xi - \zeta)^2 - (t_i - t_j)^2})\right. \\
 &\qquad\qquad\qquad \left. (A_0(m, \xi, \zeta, t_i, t_j) + 1) \chi_x(x_j - \zeta)\right),
 \end{aligned}$$

where:

$$A_0(m, \xi, \zeta, t_i, t_j) = I_0(m\sqrt{(\xi - \zeta)^2 - (t_i - t_j)^2}) - 1, \quad (3.7)$$

$$A_{2k}(m, \xi, \zeta, t_i, t_j) = \frac{2}{k} I_{2k}(m\sqrt{(\xi - \zeta)^2 - (t_i - t_j)^2}), \quad (3.8)$$

with the usual definition of the square root.

The result follows from the well known behaviour of the functions  $I_p(z)$  for small  $z$  (see eg. [20]).

Secondly we remark that for any collection  $\{f_i\}_{i=1\dots n}$  of functions of  $D_{\mathbb{R}}(\mathbb{R})$ :

$$\sum_{i,j=1}^n \int_{\mathbb{R}} \int_{\mathbb{R}} d\xi d\zeta f_i(\xi) \operatorname{Re} (\Delta_F^{(m)}(\xi - \zeta; t_i - t_j)) f_j(\zeta) \geq 0. \tag{3.9}$$

Hence we have the inequality:

$$|F_{xm}^{(3)}(\omega = (n, t_i, \varepsilon_i, x_i))| \leq \exp\left(\frac{n\alpha^2}{4} \int_{\mathbb{R}} \frac{dk |\tilde{\chi}_x(k)|^2}{\sqrt{k^2 + \mu^2}}\right). \tag{3.10}$$

This last expression defines a function on  $\Omega$  which is  $P_{\Lambda|\lambda'}$  integrable.

Finally the proposition follows from Lebesgue's dominated convergence theorem applied to the integral with respect to the bounded positive measure  $P_{\Lambda|\lambda'}$ .

#### § 4. REMOVAL OF ULTRAVIOLET DIVERGENCIES

In this section we remove the ultraviolet cut-off by letting  $\varkappa$  going to zero, in the expression we have obtained in the previous section, for  $m=0$ :

$$(\Omega_F | \exp(iH_0 T) \exp(-i(H_0 + H_1)T) \Omega_F) = \int_{\Omega} P_{\Lambda|\lambda'}(d\omega) F^{(1)}(\omega) F_x^{(3)}(\omega) \tag{4.1}$$

where

$$\lambda' = \lambda \left(\frac{2}{\mu c}\right)^{\alpha^2/4\pi} \tag{4.2}$$

$$F^{(1)}(\omega = (n, t_i, \varepsilon_i, x_i)) = \left(-\frac{i\lambda'}{|\lambda'|}\right)^n \tag{4.3}$$

$$F_x^{(3)}(\omega = (n, t_i, \varepsilon_i, x_i)) = \exp\left(\frac{n\alpha^2}{4} \int_{\mathbb{R}} dk \left(|d\tilde{\chi}_x(k)|^2 - \frac{1}{2\pi}\right) \left(\frac{1}{\sqrt{k^2 + \mu^2}} - \frac{1}{|k|}\right)\right) \tag{4.4}$$

$$\exp\left(\frac{\alpha^2}{4\pi} \sum_{i \neq j} \varepsilon_i \varepsilon_j \int_{\mathbb{R}} \int_{\mathbb{R}} d\xi d\zeta \chi_x(x_i - \xi) C_F(\xi - \zeta; t_i - t_j) \chi_x(x_j - \zeta)\right)$$

$$\text{if } \sum_{i=1}^n \varepsilon_i = 0, \quad = 0, \quad \text{otherwise.}$$

$$C_F(x, t) = \begin{cases} \frac{1}{2} \ln(t^2 - x^2) + \frac{i\pi}{2}, & \text{if } 0 < |x| < |t|, \\ \frac{1}{2} \ln(x^2 - t^2), & \text{if } 0 < |t| < |x|. \end{cases} \tag{4.5}$$

First we prove that (4.1) is uniformly bounded for  $\varkappa \in ]0, 1]$ , indeed:

$$\left| \int_{\Omega} P_{\Lambda|\lambda'}(d\omega) F^{(1)}(\omega) F_{\varkappa}^{(3)}(\omega) \right| \leq \int_{\Omega} P_{\Lambda+2\varkappa|\lambda'}(d\omega) C^{(1)}(\omega) I^{(3)}(\omega), \quad (4.6)$$

where

$$C^{(1)}(\omega = (n_i, t_i, \varepsilon_i, x_i)) = \exp\left(\frac{n\alpha^2}{4} \int_{\mathbb{R}} dk \left( |\tilde{\chi}(k)|^2 - \frac{1}{2\pi} \left( \frac{1}{\sqrt{k^2 + \mu^2}} - \frac{1}{|k|} \right) \right)\right), \quad (4.7)$$

$$I^{(3)}(\omega = (n, t_i, \varepsilon_i, x_i)) = \exp\left(\frac{\alpha^2}{4\pi} \sum_{i \neq j}^n \varepsilon_i \varepsilon_j \operatorname{Re} \{ C_F(x_i - x_j; t_i - t_j) \}\right),$$

$$\text{if } \sum_{i=1}^n \varepsilon_i = 0, \quad = 0, \quad \text{otherwise,} \quad (4.8)$$

which is an easy consequence of the Jensen's inequality, [21]. However,

$$\operatorname{Re} C_F(x_i - x_j, t_i - t_j) = \frac{1}{2} \ln(|x_i - x_j - t_i + t_j|) + \frac{1}{2} \ln(|x_i - x_j + t_i - t_j|). \quad (4.9)$$

Hence by the Cauchy-Schwartz inequality:

$$\int_{\Omega'} P_{\Lambda+2\varkappa|\lambda'}(d\omega) C^{(1)}(\omega) I^{(3)}(\omega) \leq \int_{\Omega'} P_{\Lambda+2\varkappa|\lambda'}(d\omega) C^{(1)}(\omega) J^{(3)}(\omega), \quad (4.10)$$

where:

$$J^{(3)}(\omega = (n, t_i, \varepsilon_i, x_i)) = \exp\left(\frac{\alpha^2}{4\pi} \sum_{i \neq j}^n \varepsilon_i \varepsilon_j \ln(|x_i - x_j - t_i + t_j|)\right),$$

$$\text{if } \sum_{i=1}^n \varepsilon_i = 0, \quad = 0, \quad \text{otherwise.} \quad (4.11)$$

However, from the results of [16] and [13], for  $\alpha^2 < 2\pi$ , this is an integrable function with respect to  $P_{\Lambda+2\varkappa|\lambda'}$ .

Consequently we have:

PROPOSITION (4.12). — For  $\alpha^2 < 2\pi$

$$\int_{\Omega} P_{\Lambda|\lambda'}(d\omega) F^{(1)}(\omega) F_{\varkappa}^{(3)}(\omega) < C$$

independently of  $\kappa \in ]0, 1]$  and:

$$F(\omega = (n, t_i, \varepsilon_i, x_i)) = \left(-\frac{i\lambda'}{|\lambda'|}\right)^n \exp\left(\frac{\alpha^2}{4\pi} \sum_{i \neq j} \varepsilon_i \varepsilon_j C_F(x_i - x_j, t_i - t_j)\right)$$

$$\text{if } \sum_{i=1}^n \varepsilon_i = 0, \quad = 0, \quad \text{otherwise} \quad (4.12)$$

is  $P_{\Lambda|\lambda'}$  integrable.

This was a crucial step to remove the ultraviolet cut-off, indeed we can proceed along the same lines as in [13]. However, we have to control the factor,

$$\exp\left(\frac{n\alpha^2}{4} \int_{\mathbb{R}} dk \left(|\tilde{\chi}_\kappa(k)|^2 - \frac{1}{2\pi}\right) \left(\frac{1}{\sqrt{k^2 + \mu^2}} - \frac{1}{|k|}\right)\right) \quad (4.13)$$

This can be done by the following observation: for  $\alpha^2 < 2\pi$ ,

$$\lim_{\kappa \rightarrow 0} \int_{\Omega} P_{\Lambda|\lambda'}(d\omega) K^{(1)}(\omega) (K^{(2)}(\omega) - 1) = 0, \quad (4.14)$$

where

$$K^{(1)}(\omega = (n, t_i, \varepsilon_i, x_i))$$

$$= \exp\left(+\frac{\alpha^2}{4\pi} \sum_{i \neq j} \varepsilon_i \varepsilon_j \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_\kappa(x_i - \xi) C_F(\xi - \zeta, t_i - t_j) \chi_\kappa(x_j - \zeta)\right) d\xi d\zeta,$$

$$\text{if } \sum_i^n \varepsilon_i = 0, \quad = 0, \quad \text{otherwise.} \quad (4.15)$$

$$K^{(2)}(\omega = (n, t_i, \varepsilon_i, x_i)) = \exp\left(\frac{n\alpha^2}{4} \int_{\mathbb{R}} dk \left(|\tilde{\chi}_\kappa(k)|^2 - \frac{1}{2\pi}\right) \left(\frac{1}{\sqrt{k^2 + \mu^2}} - \frac{1}{|k|}\right)\right) \quad (4.16)$$

Indeed, again by Jensen's inequality

$$\left| \int_{\Omega} P_{\Lambda|\lambda'}(d\omega) K^{(1)}(\omega) (K^{(2)}(\omega) - 1) \right| \leq \int_{\Omega} P_{\Lambda+2\kappa|\lambda'}(d\omega) I^{(3)}(\omega) (K^{(2)}(\omega) - 1) \quad (4.17)$$

where  $I^{(3)}$  was defined in (4.8). However, the integrand can be easily majorized for  $\alpha^2 < 2\pi$  by an integrable function, moreover  $K^{(2)} - 1$  tends pointwise to zero for  $\kappa \rightarrow 0$ .

Now we are able to proceed to prove the convergence of the above matrix element. Namely, one has (see [13]):

$$\begin{aligned} \int_{\Omega} P_{\Lambda|\lambda'}(d\omega)(F(\omega) - G_x(\omega)) &= \frac{\alpha^2}{4\pi} \sum_{n \geq 0} \frac{1}{2n!} \left(-\frac{i\lambda'}{2}\right)^{2n} \int_0^T dt_{2n} \dots \int_0^T dt_1 \int_{\Lambda} dx_{2n} \\ &\dots \int_{\Lambda} dx_1 \sum_{\substack{\varepsilon_i = \pm 1 \\ \sum \varepsilon_i = 0}} \int_0^1 ds \left( \sum_{i \neq j} \varepsilon_i \varepsilon_j \int_{\mathbb{R}} \int_{\mathbb{R}} d\xi d\zeta \chi_x(x_i - \xi) (C_F(x_i - x_j; t_i - t_j) \right. \\ &\quad \left. - C_F(\xi - \zeta; t_i - t_j)) \chi_x(x_j - \zeta) \right) \quad (4.18) \\ &\exp\left(\frac{\alpha^2}{4\pi} s \sum_{i \neq j} \varepsilon_i \varepsilon_j C_F(x_i - x_j; t_i - t_j)\right) \\ &\exp\left(\frac{\alpha^2}{4\pi} (1-s) \sum_{i \neq j} \varepsilon_i \varepsilon_j \int_{\mathbb{R}} \int_{\mathbb{R}} d\xi d\zeta \chi_x(x_i - \xi) C_F(\xi - \zeta, t_i - t_j) \chi_x(x_j - \zeta)\right) \end{aligned}$$

where

$$\begin{aligned} G_x(\omega = (n, t_i, \varepsilon_i, x_i)) &= \left(\frac{-i\lambda'}{|\lambda'|}\right)^n \exp\left(\frac{\alpha^2}{4\pi} \sum_{i \neq j} \varepsilon_i \varepsilon_j \int_{\mathbb{R}} \int_{\mathbb{R}} d\xi d\zeta \chi_x(x_i - \xi) C_F(\xi - \zeta, t_i - t_j) \chi_x(x_j - \zeta)\right), \\ &= 0, \quad \text{if } \sum_{i=1}^n \varepsilon_i = 0, \quad = 0, \quad \text{otherwise.} \quad (4.19) \end{aligned}$$

However, there exists an  $s_0 \in [0, 1]$  such that (see [13]):

$$\begin{aligned} \left| \int_{\Omega} P_{\Lambda|\lambda'}(d\omega)(F(\omega) - G_x(\omega)) \right| &\leq \int_{\Omega} P_{\Lambda|\lambda'}(d\omega) |F_{\alpha\sqrt{s_0}}(\omega)| |G_{x, \alpha\sqrt{1-s_0}}(\omega)| |L_x(\omega)|, \quad (4.20) \end{aligned}$$

with the obvious definitions of  $F_{\alpha\sqrt{s_0}}$  and  $G_{x, \alpha\sqrt{1-s_0}}$ , and

$$\begin{aligned} L_x(\omega = (n, t_i, \varepsilon_i, x_i)) &= \left(-\frac{i\lambda'}{|\lambda'|}\right)^n \frac{\alpha^2}{4\pi} \sum_{i \neq j} \varepsilon_i \varepsilon_j \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_x(x_i - \xi) (C_F(\xi - \zeta, t_i - t_j) \\ &\quad - C_F(x_i - x_j; t_i - t_j)) \chi_x(x_j - \zeta), \quad \text{if } \sum_i^n \varepsilon_i = 0, \quad = 0, \quad \text{otherwise.} \quad (4.21) \end{aligned}$$

Now if we choose  $\delta > 0$ , such that:

$$\frac{\alpha^2}{2\pi} (1 + \delta) < 1, \quad (4.22)$$

by Hölder inequality,

$$\begin{aligned} & \left| \int_{\Omega} P_{\Lambda|\lambda'}(d\omega)(F(\omega) - G_{\kappa}(\omega)) \right| \\ & \leq \left( \int_{\Omega} P_{\Lambda|\lambda'}(d\omega) |F_{\alpha\sqrt{1+\delta}}(\omega)| \right)^{\frac{s_0}{1+\delta}} \left( \int_{\Omega} P_{\Lambda|\lambda'}(d\omega) |G_{\kappa\alpha\sqrt{1+\delta}}(\omega)| \right)^{\frac{1-s_0}{1+\delta}} \quad (4.23) \\ & \times \left( \int_{\Omega} P_{\Lambda|\lambda'}(d\omega) |L_{\kappa}(\omega)|^{1+\frac{1}{\delta}} \right)^{\frac{\delta}{1+\delta}} \end{aligned}$$

By proposition (4.12) the second term is uniformly bounded in  $\kappa$ .  
By Minkowski inequality, for  $N > 1$  (see e. g. [22]):

$$\begin{aligned} & \left( \int_{\Omega} P_{\Lambda|\lambda'}(d\omega) |L_{\kappa}(\omega)|^N \right)^{1/N} \\ & \leq \sum_{n \geq 1} \frac{\alpha^2 |\lambda'|^{2n/N}}{4\pi \ 2n!} \sum_{i+j} \left[ \int_0^T dt_{2n} \dots \int_0^T dt_1 \int_{\Lambda} dx_{2n} \dots \int_{\Lambda} dx_1 \right] \quad (4.24) \end{aligned}$$

$$\begin{aligned} & \left| \int_{\mathbb{R}} \int_{\mathbb{R}} d\xi d\zeta \chi_{\kappa}(x_i - \xi) (C_F(x_i - x_j; t_i - t_j) - C_F(\xi - \zeta; t_i - t_j)) \chi_{\kappa}(x_j - \zeta) \right|^N \\ & = \frac{\alpha^2}{2\pi} (A_{\kappa})^{1/N} \sum_{n \geq 1} \frac{|\lambda'|^{2n/N}}{2n!} n(2n-1) T^{\frac{2(n-1)}{N}} \Lambda^{\frac{2(n-1)}{N}}, \quad (4.25) \end{aligned}$$

where:

$$\begin{aligned} A_{\kappa} & = \int_0^T dt_1 \int_0^T dt_2 \int_{\Lambda} dx_1 \int_{\Lambda} dx_2 \left| \int_{\mathbb{R}} d\xi \int_{\mathbb{R}} d\zeta \right. \\ & \quad \cdot \chi_{\kappa}(x_1 - \xi) (C_F(\xi - \zeta; t_1 - t_2) - C_F(x_1 - x_2; t_1 - t_2)) \chi_{\kappa}(x_2 - \zeta) \left. \right|^N. \quad (4.26) \end{aligned}$$

However, the series in (4.25) is convergent and  $A_{\kappa}$  tends to zero for  $\kappa$  tending to zero. Then we can state the theorem:

**THEOREM (4.27).** — Let  $\phi_{\kappa}$  be the free relativistic Bose field in two space time dimensions of mass  $m$  at time zero and with ultra violet cut off.

Let  $H_0$  be the corresponding free relativistic Hamiltonian.

Let  $H_1$  be the Sine Gordon interaction viz.

$$H_1 = \lambda \int_{\Lambda} dx : \cos(\alpha\phi_{\kappa}(x)) :_{\mu}, \quad \alpha \text{ real,}$$

with  $\mu$  a strictly positive constant, and the Wick order being taken w. r. t.  $\Delta_F^{(\mu)}$ .

Let  $\Omega_F$  be the Fock vacuum state corresponding to the mass  $m$ .

Then for  $\alpha^2 < 2\pi$  the following limit exists

$$\lim_{\kappa \rightarrow 0} \lim_{m \rightarrow 0} (\Omega_F | \exp(-iT(H_0 + H_I)) \Omega_F) = \int_{\Omega} P_{\Lambda|\lambda'}(d\omega) \cdot F(\omega)$$

where  $P_{\Lambda|\lambda'}$  is the (unnormalized) Poisson measure with constant  $|\lambda'| = |\lambda| \left(\frac{2}{\mu c}\right)^{\alpha^2/4\pi}$  and  $F$  is

$$F(\omega = (n, t_i, \varepsilon_i, x_i)) = \left(-\frac{i\lambda'}{|\lambda'|}\right)^n \exp\left(\frac{\alpha^2}{4\pi} \sum_{i \neq j} \varepsilon_i \varepsilon_j C_F(x_i - x_j; t_i - t_j)\right) \\ \text{if } \sum_{i=1}^n \varepsilon_i = 0, \quad = 0, \quad \text{otherwise.}$$

The general matrix elements of  $\exp(iH_0 T) \exp(-i(H_0 + H_I)T)$  in between coherent states are treated by the same techniques except that one has to take care of the infrared divergence in the choice of the functions defining the coherent states.

#### ACKNOWLEDGMENTS

It is a special pleasure to thank Prof. Sergio Albeverio, Jean Bellissard, Philippe Blanchard, Franchesco Guerra and Giovanni Jona-Lasinio for discussions. One of us (M. S.) express his deep thanks to the members of the Fakultät für Physik of the university of Bielefeld for their warm hospitality. Finally two of us (M. S. C. and Ph. C.) acknowledge the financial support of the Fakultät für Physik of the university of Bielefeld where the last stage of this work was completed.

#### REFERENCES

- [1] Ph. COMBE, R. HØEGH-KROHN, R. RODRIGUEZ, M. SIRUGUE, M. SIRUGUE-COLLIN, Feynman Path integral and Poisson Processes with piecewise classical Paths. *J. Math. Phys.*, t. **23**, 1982, p. 405.
- [2] Ph. COMBE, R. HØEGH, R. RODRIGUEZ, M. SIRUGUE, M. SIRUGUE-COLLIN, Generalized Poisson Processes in Quantum Mechanics and Field theory, *Physics Reports*, t. **77**, 1981, p. 221.
- [3] R. HØEGH-KROHN, *Commun. Math. Phys.*, t. **21**, 1971, p. 244.
- [4] T. H. R. SKYRME, *Proc. Roy. Soc.*, t. **A 247**, 1958, p. 260; see also R. DASHEN, B. HASSLACHER, A. NEVEU, *Phys. Rev.*, t. **D 10**, 1974, p. 4114.
- [5] W. THIRRING, *Ann. Phys. (N. Y.)*, t. **3**, 1958, p. 91.
- [6] S. COLEMAN, *Phys. Rev.*, t. **11 D**, 1975, p. 2088.
- [7] S. COLEMAN, in *Proceedings of the 1975 International School of Subnuclear Physics*, Ettore Majorana, A. Zichichi Ed., Plenum Press, New York.
- [8] S. ALBEVERIO, R. HØEGH-KROHN. *J. Funct. Anal.*, t. **16**, 1974, p. 39; *Helv. Phys. Acta*, t. **46**, 1973, p. 504; *Helv. Phys. Acta*, t. **46**, 1973, p. 535.

- [9] S. ALBEVERIO, in *Functional Analysis and Probabilistic Methods in Quantum Field Theory*, Acta Univ. Wratislawa. 1976, B. Jancewicz, Ed.
- [10] J. FRÖHLICH, Y. M. PARK, *Helv. Phys. Acta*, t. **50**, 1977, p. 315.
- [11] S. ALBEVERIO, G. GALLAVOTTI, R. HØEGH-KROHN, *Commun. Math. Phys.*, t. **70**, 1979, p. 187.
- [12] S. ALBEVERIO, R. HØEGH-KROHN, *Commun. Math. Phys.*, t. **68**, 1979, p. 95.
- [13] J. FRÖHLICH, *Commun. Math. Phys.*, t. **47**, 1976, p. 233.
- [14] J. FRÖHLICH, *Renormalization theory, International School of Mathematical Physics*, G. Velo, A. S. Wightman Ed.
- [15] K. OSTERWALDER, in *Constructive Quantum Field Theory*, G. Velo and A. S. Wightman. Ed. Springer Verlag, 1973.
- [16] C. DEUTSCH, M. LAVAUD, *Phys. Rev.*, t. **A 9**, 1974, p. 2598.
- [17] B. SIMON, *The  $P(\phi_2)$  Euclidean (Quantum) Field Theory*, Princeton Series in Physics, Princeton, Princeton University Press, 1974.
- [18] R. MINLOS, *Tr. Mosk. Obs.*, t. **8**, 1959, p. 471.
- [19] V. P. MASLOV, A. M. CHEBOTAREV, *Vinit. Itogi Nauki*, t. **15**, 1978.
- [20] W. MAGNUS, F. OBERHETTINGER, R. P. SONI, *Formulas and Theorems on the special functions of Mathematical Physics* (3<sup>e</sup> ed.), Springer-Verlag, Berlin, 1966.
- [21] D. RUELLE, *Statistical Mechanics*, New York, Benjamin, 1969.
- [22] G. HARDY, J. LITTLEWOOD, G. POLYA, *Inequalities*. Cambridge University Press. 1967.

(Manuscrit reçu le 12 octobre 1981)