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BERNARD DIU

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Section A :
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Note on a recent proposal concerning statistical inference in quantum theory

by

Bernard DIU
Laboratoire de Physique Théorique
et Hautes Énergies (*), Paris (**)

RÉSUMÉ. — Une proposition récente concernant la solution du problème d'inférence statistique en mécanique quantique est examinée et s'avère inadéquate : son application à un cas simple mais non-trivial conduit à une famille infinie de résultats possibles, qui peuvent être assez différents de la réponse donnée par le postulat de projection habituel.

ABSTRACT. — A new solution recently proposed for the statistical inference problem in quantum mechanics is shown to be inadequate: its application to a simple but non-trivial case yields an infinite family of possible results, which can be quite different from the answer based on the usual projection postulate.

In a recent work [*1*], N. Hadjisavvas studies a particular « distance » between quantum states [*2*], and suggests to use it as a starting point to solve the statistical inference problem in quantum theory. The aim is ambitious, since Hadjisavvas proposes to replace the usual « projection postulate » of quantum mechanics by a hopefully much more general one, based on the concept of distance between states, which, he shows, implies the former postulate in the particular case of pure states.

(*) Laboratoire associé au C. N. R. S., n° LA 280.
(**) *Postal address :* Université Paris VII, Tour 33, 1er étage, 2, place Jussieu, 75251 Paris Cedex-05, France.

The purpose of the present note is to show that the new postulate presented in ref. 1 is actually ambiguous, and cannot then be accepted as it stands. Section 1 will briefly recall the statistical inference problem under discussion (§ 1.1), summarize the solution proposed by Hadjisavvas (§ 1.2), and compare it with the usual prescription (§ 1.3). Section 2 will then analyze a particular and very simple example (§ 2.1), to which the new proposal can rather easily be applied (§ 2.2), leading to a whole family of possible solutions that can be quite different from one another and from the usual result (§ 2.3). The conclusions are finally stated in section 3.

1) BRIEF DESCRIPTION OF THE NEW PROPOSAL

1.1) The statistical inference problem in quantum mechanics.

The so-called « statistical inference » problem is concerned with the determination of the density operator describing the state of a quantum system, given some specific information about the preparation of this state.

To be more precise, let us concentrate on the following simple and classical situation. Suppose the state of the system is known to be described by the density operator W_0, just before a measurement is performed on the system. This measurement yields a particular result, which one can characterize, as usual [3], by a particular projector E onto a subspace of the Hilbert space of states. One assumes that the measurement does not destroy the system, and that, if it were immediately repeated, it would give the same result again, with absolute certainty (« measurement of the first kind » [3]).

The question is then: given W_0 and E, what is the density operator W_1 describing the state of the system immediately after the measurement ?

1.2) The solution proposed by N. Hadjisavvas.

The proposal presented in ref. 1 is based on a particular definition of a distance between quantum states. This definition had already been introduced, for a different purpose, by Jauch, Misra and Gibson [2]. N. Hadjisavvas shows in ref. 1 that it has a formally simple expression in terms of the Hilbert space characterization of quantum states. If W and W′ are two density operators (associated to different states of the same system), the distance between the two corresponding states is:

$$d(W, W') = \frac{1}{2} || W - W' ||_1 \qquad (1)$$

where the norm $|| \, . \, ||_1$ is that given by the trace [4] :

$$|| A ||_1 = \mathrm{Tr} \, | A | . \tag{2}$$

In practical terms, when A is a hermitian operator, $\mathrm{Tr} \, | A |$ is just the sum of the absolute values of its eigenvalues λ_n :

$$|| A ||_1 = \sum_n | \lambda_n | . \tag{3}$$

On the basis of this distance, N. Hadjisavvas proposes the following solution to the problem outlined in § 1.1 (*).

Let S be the set of all density operators W which are compatible with the result of the measurement, i. e. such that:

$$\mathrm{Tr} \, (WE) = 1 . \tag{4}$$

The final density operator W_1 one is looking for must belong to the set S. According to ref. 1, one should choose as W_1 the member of this set which is closest to the initial density operator W_0, with respect to the distance previously defined:

$$W_1 : \begin{cases} \mathrm{Tr} \, (W_1 E) = 1 \\ d(W_1, W_0) = \inf \{ d(W, W_0) : \mathrm{Tr} \, (WE) = 1 \} . \end{cases} \tag{5}$$

1.3) **Comparison with the usual prescription.**

Within the usual framework of quantum mechanics, the answer to the problem set in § 1.1 is both unique and simple. It is derived from the *projection postulate*, which is included in the set of axioms.

The projection postulate is formulated for pure states: if, just before the measurement, the system is in a state described by a (normed) vector $| \psi_0 \rangle$, its state after the measurement will be described by the vector $| \psi_1 \rangle$ obtained by projecting $| \psi_0 \rangle$ onto the subspace associated with the result found:

$$| \psi_1 \rangle = \frac{E | \psi_0 \rangle}{[\langle \psi_0 | E | \psi_0 \rangle]^{1/2}} . \tag{6}$$

This formulation is sufficient, since it can be unambiguously applied also to the case of statistical mixtures of pure states. If the state of the

(*) Hadjisavvas's proposal is meant to apply in a more general context (see ref. 1). For simplicity, only its application to the simple situation presented in § I-1 is formulated here.

system is described by a density operator W_0 before the measurement. one can decompose it as:

$$W_0 = \sum_i p_i | u_i \rangle \langle u_i | \tag{7}$$

with

$$0 \leqslant p_i \leqslant 1 \tag{8 a}$$

$$\sum_i p_i = 1 . \tag{8 b}$$

This expression is given the following physical interpretation: the system has probability p_i to be in the pure state $| u_i \rangle$. One is then led uniquely to the result that the density operator W_1 describing the system after the measurement is:

$$W_1 = \frac{EW_0E}{\mathrm{Tr}\,(W_0E)}. \tag{9}$$

Hadjisavvas's proposal consists in discarding the projection postulate, to replace it by a more general one (see § 1.2 and ref. 1). The projection « postulate » for pure states is then derived from the new prescription (see ref. 1, theorem 6 and corollary 3), thus becoming a « projection theorem ».

2) ANALYSIS OF A SIMPLE SITUATION

2.1) The example considered.

In order to test the new prescription, let us analyze the following very simple case. Let us assume that the initial density operator W_0 and the projector E associated to the result of the measurement have a common set of eigenstates. To avoid unnecessary complications, while retaining the essential features of the problem, let us concentrate on a 3-dimensional state-space and write W_0 and E as:

$$W_0 = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix} \tag{10 a}$$

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{10 b}$$

Formula (9) immediately yields the final density operator W_1 derived from the usual projection postulate:

$$W_1 = \begin{pmatrix} \dfrac{p_1}{p_1 + p_2} & 0 & 0 \\ 0 & \dfrac{p_2}{p_1 + p_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

2.2) **Application of the new proposal.**

Consider now the procedure proposed in ref. 1 and recalled in § 1.2. The most general density operator W compatible with the result found in the measurement, i. e. verifying condition (4), is easily shown to be of the form:

$$W = \left(\begin{array}{cc|c} & & 0 \\ & w & 0 \\ 0 & 0 & 0 \end{array}\right) \tag{12}$$

where w is a 2×2 density matrix. W thus depends on 3 real parameters, for instance:

$$w = \frac{1}{2}\,[\mathbb{1} + \vec{P} \,.\, \vec{\sigma}\,]$$

with: $$|\vec{P}| \leqslant 1 \tag{14}$$

($\vec{\sigma}$ denotes the three Pauli matrices).

To calculate the distance $d(W, W_0)$, one has to find the eigenvalues of $W - W_0$, which writes here:

$$W - W_0 = \left(\begin{array}{cc|c} & & 0 \\ & w - w_0 & 0 \\ 0 & 0 & -p_3 \end{array}\right) \tag{15}$$

where:

$$w_0 = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}. \tag{16}$$

One of the eigenvalues of $W - W_0$ is evidently $-p_3$. Calling λ_+ and λ_- those of $w - w_0$, one gets from formulae (1)-(3):

$$d(W, W_0) = \frac{1}{2}\,[p_3 + |\lambda_+| + |\lambda_-|]. \tag{17}$$

One should then find the minimal value of $d(W, W_0)$, and the density

operator $W = W_1$ which reaches it. In the present simple case, one knows that:

$$\lambda_+ + \lambda_- = p_3 \tag{18}$$

(This is because the difference $W - W_0$ of two density operators necessarily has a vanishing trace). p_3 being positive (or zero), so is at least one of the eigenvalues $\lambda_\pm$, say:

$$\lambda_+ \geqslant 0 . \tag{19}$$

Formula (17) then becomes:

$$d(W, W_0) = \frac{1}{2} [p_3 + \lambda_+ + |\lambda_-|] . \tag{20}$$

This implies in turn:

$$d(\hat{W}, W_0) \geqslant \frac{1}{2} [p_3 + \lambda_+ + \lambda_-] , \tag{21}$$

or, using equation (18):

$$d(W, W_0) \geqslant p_3 .$$

One sees at the same time that this minimal value of the distance is attained whenever:

$$\lambda_- \geqslant 0 . \tag{23}$$

2.3) **Discussion.**

That there exists an infinite set of density operators W verifying this last condition is easily demonstrated.

Restricting for simplicity to the one-parameter subset:

$$W = \begin{pmatrix} q & 0 & 0 \\ 0 & 1-q & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{24}$$

with:

$$0 \leqslant q \leqslant 1 , \tag{25}$$

one sees that the inequalities (19) and (23) are satisfied for all q such that:

$$p_1 \leqslant q \leqslant 1 - p_2 . \tag{26}$$

W_1, as given by formula (11), belongs to this family, but the interval $[p_1, 1 - p_2]$ shrinks down to a point only in the special case when p_3 vanishes —in which case the example looses all interest since the measurement brings no new information and the initial density operator W_0 just goes through unmodified.

Of course, it is also easy to exhibit non-diagonal density operators verifying condition (23). Furthermore, it should be clear that the arguments presented above are not restricted to 3-dimensional state spaces and are actually general.

To get a more quantitative feeling about how different from W_1 these density operators can be, one can consider for instance the case:

$$p_1 = p_2 = p_3 = \frac{1}{3}. \tag{27}$$

W_1 is then of the form (24), with $q = q_1$ such that:

$$q_1 = 1 - q_1 = \frac{1}{2}. \tag{28}$$

On the other hand, the inequalities (26) allow:

$$q = \frac{1}{3}; \qquad 1 - q = \frac{2}{3} \tag{29}$$

(or *vice versa*). The ratio of the two final probabilities can then reach 2, when it should be 1. This maximal discrepancy increases of course if the relative importance of p_3 increases. For instance, if:

$$p_1 = p_2 = \frac{1}{6}; \qquad p_3 = \frac{2}{3}, \tag{30}$$

the ratio $q/(1 - q)$ can vary between 1/5 and 5, when it should still be 1.

3) CONCLUSION

The point of view presented in ref. 1 on the statistical inference problem in quantum mechanics has been shown to be inadequate, or at least incomplete. The aim was to replace the well-known projection postulate by a new, hopefully more general one, which contains the former as a special case. However, the simple example analyzed above proves that the new postulate is actually ambiguous when applied to density operators instead of state vectors.

One could try to complement the prescription in such a way that the result be finally unique. The above example could help in choosing the complementary condition. It is however not clear whether such an attempt is really worthwhile: the expression (1) for the distance between states, although formally compact, is difficult to apply in practice (this is apparent in § 2.2 above, in spite of the extreme simplicity of the case considered); this complication is to be contrasted with the straightforwardness of the usual projection prescription. It would then seem more reasonable to take the latter as a starting point, and to try eventually to generalize it, if one really felt the necessity of doing so.

REFERENCES

[1] N. HADJISAVVAS, *Ann. I. H. P.*, t. **A 35**, n° 4 (To be published) and Thèse, Université de Reims, octobre 1981.
[2] J. M. JAUCH, B. MISRA and A. G. GIBSON, *Helv. Phys. Acta*, t. **41**, 1968, p. 513.
[3] J. VON NEUMANN, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, 1955.
[4] R. SCHATTEN, *Theory of cross-norm spaces*, Princeton University Press, 1950.

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