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Regular operators on partial inner product spaces

by

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ABSTRACT. — A regular operator on a partial inner product space V is an operator which, together with its adjoint, is defined on the whole space V. The set of all regular operators on V is a $*$-algebra, isomorphic to a $*$-algebra of unbounded operators on the dense domain $V^{\#}$. Their spectral properties are investigated; in particular, criteria are given for a symmetric regular operator to be essentially self-adjoint and for its self-adjoint closure to have regular spectral projections. Some applications are discussed, in quantum mechanics and in the representation theory of Lie groups or algebras.

RÉSUMÉ. — Un opérateur régulier dans un espace à produit interne partiel V est un opérateur défini, ainsi que son adjoint, sur l'espace V tout entier. L'ensemble des opérateurs réguliers sur V est une $*$-algèbre, isomorphe à une $*$-algèbre d'opérateurs non bornés sur le domaine dense $V^{\#}$. On étudie leurs propriétés spectrales ; en particulier, on donne des critères pour qu'un opérateur régulier symétrique soit essentiellement auto-adjoint et pour que les projecteurs spectraux de sa fermeture auto-adjointe soient eux-mêmes des opérateurs réguliers. Ces notions sont appliquées en mécanique quantique et dans la théorie des représentations de groupes et algèbres de Lie.

1. INTRODUCTION

In a theory which is formulated in a Hilbert space, it happens frequently that interesting operators behave very badly: they may be unbounded, non densely defined, or worse.

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Quantum theories, and especially Quantum Field Theory, are typical in this respect. In such a case it is often convenient to build some sort of superstructure « around » the Hilbert space $\mathcal{H}$, by extending, so to speak, the domain of definition of the inner product beyond $\mathcal{H}$. Various structures have emerged in this way, which indeed can accomodate very singular operators. Such are, for instance, scales of Hilbert or Banach spaces (like the Sobolev spaces), rigged Hilbert spaces (as in the theory of distributions) or nested Hilbert spaces. The latest and, in a sense, the most extreme proposal along this trend is the concept of partial inner product space (PIP-space), which can be seen as a general framework embodying the various structures just listed. PIP-spaces have been introduced and studied systematically by Grossmann and one of us in a series of papers [*1-4*], to which we refer for further information and full references.

Given a PIP-space V around $\mathcal{H}$, operators on V may be defined and handled in a natural fashion (see [*2*]), and they can be very singular objects indeed, when viewed from $\mathcal{H}$. The set of all operators on V is a vector space (since operators are basically extensions of sesquilinear forms), and its carries an involution $A \leftrightarrow A^{\times}$, where $A^{\times}$ is the adjoint (or transpose) of A [*Caution* : this notation differs from the one used in [*1-4*]. But two operators may not always be multiplied one by another. Obviously this is a nuisance (think of a representation of a group or an algebra !), and a more restricted class of operators should be considered.

We will characterize such a class in the present paper, the so-called *regular operators.* They are exactly those operators A on V such that both A and its adjoint $A^{\times}$ have as domain the whole of V. Such operators can be multiplied freely and indeed constitute a $*$-algebra. Considered in $\mathcal{H}$, regular operators are, together with their adjoints, closable operators with a common dense invariant domain. Thus the set of all regular operators on V, which we will denote by Reg (V), may be identified with an algebra of unbounded operators in the sense of Powers [*5*] and Lassner [*6*]. In fact, as we show in Section 2, the two concepts are essentially identical. This result is extremely important for the development of PIP-space theory and its applications, in view of the immense wealth of information on algebras of unbounded operators that has been obtained in the recent years by a number of authors [*5-9*]. On the other hand, the PIP-space point of view may shed new light on some questions about unbounded operator algebras.

Spectral properties of regular operators is an obvious problem to look at, and the rest of this paper is devoted to it. Some steps in that direction have been taken recently by Epifanio and Trapani [*10*]. We generalize some of their results and put them in proper perspective, within the PIP-space framework. Basically we answer two questions:

i) When is a symmetric ($T = T^{\times}$) regular operator essentially self-adjoint? (Sec. 4).

ii) Assuming it is, when are the spectral projections of its closure themselves regular operators? (Sec. 5).

For answering question (*i*), it proves useful to consider a notion of spectrum different from the familiar Hilbert space spectrum. Actually, this is formulated in Sec. 3 for an arbitrary algebra of unbounded operators, not only Reg (V), and so is the answer to question (*i*).

The analysis of question (*ii*) points towards an extension of those results, by considering a special class of PIP-spaces. The lesson of (*ii*) is, indeed, the more a given operator is « matched » with the structure of the PIP-space, the better are its spectral properties. So the best results are to be expected when the PIP-space itself is generated from the given algebra of unbounded operators. Such PIP-spaces are frequent in applications, and will be studied systematically in another publication [*11*].

The last section of the paper is devoted to applications. First, we show that the concept of regular operator fits perfectly in the rigged Hilbert space formulation of Quantum Mechanics, as described some time ago by Böhm [*12*], Roberts [*13*] and one of us [*14*]. This applies in particular to the so-called Dirac bra-ket formalism, which is based on the nuclear spectral theorem. Second, we illustrate some of the concepts and results of this paper in the context of representations of Lie algebras and their enveloping algebras.

Finally we collect in a short appendix some possible alternative definitions.

2. THE $*$-ALGEBRA OF REGULAR OPERATORS

Throughout the paper V_I will denote a nondegenerate PIP-space, with index set I. By this we mean either a general PIP-space, as in [*1*] or an indexed PIP-space, as in [*4*]. In the first case, $I = F$ and $\{ V_r \mid r \in I \}$ is the set of *all* assaying subspaces. In the second case ($I \neq F$), we always mean that the extreme assaying subspaces, $V^{\#}$ and V, belong to the defining family $\{ V_r \mid r \in I \}$. When necessary, the other assaying subspaces will be called *proper*, and their set denoted by I_0. With that convention, we may drop the word « indexed » and get unified statements about operators.

We will assume furthermore that the partial inner product (pip) $\langle . \mid . \rangle$ is positive definite and that V_I possesses a central Hilbert space $\mathscr{H}$, namely the completion of $V^{\#}$ with respect to the pip-norm $\langle f \mid f \rangle^{1/2}$, $f \in V^{\#}$. As shown in [*4*], the last requirement follows from the positive definiteness of the pip whenever V is quasi-complete for its Mackey topology $\tau(V, V^{\#})$.

The nondegeneracy assumption made above can be formulated in two ways. If V_I is assumed to be nondegenerate, i. e. $(V^{\#})^{\perp} = \{ 0 \}$, then each pair $\langle V_r, V_{\bar{r}} \rangle$ of assaying subspaces is a dual pair with respect to the partial inner product. Conversely, let each *proper* pair $\langle V_r, V_{\bar{r}} \rangle$, $r \in I_0$, be nondegenerate for $\langle . | . \rangle$, so that Mackey topologies $\tau(V_r, V_{\bar{r}})$ may be defined. Then $V = \operatorname{ind\,lim}_{r \in I_0} V_r$ and $V^{\#} = \operatorname{proj\,lim}_{r \in I_0} V_r$ are also a dual pair for $\langle . | . \rangle$ and $\tau(V, V^{\#}) = \operatorname{ind\,lim}_{r \in I_0} \tau(V_r, V_{\bar{r}})$. For the record we state this observation as a lemma.

LEMMA 2.1. — A PIP-space V_I is nondegenerate iff the partial inner product is nondegenerate on each pair of compatible, proper, assaying subspaces. □

The condition applies to all particular types of indexed PIP-spaces introduced in [*4*]: reflexive, type (B), type (H), where of course the qualification applies to *proper* assaying subspaces only.

We turn now to operators, as defined in [*2*] and [*4*]. An operator on a PIP-space V_I is a map $A : \mathcal{D}(A) \to V$, where $\mathcal{D}(A)$ is the largest union of assaying subsets (actually subspaces) of V such that the restriction of A to any of them is linear and continuous into V (each space with its own Mackey topology). The domain $\mathcal{D}(A)$ is a vector subspace in most cases, for instance when V_I is of type (B) or (H) (see [*4*]). As usual we denote by Op (V_I) the set of all operators on V_I. The set Op (V_I) carries an involution $A \leftrightarrow A^{\times}$ (the adjoint of A), and it is a vector space, but in general it is not an algebra, for the product of two operators is not always defined (such a set has been called a *partial ∗-algebra* by Borchers [*15*]). In order to avoid this undesirable feature, we introduce a smaller class of maps.

DEFINITION 2.2. — An operator A on a PIP-space V_I is said to be *regular* if $\mathcal{D}(A) = \mathcal{D}(A^{\times}) = V$.

Thus a regular operator is a continuous linear map of V into itself, (the restriction of) which maps $V^{\#}$ into itself continuously (each space with its own Mackey topology). Alternatively, a regular operator is a linear map $A : V \to V$, such that $AV^{\#} \subseteq V^{\#}$, the transpose of which (in the dual pair $\langle V^{\#}, V \rangle$) $A^{\times}$ has the same properties; indeed $AV \subseteq V$ and $A^{\times}V^{\#} \subseteq V^{\#}$ imply that both maps are weakly continuous, equivalently Mackey continuous, and similarly for $AV^{\#} \subseteq V^{\#}$ and $A^{\times}V \subseteq V$. We denote by Reg (V) the set all regular operators on V_I. Another definition, slightly more restrictive, is also possible; we have relegated it to the Appendix. Here we will stick to Definition 2.2. Actually regular operators were briefly considered by Schwartz [*16*], under the name of « continuable kernels » (« noyaux prolongeables »). In the particular case of distributions, he calls them « regular compact kernels », so that our terminology generalizes his (it also supersedes the term « good operators » which was used in earlier works of ours).

The following properties are immediate:

PROPOSITION 2.3. — Let V_I be a PIP-space, Reg (V) the set of its regular operators. Then:

a) Given $A \in$ Reg (V), the products AB and BA are well-defined for any operator B.

b) Reg (V) is a $*$-algebra. □

Notice that Reg (V) depends only on the dual pair $\langle V^{\#}, V \rangle$, not on the compatibility $\#$ itself. Any compatibility on V comparable to $\#$, including the trivial one, yields the same space $V^{\#}$ (see Prop. 5.3 of [*3*]), thus the same regular operators. This justifies the notation Reg (V) instead of Reg (V_I). For the operators themselves, the situation is not that simple. It is true that every operator B is uniquely determined by its restriction to $V^{\#}$, i. e. as a map $B : V^{\#} \to V$, but the product of two operators on V_I may be defined for the given compatibility $\#$, and cease to be for a coarser compatibility $\hat{\#}$. Thus the two sets Op (V_I) and Op ($V_{\hat{I}}$), corresponding to $\#$ and $\hat{\#}$, are both in bijection with the space $\mathscr{L}(V^{\#}, V)$, but their algebraic structure may be different. Thus we keep the notation Op (V_I).

Before continuing let us give a few examples.

i) For $V = \omega$, the space of all complex sequences ($V^{\#} = \varphi$, the finite sequences), operators are arbitrary infinite matrices, regular operators are infinite matrices with finite rows and finite columns.

ii) For $V = \mathscr{S}'$, the tempered distributions, operators are arbitrary tempered kernels, whereas regular operators are those (regular) kernels that can be extended to $\mathscr{S}'$ and map $\mathscr{S}$ into itself; a typical example is the Fourier transformation.

iii) In Bargmann's space $\mathfrak{E}'$ of entire functions [*17*], regular operators are called properly bounded: such are e. g. multiplication by z and derivation $\frac{d}{dz}$.

Notice that in each case, the algebra Reg (V) is fairly large, in particular it is nonabelian, but contains large abelian subalgebras: the set of all diagonal matrices for ω, the algebra $\mathfrak{P}[x]$ of all polynomials in x, and also the algebra $\mathfrak{P}\left[\frac{d}{dx}\right]$ in the case of $\mathscr{S}'$, the algebras $\mathfrak{P}[z]$ and $\mathfrak{P}\left[\frac{d}{dz}\right]$ for $\mathfrak{E}'$.

REMARKS 2.4. — *a*) For every subspace V_r such that $V_{\bar{r}} \subset \mathscr{H} \subset V_r$ one may consider in the same way the set Reg (V_r) of those operators that map V_r *and* $V_{\bar{r}}$ into themselves continuously, which is again a $*$-algebra. However these $*$-algebras, including Reg (V), need not be comparable to each other. In fact, each Reg (V_r) is extremal as a $*$-algebra of operators, but none is maximal, in the sense that it would contain all of them. However, if V_r is a Hilbert space, then by interpolation theorems, every element of Reg (V_r) is a bounded operator. Similarly in a scale of Hilbert spaces $V_{\bar{r}} \subseteq V_{\bar{s}} \subseteq \mathscr{H} \subseteq V_s \subseteq V_r$, one has Reg ($V_r$) $\subseteq$ Reg (V_s) $\subseteq \mathscr{B}(\mathscr{H})$. For this reason, these $*$-algebras are not very interesting.

b) One defines similarly regular operators from a PIP-space V into another one Y: A $\in$ Reg (V, Y) iff $\mathscr{D}(A) = V$ and $\mathscr{D}(A^\times) = Y$, or iff A: V $\to$ Y and A: $V^\# \to Y^\#$, both continuously.

The examples given above show that a regular operator A on V_I may be identified with an operator in the central Hilbert space $\mathscr{H}$ (also denoted A), such that both A and its Hilbert space adjoint A^* are defined on $V^\#$ and leave it invariant; their restrictions to $V^\#$ are both closable, but not necessarily bounded. This is precisely the kind of operators studied systematically in the theory of the so-called algebras of unbounded operators [*5-9*]. More precisely, given any dense domain $\mathscr{D} \subseteq \mathscr{H}$, let A be an operator with domain $\mathscr{D}$, such that A^* has a domain containing $\mathscr{D}$ and $A\mathscr{D} \subseteq \mathscr{D}$, $A^*\mathscr{D} \subseteq \mathscr{D}$. Equivalently the restriction of A and A^* to $\mathscr{D}$ are continuous for either the weak topology $\sigma(\mathscr{D}, \mathscr{D})$ or the Mackey topology $\tau(\mathscr{D}, \mathscr{D})$. The set of all such operators is a $*$-algebra, denoted $L^+(\mathscr{D})$ by Lassner [*6*], and $C_{\mathscr{D}}$ by Epifanio [*7*]. An *Op*-algebra* on $\mathscr{D}$ is a $*$-subalgebra with unit of $L^+(\mathscr{D})$.

As remarked already a regular operator on V may be identified with its restriction to $V^\#$. Then we have:

PROPOSITION 2.5. — Let V_I be a nondegenerate PIP-space with positive definite pip. Then:

i) Reg (V) is isomorphic to an Op*-algebra on $V^\#$;

ii) if V is quasi-complete for its Mackey topology, Reg (V) is isomorphic to $L^+(V^\#)$.

Proof. — Let A $\in$ Reg (V). Both A and $A^\times$ are continuous on V $[\tau(V, V^\#)]$ and both leave $V^\#$ invariant. Hence their restrictions to $V^\#$ are continuous for the induced topology, that is, $\tau(V^\#, V^\#)$. This proves (*i*). Conversely, let $B \in L^+(V^\#)$, i. e. B and $B^+ \equiv B^* \upharpoonright V^\#$ map $V^\#$ into itself, continuously for the topology $\tau(V^\#, V^\#)$ induced by V. This implies that $B \in Op(V_I)$ and $B^+ = B^\times \upharpoonright V^\#$. Let now V be quasi-complete for its Mackey topology $\tau(V, V^\#)$. Then B and B^+ may be extended by continuity to the quasi-completion of $V^\#$, that is, V, so that B is indeed regular. □

The condition of Mackey quasi-completeness of V (the same one used in [*4*], Sec. 7 for the construction of $\mathscr{H}$ as the (quasi)-completion of V) is actually satisfied in almost all examples; the only known exceptions are quite pathological [*11*] [*18*].

It is of course automatic if V is Mackey-complete, e. g. if it is a (possibly nonreflexive) Banach or Fréchet space, or the dual of a Fréchet space. It is also verified if V is reflexive (in that case, V may fail to be Mackey-complete). Thus for one or another reason, the following familiar examples are covered: spaces of distributions ($\mathscr{S}'$, $\mathfrak{E}'$), spaces of sequences (ω, s), the Banach scales $\{\ell^p\}$, $\{\mathscr{C}^p\}$, $\{L^p([0, 1])\}$, $1 \leqslant p \leqslant \infty$; the Banach lattice generated by $\{L^p(X, \mu)\}$, $1 \leqslant p \leqslant \infty$, the spaces L^p_{loc}, $1 \leqslant p \leqslant \infty$ (which are Fréchet spaces, reflexive for $1 < p < \infty$ only). All these examples may be found in detail in [*3*] and [*4*].

If V is *not* quasi-complete we feel that the PIP-space structure is not natural, and that V should simply be replaced by its quasi-completion $\overline{V}$. Let indeed $V \neq \overline{V}$, and assume that V_I does not have a central Hilbert space. Then the argument of [*4*], Prop. 7.1, shows that the completion $\mathscr{H}$ of $V^\#$ in the pip-norm $\| f \| = \langle f | f \rangle^{1/2}$ is a dense subspace of $\overline{V}$, not necessarily contained in V. Similarly, every element A of $L^+(V^\#)$ extends by continuity to a continuous linear map $A : \overline{V} \to \overline{V}$, which need not leave V invariant, unless, of course, it is continuous for the stronger topology $\tau(V^\#, V)$. We will come back to this point elsewhere [*11*], but for the moment, we will not require V_I to be Mackey-quasi-complete (we still do require, of course, the presence of the central Hilbert space $\mathscr{H}$).

Thus given a PIP-space V_I, one can simply identify the $*$-algebra Reg (V) with an Op*-algebra, in particular, if V is quasi-complete we identify Reg (V) and $L^+(V^\#)$. Conversely, any Op*-algebra defines a PIP-space structure.

PROPOSITION 2.6. — Let $\mathfrak{A}$ be any Op*-algebra on the dense domain $\mathscr{D} \subset \mathscr{H}$. Then there exists a PIP-space V_I such that $\mathscr{D} \subseteq V^\#$ and $\mathfrak{A} \subseteq \text{Reg}(V)$.

Proof. — This is a standard construction, due to Roberts [*13*] and developed by Friedrich and Lassner [*18*]. First one equips $\mathscr{D}$ with the so-called $\mathfrak{A}$-topology, i. e. the projective topology defined by all semi-norms $f \mapsto \| Af \|$, $A \in \mathfrak{A}$. One may assume that $\mathscr{D}$ is complete in this topology, for if it is not, the algebra $\mathfrak{A}$ extends by continuity to an Op*-algebra $\overline{\mathfrak{A}}$ (its closure), defined on the completion of $\mathscr{D}$, namely $\overline{\mathscr{D}} = \bigcap_{A \in \mathfrak{A}} D(\overline{A})$ [*5*]. Then by transposition one gets a triplet $\mathscr{D} \subset \mathscr{H} \subset \mathscr{D}'$, where $\mathscr{D}'$ is the dual of $\mathscr{D}$. As usual such a triplet defines a (trivial) compatibility relation on $\mathscr{D}' : f \# g$ iff either one of them belongs to $\mathscr{D}$, or both belong to $\mathscr{H}$. Thus $(\mathscr{D}')^\# = \mathscr{D}$, $\mathscr{H}^\# = \mathscr{H}$. Clearly any element of $\mathfrak{A}$ defines a regular operator, so that we have $\mathfrak{A} \subseteq \text{Reg}(\mathscr{D}') \subseteq L^+(\mathscr{D})$ (equality is not guaranteed [*18*]). □

The standard example is, of course, that of canonical commutation relations for a quantum mechanical system. Let $\mathscr{H} = L^2(\mathbb{R}^n)$, $\mathscr{D} = C_0^\infty(\mathbb{R}^n)$, $\mathfrak{A}$ the noncommutative algebra generated by q_j = multiplication by x_j, $p_j = -i \dfrac{d}{dx_j}$ $(j = 1, \dots, n)$, all restricted to $\mathscr{D}$.

The $\mathfrak{A}$-topology on $\mathcal{D}$ is the usual Schwartz topology and $\overline{\mathcal{D}} = \mathcal{S}(\mathbb{R}^n)$. Thus one gets the familiar triplet

$$\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \qquad \text{with} \qquad \mathfrak{A} \subseteq \text{Reg}\,(\mathcal{S}') \simeq L^+(\mathcal{S}).$$

Remark. — In fact one can go further: the Op*-algebra $\mathfrak{A}$ defines a canonical structure of nested Hilbert space on $\mathcal{D}'$. This will be discussed at length in another publication [*11*].

The conclusion of Propositions 2.6-2.7 is that an Op*-algebra and the algebra of regular operators on a PIP-space are essentially identical concepts. Now the former have received considerable attention [5-10, 18]: several canonical topologies have been defined on them, the structure of their state space is fairly well understood. Thus all those results can be applied directly to regular operators. In a similar fashion the space Op (V_I) may be identified with $\mathcal{L}(V^\#, V) \simeq \mathcal{B}(V^\#, V^\#)$, the space of separately continuous sesquilinear forms on $V^\#$, and thus it can be topologized in a natural way. However we will not discuss this topic any further in this paper (see however the Appendix).

3. SPECTRAL PROPERTIES OF REGULAR OPERATORS

Let again V_I be a PIP-space, and $\mathfrak{A}$ an arbitrary Op*-algebra on $V^\#$. We want to study the spectral properties of the elements of $\mathfrak{A}$. Of course the most interesting case for applications is $\mathfrak{A} = \text{Reg}\,(V)$, but all concepts and results apply to any $\mathfrak{A}$ as well, in particular to $L^+(V^\#)$, if the latter is different from Reg (V). Thus for the sake of generality we will keep $\mathfrak{A}$ general.

Some results in this direction have been obtained recently by Epifanio and Trapani [*10*] for the case $\mathfrak{A} = L^+(V^\#)$. We will generalize them to arbitrary $\mathfrak{A}$, and also rephrase the problem in PIP-space language, which seems more natural.

The first step is to define properly the spectrum of an operator $T \in \mathfrak{A}$. There are several natural possibilities. On one hand, T is a closable operator in $\mathcal{H}$, with closure $\overline{T}$, so that the usual Hilbert space concepts may be used [*19*]:

$$\begin{aligned} &\text{resolvent set: } \rho(T) \equiv \rho(\overline{T}) = \{\, \lambda \in \mathbb{C} \mid (\overline{T} - \lambda)^{-1} \in \mathcal{B}(\mathcal{H}) \,\} \\ &\text{spectrum:} \quad \sigma(T) \equiv \sigma(\overline{T}) = \mathbb{C} \backslash \rho(T). \end{aligned} \tag{3.1}$$

These sets have the usual properties: $\sigma(T)$ is closed, $\rho(T)$ is open, the resolvent operator $(\overline{T} - \lambda)^{-1}$ is a holomorphic function of λ in every connected component of $\rho(T)$ and verifies the resolvent equation.

On the other hand, T belongs to the Op*-algebra $\mathfrak{A}$, so that another definition suggests itself:

$$\begin{aligned} &.\ \text{resolvent set: } \rho_{\mathfrak{A}}(T) = \{\, \lambda \in \mathbb{C} \mid (T - \lambda)^{-1} \in \mathfrak{A} \,\} \\ &.\ \text{spectrum:} \quad \sigma_{\mathfrak{A}}(T) = \mathbb{C} \backslash \rho_{\mathfrak{A}}(T). \end{aligned} \tag{3.2}$$

As pointed out by Schaefer [*20*], in the case of an arbitrary topological algebra, this purely algebraic definition is not very useful, and may lead to pathologies (see the examples below). Instead he defines the resolvent set as the largest open set in which $(\overline{T} - \lambda)^{-1}$ is holomorphic in λ. In the present case, however, this is exactly the Hilbert space resolvent set $\rho(T)$.

Now the two sets $\rho(T)$, $\rho_{\mathfrak{A}}(T)$ are not necessarily comparable (see the examples below). This fact suggests to define as resolvent the smaller set (this is the definition used by Epifanio and Trapani [*10*], in the case $\mathfrak{A} = L^+(V^\#)$):

$$\begin{aligned}\hat{\rho}_{\mathfrak{A}}(T) &= \rho(T) \cap \rho_{\mathfrak{A}}(T) \\ &= \{ \lambda \in \mathbb{C} \mid (T - \lambda)^{-1} \in \mathfrak{A}_b \equiv \mathfrak{A} \cap \mathscr{B}(\mathscr{H}) \}\end{aligned}$$

and therefore

$$\hat{\sigma}_{\mathfrak{A}}(T) = \mathbb{C} \backslash \hat{\rho}_{\mathfrak{A}}(T) = \sigma(T) \cup \sigma_{\mathfrak{A}}(T).$$

These sets indeed have properties very similar to those of their Hilbertian counterparts $\rho(T)$, $\sigma(T)$. However the purely algebraic concepts $\rho_{\mathfrak{A}}(T)$, $\sigma_{\mathfrak{A}}(T)$ have interesting properties, too. In particular, they yield a handy criterion of essential self-adjointness for symmetric operators (see Sec. 4).

The following facts are straighforward consequences of the definition.

PROPOSITION 3.1. — Let $T \in \mathfrak{A}$, $\rho(T)$, resp. $\rho_{\mathfrak{A}}(T)$ the resolvent set defined in (3.1), resp (3.2), $\sigma(T)$, resp. $\sigma_{\mathfrak{A}}(T)$ the corresponding spectrum. Then one has:

i) The eigenvalues of T belong to $\sigma_{\mathfrak{A}}(T)$.

ii) The residual spectrum of T (in the usual sense [*19*]) belongs to $\sigma_{\mathfrak{A}}(T)$.

iii) If $\lambda \in \rho_{\mathfrak{A}}(T) \backslash \rho(T)$, then λ belongs to the continuous spectrum $\sigma_c(T)$ (in the usual sense); in other words, $\sigma(T) \backslash \sigma_{\mathfrak{A}}(T) \subseteq \sigma_c(T)$.

iv) $\lambda \in \rho_{\mathfrak{A}}(T)$ iff $\lambda^* \in \rho_{\mathfrak{A}}(T^\times)$, i. e. $\rho_{\mathfrak{A}}(T^\times) = [\rho_{\mathfrak{A}}(T)]^*$.

v) If $T = T^\times$, then $\rho_{\mathfrak{A}}(T)$ and $\sigma_{\mathfrak{A}}(T)$ are symmetric with respect to the real axis. In particular, all eigenvalues are real. Also $\rho_{\mathfrak{A}}(T) \backslash \rho(T) \subseteq \mathbb{R}$ by (*iii*). □

Remarks. — *a*) In (*iv*) we should really write $\rho_{\mathfrak{A}}(T^+)$, where

$$T^+ \equiv T^\times \upharpoonright V^\# \in \mathfrak{A},$$

but we find the formulation above more suggestive, at the risk of a slight abuse of language.

b) Property (*v*) is, of course, true for $\rho(T)$, $\sigma(T)$ also, since T is then a symmetric operator in $\mathscr{H}$, hence for $\hat{\rho}_{\mathfrak{A}}(T)$, $\hat{\sigma}_{\mathfrak{A}}(T)$ as well, as shown in [*10*].

In the sequel we will consider only symmetric operators, $T = T^\times$, since these are the most interesting ones for applications. Before doing that, let us describe a few examples that illustrate the discussion above, in the case $\mathfrak{A} = \mathrm{Reg}(V) = L^+(V^\#)$. For convenience we write

$$\rho_\#(T) \equiv \rho_{L^+(V^\#)}(T), \qquad \sigma_\#(T) \equiv \sigma_{L^+(V^\#)}(T).$$

(1) Let $\overline{T}$ be compact, thus $\sigma(T)$ consists of the point $\lambda = 0$ and a set of eigenvalues. So if 0 is an eigenvalue, one has $\rho_{\#}(T)=\rho(T)$, $\sigma_{\#}(T)=\sigma(T)$. If 0 is not an eigenvalue, but merely a limit point of such, then T^{-1} may or may not be regular. In the latter case, we have again $\rho_{\#}(T) = \rho(T)$, but in the former, we get instead $\rho_{\#}(T)=\rho(T) \cup \{0\}$, $\sigma(T)=\sigma_{\#}(T) \cup \{0\}$.

This last example exhibits all pathologies announced above: $\rho_{\#}(T)$ is not open, $\sigma_{\#}(T)$ is not closed, the resolvent $(\overline{T} - \lambda)^{-1}$ is not holomorphic in $\rho_{\#}(T)$ (a similar example is given by Schaefer [*20*]).

(2) In the PIP-space $\mathscr{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathscr{S}'(\mathbb{R})$, let $T = -\dfrac{d^2}{dx^2}$. Then $T-\lambda$ maps $\mathscr{S}$ into itself for any $\lambda \in \mathbb{C}$. For λ non real, as well as for $\lambda < 0$, $(T-\lambda)^{-1}$ is bounded and regular; but for $\lambda \geqslant 0$, $(T-\lambda)^{-1}$ is unbounded, and does not map $\mathscr{S}$ into itself. Thus in this example $\rho_{\#}(T) = \rho(T) = \mathbb{C}\backslash[0, \infty)$.

(3) A third example may be found in [*10*]. Let $V^{\#}$ be the vector subspace of $L^2(\mathbb{R})$ generated by the Hermite functions:

$$V^{\#} = \{ f(x) \mid f \in L^2(\mathbb{R}), \quad f(x) = p(x)e^{-\frac{x^2}{2}},\ p(x) \text{ polynomial} \}$$

$$= \left\{ f(x) \mid f = \sum_{n=0}^{N} c_n\varphi_n(x), \quad c_n \in \mathbb{C}, \quad \varphi_n \text{ the } n^{\text{th}} \text{ Hermite function} \right\}.$$

This space is naturally isomorphic to the space φ of all finite sequences, so that we are in the PIP-space $\varphi \subset \ell^2 \subset \omega$ corresponding to the Hermite basis of $L^2(\mathbb{R})$. Take T = multiplication by x. Then $\rho(T)=\mathbb{C} \setminus \mathbb{R}$, but $\rho^{\#}(T)=\emptyset$. In this case, the origin of the pathology is clear: the domain $V^{\#}$ is too small. If we take instead $V^{\#} = \mathscr{S}(\mathbb{R})$, we get the triplet $s \subset \ell^2 \subset s'$, and there $\rho_{\#}(T) = \rho(T) = \mathbb{C} \setminus \mathbb{R}$.

In those three examples, $\rho_{\#}(T)$ is, respectively, larger than, equal to, and smaller than $\rho(T)$. In general, however, the two sets need not be comparable, although we have no example of a regular operator T on some PIP-space V such that $(T - \lambda)^{-1}$ would be a regular, but unbounded operator for $\lambda \in \rho_{\#}(T) \setminus \rho(T)$, and a bounded, but not regular operator for $\lambda \in \rho(T) \setminus \rho_{\#}(T)$.

4. ESSENTIALLY SELF-ADJOINT REGULAR OPERATORS

Let $T = T^{\times}$ be a symmetric regular operator on V_I. Its restriction to $V^{\#}$ is a symmetric operator in the Hilbert space $\mathscr{H}$, with dense domain $V^{\#}$. For many applications the crucial question is whether this operator is essentially self-adjoint. The usual criterion [*19*] is, of course, that $(T \pm i)^{-1}$ be densely defined, which implies $\pm i \in \rho(T)$, i. e. $(\overline{T} \pm i)^{-1} \in \mathscr{B}(\mathscr{H})$. But we still get a sufficient condition if we replace $\rho(T)$ by the algebraic resolvent set defined in Sec. 3. Again the result holds true for an arbitrary Op*-algebra $\mathfrak{A}$ on $V^{\#}$.

PROPOSITION 4.1. — Let $T = T^{\times}$ be a symmetric element of the Op*-algebra $\mathfrak{A}$ on $V^{\#}$. If $(T \pm i)^{-1} \in \mathfrak{A}$, then T is essentially self-adjoint in $\mathscr{H}$.

Proof. — We known that T, defined on $V^{\#}$, is essentially self-adjoint in $\mathscr{H}$ iff Ran $(T \pm i)$ are dense in $\mathscr{H}$. Assume the contrary. Then $(T \pm i)^{-1}$ are not densely defined (equivalently $\pm i \in \sigma_{res}(T) \subseteq \sigma_{\mathfrak{A}}(T)$, by Proposition 3.1 (*ii*)), and *a fortiori*, $(T \pm i)^{-1} \notin \mathfrak{A}$. □

Of course, the weaker condition $(T \pm i)^{-1} \in L^{+}(V^{\#})$ is already sufficient, but the formulation above is more in the spirit of a spectral theory *within* $\mathfrak{A}$. Notice that the same result is proved in [*10*] under the stronger condition $\pm i \in \hat{\rho}_{\#}(T)$. Since the case $\mathfrak{A} = \text{Reg}(V)$ is the most useful for applications we reformulate the statement explicitly:

COROLLARY 4.2. — Let $T = T^{\times}$ be a symmetric regular operator on V_I. If $(T \pm i)^{-1}$ are regular operators, then T is essentially self-adjoint in $\mathscr{H}$. □

Since the only condition is that $(T \pm i)^{-1}$ be densely defined, the criterion of Proposition 4.1 extends immediately to all those symmetric operators on V_I, regular or not, that are at the same time operators on $\mathscr{H}$ (with domain containing $V^{\#}$).

PROPOSITION 4.3. — Let $T = T^{\times}$ be a symmetric operator on V_I, such that $TV^{\#} \subseteq \mathscr{H}$. Then T is essentially self-adjoint in $\mathscr{H}$ iff $(T \pm i)^{-1} \in \text{Op}(V_I)$ with $(T \pm i)^{-1} V^{\#} \subseteq \mathscr{H}$. □

The « only if » part follows from the fact that $\bar{T}$ self-adjoint means $(\bar{T} \pm i)^{-1} \in \mathscr{B}(\mathscr{H})$, which implies that $(T \pm i)^{-1}$ are defined as operators on V_I. But, even if T is regular, $(T \pm i)^{-1}$ need not be, so the sufficient condition of Corollary 4.2 is *a priori* too strong. Yet we will see in Sect. 5 below that it is in fact necessary for certain regular operators T (Corollary 5.3).

Of course the same results hold true if $\pm i$ is replaced by an arbitrary non real pair (λ, λ^*). This, together with Proposition 3.1 (*v*), implies that for a symmetric element of $\mathfrak{A}$, $\rho(T)$ and $\rho_{\mathfrak{A}}(T)$ differ at most on the real axis. Also, if T is a positive operator, $(T \pm i)^{-1}$ may be replaced in all statements by $(T + 1)^{-1}$, as usual.

We illustrate the criterion of Propositions 4.1-4.3 by an example and a counterexample.

EXAMPLE 4.4. — Let T be the operator of multiplication by x in $\mathscr{H} = L^2(\mathbb{R})$, and V_I be the scale of Hilbert spaces built on T:

. $V_n \equiv D(T^n) = \{ f \in L^2(\mathbb{R}) \mid x^j f \in L^2,\ j = 1, 2, \ldots, n \}$ for $n = 1, 2 \ldots$

. V_{-n} = dual of V_n

. $V^{\#} = \bigcap_{n \in \mathbb{Z}} V_n, \qquad V = \bigcup_{n \in \mathbb{Z}} V_n.$

For every $n \in \mathbb{Z}$, T maps V_{n+1} continuously into V_n, and so does $T^{\times} = T$ (of course we know that T is self-adjoint !). Hence T is regular, and so are $(T \pm i)$. Furthermore, for every $n \in \mathbb{Z}$, $T \pm i$ are in fact bijections from V_{n+1} onto V_n (exactly as in Example (2) of Sec. 3). Thus $(T \pm i)^{-1}$ are bounded from V_n onto V_{n+1}, i. e. $(T \pm i)^{-1}$ are regular.

EXAMPLE 4.5. — As a counterexample, we consider a symmetric regular operator which is *not* essentially self-adjoint, and show that indeed $(T+i)^{-1}$ is not an operator on V_I. Let $T = -i\dfrac{d}{dx}$ acting in $\mathscr{H} = L^2([0, \infty))$, and again let V_I be the scale built on T whith:

$$V_n \equiv D(T^n) = \{ f \in \mathscr{H} \mid f^{(j)} \in \mathscr{H},\ f^{(j)}(0) = 0,\ j = 1, 2 \ldots n \}.$$

On the domain $V_1 = D(T)$, T is a closed symmetric operator, with defect indices (0, 1), since $\mathrm{Ker}(T^* - i) = \{ \lambda e^{-x}, \lambda \in \mathbb{C} \}$. The adjoint T^* is equal to $-i\dfrac{d}{dx}$ on the domain:

$$\hat{V}_1 \equiv D(T^*) = \{ f \in \mathscr{H} \mid f^{(1)} \in \mathscr{H} \},$$

so that, indeed, $T = T^{**} \subset T^*$. By construction, T is a regular operator on V_I, which maps V_{n+1} into V_n, for each $n \in \mathbb{Z}$. So are $T \pm i$. More precisely, since $\mathrm{Ker}(T^* - i) = [\mathrm{Ran}(T+i)]^{\perp} = \{ \lambda e^{-x} \} \in V_n$, $\forall n \leqslant 0$, the mapping $T+i$: $V_{n+1} \to V_n$ is, for every n, injective and bounded, but not surjective (for instance, 0 is never in its range). Moreover it has a dense range for $n \geqslant 1$ only. Thus the inverse $(T + i)^{-1}$: $V_n \to V_{n+1}$ is always unbounded, and densely defined for $n \geqslant 1$ only. Therefore $(T + i)^{-1}$ cannot be an operator on V_I. See [*21*] for more details on this example.

5. SELF-ADJOINT OPERATORS WITH REGULAR SPECTRAL PROJECTIONS

Let $T = T^{\times}$ be a symmetric operator in V_I, leaving $V^{\#}$ invariant, and essentially self-adjoint in $\mathscr{H}$. Let $\overline{T} = \int \lambda dE(\lambda)$ be the spectral decomposition of its unique self-adjoint extension. When does each $E(\lambda)$ map $V^{\#}$ into itself? (Such operators T are called $V^{\#}$-spectral in [*10*]). If, in addition, T is assumed to be regular, when is each $E(\lambda)$ itself regular? (The two problems are identical if V is quasi-complete). Obviously the answer will depend on how well the operator T and the PIP-space V_I are matched together, more precisely, how V_I and the canonical scale built on T are related.

First we need some definitions. Given any dense domain $\mathscr{D}$ in a Hilbert space $\mathscr{H}$, let $\mathcal{O}$ be a $*$-invariant family of closable operators defined on $\mathscr{D}$: $\mathscr{D} \subset D(A) \cap D(A^*)$, for every $A \in \mathcal{O}$ (notice that we do *not* require $A^{(*)}\mathscr{D} \subseteq \mathscr{D}$). Then [*15*] the *strong commutant* of $\mathcal{O}$ is the set:

$$(\mathcal{O}, \mathscr{D})'_s \equiv \mathcal{O}'_s = \{ B \in \mathscr{B}(\mathscr{H}) \mid B\mathscr{D} \subseteq \mathscr{D},\ BAf = ABf,\ \forall A \in \mathcal{O},\ \forall f \in \mathscr{D} \}.$$

This set is an algebra, but is in general not $*$-invariant. On the other hand, the *weak commutant* [*5*] [*15*]:

$$(\mathcal{O}, \mathscr{D})'_w \equiv \mathcal{O}'_w = \{ C \in \mathscr{B}(\mathscr{H}) \mid \langle Cf \mid Ag \rangle = \langle A^*f \mid C^*g \rangle,\ \forall A \in \mathcal{O},\ \forall f, g \in \mathscr{D} \}$$

is $*$-invariant but not an algebra. Of course $\mathcal{O}'_s \subseteq \mathcal{O}'_w$. Notice that if $\mathscr{D}_0 \subset \mathscr{D}$ is another dense domain, one has $(\mathcal{O}, \mathscr{D})'_w \subseteq (\mathcal{O}, \mathscr{D}_0)'_w$. Let now R be a self-adjoint operator in $\mathscr{H}$, $\mathscr{R}$ the abelian $*$-algebra generated by the restriction of R to the invariant domain $D^\infty(R) \equiv \bigcap_{n=0}^{\infty} D(R^n)$. This algebra is self-adjoint in the sense of Powers [*5*], hence $\mathscr{R}'_s = \mathscr{R}'_w$ and this commutant is a von Neumann algebra. On the other hand, the usual commutant $\mathscr{R}' \equiv \{ F(\lambda) \}'$, where $R = \int \lambda dF(\lambda)$, is also a von Neumann algebra. Its commutant (in the usual sense) $\mathscr{R}''$ contains $F(\lambda)$, $\forall \lambda$, and $(R \pm i)^{-1}$. In fact there is only one commutant in this case:

LEMMA 5.1. — Let R a self-adjoint operator on $\mathscr{H}$, $\mathscr{R}$ the algebra generated by the restriction of R to $D^\infty(R)$. Then

i) $\mathscr{R}' = \mathscr{R}'_s = \mathscr{R}'_w$;

ii) Let $\mathscr{D}$ be any invariant core for R and let

$$\mathring{\mathscr{R}} \equiv \mathscr{R} \upharpoonright \mathscr{D}. \qquad \text{Then} \qquad \mathring{\mathscr{R}}'_w = \mathscr{R}'.$$

Proof. — *i*) $\mathscr{R}' \subseteq \mathscr{R}'_s$ follows from the spectral theorem: $B \in \mathscr{R}'$ leaves each $D(R^n)$ invariant and commutes with R, hence $B \in \mathscr{R}'_s$. Conversely, let $B \in \mathscr{R}'_s$: $BRf = RBf$, $\forall f \in D^\infty(R)$. This implies $B(R+i)^{-1}g = (R+i)^{-1}Bg$, $\forall g \in D(R + i)^{-1}) = \mathscr{H}$, which in turn implies that B commutes with R ($BR \subseteq RB$), and thus with every $F(\lambda)$, so that $B \in \mathscr{R}'$. The other equality is standard.

ii) By the remark above, since $\mathscr{D} \subset D^\infty(R)$, $\mathscr{R}' = \mathscr{R}'_w \subseteq \mathring{\mathscr{R}}'_w$. The other inclusion follows from the argument of [*10*] (Theor. 13) and the fact that $\mathscr{D}$ is dense in D(R) in the graph topology. Let $C \in \mathring{\mathscr{R}}'_w$: $\langle Cf \mid Rg \rangle = \langle CRf \mid g \rangle$, $\forall f, g \in \mathscr{D}$, implies as in (*i*) that C commutes with R, and thus $C \in \mathscr{R}'_s = \mathscr{R}'_w = \mathscr{R}'$. □

We note, however, that $\mathring{\mathscr{R}}'_s \subset \mathscr{R}'$ in general, since an arbitrary operator in $\mathscr{R}'$ need not leave $\mathscr{D}$ invariant.

In particular, the bicommutant $\mathscr{R}'' = (\mathscr{R}')'$, which is uniquely defined, is contained in $\mathscr{R}'$, since $\mathscr{R}$ is abelian, but need not be contained in $\overset{\circ}{\mathscr{R}}{}'_s$, nor in the strong commutant of any family of operators defined on a smaller domain $\mathscr{D} \subset D^\infty(R)$.

Let us now come back to our problem. $T = T^\times \in L^+(V^\#)$ is an essentially self-adjoint operator on V_I, with unique self-adjoint extension $\overline{T} = \int \lambda dE(\lambda)$. As above, $E(\lambda) \in \mathscr{C}''$, $(T \pm i)^{-1} \in \mathscr{C}''$ where $\mathscr{C}$ is the algebra over $D^\infty(\overline{T})$ generated by T. In general, these operators need not leave $V^\#$ invariant. A sufficient condition for that would be that they belong to the strong commutant $(\mathcal{O}, V^\#)'_s$ of some operator family $\mathcal{O}$ defined on $V^\#$. Notice that $(\mathcal{O}, V^\#)'_s$ need not be contained in $L^+(V^\#)$, since the strong commutant is not necessarily $*$-invariant, but any $*$-invariant subset of $(\mathcal{O}, V^\#)'_s$ is contained in $L^+(V^\#)$. In particular, if $\mathcal{O}$ is reduced to the identity operator, restricted to $V^\#$, then $(\mathbb{1}, V^\#)'_s \cap (\mathbb{1}, V^\#)'^*_s$ is exactly the set of all bounded elements of $L^+(V^\#)$. Thus a sufficient condition for each $E(\lambda)$ to leave $V^\#$ invariant is that $\mathscr{C}''$ (which is $*$-invariant) be contained in the strong commutant of some family $(\mathcal{O}, V^\#)$.

PROPOSITION 5.2. — Let $T = T^\times \in L^+(V^\#)$ be essentially self-adjoint, $\overline{T} = \int \lambda dE(\lambda)$ the spectral decomposition of its closure $\overline{T}$. Let $\mathscr{C}$ be the abelian algebra generated by T. Assume there exists a family $\mathcal{O}$ of closable operators defined on $V^\#$, such that $\mathscr{C}'' \equiv (\mathcal{O}, V^\#)'_s$. Then all spectral projections $E(\lambda)$, as well as the resolvents $(T \pm i)^{-1}$, belong to $L^+(V^\#)$. □

As announced above, the sufficient condition of Corollary 4.2 is also necessary if T satisfies the condition of Proposition 5.2:

COROLLARY 5.3. — Let V be quasi-complete and T an essentially self-adjoint regular operator. Assume $\mathscr{C}'' \subseteq (\mathcal{O}, V^\#)'_s$ for some family $\mathcal{O}$. Then $E(\lambda)$ and $(T \pm i)^{-1}$ are regular operators. □

In fact $E(\lambda)$ and $(T \pm i)^{-1}$ belong to $\overset{\circ}{\mathscr{C}}{}'_s$, where $\overset{\circ}{\mathscr{C}} \equiv \mathscr{C} \upharpoonright V^\#$. But $\mathscr{C}''$ is in general not contained in $\overset{\circ}{\mathscr{C}}{}'_s$, unless $V^\# = D^\infty(\overline{T})$, i. e. when the PIP-space is the scale built on T (see Example 5.5 (*a*) below).

Proposition 5.2 was obtained by Epifanio and Trapani [*10*] under the condition that $\mathcal{O}$ be a self-adjoint $*$-algebra on $V^\#$, i. e. $V^\# = \bigcap_{A \in \mathcal{O}} D(A^*)$, but this condition is unnecessarily strong.

In fact, the smaller the family $\mathcal{O}$, the larger its commutant, hence the less restrictive the condition $\mathscr{C}'' \subseteq \mathcal{O}'_s$. It is true, however, that the most interesting example is that where $\mathcal{O}$ is, or generates, a closed Op*-algebra $\mathfrak{A}$ on $V^\#$ (not necessarily self-adjoint), $V^\# = \bigcap_{A \in \mathfrak{A}} D(\overline{A})$. Such PIP-spaces, generated by an Op*-algebra, will be studied elsewhere [11], and it turns out that Theorem 5.2 may be strengthened substantially in that case.

We conclude this section with a few examples, all for quasi-complete PIP-spaces.

EXAMPLES 5.5. — *a*) The simplest case is that of the scale built on a self-adjoint operator T. Then $V^{\#} = D^{\infty}(T) = \bigcap_{n\geq 0} D(T^n) = \bigcap_{S\in\mathscr{C}} D(\bar{S})$. The algebra $\mathscr{C}$ generated by T is abelian, i. e. $\mathscr{C}'' \subseteq \mathscr{C}' = \mathscr{C}'_s$, and self-adjoint, so Theorem 5.2 applies with $\mathcal{O} = \mathscr{C}$ or oven $\mathcal{O} = \{ T^n, n = 1, 2 \dots \}$. Indeed in this case, $(T \pm i)^{-1}$ and every $E(\lambda)$ are regular.

b) Take the PIP-space $\mathscr{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathscr{S}'(\mathbb{R})$ and $T = q$. Then $\mathscr{C}'' = L^{\infty}(\mathbb{R})$ consists of operators of multiplication by arbitrary bounded functions. These of course do not leave $V^{\#} \equiv \mathscr{S}(\mathbb{R})$ invariant, so that $\mathscr{C}''$ cannot be contained in any strong commutant $(\mathcal{O}, V^{\#})'_s$. And indeed the spectral projections of q, namely $E(\lambda)$ = multiplication by the characteristic function of $(-\infty, \lambda]$, are not regular, although $(q \pm i)^{-1}$ are. Notice that in this case also, the PIP-space is generated by a self-adjoint Op*-algebra, and even two: the CCR algebra generated by p and q, which is irreducible, i. e. has a trivial commutant $\mathfrak{A}'_s = \{ \lambda \mathbb{1} \}$, and also the algebra generated by the single operator $H_{osc} = p^2 + q^2$, which has a nontrivial commutant (not containing $\{ q \}''$ of course).

c) Let $V_I = \omega$ and T a diagonal real matrix, so that $T = T^{\times} \in \operatorname{Reg}(\omega)$. Assume all diagonal elements of T different from each other. Then $\mathscr{C}''$ consists of all diagonal matrices with bounded entries. Let $\mathcal{O}$ consist of all infinite diagonal matrices. Then $(\mathcal{O}, \varphi)'_s$ coincides with $\mathscr{C}''$, so that all spectral projections of T are regular.

d) A similar example is obtained in the space $\mathfrak{s}'$ of tempered sequences (see Appendix B). Let T be as in (*c*) and tempered. Then $T = T^{\times} \in \operatorname{Reg}(\mathfrak{s}')$ and $\mathscr{C}''$ consists again of all diagonal bounded matrices. Let $\mathcal{O}$ be the family of all diagonal tempered matrices. Then again $(\mathcal{O}, \mathfrak{s})'_s = \mathscr{C}''$ and the same conclusion follows. Less trivial examples would result from a better choice of $\mathcal{O}$. The optimal choice $\mathcal{O} = \{ \mathbb{1}, \mathfrak{s} \}$ gives $\mathcal{O}'_s \cap \mathcal{O}'^{*}_s = \operatorname{Reg}(\mathfrak{s}') \cap \mathscr{B}(\ell^2)$, that is, the set of all matrices in $\operatorname{Reg}(\mathfrak{s}')$ which correspond to bounded operators in ℓ^2. This would allow cases where $\mathscr{C}''$ contains nondiagonal matrices as well, for instance when the diagonal elements of T are not all different from each other.

6. APPLICATIONS

Clearly regular operators will play an important rôle in many applications of PIP-spaces. We will discuss here two (related) such situations, the rigged Hilbert space formulation of Quantum Mechanics and the representation theory of Lie algebras and their enveloping algebras. Of course most of these results are well-known, but we want to stress that PIP-spaces and regular operators provide a natural framework for these problems.

A. **Quantum Mechanics and the nuclear spectral theorem.**

It is has been known for a long time that Dirac's bra and ket formalism in Quantum Mechanics can be put on a sound mathematical basis by using a rigged Hilbert space $\Phi \subset \mathscr{H} \subset \Phi^{\times}$, where Φ is a nuclear space; see the works of Böhm [*12*], Roberts [*13*] and one of us [*14*]. In fact it is easily seen that the two basic ingredients of Quantum Mechanics, namely the principle of linear superposition of states and the rôle of sesquilinear transition amplitudes, precisely imply that the space of states should be a PIP-space.

The starting point of that approach is the concept of *labeled observables* characterizing a physical system: a set of observables for which one is given both a mathematical definition, as a self-adjoint operator in the Hilbert space of the system, and a physical definition, i. e. essentially a prescription as how to measure them. Such are for instance, positions and momenta of particles, energy, angular momentum, etc. In fact, most of them can be derived from the symmetry group of the system [*14*].

One crucial assumption is needed: the family $\mathcal{O}$ of labeled observables should have a common, dense, invariant domain $\mathscr{D}_0$ (instead of self-adjoint operators, one may start from a $*$-invariant family of closed operators [*13*]). Then this family generates, by restriction to $\mathscr{D}_0$ and closure, a $*$-algebra $\mathfrak{A}$ with unit of *closed* operators, with invariant domain $\mathscr{D} = \bigcap_{A\in\mathfrak{A}} D(A) \supseteq \mathscr{D}_0$. Thus we are exactly in the situation of Sec. 2: the system is characterized by a closed Op*-algebra $\mathfrak{A}$ on the domain $\mathscr{D}$, generated by its labeled observables. Applying Proposition 2.6 we get a PIP-space V_I, with $V^{\#} = \mathscr{D}$, in which all labeled observables are regular operators: $V^{\#} \subset \mathscr{H} \subset V$. The extreme spaces may then be given a physical meaning [*14*]: $V^{\#}$ represents physical states that can be prepared in the laboratory, whereas V corresponds to idealized states, to be linked to measurement processes.

The first step is to check that all labeled observables are essentially self-adjoint on $V^{\#}$; since they belong to $\mathfrak{A}$, Proposition 4.1 provides a possible criterion.

The key to the justification of the Dirac formalism is the nuclear spectral theorem [*23*] [*24*], which basically requires the embedding $V^{\#} \to \mathscr{H}$ to be a nuclear map. Now $V^{\#}$ has two natural topologies, for both of which it is complete: the $\mathfrak{A}$-topology (see Prop. 2.6), and the Mackey topology $\tau(V^{\#}, V)$, which is finer. Clearly it is enough to require $V^{\#}$ to be nuclear for the $\mathfrak{A}$-topology. A criterion for this is given by the following theorem of Roberts [*13*].

THEOREM 6.1 (Roberts). — $V^{\#}$ is nuclear for the $\mathfrak{A}$-topology iff there exists an operator A in $\mathfrak{A}$, essentially self-adjoint on $V^{\#}$, such that A^{-1} is a nuclear operator. □

Since $\mathfrak{A} \subseteq \text{Reg}(V)$, the operator A is regular. In fact, it will happen often that the algebra $\mathfrak{A}$ defines the same topology as the full algebra Reg (V); then the theorem is valid with the weaker condition $A \in \text{Reg}(V)$.

It is interesting to remark that several properties of the algebra $\mathfrak{A} \subseteq \text{Reg}(V)$, such as density of finite rank operators or normality of states, can be characterized by the existence in $\mathfrak{A}$ of an operator of a certain type. See [*22*] for a full discussion.

Assuming the embedding $V^{\#} \to \mathscr{H}$ to be nuclear, the spectral theorem may be applied; we emphasize that again it is a statement about regular operators (in the terminology of [*13*], an operator *with a conjugate* means a regular operator, a *real* operator means a symmetric regular operator).

THEOREM 6.2. — Consider the triplet $V^{\#} \subset \mathscr{H} \subset V$, with a nuclear embedding $V^{\#} \to \mathscr{H}$. Then, for any regular operator A, essentially self-adjoint on $V^{\#}$, there exists an integral decomposition $\mathscr{H} = \int_{\Lambda}^{\oplus} \mathscr{H}(\lambda) d\mu(\lambda)$ such that $\mathscr{H}(\lambda)$ may be identified with a subspace of V, consisting of generalized eigenvectors χ_λ of A: $A\chi_\lambda = \lambda\chi_\lambda$ $(\lambda \in \Lambda)$. Furthermore the system of all generalized eigenvectors of A is complete and normalized in the following sense:

$$\langle \varphi \mid \psi \rangle = \int_{\Lambda} d\mu(\lambda) \langle \chi_\lambda \mid \varphi \rangle^{*} \langle \chi_\lambda \mid \psi \rangle, \qquad \forall \varphi, \psi \in V^{\#} \tag{6.1}$$

□

As usual with such theorems, one would like to get more detailed informations about the eigenvectors χ_λ. This is possible if the PIP-space at hand is a nested Hilbert space (and we will see in [*11*] that such is always the case), since then the Hilbert-Schmidt variant of the spectral theorem [*23*] [*24*] may be used:

THEOREM 6.3. — Let V_I be a nested Hilbert space, and $A = A^{\times}$ a symmetric operator, essentially self-adjoint on $V^{\#}$.

Assume there exists an assaying subspace V_r, such that:

i) $A : V^{\#} \to V_r$ is continuous;

ii) the embedding $V_r \to \mathscr{H}$ is Hilbert-Schmidt.

Then A has in $V_{\bar{r}}$ a complete system of generalized eigenvectors, i. e. eq. (6.1) holds with $\chi_\lambda \in V_{\bar{r}}$ and $\varphi, \psi \in V_r$. □

As a very simple example, we consider the Schwartz scale built on the hamiltonian of the one-dimensional harmonic oscillator $H_{osc} \equiv p^2 + q^2$. For each $\alpha \geqslant 0$, let $V_\alpha = D(H^\alpha_{osc})$, with the norm $\|f\|_\alpha = \|H^\alpha_{osc}f\|$, $V_{\bar{\alpha}}$ its dual ($\bar{\alpha} \equiv -\alpha$). Then

$$\mathscr{H} = L^2(\mathbb{R}), \quad V^\# = \bigcap_{\alpha\in\mathbb{R}} V_\alpha = \mathscr{S}(\mathbb{R}), \quad V = \bigcup_{\alpha\in\mathbb{R}} V_\alpha = \mathscr{S}'(\mathbb{R}).$$

One sees easily that the embedding $V_\alpha \to \mathscr{H}$ is Hilbert-Schmidt for $\alpha > \frac{1}{2}$. Let A be the operator $p \equiv -i\frac{d}{dx}$, which is regular. Then the map $A: V^\# \to V_\beta$ is continuous for every β. The theorem says that A has a complete system of eigenvectors, namely $\{ e^{ikx}, k \in \mathbb{R} \}$, in $V_{\bar{\alpha}}$, with $\alpha > \frac{1}{2}$. This can be checked directly.

B. Representations of enveloping algebras.

Let U be a strongly continuous unitary representation of a Lie group G in a Hilbert space $\mathscr{H}$. Then U generates in a well-known fashion a representation of the Lie algebra $\mathfrak{g}$ of G, and of its enveloping algebra $\mathscr{E}$, by operators on the Gårding domain $\mathscr{D}_G$ of U, that is, the linear span of all vectors $U(f)x \equiv \int dg f(g)U(g)x$, $x \in \mathscr{H}$, $f \in C_0^\infty(G)$. The analysis of these representations is, in fact, an exercise on regular operators, as results from the work of Nelson, Goodman and others (we refer to the monograph of Barut and Raczka [*25*] for a survey as well as references to the original papers).

Let $X_1, \ldots, X_d$ be a basis of $\mathfrak{g}$, and $\Delta = X_1^2 + \ldots + X_d^2$ the so-called Nelson operator. Then $U(\Delta)$ is positive and essentially self-adjoint on the Gårding domain. Let V_I be the scale built on the powers of $A \equiv \overline{U(\Delta)} + 1$: $V_n = D(A^n)$, $n = 1, 2 \ldots$; $V_{\bar{n}} = (V_n)^\times$; and $V^\# = D^\infty(A)$ coincides with the space of C^∞-vectors for U, which contains $\mathscr{D}_G$. Then the following is true:

i) For every $g \in G$, the operator $U(g)$ maps each V_n continuously into itself, hence also every $V_{\bar{n}}$ (i. e. $U(g) \in \mathscr{A}$, in the terminology of [*26*]); in particular $U(g)$ extends to a regular operator on V_I.

ii) For every $L \in \mathscr{E}$, the representative $U(L)$, originally defined on $\mathscr{D}_G$, extends to a regular operator on V_I.

Thus the representation U extends to a $*$-representation of the whole enveloping algebra $\mathscr{E}$ (and in particular the Lie algebra $\mathfrak{g}$), by regular operators on the Nelson scale V_I:

$$U(L_1L_2) = U(L_1)U(L_2)$$
$$U(L^\dagger) = U(L)^\times$$

where $L \leftrightarrow L^\dagger$ is the involution on $\mathscr{E}$. Considering as usual the restriction of U(L) to $V^\#$, the standard question arises: when is it essentially self-adjoint in $\mathscr{H}$? More generally, when do we have $\overline{U(L)^\dagger} = U(L)^*$? Two cases are known [*25*] [*27*]:

a) L is elliptic (example: $L = \Delta$)

b) $L^\dagger L$ commutes with a symmetric elliptic element $M = M^\dagger \in \mathscr{E}$.

The proof of (*a*) is in two steps. First, if $L = K^\dagger K$, one shows that $U(L)+1$ is bijective on $V^\#$, i. e. $(U(L) + 1)^{-1}$ is regular (cf. Sec. 4). Then for a general elliptic L, the result follows from the following useful lemma [*27*]:

LEMMA 6.4 (Nelson-Stinespring). — Given any dense domain $\mathscr{D}$, and $T \in L^+(\mathscr{D})$, if $T^+ T$ is essentially self-adjoint, then $\overline{T^+} = T^*$. □

As for (*b*), it yields interesting results when one chooses $M = \Delta$. Indeed if $L^\dagger L$ commutes with Δ, then the proof of [*27*] shows that $U(L^\dagger L)$ is essentially self-adjoint, and its closure commutes strongly with $A \equiv \overline{U(\Delta)}+1$, i. e. their spectral projections commute. In other words, $\overline{U(L^\dagger L)}$ is affiliated to the von Neumann algebra $\{ A \}'$. The same is true of $\overline{U(L)}$ whenever $L^\dagger = L$.

Finally we come to the integrability problem for representations of Lie algebras. Combining the results of Nelson [*25*] [*27*] with those of Sections 4 and 5, we may summarize the analysis in the following theorem:

THEOREM 6.5. — Let $\mathfrak{g}$ be a real Lie algebra, with a basis $X_1, \ldots, X_d$, G the simply connected Lie group with Lie algebra $\mathfrak{g}$, $\mathscr{H}$ a Hilbert space. Let ρ be a representation of $\mathfrak{g}$ by symmetric operators on the dense domain $D^\infty(A)$, where $A = \overline{\rho(X_1)^2 + \ldots + \rho(X_d)^2} + 1$. Let V_I be the scale built on A with $V^\# = D^\infty(A)$. Then the following statements are equivalent:

i) There is a unitary representation U of G on $\mathscr{H}$ such that $U(X) = \rho(X)$ for all $X \in \mathfrak{g}$ and $V^\# = D^\infty(A)$ consists of all C^∞-vectors for U.

ii) A is essentially self-adjoint on $V^\#$;

iii) $(A + 1)^{-1}$ is a regular operator on V_I. □

The implication (*ii*) ⇒ (*iii*) follows from Corollary 5.3, since here $\mathfrak{A}'' \subseteq \mathfrak{A}' = \mathfrak{A}'_s$, where $\mathfrak{A}$ is the self-adjoint algebra generated by A on $D^\infty(A)$.

APPENDIX

We present in this Appendix some alternative definitions which may prove useful for applications.

A. I-REGULAR OPERATORS

For an indexed PIP-space V_I, a definition slightly more restrictive than 2.2 may be given, similar to the definition of operator given in [*4*] (Def. 6.1).

DEFINITION A.1. — An operator A on V_I is said to be *I-regular* if the following two conditions hold:

i) for every $r \in I_0$, there exists $q \in I_0$ with $A: V_r \to V_q$ continuously;
ii) for every $s \in I_0$, there exists $t \in I_0$ with $A: V_t \to V_s$ continuously.

Clearly an operator is I-regular iff it maps $V^\# [t_{\text{proj}}]$ continuously into itself, where $V^\# [t_{\text{proj}}]$ denotes $V^\#$ with the topology $\text{proj.}\lim_{r \in I_0} \tau(V_r, V_{\bar{r}})$, which may be coarser than $\tau(V^\#, V)$ (see [*4*], Secs. 4-5). Thus every I-regular operator is regular, but the converse need not be true, except of course if the projective and the Mackey topology coincide on $V^\#$. Such is the case, in particular, if $V^\# [t_{\text{proj}}]$ is metrizable, i. e. if I_0 contains a cofinal countable subset, for instance in the case of a chain of Banach or Hilbert spaces. In the general case, a regular, but not I-regular, operator would be an operator for which there exists $r \in I_0$ such that $A: V_r \to V$, but no $q \in I_0$ such that $A: V_r \to V_q$, with both maps continuous. This cannot happen in the countable case: every continuous map $u: V_r \to V = \text{ind}\lim_{s \in I_0} V_s$ factorizes through some V_q, if every V_s is a Fréchet space (Grothendieck's factorization theorem: Ref. 28, I. p. 16). We hasten to point out that no example of such an operator is known to us.

B. VERY REGULAR OPERATORS

One can go one step further and call *very regular* every (regular) operator that maps the whole of V into $V^\#$. The very regular operators form also a $*$-algebra Vrg (V). For instance, a very regular operator on ω is just a finite matrix. Also a finite rank projection in an arbitrary PIP-space is always very regular [*29*].

Finally it is interesting to relate all three types of operators to the familiar concepts of continuous linear maps and separately continuous sesquilinear maps, simply by restricting an operator $A \in \text{Op}(V_I)$ to $V^\#$:

$$\begin{aligned} \text{Op}(V_I) \ni A &\Leftrightarrow A \upharpoonright V^\# \in \mathscr{L}(V^\#, V) \\ &\Leftrightarrow \langle \,.\, | A \,.\, \rangle \upharpoonright V^\# \times V^\# \in \mathscr{B}(V^\#, V^\#). \end{aligned}$$

Similarly:

$$\begin{aligned} \text{Reg}(V) &\simeq \mathscr{L}(V^\#, V^\#) \cap \mathscr{L}(V, V) \\ &\simeq \mathscr{B}(V^\#, V) \cap \mathscr{B}(V, V^\#) \\ \text{Vrg}(V) &\simeq \mathscr{L}(V, V^\#) \simeq \mathscr{B}(V, V) \quad . \end{aligned}$$

Applying this, for instance, to the space s' of tempered sequences, we can identify an operator B on s' with an infinite matrix (b_{mn}) and get the following characterizations:

. B *operator on* $\mathscr{s}'$:

$\exists\alpha, \beta \in \mathbb{R}$ and $C > 0$ such that, for all m, n :

$$|b_{mn}| \leqslant C(1+n)^{\alpha}(1+m)^{\beta}$$

. B *regular operator on* $\mathscr{s}'$:

$\forall\alpha \in \mathbb{R}, \exists\beta, \beta' \in \mathbb{R}$, C and $C' > 0$, such that:

. for all n, $\sup_m (1+m)^{\alpha}|b_{mn}| \leqslant C(1+n)^{\beta}$

. for all m, $\sup_n (1+n)^{\alpha}|b_{mn}| \leqslant C'(1+m)^{\beta'}$

. B *very regular operator on* $\mathscr{s}'$:

$\forall\alpha, \beta \in \mathbb{R}, \sup_{m,n} (1+n)^{\alpha}(1+m)^{\beta}|b_{mn}| < \infty.$

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