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I. M. BENN

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Conservation laws in arbitrary space-times

by

I. M. BENN
Department of Physics, University of Lancaster,
Lancaster, U. K.

ABSTRACT. — Definitions of energy and angular momentum are discussed within the context of field theories in arbitrary space-times. It is shown that such definitions rely upon the existence of symmetries of the space-time geometry. The treatment of that geometry as a background is contrasted with Einstein's theory of dynamical geometry: general relativity. It is shown that in the former case symmetries lead to the existence of closed 3-forms, whereas in the latter they lead to the existence of 2-forms which are closed in source-free regions, thus allowing mass and angular momentum to be defined as de Rham periods.

INTRODUCTION

The concepts of energy and angular momentum are perhaps two of the most deeply rooted concepts of physics, yet in general relativity their exact role is still open to question [*1*] [*2*]. Their unambiguous definition is related to their being conserved. Thus this paper discusses jointly the definition and conservation of energy and angular momentum within the context of field theories in arbitrary space-times.

Since conserved quantities are usually defined as integrals the use of differentials forms is appropriate for their description. Further, for the theorems of Stokes and de Rham to be accessible the use of differential forms is essential. In this paper the conventional Cartan calculus is used [*3*].

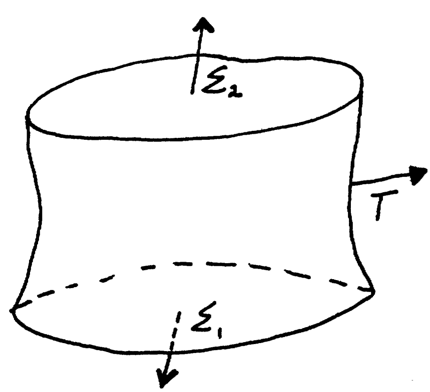
In section 1 it is shown in what sense the existence of a closed 3-form gives a conservation law, and this is contrasted in section 2 with the definition of electric charge as a de Rham period. Section 3 shows that symmetries of a background geometry lead to the existence of closed 3-forms, and a resulting conservation law. Section 4 contrasts this with the definition of mass in Newtonian gravity as a de Rham period. It is shown in section 6 that in general relativity, for spaces admitting symmetries, both mass and angular momentum are definable as de Rham periods.

1. CLOSED 3-FORMS AND CONSERVATION LAWS

The notion of a conservation law encompasses several related but distinct concepts. In the context of energy and angular momentum the existence of a closed 3-form is interpreted as a conservation law (see e. g. ref. 4)

i. e. $$\exists J \in \Lambda^3(M) : dJ = 0 \tag{1.1}$$

(M is the space-time manifold).
This is interpreted as meaning that the total 'flux' of energy (angular momentum) entering and leaving a four dimensional region is zero. It is then argued that if the fields that contribute to J vanish at (spatial) infinity then we obtain a constant of the motion. Assume that J is closed on a domain whose boundary is $\Sigma_1 + \Sigma_2 + T$, where $\Sigma_1(\Sigma_2)$ are spacelike hypersurfaces at $t_1(t_2)$ and T is the timelike surface connecting them.

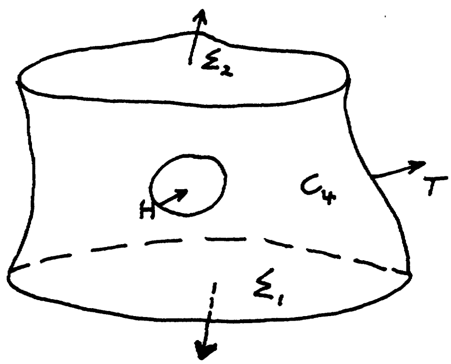


Since J is closed $\int_{\Sigma_1+\Sigma_2+T} J = 0$, and if T can be chosen such that $\int_T J = 0$, which follows from the assumed boundary conditions, then $\int_{\Sigma_1+\Sigma_2} J = 0$.

That is $Q \equiv \int_{\Sigma(t)} J$ is a constant of the motion. In fact if J were closed on this domain then it would be exact.

De Rham's theorem [5] gives the necessary and sufficient conditions for a closed form to be exact. A consequence is that if a 3-form is closed on a region that admits no non-trivial (1) 3-cycles then it will be exact. In general we will only obtain a constant of the motion from a closed 3-form if it is also exact. Suppose J to be closed on the region shown, C_4.

$$\partial C_4 = \Sigma_1 + \Sigma_2 + T + H$$
$$\partial(\Sigma_1 + \Sigma_2 + T) = 0$$



If $\int_{\Sigma_1+\Sigma_2+T} J = P$, then J is exact on C_4 iff $P = 0$. If the boundary conditions are such that $\int_T J = 0$ then $\int_{\Sigma_1+\Sigma_2} J = P$, and thus $Q \equiv \int_{\Sigma(t)} J$ is a constant of the motion only if J is exact. If J is exact, $J = dj$ say, then $\int_\Sigma J = \int_{\partial\Sigma} j$. However, in general $\partial\Sigma$ will not merely consist of one chain with the topology of a 2-sphere, S^2, and Ω will not be expressible as a flux integral at infinity.

An illustration of these ideas is provided by the electromagnetic field system generated by an electric pole of strength q and a magnetic pole of strength Q separated by a distance 'a' .If the poles are taken to lie on the z-axis with the magnetic pole at the origin then the electromagnetic 2-form, F, is given by

$$F = Q \sin\theta d\theta \wedge d\phi + q \frac{\{(r - a\cos\theta)dt \wedge dr + ar\sin\theta dt \wedge d\theta\}}{\{r^2 - 2ar\cos\theta + a^2\}^{3/2}} \quad (1.2)$$

The 3-form

$$J_z = \frac{2Qqa\sin^3\theta r dr \wedge d\theta \wedge d\phi}{(r^2 - 2ar\cos\theta + a^2)^{3/2}} \quad (1.3)$$

(1) A non-trivial cycle is one which is not a boundary.

is conventionally interpreted as being the density of field angular momentum about that axis [*6*]. (Motivation for this identification is provided in section 3 (²).

Here (r, θ, ϕ) are the usual spherical polar coordinates in Minkowski space. J_z is closed everywhere except at the two poles where it is singular. Because this region admits no non-trivial 3-cycles then by de Rham's theorem J_z is also exact on this region. So we may put

$$J_z = dj \tag{1.4}$$

where a suitable 2-form is

$$j = \frac{2Qq \sin\theta\,(r\cos\theta - a)d\theta \wedge d\phi}{(r^2 - 2ar\cos\theta + a^2)^{1/2}} \tag{1.5}$$

Let Σ be a constant time hypersurface with a 3-ball of radius λ surrounding q, and a 3-ball of radius μ surrounding Q, removed. Then

$$Q_z \equiv \int_\Sigma J_z \tag{1.6}$$

The 2-form j can be used to evaluate this integral

$$Q_z = \int_{\partial\Sigma} j = \int_{c_1} j + \int_{c_2} j + \int_{c_3} j \tag{1.7}$$

where c_1, c_2, c_3 are as shown in figure 1.

It can be checked that

$$\int_{c_1} j = -\frac{16\pi}{3}\frac{Qqa}{r_0} \tag{1.8}$$

$$\int_{c_2} j = \frac{8\pi}{3}\frac{Qq\lambda^2}{a^2} \tag{1.9}$$

$$\int_{c_3} j = -4\pi Qq\left(\frac{2\mu^2}{3a^2} - 2\right) \tag{1.10}$$

Thus

$$\lim_{\substack{r_0 \to \infty \\ \lambda \to 0 \\ \mu \to 0}} \int_\Sigma J_z = 8\pi Qq \tag{1.11}$$

the value usually assigned to the field angular momentum. Note that

(²) In fact this identification needs to be treated with circumspection. In section 3 reliance upon the action principle is made in constructing closed 3-forms related to symmetries of a background geometry. If Maxwell's (empty space) equations are obtained from the usual action principle then they are $d * dA = 0$, rather than $dF = d * F = 0$. Thus the field system under consideration would only be a solution to Maxwell's equations on a region with a Dirac string removed.

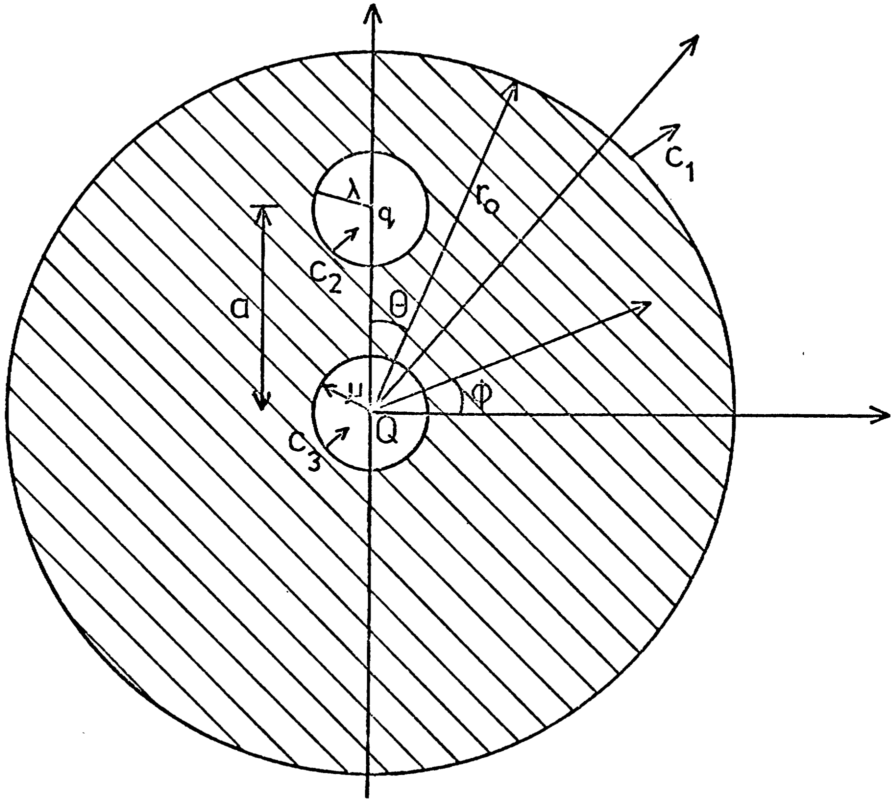


FIG. 1.

whereas here $\int_{c_3} j$ is the only non-vanishing surface term this is easily altered by adding closed 2-forms to j

e. g. let
$$j' \equiv j + 2q\mathrm{F}$$
such that
$$dj' = \mathrm{J}_z \tag{1.12}$$
Then
$$\lim_{r_0 \to \infty} \int_{c_1} j' = 8\pi q \mathrm{Q} \tag{1.13}$$
$$\lim_{\lambda \to 0} \int_{c_2} j' = 0 \tag{1.14}$$
$$\lim_{\mu \to 0} \int_{c_3} j' = 0 \tag{1.15}$$
Similarly if
$$j'' \equiv j' - 2\mathrm{Q} * \mathrm{F}$$

such that

$$dj'' = J_z \tag{1.16}$$

then

$$\lim_{r_0 \to \infty} \int_{c_1} j'' = 0 \tag{1.17}$$

$$\lim_{\lambda \to 0} \int_{c_2} j'' = 8\pi Q q \tag{1.18}$$

$$\lim_{\mu \to 0} \int_{c_3} j'' = 0 \tag{1.19}$$

Thus it is necessary to consider the three 2-cycles that make up $\partial\Sigma$ in order to express Q_z as the integral of a 2-form over a 2-chain. Further, we may write J_z as the exterior derivative of different 2-forms to redistribute the 'flux' between these three 2-cycles.

2. CONSERVATION OF ELECTRIC CHARGE

Consider firstly the case of classical fields interacting with some prescribed background electromagnetic field. If the field equations are obtainable from an action principle then the action-density 4-form (on M), Λ, will be a functional of A and ϕ^i where A is the electromagnetic connection 1-form and ϕ^i are some dynamical fields. If the interaction is assumed to be U(1) gauge invariant then

$$\int_M \delta_g \Lambda = \int_M (J \wedge \delta_g A + \Sigma_i \wedge \delta_g \phi^i) = 0 \tag{2.1}$$

where δ_g denotes an infinitesimal gauge transformation and J and Σ_i are the coefficients of arbitrary variations in A and ϕ^i respectively. Since the ϕ^i are all assumed dynamical then their field equations are

$$\Sigma_i = 0 \tag{2.2}$$

and when these are satisfied (2.1) gives

$$\int J \wedge \delta_g A = 0 \tag{2.3}$$

Substituting in the form of $\delta_g A$ gives

$$\int J \wedge d\chi = 0 \tag{2.4}$$

where χ is an arbitrary function. Thus

$$\int dJ\chi - \int d(J\chi) = 0 \tag{2.5}$$

and so we require

$$\int dJ\chi = 0 \qquad \forall\chi \tag{2.6}$$

if we make the usual assumption about boundary terms. This gives

$$dJ = 0 \tag{2.7}$$

Thus the U(1) gauge invariant coupling of a background electromagnetic field implies the existence of a closed 3-form; and hence a conservation law in the sense previously discussed.

When the electromagnetic field is treated as dynamical the notion of conservation of charge changes. In this situation we obtaine a field equation by requiring the action to be invariant under arbitrary variations in A,

$$d * F = J \tag{2.8}$$

where $F = dA$.

The form of the left hand side is obviously a consequence of the particular choice of the Maxwell action. Now J is exact, and hence trivially closed, wherever (2.8) is satisfied. Further, the definition of electric charge is now

$$Q \equiv \int_{s^2} * F \tag{2.9}$$

If T is a timelike 3-chain whose boundary consists of a spacelike s^2 at time t_1, and another at time t_2 then

$$Q(t_2) - Q(t_1) = \int_T J \tag{2.10}$$

Thus if T can be chosen such that $\int_T J = 0$ then the charge will be a constant of the motion. Perhaps the most important facet of electromagnetism is that for regions where $J = 0$ Q is thus defined as *a* de Rham period; that is, the integral of a closed 2-form over a 2-cycle. Thus the value of Q depends on the s^2 chosen only up to a homology class; which is just a statement of Gauss's law, as treated in elementary electrostatics. Empirical justification for claiming that the most important facet of the Maxwell equations is the source free case is that when we physically measure a charge it is by probing F in a source free region. We neither know nor care whether $d * F = J$ everywhere when, for example, we measure the charge of a charged conductor.

This concept of electric charge, as a period of a 2-form, is quite distinct from the concept of charge discussed in section I; as the integral of a closed 3-form over a 3-chain which is not closed. Formally it is a consequence of treating the electromagnetic field as dynamical, and of the particular form of the theory chosen to describe that dynamics.

3. BACKGROUND GEOMETRY

A theory of 'matter' fields in a background geometry will be assumed to be specified by an action-density 4-form (on M)

$$\Lambda_m = \Lambda_m(e^\mu, \omega^\mu{}_\nu, \phi^i)$$

The ϕ^i are any number of dynamical fields; that is, fields whose equations of motion are obtained by a variational principle. The e^μ are the g_M-orthonormal co-frames, where g_M is the space-time metric, and the $\omega^\mu{}_\nu$ are the connection 1-forms. The connection will be assumed to be metric compatible, and Λ_m to be locally invariant under the resulting gauge group of orthonormal frame transformations SO(3, 1), or its covering SL(2, C) (see ref. 7 for more details).

If X is an arbitrary vector field on M

$$\int \pounds_X \Lambda_m = \int \{ \tau_\mu \wedge \pounds_X e^\mu + S_\mu{}^\nu \wedge \pounds_X \omega^\mu{}_\nu + \Sigma_i \wedge \pounds_X \phi^i \}$$

where τ_μ, $S_\mu{}^\nu$, Σ_i are the coefficients of arbitrary variations of e^μ, $\omega^\mu{}_\nu$, ϕ^i, and $\pounds_X$ denotes the Lie derivative. The field equations for the ϕ^i are $\Sigma_i = 0$, so « on shell ».

$$\int \pounds_X \Lambda_M = \int \{ \tau_\mu \wedge \pounds_X e^\mu + S_\mu{}^\nu \wedge \pounds_X \omega^\mu{}_\nu \} \tag{3.1}$$

$$\pounds_X e^\mu = i_X de^\mu + di_X e^\mu ,$$

where i_X denotes the interior product

$$= i_X(T^\mu - \omega^\mu{}_\nu \wedge e^\nu) + di_X e^\mu$$

where T^μ are the torsion 2-forms.
So

$$\begin{aligned} \pounds_X e^\mu &= i_X T^\mu - i_X \omega^\mu{}_\nu \wedge e^\nu + \omega^\mu{}_\nu i_X e^\nu + di_X e^\mu \\ &= i_X T^\mu + D i_X e^\mu - i_X \omega^\mu{}_\nu \wedge e^\nu , \end{aligned}$$

D denoting the gauge covariant exterior derivative.

But $i_X \omega^\mu{}_\nu \in \mathscr{L}SO(3, 1)$, so

$$i_X \omega^\mu{}_\nu e^\nu = \delta_g e^\mu$$

an infinitesimal gauge transformation of e^μ with parameters $i_X \omega^\mu{}_\nu$.

Thus
$$\pounds_X e^\mu = i_X T^\mu + D i_X e^\mu - \delta_g e^\mu \tag{3.2}$$

Similarly

$$\begin{aligned} \pounds_X \omega^\mu{}_\nu &= i_X d\omega^\mu{}_\nu + di_X \omega^\mu{}_\nu \\ &= i_X(R^\mu{}_\nu - \omega^\mu{}_\alpha \wedge \omega^\alpha{}_\nu) + di_X \omega^\mu{}_\nu \end{aligned}$$

where $R^{\mu}{}_{\nu}$ are the curvature 2-forms

$$= i_X R^{\mu}{}_{\nu} - i_X\omega^{\mu}{}_{\alpha} \wedge \omega^{\alpha}{}_{\nu} + \omega^{\mu}{}_{\alpha} \wedge i_X\omega^{\alpha}{}_{\nu} + d i_X \omega^{\mu}{}_{\nu}$$

$$\pounds_X \omega^{\mu}{}_{\nu} = i_X R^{\mu}{}_{\nu} - \delta_g \omega^{\mu}{}_{\nu} \tag{3.3}$$

Putting (3.2) and (3.3) into (3.1) gives

$$\int \pounds_X \Lambda_m = \int \{ \tau_\mu \wedge i_X T^\mu + \tau_\mu \wedge D i_X e^\mu - \tau_\mu \wedge \delta_g e^\mu + S_\mu{}^\nu \wedge i_X R^\mu{}_\nu - S_\mu{}^\nu \wedge \delta_g \omega^\mu{}_\nu \}$$

But $$\int \delta_g \Lambda_m = \int \tau_\mu \wedge \delta_g e^\mu + S_\mu{}^\nu \wedge \delta_g \omega^\mu{}_\nu + \Sigma_i \wedge \delta_g \phi^i$$

$$= \int \tau_\mu \wedge \delta_g e^\mu + S_\mu{}^\nu \wedge \delta_g \omega^\mu{}_\nu \tag{3.4}$$

« on shell »

$= 0$, since we have assumed gauge invariance.

So we have

$$\int \pounds_X \Lambda_m = \int \{ \tau_\mu \wedge i_X T^\mu + \tau_\mu \wedge D i_X e^\mu + S_\mu{}^\nu \wedge i_X R^\mu{}_\nu \}$$

$$= \int \{ \tau_\mu \wedge i_X T^\mu + S_\mu{}^\nu \wedge i_X R^\mu{}_\nu - D(\tau_\mu i_X e^\mu) + D\tau_\mu i_X e^\mu \}$$

$$= \int \{ \tau_\mu \wedge i_X T^\mu + S_\mu{}^\nu \wedge i_X R^\mu{}_\nu + D\tau_\mu i_X e^\mu \}$$

since $D(\tau_\mu i_X e^\mu) = d(\tau_\mu i_X e^\mu)$

$$= \int \{ \tau_\mu \wedge i_\alpha T^\mu + S_\mu{}^\nu \wedge i_\alpha R^\mu{}_\nu + D\tau_\alpha \} X^\alpha$$

where $i_\alpha e^\mu = \delta^\mu{}_\alpha$.

But $\pounds_X \Lambda_m = d i_X \Lambda_m$, since Λ_m is a 4-form, and hence

$$\int \pounds_X \Lambda_m = 0,$$

and we have $$\int \{ \tau_\mu \wedge i_\alpha T^\mu + S_\mu{}^\nu \wedge i_\alpha R^\mu{}_\nu + D\tau_\alpha \} X^\alpha = 0$$

for *any* X.

So we must have

$$\boxed{D\tau_\alpha + \tau_\mu \wedge i_\alpha T^\mu + S_\mu{}^\nu \wedge i_\alpha R^\mu{}_\nu = 0} \tag{3.5}$$

Putting the explicit form of $\delta_g e^\mu$, $\delta_g \omega^\mu{}_\nu$ in (3.4) gives

$$\int \{ \tau_\mu \wedge \Lambda^\mu{}_\nu e^\nu + S_\mu{}^\nu \wedge (\Lambda^\mu{}_\alpha \omega^\alpha{}_\nu - \omega^\mu{}_\alpha \Lambda^\alpha{}_\nu - d\Lambda^\mu{}_\nu) \} = 0$$

where $\Lambda^{\mu}{}_{\nu} \in \mathscr{L}SO(3,1)$

$$\int \{ \tau_{\mu} \wedge e^{\nu}\Lambda^{\mu}{}_{\nu} + S_{\mu}{}^{\alpha} \wedge \omega^{\nu}{}_{\alpha}\Lambda^{\mu}{}_{\nu} - S_{\alpha}{}^{\nu} \wedge \omega^{\alpha}{}_{\mu}\Lambda^{\mu}{}_{\nu} + d(S_{\mu}{}^{\nu}\Lambda^{\mu}{}_{\nu}) - dS_{\mu}{}^{\nu}\Lambda^{\mu}{}_{\nu} \} = 0$$

$$\int \left\{ \frac{1}{2}(\tau_{\mu} \wedge e^{\nu} - \tau^{\nu} \wedge e_{\mu}) - dS_{\mu}{}^{\nu} - \omega_{\mu}{}^{\alpha} \wedge S_{\alpha}{}^{\nu} - S_{\mu}{}^{\alpha}\omega_{\alpha}{}^{\nu} \right\} \Lambda^{\mu}{}_{\nu} = 0$$

where $e_{\mu} \equiv n_{\mu\nu}e^{\nu}$.

Since this is true $\forall \Lambda^{\mu}{}_{\nu} \in \mathscr{L}SO(3,1)$ we must have

$$\boxed{DS_{\mu}{}^{\nu} = \frac{1}{2}(\tau_{\mu} \wedge e^{\nu} - \tau^{\nu} \wedge e_{\mu})} \qquad (3.6)$$

Equations (3.5) and (3.6) are usually referred to as 'covariant conservation laws'. However, it must be stressed that they are not conservation laws, but are merely identities which follow from the assumptions made at the beginning of this section. Further, it should be emphasised that these relations put no restrictions on the form of the gravitational action should we wish to consider dynamical geometry.

In view of these comments the presentation of these covariant identities, as they stand, serves no more use than making contact with similar expressions encountered in the literature, and making clear what conditions are necessary for their derivation. However, if the background geometry admits symmetries then these identities can be used to construct closed 3-forms in terms of τ_{μ}, $S_{\mu}{}^{\nu}$.

Before proceeding further two preliminary results are needed. Their proof is straightforward but is included for completeness.

LEMMA A. — $£_X e^{\mu} = \delta_g e^{\mu}$ and $£_X \omega^{\mu}{}_{\nu} = \delta_g \omega^{\mu}{}_{\nu}$ if, and only if, $£_X g_M = £_X T = 0$. T is the (2, 1) torsion tensor, that is $T = T^{\mu} \otimes b_{\mu}$ where b_{μ} is the tangent frame dual to e^{μ}

$$e^{\mu}(b_{\nu}) = \delta^{\mu}{}_{\nu}$$

Proof. — The only if part of the proof is trivial since g_M and T are gauge scalars, and so if $£_X e^{\mu} = \delta_g e^{\mu}$, $£_X \omega^{\mu}{}_{\nu} = \delta_g \omega^{\mu}{}_{\nu}$ we must have $£_X g_M = £_X T = 0$.

We may write $g_M = \eta_{\alpha\beta} e^{\alpha} \otimes e^{\beta}$ where $\eta_{\alpha\beta}$ is diag $(-1, 1, 1, 1)$. So if $£_X g_M = 0$,

$$£_X \eta_{\alpha\beta} e^{\alpha} \otimes e^{\beta} + \eta_{\alpha\beta} £_X e^{\alpha} \otimes e^{\beta} + \eta_{\alpha\beta} e^{\alpha} £_X e^{\beta} = 0$$

Writing

$$£_X e^{\alpha} = \Lambda^{\alpha}{}_{\beta} e^{\beta},$$

for some $\Lambda^{\alpha}{}_{\beta}$ we have

$$\eta_{\alpha\beta}\Lambda^{\alpha}{}_{\mu} e^{\mu} \otimes e^{\beta} + \eta_{\alpha\beta} e^{\alpha} \otimes \Lambda^{\beta}{}_{\mu} e^{\mu} = 0$$

$$(\eta_{\mu\beta}\Lambda^{\mu}{}_{\alpha} + \eta_{\alpha\mu}\Lambda^{\mu}{}_{\beta}) e^{\alpha} \otimes e^{\beta} = 0$$

giving

$$\eta_{\mu\beta}\Lambda^{\mu}{}_{\alpha} + \eta_{\alpha\mu}\Lambda^{\mu}{}_{\beta} = 0$$

or $\Lambda_{\beta\alpha} + \Lambda_{\alpha\beta} = 0$, with the usual lowering convention. So it has been shown that $\pounds_X g_M = 0$ gives $\pounds_X e^{\alpha} = \Lambda^{\alpha}{}_{\beta} e^{\beta}$ with $\Lambda^{\alpha\beta} \in \mathscr{L}SO(3, 1)$, i. e. $\pounds_X e^{\alpha} = \delta_g e^{\alpha}$.

Similarly
$$\pounds_X T = \pounds_X T^{\mu} \otimes b_{\mu} + T^{\mu} \otimes \pounds_X b_{\mu} = 0$$
$$\pounds_X T^{\mu} \otimes b_{\mu} + T^{\mu} \otimes (-\Lambda^{\beta}{}_{\mu} b_{\beta}) = 0$$

with $\Lambda^{\beta}{}_{\mu} \in \mathscr{L}SO(3, 1)$ since $\pounds_X g_M = 0$ and b_{μ} transforms contragradiently to e^{μ}.

So
$$(\pounds_X T^{\mu} - \Lambda^{\mu}{}_{\beta} T^{\beta}) \otimes b_{\mu} = 0$$

giving
$$\pounds_X T^{\mu} = \Lambda^{\mu}{}_{\beta} T^{\beta} .$$

that is, an infinitesimal gauge transformation of the torsion forms. But

$$\begin{aligned}
\pounds_X T^{\mu} &= \pounds_X de^{\mu} + \pounds_X(\omega^{\mu}{}_{\nu} \wedge e^{\nu}) \\
&= d\pounds_X e^{\mu} + \pounds_X \omega^{\mu}{}_{\nu} \wedge e^{\nu} + \omega^{\mu}{}_{\nu} \wedge \pounds_X e^{\nu} \\
&= d(\Lambda^{\mu}{}_{\nu} e^{\nu}) + \pounds_X \omega^{\mu}{}_{\nu} \wedge e^{\nu} + \omega^{\mu}{}_{\nu} \Lambda^{\nu}{}_{\alpha} e^{\alpha} \\
&= d\Lambda^{\mu}{}_{\nu} \wedge e^{\nu} + \Lambda^{\mu}{}_{\nu} de^{\nu} + \pounds_X \omega^{\mu}{}_{\nu} \wedge e^{\nu} + \omega^{\mu}{}_{\nu} \Lambda^{\nu}{}_{\alpha} e^{\alpha}
\end{aligned}$$

Thus

$$\begin{aligned}
d\Lambda^{\mu}{}_{\nu} e^{\nu} + \Lambda^{\mu}{}_{\beta} de^{\beta} + \pounds_X \omega^{\mu}{}_{\nu} \wedge e^{\nu} + \omega^{\mu}{}_{\nu} \Lambda^{\nu}{}_{\alpha} e^{\alpha} \\
= \Lambda^{\mu}{}_{\beta} de^{\beta} + \Lambda^{\mu}{}_{\beta} \omega^{\beta}{}_{\nu} e^{\nu}
\end{aligned}$$
$$\pounds_X \omega^{\mu}{}_{\nu} \wedge e^{\nu} = \Lambda^{\mu}{}_{\beta} \omega^{\beta}{}_{\nu} e^{\nu} - \omega^{\mu}{}_{\beta} \Lambda^{\beta}{}_{\nu} \wedge e^{\nu} - d\Lambda^{\mu}{}_{\nu} \wedge e^{\nu}$$

These 24 equations give

$$\pounds_X \omega^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\beta} \omega^{\beta}{}_{\nu} - \omega^{\mu}{}_{\beta} \Lambda^{\beta}{}_{\nu} - d\Lambda^{\mu}{}_{\nu} ,$$

that is
$$\pounds_X \omega^{\mu}{}_{\nu} = \delta_g \omega^{\mu}{}_{\nu} ,$$

completing the proof of the lemma.

COROLLARY. — $i_{\nu} L_X e^{\mu} \in \mathscr{L}SO(3, 1)$ iff $\pounds_X g_M = 0$.

L_X is the covariant Lie derivative

$$\begin{aligned}
L_X &\equiv i_X D + D i_X \\
L_X e^{\mu} &= i_X D e^{\mu} + D i_X e^{\mu} \\
&= i_X(de^{\mu} + \omega^{\mu}{}_{\nu} e^{\nu}) + d i_X e^{\mu} + \omega^{\mu}{}_{\nu} i_X e^{\nu} \\
&= \pounds_X e^{\mu} + i_X \omega^{\mu}{}_{\nu} e^{\nu} \\
&= (\Lambda^{\mu}{}_{\nu} + i_X \omega^{\mu}{}_{\nu}) e^{\nu} , \\
&= (\Lambda^{\mu}{}_{\nu} + i_X \omega^{\mu}{}_{\nu}) e^{\nu} ,
\end{aligned}$$

putting
$$\pounds_X e^{\mu} = \Lambda^{\mu}{}_{\nu} e^{\nu}$$
$$i_{\nu} L_X e^{\mu} = \Lambda^{\mu}{}_{\nu} + i_X \omega^{\mu}{}_{\nu} ,$$

and so $\quad i_{\nu} L_X e^{\mu} \in \mathscr{L}SO(3, 1) \quad$ iff $\quad \Lambda^{\mu}{}_{\nu} \in \mathscr{L}SO(3, 1)$,

but from the lemma above this follows if, and only if $\pounds_X g_M = 0$.

Lemma B. — If $\pounds_X g_M = \pounds_X T = 0$ then $D i_\mu L_X e^\nu + i_X R^\nu{}_\mu = 0$

$$
\begin{aligned}
D i_\mu L_X e^\nu &= d i_\mu L_X e^\nu + \omega_\mu{}^\alpha i_\alpha L_X e^\nu - i_\mu L_X e^\alpha \omega_\alpha{}^\nu \\
&= d i_\mu(\pounds_X e^\nu + i_X \omega^\nu{}_\beta e^\beta) + \omega_\mu{}^\alpha i_\alpha(\pounds_X e^\nu + i_X \omega^\nu{}_\beta e^\beta) \\
&\qquad - i_\mu(\pounds_X e^\alpha + i_X \omega^\alpha{}_\beta e^\beta)\omega_\alpha{}^\nu \\
&= d(i_\mu \pounds_X e^\nu + i_X \omega^\nu{}_\beta \delta^\beta{}_\mu) + \omega_\mu{}^\alpha(i_\alpha \pounds_X e^\nu + i_X \omega^\nu{}_\beta \delta^\beta{}_\alpha) \\
&\qquad - i_\mu \pounds_X e^\alpha \omega_\alpha{}^\nu - i_X \omega^\alpha{}_\beta \delta^\beta{}_\mu \omega_\alpha{}^\nu \\
&= d i_\mu \pounds_X e^\nu + \pounds_X \omega^\nu{}_\mu - i_X d\omega^\nu{}_\mu + \omega_\mu{}^\alpha i_\alpha \pounds_X e^\nu \\
&\qquad + \omega_\mu{}^\alpha i_X \omega^\nu{}_\alpha - i_\mu \pounds_X e^\alpha \omega_\alpha{}^\nu - i_X \omega^\alpha{}_\mu \omega_\alpha{}^\nu \\
&= \pounds_X \omega^\nu{}_\mu - (i_\alpha \pounds_X e^\nu \omega^\alpha{}_\mu - \omega^\nu{}_\alpha i_\mu \pounds_X e^\alpha - d i_\mu \pounds_X e^\nu) \\
&\qquad - i_X(d\omega^\nu{}_\mu + \omega^\nu{}_\alpha \wedge \omega^\alpha{}_\mu)
\end{aligned}
$$

If $\pounds_X g_M = 0$ then $i_\alpha \pounds_X e^\nu = \Lambda^\nu{}_\alpha \in \mathscr{L}SO(3,1)$ and the term in the first bracket is an infinitesimal gauge transformation of $\omega^\nu{}_\mu$ with parameters $\Lambda^\nu{}_\alpha$. If $\pounds_X T = 0$ then this will cancel $\pounds_X \omega^\nu{}_\mu$, by the first lemma. This completes the proof.

These two lemmas; one rather trivial and the other obscure; can be combined with the covariant identities (3.5) and (3.6) to construct a closed 3-form for every Killing vector of the geometry, that is an X such that $\pounds_X g_M = \pounds_X T = 0$

$$
\begin{aligned}
d(i_X e^\mu \tau_\mu) &= D(i_X e^\mu \tau_\mu) \\
&= D i_X e^\mu \wedge \tau_\mu + i_X e^\mu D\tau_\mu \\
&= L_X e^\mu \wedge \tau_\mu - i_X T^\mu \wedge \tau_\mu + i_X e^\mu D\tau_\mu \\
&= L_X e^\mu \wedge \tau_\mu - S_\mu{}^\nu \wedge i_X R^\mu{}_\nu \qquad \text{by (3.5)} \\
&= e^\nu i_\nu L_X e^\mu \wedge \tau_\mu - S_\mu{}^\nu \wedge i_X R^\mu{}_\nu
\end{aligned}
$$

since $e^\nu i_\nu$ is the identify operator on 1-forms

$$= - i_\nu L_X e^\mu \wedge \tau_\mu \wedge e^\nu - S_\mu{}^\nu \wedge i_X R^\mu{}_\nu$$

If $\pounds_X g_M = 0$ then $i_\nu L_X e^\mu \in \mathscr{L}SO(3,1)$ by the corollary to lemma A, and so then

$$
\begin{aligned}
d(i_X e^\mu \tau_\mu) &= - i_\nu L_X e^\mu \frac{1}{2}(\tau_\mu \wedge e^\nu - \tau^\nu \wedge e_\mu) - S_\mu{}^\nu \wedge i_X R^\mu{}_\nu \\
&= - i_\nu L_X e^\mu D S_\mu{}^\nu - S_\mu{}^\nu \wedge i_X R^\mu{}_\nu \qquad \text{by (3.6)}
\end{aligned}
$$

If $\pounds_X T = 0$ as well then

$$d(i_X e^\mu \tau_\mu) = - D i_\nu L_X e^\mu \wedge S_\mu{}^\nu - i_\nu L_X e^\mu D S_\mu{}^\nu$$

by lemma B,

$$
\begin{aligned}
&= - D(i_\nu L_X e^\mu S_\mu{}^\nu) \\
&= - d(i_\nu L_X e^\mu S_\mu{}^\nu)
\end{aligned}
$$

So finally, if $\pounds_X g_M = \pounds_X T = 0$ then

$$\boxed{d(i_X e^\mu \tau_\mu + i_\nu L_X e^\mu S_\mu{}^\nu) = 0} \tag{3.7}$$

That symmetries lead to the existence of closed 3-forms was first shown by Trautman [*8*].

The theory of special relativity assumes a background geometry of Minkowski space; that is, the space-time with zero torsion and curvature. This space admits a ten-dimensional symmetry group: the Poincaré group. Thus (3.7) can be used to construct ten closed 3-forms. It should be noted that since (3.7) is gauge invariant and makes no reference to any preferred coordinates it is not necessary to use Minkowskian coordinates to compute angular momentum and momentum densities. This affords the computational advantage of allowing coordinates and frames to be adopted so as to exploit any symmetries a field system may possess.

Hehl and others [*9*] have argued that since the conservation laws for momentum and angular momentum in special relativity result from invariance under the Poincaré group this group must be fundamental to any gauge approach to gravity. Irrespective of any virtues or failings in attempting to formulate gravity as a Poincaré gauge theory I believe this motivation to be ill conceived. Gauge theories of gravity; including Einstein's; treat the geometry as dynamical, that is they give field equations for the metric and connection. Minkowski space is usually required to be one solution of the source free equations. As has been stresses above, the conservation laws in special relativity follow from the invariance of the Minkowski metric under a group of diffeomorphisms; The Poincaré group. It appears to me illogical to tie the structure group of a dynamical theory of geometry to the properties of one particular solution.

4. MASS AS THE PERIOD OF 2-FORM IN NEWTONIAN GRAVITY

In section 2 it was shown that the coupling of a field theory to a background electromagnetic field in a U(1) gauge invariant fashion lead to the existence of a closed 3-form. This gave a conservation law in the sense discussed in section 1. It was then argued that treating the electromagnetic field as dynamical introduced a different concept of electric charge. In particular, in source free regions, charge could be identified as the period of a 2-form constructed out of the dynamical electromagnetic fields; namely $*F$. It has been shown in the preceding section how the SL(2, C) gauge invariant coupling of a relativistic field theory to a background geometry with symmetries leads to the existence of closed 3-forms. Section 6 will exhibit that if the geometry is treated as dynamical, described by Einstein's theory, then mass and spin are definable as periods of 2-forms (in source free regions). Here it is shown how the concept of mass as a period of a 2-form is present in Newtonian gravity.

The Newtonian potential, ϕ, is a 0-form on a three dimensional Euclidean manifold. It is assumed to satisfy Poisson's equation,

$$d * d\phi = * \rho \tag{4.1}$$

where ρ is the (mass) density. (Here all exterior derivatives and Hodge duals act on forms on the 3-D manifold). Mass can be defined by

$$\mathrm{M_N} \equiv \int_{\mathrm{S}^2} * d\phi , \tag{4.2}$$

and is thus a period in source free regions. Since the Newtonian potential for a sphere does not depend on whether or not the sphere is spinning, there is no analogous period for angular momentum in Newton's theory.

5. PSEUDOTENSORS AND SUPERPOTENTIALS

Einstein's equations may be derived from an action principle with action density

$$\Lambda = \lambda \Lambda_{\mathrm{E}} + \Lambda_m \tag{5.1}$$

where

$$\Lambda_{\mathrm{E}} = \mathrm{R}_{\mu\nu} \wedge *(e^{\mu} \wedge e^{\nu}) , \tag{5.2}$$

λ being some coupling constant, and Λ_m the action density for the 'matter' fields. Variation of the orthonormal frames in (5.1) yields the field equation

$$\lambda \mathrm{G}_{\mu} = \tau_{\mu} \tag{5.3}$$

where G_{μ}, the Einstein 3-form, is given by

$$\begin{aligned} \mathrm{G}_{\mu} &= \mathrm{R}_{\alpha\beta} i_{\mu} *(e^{\alpha} \wedge e^{\beta}) \\ &= \mathrm{R}_{\alpha\beta} \wedge *(e^{\alpha} \wedge e^{\beta} \wedge e_{\mu}) \end{aligned} \tag{5.4}$$

and τ_{μ} was defined in section 3.

When discussing conservation laws in general relativity it is customary to draw comparisons with Maxwell's equations. To this end we may write

$$\lambda \mathrm{G}_{\mu} = d\mathrm{S}_{\mu} - t_{\mu} \tag{5.5}$$

for some 2-form S_{μ}, and 3-form t_{μ}; putting (5.3) in the form

$$d\mathrm{S}_{\mu} = \tau_{\mu} + t_{\mu} \tag{5.6}$$

resembling Maxwell' equations. S_{μ} is designated the superpotential and t_{μ} the pseudotensor; the prefix pseudo-indicating that neither transform as tensors under SL(2, C). Mimicking Maxwell's equations $\tau_\mu + t_\mu$ is identified as the current of total energy-momentum, t_μ being the contribution from the gravitational field. Of course there are myriad decompositions of the form (5.5). Those most frequently encountered have been reviewed by Thirring and Wallner [*10*]. No such explicit expressions will be exhibited here; all suffer from a common malady.

The total 4-momentum of the gravitating system is identified as

$$P_\mu \equiv \int_{S^2} S_\mu \tag{5.7}$$

Since S_μ does not transform as an SL(2, C) tensor there is the problem of deciding in which gauge to evaluate this integral. It is usually argued [*4*] that this definition only makes sense if we have asymptotic flatness, when it is possible to define 'asymptotically global Minkowskian frames'. It is then stated that (5.7) must be evaluated within this restricted class of gauges, and the limit taken in which the S^2 extends to infinity. In Minkowski space there exist coordinates x^μ such that we may choose $e^\mu = dx^\mu$; a frame that asymptotically takes this form will be deemed 'asymptotically globally Minkowskian'. Such frames are asymptotically related by a global Lorentz transformation and thus the P_μ will transform as a Lorentz vector under this residual gauge freedom.

If, allowing for the possibility of spinors in our theory, we treat the metric and connection as independent variables then we will get, in addition to (5.3), the field equation

$$\lambda D * (e_\mu \wedge e^\nu) = S_\mu{}^\nu \tag{5.8}$$

with $S_\mu{}^\nu$ as defined in section 3. This equation may also be written expressing a 3-form as the exterior derivative of a 2-form: say

$$d * (e_\mu \wedge e^\nu) = \frac{1}{\lambda} S_\mu{}^\nu - \omega_\mu{}^\alpha \wedge *(e_\alpha \wedge e^\nu) + *(e_\mu \wedge e^\alpha) \wedge \omega_\alpha{}^\nu \tag{5.9}$$

If we pursue the approach taken above then we may (3) identify

$$J_\mu{}^\nu \equiv \int_{S^2} *(e_\mu \wedge e^\nu) \tag{5.10}$$

as the angular momentum of thesystem. Here also the frames must be restricted to be 'asymptotically globally Minkowskian' (with the pre-

(3) In fact this is not what is usually done. It is usual to assume the asymptotic form of the metric and identify certain terms as being the angular momentum [*4*]. If the metric does approach this form then it can be checked that this approach will agree (up to a constant !) with (5.10).

requisite of asymptotic flatness), and the limit taken in which the S^2 extends to infinity.

There are many unsatisfactory features of this approach to conservation laws in general relativity. Firstly, decompositions of the form (5.5) are not unique, and it would appear merely fortuitous that those commonly made all give the same result for simple examples like the Schwartzschild solution. Secondly, such definitions would appear inadequate if, for example, gravitational radiation were considered. If it were necessary to require asymptotic flatness then it would be more palatable if this requirement were formulated in a less ad hoc fashion than the prescription given for choosing the frames in which to evaluate (5.7) and (5.10).

In section 3 it was shown that for a field theory in a background geometry conservation laws were only obtained if that geometry admitted symmetries. It is therefore perhaps a little optimistic to expect to define conservation laws in arbitrary spaces if that geometry is treated as dynamical. This indicates an ingredient missing, or not overtly present, in the above approach. That approach places reliance on parallels with Maxwell's theory. However, it does not parallel that crucial facet of electromagnetism: the definition of charge as a de Rham period in source free regions.

6. MASS AND ANGULAR MOMENTUM AS PERIODS IN EINSTEIN'S THEORY OF GRAVITY

If X is a vector field then its metric dual, $\tilde{X}$, is defined by

$$\tilde{X}(Y) = g_M(X, Y), \qquad \forall Y \in \mathscr{I}_1(M) \tag{6.1}$$

If K is a Killing vector then I shall call $* d\tilde{K}$ a 'Komar' (4) 2-form.

THEOREM. — If a solution to the vacuum Einstein equations is a space-time which admits a Killing vector then the associated Komar 2-form is closed.

Proof. —

$$\begin{aligned}
\tilde{K} &\equiv i_K e_\mu e^\mu \\
d\tilde{K} &= D\tilde{K} \\
&= D i_K e_\mu \wedge e^\mu && \text{when} \quad T^\mu = 0 \\
&= L_K e_\mu \wedge e^\mu, && \text{for} \quad T^\mu = 0 \\
&= i_\nu L_K e_\mu \wedge e^\nu \wedge e^\mu, && \text{since} \quad e^\mu i_\mu \omega \equiv p\omega \quad \forall \omega \in \Lambda^p(M)
\end{aligned}$$

(4) See appendix A.

$$\text{So} \qquad *\, d\tilde{K} = i_\nu L_K e_\mu * (e^\nu \wedge e^\mu)$$

$$\begin{aligned}
\text{and} \quad d * d\tilde{K} &= D * d\tilde{K} \\
&= D i_\nu L_K e_\mu * (e^\nu \wedge e^\mu) && \text{for } T^\mu = 0 \\
&= - i_K R_{\mu\nu} \wedge *(e^\nu \wedge e^\mu), && \text{by lemma B of section 3} \\
&= i_K R_{\mu\nu} \wedge *(e^\mu \wedge e^\nu) \\
&= i_K [R_{\mu\nu} \wedge *(e^\mu \wedge e^\nu)] - R_{\mu\nu} \wedge i_K * (e^\mu \wedge e^\nu) \\
&= i_K\left[R_{\mu\nu} \wedge \frac{1}{2} e^\alpha i_\alpha * (e^\mu \wedge e^\nu) \right] - R_{\mu\nu} \wedge i_K e^\alpha i_\alpha * (e^\mu \wedge e^\nu) \\
&= \frac{1}{2} i_K (e^\alpha \wedge G_\alpha) - i_K e^\alpha \wedge G_\alpha \quad \text{by (5.4)} && (6.2)
\end{aligned}$$

Thus $d * d\tilde{K} = 0$ when $G_\mu = T^\mu = 0$.

It is suggested that the mass and angular momentum of a gravitating system should be defined as periods of the Komar 2-forms associated with commuting timelike and spacelike Killing vectors. For these definitions to be applicable we need some source free region of spacetime; containing closed two dimensional surfaces; which is stationary and (at least) axially symmetric. The topological requirement is usually met by physically interpretable solutions, but renders these definitions inapplicable to, for example, the Taub N. U. T. [*11*] solutions. Of course not all solutions to Einstein's source free equations will be stationary and axially symmetric, but as stressed earlier we must be prepared to forego the definition of mass and angular momentum in arbitrary spaces. It is possible that asymptotically flat spaces allow the introduction of asymptotic Killing vectors [*12*] [*17*] and hence the application of these definitions, but no such attempt will be made here. Potentially more worrying than a lack of Killing vectors; when trying to identify mass and angular momentum as periods of Komar 2-forms; it a surfeit of Killing vectors: for then how are we to interpret all the periods? A partial answer is found in the theorem below; but first we need a preliminary result.

LEMMA. — If $£_{K_i} g_M = 0\ \forall K_i$, $i = 1, \ldots, n$ and $[K_i, K_j] = C^K_{ij} K_k$, where the C^K_{ij} are constants, then $£_{k_i} \tilde{K}_j = C^K_{ij} \tilde{K}_k$.

Proof. — $\tilde{K}_j(Y) = g_M(K_j, Y)\ \forall Y \in \mathscr{I}_1(M)$, by definition (6.1).

$$\text{So} \qquad (£_{k_i} \tilde{K}_j)(Y) + \tilde{K}_j(£_{k_i} Y) = £_{k_i} g_M(K_j, Y) + g_M(£_{k_i} K_j, Y) + g_M(K_j, £_{k_i} Y)$$

$$(£_{k_i} \tilde{K}_j)(Y) + \tilde{K}_j(£_{k_i} Y) = C^K_{ij} g_M(K_k, Y) + \tilde{K}_j(£_{k_i} Y)$$

using definition (6.1) and $£_{k_i} g_M = 0$.

$$\text{Thus} \qquad (£_{k_i} \tilde{K}_j)(Y) = C^K_{ij} \tilde{K}_k(Y), \qquad \forall Y$$

$$\text{giving} \qquad £_{k_i} \tilde{K}_j = C^K_{ij} \tilde{K}_k$$

THEOREM. — The periods of Komar 2-forms associated with an algebra of Killing vectors are related by the structure constants of that group. That is, if $\pounds_{k_i} g_M = 0 \ \forall K_i$, $i = 1, \ldots, n$ and $[K_i, K_j] = C^K_{ij} K_k$, with the C^K_{ij} constants,

$$P_k \equiv \int_{S^2} * d\tilde{K}_k, \qquad \text{where } d * d\tilde{K}_k = 0, \qquad \text{then } C^K_{ij} P_k = 0 \qquad \forall_{i,j}$$

Proof. —

$$\begin{aligned} \pounds_{k_i} * d\tilde{K}_j &= i_{k_i} d * d\tilde{K}_j + d i_{k_i} * d\tilde{K}_j \\ &= d i_{k_i} * d\tilde{K}_j \end{aligned}$$

But $\pounds_{k_i} * d\tilde{K}_j = * d \pounds_{k_i} \tilde{K}_j$, since the Lie derivative commutes with the Hodge dual for Killing vectors, and it always commutes with d.

So use of the lemma gives

$$C^K_{ij} * d\tilde{K}_k = d i_{k_i} * d\tilde{K}_j$$

Thus

$$\begin{aligned} C^K_{ij} \int_{S^2} * d\tilde{K}_k &= \int_{S^2} d i_{k_i} * d\tilde{K}_j \\ &= 0, \qquad \text{by Stoke's theorem}. \end{aligned}$$

Hence $$\underline{C^K_{ij} P_K = 0, \qquad \forall i, j.}$$

Determining the full extent to which this relation limits the number of independent periods is work for the future. One simple consequence is that Komar 2-forms associated with Killing vectors which generate the rotation group have vanishing periods. This is reassuring if we wish to identify angular momentum as the period of a Komar 2-form associated with a spacelike Killing vector.

Periods of Komar 2-forms are certainly useful characterisations of a spacetime, although it might be thought inappropriate to label them mass and angular momentum. In section 4 it was shown that mass is definable as a period in Newtonian gravity; thus defining mass as a period of a Komar form in general relativity gives a Newtonian limit. In a static spacetime with Killing vector

$$K_t \equiv \frac{\partial}{\partial t}$$

we may choose a gauge in which

$$e^0 = f dt, \qquad \text{and} \qquad \frac{\partial f}{\partial t} = 0$$

Then

$$\begin{aligned} \tilde{K}_t &= - f^2 dt \\ d\tilde{K}_t &= - d_{(3)} f^2 \wedge dt, \end{aligned}$$

where $d_{(3)}$ is the exterior derivative restricted to the three dimensional spacelike submanifold.

Thus $* d\tilde{K}_t = \underset{(3)}{*} d_{(3)} f^2$, where $\underset{(3)}{*}$ is the similarly restricted Hodge dual.

So if
$$P_t \equiv \int_{S^2} * d\tilde{K}_t$$
$$P_t \equiv \int_{S^2} \underset{(3)}{*} d_{(3)} f^2$$

If the spacetime is asymptotically flat then this expression will meet ([5]) the Newtonian one, (4.2), as the S^2 extends to infinity if

$$f^2 = \lambda(\phi + c) \tag{6.4}$$

with λ and c constants. This is in accord with the usual identification. It is explicity demonstrated in Appendix B that the constant appearing in the Kerr metric wich is conventionally interpreted as being the angular momentum is the period of the Komar 2-form associated with the axial symmetry.

The Komar 2-forms will not be closed in regions where sources are present. Thus the definitions of mass and angular momentum as periods are lost in these regions. If we continue the analogy with electromagnetism that has so far been relentlessly pursued then we will extend these definitions to regions where they no longer define periods. That is, we identify mass and angular momentum with $\int_{S^2} * d\tilde{K}_i$, for the appropriate K_i, even when $d * dK_i \neq 0$.

However, it is possible to argue that the concepts of mass and angular momentum derive their usefulness from being defined as periods; in which case it is natural to look for modified 2-forms that are closed even when sources are present. In general such 2-forms cannot be found. When the only source is an electromagnetic field then, with one caveat, such an extension can be made. Using (5.3) in (6.2) gives

$$\lambda d * d\tilde{K} = \frac{1}{2} i_K(e^\mu \wedge \tau_\mu) - i_k e^\mu \tau_\mu \tag{6.5}$$

The Maxwell stress form is given by

$$\tau_\mu = i_\mu F \wedge * F - F \wedge i_\mu * F \tag{6.6}$$

and so $e^\mu \tau_\mu = 0$, rembering that $e^\mu i_\mu \omega = p\omega \ \forall p$-forms ω.

Thus for an electromagnetic source

$$\lambda d * d\tilde{K} + i_k F \wedge * F - F \wedge i_k * F = 0$$
$$\lambda d * d\tilde{K} + i_k dA \wedge * F - dA \wedge i_k * F = 0, \qquad \text{putting } F = dA$$
$$\lambda d * d\tilde{K} + \pounds_k A \wedge * F - d(i_k A \wedge * F) + i_k A \wedge d * F$$
$$- d(A \wedge i_k * F) - A \wedge (\pounds_k * F - i_k d * \tilde{K}) = 0 .$$

([5]) In fact this will give $P_t = \lambda M_N$, but (4.2) and (6.3) are only trivially modified by a change of normalisation.

So $\lambda d * d\tilde{K} + £_k A * F - d(i_k A \wedge * F + A \wedge i_k * F) - A \wedge £_k * F = 0$ (6.7)

if we use the Maxwell equations.

For general solutions of the Einstein-Maxwell equations symmetries of the geometry will not be symmetries of the electromagnetic field [13] [14]. If we make the additional assumption that

$$£_k F = 0 \tag{6.8}$$

then

$$£_k A = d\chi \tag{6.9}$$

for some 0-form χ.

Putting (6.8) and (6.9) in (6.7), and again using the Maxwell equations, gives

$$\lambda d * d\tilde{K} + d(\chi * F - i_k A * F - A \wedge i_k * F) = 0$$

That is, if

$$j_k \equiv \lambda * d\tilde{K} + \chi * F - i_k A * F - A \wedge i_k * F, \tag{6.10}$$

then j_k is closed whenever we have a solution of the Einstein-Maxwell equations admitting a symmetry (generated by K) of the metric and the electromagnetic field ([6]). Since j_k explicitly contains A its behaviour under U(1) transformations may be suspect. However, (6.5) is manifestly U(1) invariant, and since we have only dropped (gauge dependent) total derivatives in obtaining (6.10) j_k can only change by a total derivative under changes of gauge. This ensures the U(1) invariance of its periods. Note, however, that j_k also explicitly contains χ; and for a given electromagnetic field (6.9) only defines χ up to a constant function. Thus the periods of j_k are only defined up to arbitrary multiples of the electric charge. This is no cause for concern if we are only interested in examining the number of independent periods definable for a given spacetime.

As was anticipated in section 2 the existence of 2-forms that are closed in source free regions of spacetime that admit symmetries is very much a property of a particular theory: in this case Einstein's. If, for example, we modify Einstein's theory by the inclusion of a cosmological constant then there is no 2-form we can add to the Komar 2-form to construct one which is closed under the above conditions ([7]). The realisation that vacuum Einstein spaces with symmetries are characterised by the periods of the Komar 2-forms suggests that it would be fruitful to look for similar expressions when considering alternative theories of gravity.

([6]) A similar analysis, with different motivation, has been carried out by carter [15]. Note, however, that Carter formulates his symmetry condition on the electromagnetic field as $£_k A = 0$, which is not a U(1) invariant requirement.

([7]) Of course, we may be able to do this for a particular solution, but what is required is a generic expression which is closed for all solutions.

CONCLUSION

It has been shown that the concepts of energy and angular momentum owe their existence to symmetries of the spacetime manifold. When the geometry is treated as being prescribed then these symmetries allow the construction of closed 3-forms. Treating the geometry as dynamical; with the dynamics described by Einstein's theory; allows mass and angular momentum to be defined (in source free regions), as periods of 2-forms. These definitions place no fundamental reliance on asymptotic flatness. It is stressed that the existence of these closed 2-forms is dependent on a particular theory of gravity, and suggested that similar 2-forms should be sought when considering alternative theories.

APPENDIX A

In this appendix some explanation of the naming of the 2-forms introduced in section 6 is given. In 1959 [*16*] Komar concluded that there existed a conservation law corresponding to an arbitrary coordinate transformation generated by a vector field ξ. That is (in Komar's notation), corresponding to every ξ is a 'generalised energy flux vector' $E^i(\xi)$, where

$$E^i(\xi) = 2(\xi^{i,n} - \xi^{n,i})_{;n} \tag{A.1}$$

such that

$$E^m(\xi)_{;m} = 0 \tag{A.2}$$

In the language of this paper (A.1) is equivalent to

$$E_\xi = d * d\tilde{\xi} \tag{A.3}$$

and (A.2) to

$$dE_\xi = 0 \tag{A.4}$$

which is of course true for any ξ.

In 1962 [*12*] Komar considered the situation where the vector field ξ is a Killing vector. In that case the components of what I have called the Komar 2-form will correspond to Komar's generalised energy flux vector'. However, reading Komar's paper with attention to the considerations of section 1 of this paper; and, for example, noting his treatment of angular momentum; should make clear some of the important distinctions between his approach and that taken here.

APPENDIX B

THE PERIOD OF THE KOMAR 2-FORM ASSOCIATED WITH THE SPACELIKE KILLING VECTOR OF THE KERR METRIC

In Boyer-Lindquist coordinates the Kerr metric is

$$g = -\frac{(\Delta - a^2 \sin^2 \theta)}{\rho^2} dt \otimes dt - \frac{2aMr \sin^2 \theta}{\rho^2}(dt \otimes d\phi + d\phi \otimes dt) + \frac{\rho^2}{\Delta} dr \otimes dr + \rho^2 d\theta \otimes d\theta + \frac{\sin^2 \theta}{\rho^2}[(r^2 + a^2)^2 - a^2\Delta \sin^2 \theta] d\phi \otimes d\phi \quad (B.1)$$

where
$$\Delta \equiv r^2 - 2Mr + a^2 \quad (B.2)$$
and
$$\rho^2 \equiv r^2 + a^2 \cos^2 \theta \quad (B.3)$$

We may choose a gauge such that

$$e^0 = \frac{\sqrt{\Delta}}{\rho}(dt - a \sin^2 \theta d\phi) \quad (B.4)$$

$$e^1 = \frac{\rho}{\sqrt{\Delta}} dr \quad (B.5)$$

$$e^2 = \rho d\theta \quad (B.6)$$

$$e^3 = \sin \theta/\rho\,[(r^2 + a^2)d\phi - adt] \quad (B.7)$$

The Killing vector associated with the axial symmetry is $\frac{\partial}{\partial \phi}$. From the definition (equation (6.1)) we have

$$\frac{\tilde{\partial}}{\partial \phi} = -\frac{2aMr \sin^2 \theta dt}{\rho^2} + \frac{\sin^2 \theta}{\rho^2}[(r^2 + a^2)^2 - a^2\Delta \sin^2 \theta] d\phi \quad (B.8)$$

$$d\frac{\tilde{\partial}}{\partial \phi} = -2aM \sin^2 \theta \left[\frac{r}{\rho^2}\right]_r dr \wedge dt - 2aMr\left[\frac{\sin^2 \theta}{\rho^2}\right]_\theta d\theta \wedge dt + \left\{\frac{\sin^2 \theta}{\rho^2}[(r^2 + a^2)^2 - a^2\Delta \sin^2 \theta]\right\}_r dr \wedge d\phi + \left\{\frac{\sin^2 \theta}{\rho^2}[(r^2 + a^2)^2 - a^2\Delta \sin^2 \theta\right\}_\theta d\theta \wedge d\phi \quad (B.9)$$

Equations (B.4) ... (B.7) can be inverted to give

$$dt = \frac{(r^2 + a^2)e^0}{\rho\sqrt{\Delta}} + \frac{a \sin \theta e^3}{\rho}$$

$$d\phi = \frac{ae^0}{\rho\sqrt{\Delta}} + \frac{e^3}{\rho \sin \theta}$$

$$d\theta = \frac{e^2}{\rho}$$

$$dr = \frac{\sqrt{\Delta}}{\rho} e^1,$$

and hence

$$dr \wedge dt = \frac{(r^2 + a^2)e^{10}}{\rho^2} + \frac{\sqrt{\Delta} a \sin\theta e^{13}}{\rho^2}$$

$$d\theta \wedge dt = \frac{(r^2 + a^2)e^{20}}{\rho^2\sqrt{\Delta}} + \frac{a \sin\theta}{\rho^2} e^{23}$$

$$dr \wedge d\phi = \frac{a}{\rho^2} e^{10} + \frac{\sqrt{\Delta}}{\rho^2 \sin\theta} e^{13}$$

$$d\theta \wedge d\phi = \frac{a e^{20}}{\rho^2\sqrt{\Delta}} + \frac{e^{23}}{\rho^2 \sin\theta}$$

We can now calculate

$$*(dr \wedge dt) = \frac{2Mra \sin\theta}{\rho^2} d\theta \wedge dt + \frac{\sin\theta}{\rho^2} [\Delta a^2 \sin^2\theta - (r^2 + a^2)^2] d\theta \wedge d\phi$$

$$*(d\theta \wedge dt) = \frac{-2Mra \sin\theta}{\rho^2\Delta} dr \wedge dt + \frac{\sin\theta}{\rho^2\Delta} [(r^2 + a^2)^2 - a^2\Delta \sin^2\theta] dr \wedge d\phi$$

$$*(dr \wedge d\phi) = \frac{(\Delta - a^2 \sin^2\theta)}{\rho^2 \sin\theta} dt \wedge d\theta - \frac{2Mar \sin\theta}{\rho^2} d\theta \wedge d\phi$$

$$*(d\theta \wedge d\phi) = \frac{(a^2 \sin^2\theta - \Delta)}{\rho^2\Delta \sin\theta} dt \wedge dr + \frac{2aMr \sin\theta}{\rho^2\Delta} dr \wedge d\phi$$

Use of these results in (B.9) gives

$$* d\frac{\tilde{\partial}}{\partial\phi} = \left\{ \frac{-2a^3M(a^2 - r^2)\sin^5\theta}{(r^2 + a^2\cos^2\theta)^2} - \frac{2aM(r^2 + a^2)(3r^2 - a^2)}{(r^2 + a^2\cos^2\theta)^2} \right\} d\theta \wedge d\phi + \dots$$

Integrating over the r, t constant hypersurface, S^2,

$$\int_{S^2} * d\frac{\tilde{\partial}}{\partial\phi} = -4\pi a^3M(a^2 - r^2) \int_0^\pi \frac{\sin^5\theta d\theta}{(r^2 + a^2\cos^2\theta)^2} - 4\pi aM(r^2 + a^2)(3r^2 - a^2) \int_0^\pi \frac{\sin^3\theta d\theta}{(r^2 + a^2\cos^2\theta)^2}$$

$$= 8\pi Maz(1 + z) \int_0^{\frac{\pi}{2}} \frac{\sin^3\theta \cos^2\theta d\theta}{(1 - z\cos^2\theta)^2} + 8\pi Ma(z - 3) \int_0^{\frac{\pi}{2}} \frac{\sin^3\theta d\theta}{(1 - z\cos^2\theta)^2}$$

where $z \equiv -\frac{a^2}{r^2}$. Reference to p. 389 of Gradshteyn and Ryzhik [*18*] then gives

$$\int_{S^2} * d\frac{\tilde{\partial}}{\partial\phi} = 4\pi Maz(1 + z)B\left(2, \frac{3}{2}\right)F\left(\frac{3}{2}, 2; \frac{7}{2}; z\right) + 4\pi Ma(z - 3)B\left(2, \frac{1}{2}\right)F\left(\frac{1}{2}, 2; \frac{5}{2}; z\right)$$

where their notation is used for the beta functions $B(x, y)$ and hypergeometric functions $F(\alpha, \beta; \gamma; z)$. Substituting the definitions of these function gives

$$\int_{S^2} * d\frac{\tilde{\partial}}{\partial\phi} = 16\pi Ma\left\{ z(1+z)\sum_{n=0}^{\infty}\frac{(n+1)z^n}{(2n+5)(2n+3)} + (z-3)\sum_{n=0}^{\infty}\frac{(n+1)z^n}{(2n+1)(2n+3)}\right\}$$

$$= 16\pi Ma\left\{\sum_{n=1}^{\infty}\frac{nz^n}{(2n+3)(2n+1)} + \sum_{n=2}^{\infty}\frac{(n-1)z^n}{(2n+1)(2n-1)} - 3\sum_{n=0}^{\infty}\frac{(n+1)z^n}{(2n+1)(2n+3)} + \sum_{n=1}^{\infty}\frac{nz^n}{(2n-1)(2n+1)}\right.$$

Thus we finally get

$$\int_{S^2} * d\frac{\tilde{\partial}}{\partial\phi} = -16\pi Ma, \qquad \text{(B.10)}$$

which will be seen to agree with the usual identification of the angular momentum (up to a multiple).

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