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Integrability for representations appearing in geometric pre-quantization

by

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ABSTRACT. — Vectorfield representations $D_\rho(\theta^\eta, \theta)$ induced from quasi-complete infinitesimal group actions (θ^η, θ) on quantizing fibre bundles are studied. Examples for non $\tilde{G}$-maximal prequantizations θ^η with G-maximal projected symmetry θ are given. The connection between geometrical properties of the prequantization procedure and integrability properties of the associated Lie algebra representation is discussed.

1. INTRODUCTION

The geometric quantization theory of Kostant and Souriau ([*3*] [*7*]) provides for the construction of skew-adjoint vectorfield representations $D_\rho(\theta^\eta, \theta)$ induced from infinitesimal group actions (θ^η, θ) on quantizing bundles $\eta = (P, \alpha, A, \lambda, M, \omega)$ [*8*]. Here $\theta : \mathfrak{g} \to \mathscr{M}(M, \omega)$ denotes a quasi-complete $\mathfrak{g}$-action on a symplectic manifold (M, ω) which can be lifted to a $\mathfrak{g}$-action *(prequantization)* θ^η on the total space P.

A previous paper [*2*] discussed the integrability properties of vectorfield representations induced from a Mackey-like quantization. It follows from a result in [*2*] that if $\rho : A \to \operatorname{Aut} V$ is faithful and unitary, then $D_\rho(\theta^\eta, \theta)$ integrates up to a group representation iff θ^η is G-maximal. Global results for complete $\mathfrak{g}$-actions on quantizing bundles are presented in [*9*].

The central results contained in [*2*] and [*9*] will be applicable to prequantizations which are not necessarily complete. Work of Palais [5] shows that if θ is complete, then G-maximality of θ implies $\tilde{G}$-maximality of θ^n, where $\tilde{G}$ denotes the universal covering group of G. For non-complete θ, however, non $\tilde{G}$-maximal prequantizations θ^n with G-maximal projected symmetry θ can be obtained. Especially, quasi-complete actions on quantum bundles are considered. We give explicit constructions for the Heisenberg algebra acting on a bundle over $(\mathbb{R}^2 - (0, 0), dx \wedge dy)$ and for Lie algebra actions on bundles over the momentum phase space. The relationship to integrability conditions for skew-adjoint Lie algebra representations is analysed.

2. A GEOMETRIC INTEGRABILITY CRITERION

Let $\mathcal{H}$ be a separable complex Hilbert space with dense domain $\vartheta \subset \mathcal{H}$. $S(\vartheta)$ denotes the Lie algebra of skew-symmetric operators in $\mathcal{H}$ with invariant domain $\vartheta \subset \mathcal{H}$ and $\mathcal{A}(\vartheta)$ is the subset of operators in $S(\vartheta)$ being essentially skew-adjoint on ϑ. A Lie algebra homomorphism

$$D : \mathfrak{g} \to S(\vartheta)$$

of a Lie algebra $\mathfrak{g}$ into $S(\vartheta)$ is called a skew-adjoint representation of $\mathfrak{g}$ on $\vartheta \supset \mathcal{H}$ if $\operatorname{Im} D \subset \mathcal{A}(\vartheta)$.

Let G be a connected Lie group with Lie algebra of left invariant vectorfields isomorphic to $\mathfrak{g}$. Denote by exp the exponential map $\mathfrak{g} \to G$. $\operatorname{Exp} : \mathcal{A}(\vartheta) \to \mathcal{U}(\mathcal{H})$ denotes the exponentiation (given by Stone's theorem) from $\mathcal{A}(\vartheta)$ into the group $\mathcal{U}(\mathcal{H})$ of unitary operators on $\mathcal{H}$.

We say that $D : \mathfrak{g} \to \mathcal{A}(\vartheta)$ is G-integrable [2] if there exists a unitary representation $U : G \to \mathcal{U}(\mathcal{H})$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{U} & \mathcal{U}(\mathcal{H}) \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \operatorname{Exp}} \\ \mathfrak{g} & \xrightarrow[D]{} & \mathcal{A}(\vartheta) \end{array}$$

commutes.

Let $C(G, e)$ be the set of closed curves in G starting and ending at e and take

$$C^{\mathfrak{g}}(G, e) = \{ (x_1, \ldots, x_k) \mid k \in \mathbb{N} - \{0\}, \quad x_i \in \mathfrak{g}, \quad \exp x_1 \ldots \exp x_k = e \}.$$

$C^{\mathfrak{g}}(G, e)$ may be regarded as a subset of $C(G, e)$ via

$$C^{\mathfrak{g}}(G, e) \xrightarrow{i} C(G, e)$$

defined by

$$(i(x_1, \ldots, x_k))(t) = \exp x_1 \ldots \exp x_{n-1} \exp (kt - n + 1)x_n$$

for $t \in \left[\frac{n-1}{k}, \frac{n}{k}\right]$, $n = 1, \ldots, k$. Denote by $P(\mathcal{U}(\mathcal{H}), 1)$ the set of curves in $\mathcal{U}(\mathcal{H})$ starting at 1. The subset of closed curves in $P(\mathcal{U}(\mathcal{H}), 1)$ will be denoted by $C(\mathcal{U}(\mathcal{H}), 1)$. We now define a map

$$\delta(D, G) : C^{g}(G, e) \to P(\mathcal{U}(\mathcal{H}), 1)$$

by putting ($t \in [0, 1]$)

$$\delta(D, G)(x_1, \ldots, x_k)(t) := \operatorname{Exp} D(x_1) \ldots \operatorname{Exp} D(x_{n-1}) \operatorname{Exp} (kt-n+1)D(x_n)$$

for $t \in \left[\frac{n-1}{k}, \frac{n}{k}\right]$, $n = 1, \ldots, k$. Because G is connected, any $g \in G$ can be written as

$$g = \exp x_1 \ldots \exp x_k$$

for suitable $x_i \in \mathfrak{g}$. Using the commutative diagram above, we get the following more geometrical result.

PROPOSITION 1 [*2*]. — Let $D : \mathfrak{g} \to \mathcal{A}(\vartheta)$ be a skew-adjoint representation. Then the following statements are equivalent:

i) D is G-integrable;
ii) $\operatorname{Im} \delta(D, G) \subset C(\mathcal{U}(\mathcal{H}), 1)$.

3. MAXIMALITY AND INTEGRABILITY

Denote by $\mathcal{M}(M)$ the Lie algebra of smooth vectorfields on M. For $\xi \in \mathcal{M}(M)$ let

$$(m, t) \in D(\xi) \subset M \times \mathbb{R} \to F(\xi)(m, t) = \varphi_t^{\xi}(m) \in M$$

be the flow of ξ. Let $D(\xi, t)$ be the set of points m of M such that (m, t) lies in $D(\xi)$. $D(\xi, t)$ is open for $t \in \mathbb{R}$ [*4*]. ξ is complete if $D(\xi, t) = M$ for $t \in \mathbb{R}$.

A *Lie algebra action* of $\mathfrak{g}$ on M (also called *infinitesimal G-action* on M) is a Lie algebra homomorphism

$$\theta : \mathfrak{g} \to \mathcal{M}(M).$$

θ is called *complete* if $\theta(x)$ is complete for $x \in \mathfrak{g}$.

$P(M, m)$ and $C(M, m)$ denote, respectively, the set of curves in M starting at m and the subset of closed curves. $\Omega(M, m)$ will denote the homotopy classes (rel. $\{0, 1\}$) of paths of $P(M, m)$. $\pi_1(M, m) \subset \Omega(M, m)$ denotes the homotopy classes of based paths in (M, m). Thus we have the following commutative diagram

$$\begin{array}{ccc} C(M, m) & \xrightarrow{c} & P(M, m) \\ \downarrow{\scriptstyle \mu_m} & & \downarrow{\scriptstyle \nu_m} \\ \pi_1(M, m) & \xrightarrow[c]{} & \Omega(M, m) \end{array}$$

where ν_m and $\mu_m = \nu_m \mid C(M, m)$ are the natural projections.

Define $C(\theta, m; G, e) \subset C^{q}(G, e) \overset{i}{\subset} C(G, e)$ as follows:

$$C(\theta, m; G, e) := \{ (x_1, \ldots, x_k) \in C^{q}(G, e) \mid \varphi_1^{\theta(x_k)} \ldots \varphi_1^{\theta(x_1)}(m) \text{ exists} \}.$$

There exist natural maps (compare the definition of $\delta(D, G)$)

$$\delta(\theta, G, m) : C(\theta, m; G, e) \to P(M, m)$$

and

$$\varepsilon(\theta, G, m) : C(\theta, m; G, e) \to \Omega(M, m)$$

such that the diagram

$$\begin{array}{ccc} & \overset{\delta(\theta,G,m)}{\nearrow} & P(M, m) \\ C(\theta, m; G, e) & & \downarrow \nu_m \\ & \underset{\varepsilon(\theta,G,m)}{\searrow} & \Omega(M, m) \end{array}$$

commutes. θ is called *G-maximal* ([*2*] [*5*] [*9*]) if for $m \in M$

$$\operatorname{Im} \delta(\theta, G, m) \subset C(M, m)$$

or equivalently

$$\operatorname{Im} \varepsilon(\theta, G, m) \subset \pi_1(M, m).$$

A Lie algebra action $\theta : \mathfrak{g} \to \mathcal{M}(M)$ is called *transitive* if for $m, m' \in M$ there exists a

$$(x_1, \ldots, x_k) \in C(\theta, m; G, e)$$

such that $\delta(\theta, G, m)(x_1, \ldots, x_k)(1) = m'$. The proof of the following result is a straightforward calculation.

PROPOSITION 2. — Let $\theta : \mathfrak{g} \to \mathcal{M}(M)$ be a transitive Lie algebra action. Then θ is G-maximal iff there is a $m \in M$ such that

$$\operatorname{Im} \delta(\theta, G, m) \subset C(M, m).$$

Now consider a covering $p : M' \to M$. Since p is a local diffeomorphism, there is a natural (injective) Lie algebra homomorphism

$$p' : \mathcal{M}(M) \to \mathcal{M}(M').$$

In this situation, a Lie algebra action $\theta' : \mathfrak{g} \to \mathcal{M}(M')$ is called a *covering* of a Lie algebra action $\theta : \mathfrak{g} \to \mathcal{M}(M)$ if

$$\begin{array}{ccc} & & \mathcal{M}(M') \\ & \overset{\theta'}{\nearrow} & \uparrow p' \\ \mathfrak{g} & \xrightarrow[\theta]{} & \mathcal{M}(M) \end{array}$$

commutes. Since a covering has unique path lifting, an application of Proposition 2 gives.

PROPOSITION 3. — Let $\theta : \mathfrak{g} \to \mathcal{M}(M)$ be a transitive G-maximal $\mathfrak{g}$-action on M and let $p' : M' \to M$ be a regular covering of M. Then the covering $\theta' : \mathfrak{g} \to \mathcal{M}(M')$ of θ is G-maximal iff there is a $m' \in M'$ such that

$$\operatorname{Im} \varepsilon(\theta, G, p(m')) \subset p_* \pi_1(M', m') .$$

Any open imbedding $i : M \subset M^*$ induces a Lie algebra homomorphism

$$i^* : \mathcal{M}(M^*) \to \mathcal{M}(M)$$

given by $i^* \xi^* = \xi^* \mid M$; here $\xi^* \mid M$ denotes the restriction of $\xi^* \in \mathcal{M}(M^*)$ to M. Now consider a Lie algebra action $\theta : \mathfrak{g} \to \mathcal{M}(M)$ and an open imbedding $i : M \subset M^*$. Then $\theta^* : \mathfrak{g} \to \mathcal{M}(M^*)$ is called an *extension* of θ if

$$\begin{array}{ccc} & & \mathcal{M}(M^*) \\ & \overset{\theta^*}{\nearrow} & \downarrow{\scriptstyle i^*} \\ \mathfrak{g} & \xrightarrow[\theta]{} & \mathcal{M}(M) \end{array}$$

commutes.

Proposition 4 [5]. — A Lie algebra action θ is $\tilde{G}$-maximal if and only if there is a complete extension of θ.

Corollary. — Any complete $\mathfrak{g}$-action is $\tilde{G}$-maximal.

As an example, consider the construction given in [2]: take $M = \mathbb{R}^2 - (0,0)$, $\mathfrak{g} = \mathbb{R}^2$ (2-dimensional Abelian Lie algebra) and define $\theta : \mathfrak{g} \to \mathcal{M}(M)$ by

$$(*) \qquad \theta(a, b) = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}$$

for $(a, b) \in \mathbb{R}^2$. θ is a non-complete $\tilde{G}$-maximal action for $\tilde{G} \cong \mathbb{R}^2$. A natural complete extension on $M^* = \mathbb{R}^2$ is given by $(*)$, too. Now consider the double covering (Riemannian sheet) M_R of $\mathbb{R}^2 - (0, 0)$. The corresponding covering

$$\theta_R : \mathbb{R}^2 \to \mathcal{M}(M_R)$$

is non $\tilde{G}$-maximal. Thus, according to Proposition 4, there is no complete extension of θ_R.

A vectorfield $\xi \in \mathcal{M}(M)$ is called *quasi-complete* if

$$E(\xi, t) := M \backslash D(\xi, t)$$

is a set of measure zero for $t \in \mathbb{R}$; note that ξ is complete iff $E(\xi, t) = \phi$ for $t \in \mathbb{R}$. So a Lie algebra action θ is *(quasi-)complete* if $\theta(x)$ has this property for any $x \in \mathfrak{g}$.

Let Ω be a volume on M; Ω is called ξ-invariant if $L_\xi \Omega = 0$. $\mathcal{M}(M, \Omega)$ will denote the Lie algebra of vectorfields $\xi \in \mathcal{M}(M)$ such that $L_\xi \Omega = 0$. We say that θ acts on (M, Ω) if $\operatorname{Im} \theta \subset \mathcal{M}(M, \Omega)$.

Denote by $\mathcal{F}_0(M, \Omega)$ the pre-Hilbert space of compactly supported complex-valued functions on M. We denote by $L^2(M, \Omega)$ the corresponding Hilbert space. A $\mathfrak{g}$-action θ on (M, Ω) induces a representation

$$D(\theta) : \mathfrak{g} \to \text{End}\, \mathscr{F}_0(M, \Omega)$$

via $(f \in \mathscr{F}_0(M, \Omega))$

$$(D(\theta)(x)f)(m) = \frac{d}{dt}\bigg|_{t=0} f(\varphi_t^{\theta(x)}(m)).$$

PROPOSITION 5 [*2*]. — Let θ be a quasi-complete $\mathfrak{g}$-action on (M, Ω). Then $D(\theta)$ is a skew-adjoint representation of $\mathfrak{g}$ on $\mathscr{F}_0(M, \Omega)$ in $L^2(M, \Omega)$. Moreover, $D(\theta) : \mathfrak{g} \to \mathscr{A}(\mathscr{F}_0(M, \Omega))$ is G-integrable if and only if θ is G-maximal.

Hence, in view of the results of Palais, a representation induced from a quasi-complete $\mathfrak{g}$-action on (M, Ω) is G-integrable iff the $\mathfrak{g}$-action can be regarded as a restriction of a complete one.

4. LIE ALGEBRA ACTIONS ON PRINCIPAL FIBRE BUNDLES

Let (P, π, M, S) denote a principal fibre bundle with projection $\pi : P \to M$ and structure group S. A $\mathfrak{g}$-action (ϑ, θ) on (P, π, M, S) consists of $\mathfrak{g}$-actions

$$\vartheta : \mathfrak{g} \to \mathscr{M}(P), \qquad \theta : \mathfrak{g} \to \mathscr{M}(M)$$

such that for $x \in \mathfrak{g}$

i) $D(\vartheta(x)) = (\pi \times id_{\mathbb{R}})^{-1} D(\theta(x))$;

ii) $\pi(F(\vartheta(x))(p, t)) = F(\theta(x))(\pi(p), t)$ for $(p, t) \in D(\vartheta(x))$;

iii) $\vartheta(x)$ is S-invariant.

Now let $\theta : \mathfrak{g} \to \mathscr{M}(M)$ be a $\mathfrak{g}$-action and let $p : M' \to M$ be a regular covering corresponding to a normal subgroup $H \subset \pi_1(M, m_0)$. Denote by

$$\theta' : \mathfrak{g} \to \mathscr{M}(M')$$

the induced covering action. Then (θ', θ) is a $\mathfrak{g}$-action on the principal fibre bundle $(M', p, M, \pi_1(M, m_0)/H)$.

For Lie algebra actions on principal fibre bundles with non-discrete structure group see § 5.

It is not hard to prove the following result.

PROPOSITION 6. — Let (ϑ, θ) be a $\mathfrak{g}$-action on (P, π, M, S). Suppose that θ is transitive and G-maximal. Then ϑ is G-maximal iff there is a $p \in P$ such that

$$\text{Im}\, \delta(\vartheta, G, p) \subset C(P, p).$$

5. VECTORFIELD REPRESENTATIONS

Let $\rho : S \to \text{Aut } V$ be a finite-dimensional representation of S in a complex vector space V. Then the ρ-bundle associated with (P, π, M, S) is the vectorbundle

$$(E_\rho, \pi_\rho, M, V),$$

where E_ρ is the orbit space of the right G-action on $P \times V$ given by letting $g \in G$ take (p, v) to $(pg, \rho^{-1}(g)v)$. The equivalence class of (p, v) is denoted by $[p, v]_\rho$. We have $\pi_\rho[p, v]_\rho = \pi(p)$. (E_ρ, π_ρ, M, V) is sometimes denoted by $E_\rho(P)$ or simply E_ρ.

Any $\mathfrak{g}$-action (ϑ, θ) on (P, π, M, S) induces a $\mathfrak{g}$-action ϑ_ρ on E_ρ via

$$F(\vartheta_\rho(x))([p, v]_\rho, t) = [F(\vartheta(x))(p, t), v]_\rho .$$

Observe that this flow is well defined since $\vartheta(x)$ is S-invariant.

Let $\Gamma_0 E_\rho$ be the space of compactly supported smooth sections in (E_ρ, π_ρ, M, V). A $\mathfrak{g}$-action (ϑ, θ) on (P, π, M, S) induces a representation —called vectorfield representation—

$$D_\rho(\vartheta, \theta) : \mathfrak{g} \to \text{End } \Gamma_0 E_\rho$$

via $(\sigma \in \Gamma_0 E_\rho)$

$$(D_\rho(\vartheta, \theta)(x)\sigma)(m) := \frac{d}{dt}\bigg|_{t=0} \left(\varphi_{-t}^{\vartheta_\rho(x)} \circ \sigma \circ \varphi_t^{\theta(x)}\right)(m) .$$

For unitary ρ, we define a pre-Hilbert structure on $\Gamma_0 E_\rho$ by setting

$$\langle \sigma_1, \sigma_2 \rangle = \int_\Omega \langle \sigma_1(m), \sigma_2(m) \rangle_m .$$

We denote by $L^2(E_\rho, \Omega)$ the corresponding Hilbert space. The following result generalizes Proposition 5.

PROPOSITION 7 [*2*]. — Let (ϑ, θ) be a $\mathfrak{g}$-action on (P, π, M, S) and let $\rho : S \to \text{Aut } V$ be a unitary faithful finite-dimensional representation. Suppose that θ is quasi-complete on (M, Ω). Then $D_\rho(\vartheta, \theta)$ is a skew-adjoint representation of $\mathfrak{g}$ on $\Gamma_0 E_\rho \subset L^2(E_\rho, \Omega)$. Moreover, $D_\rho(\vartheta, \theta) : \mathfrak{g} \to \mathscr{A}(\Gamma_0 E_\rho)$ is G-integrable if and only if ϑ is G-maximal.

6. APPLICATION TO GEOMETRIC PRE-QUANTIZATION

(P, A, M) will denote a smooth principal fibre bundle with abelian structure group A over a connected manifold M. π will denote the projection $P \to M$. Let α be a connection form on P and let ω be a symplectic structure on M. Given a linear injective map $\lambda : \mathbb{R} \to \alpha$ from the real numbers into the Lie algebra of A, we say that

$$(P, \alpha, A, \lambda, M, \omega)$$

is a quantizing bundle [8] if

$$d\alpha = \lambda\pi^*\omega .$$

We remark that this definition includes (up to association) Kostant's Hermitian line bundle [3] and Souriau's *espace fibré quantifiant* [7].

Let $\{ , \}$ be the Lie algebra structure on the space $\mathcal{F}(M, \omega)$ of smooth real-valued functions on M defined by

$$\{ \varphi, \psi \} = \xi_\varphi \psi = \omega(\xi_\psi, \xi_\varphi)$$

where ξ_φ is the Hamiltonian vectorfield corresponding to $\varphi \in \mathcal{F}(M, \omega)$. Suppose we are given a Lie algebra homomorphism

$$\phi : \mathfrak{g} \to \mathcal{F}(M, \omega) .$$

For any quantizing bundle $\eta = (P, \alpha, A, \lambda, M, \omega)$ over (M, ω) ϕ induces a $\mathfrak{g}$-action $(\theta_\phi^\eta, \theta_\phi)$ on η as follows:

$$\theta_\phi : \mathfrak{g} \to \mathcal{M}(M)$$

is given by $\theta_\phi(x) := \xi_{\phi(x)}$, and

$$\theta_\phi^\eta : \mathfrak{g} \to \mathcal{M}(P)$$

is defined via the flows of $\theta_\phi^\eta(x)$ for $x \in \mathfrak{g}$:

$$F(\theta_\phi^\eta(x))(p, t) := F(\theta_\phi^\alpha(x))(p, t) \exp - t\lambda(\phi(x))(\pi(p)) .$$

Here $\theta_\phi^\alpha(x) \in \mathcal{M}(P)$ denotes the horizontal lift of $\theta_\phi(x) \in \mathcal{M}(M)$ with respect to α. The Lie algebra action $(\theta_\phi^\eta, \theta_\phi)$ is called prequantization.

Suppose that (M, ω) is (A, λ)-quantizable. Denote by $\mathbb{Q}(A, \lambda, M, \omega)$ the set of equivalence classes of (A, λ, M, ω)-bundles. Then there is a free and transitive action

$$\mathbb{Q}(A, \lambda, M, \omega) \times \pi_1^A(M, m_0) \to \mathbb{Q}(A, \lambda, M, \omega)$$

of the group $\pi_1^A(M, m_0)$ of group homomorphisms $\pi_1(M, m_0) \to A$ on $\mathbb{Q}(A, \lambda, M, \omega)$ (see e. g. [9]). Denote by $a_m^\alpha(\gamma) \in A$ the parallel displacement along $\gamma \in C(M, m)$ with respect to the connection form α of $\eta = (P, \alpha, A, \lambda, M, \omega)$. We know [9] that

$$(*) \qquad a_{m_0}^{\alpha\chi}(\gamma) = a_{m_0}^{\alpha}(\gamma)\chi[\gamma]$$

for $\gamma \in C(M, m_0)$, $\chi \in \pi_1^A(M, m_0)$, where $\alpha\chi$ denotes the connection form of the quantizing bundle $\eta\chi$.

We shall now discuss the maximality properties of $(\theta_\phi^\eta, \theta_\phi)$.

$$\mu(\phi, G, m) : C(\theta_\phi, m; G, e) - A$$

be the association given by

$$(x_1, \ldots, x_k) \to \exp - \lambda \sum_{i=1}^{k} (\phi(x_i))(\varphi_1^{\theta_\phi(x_i)} \ldots \varphi_1^{\theta_\phi(x_1)}(m)).$$

Suppose that θ_ϕ is G-maximal. Then

$$\delta(\theta_\phi^\eta, \mathrm{G}, p)(\overline{x})(1) = pa^\alpha_{\pi(p)}(\delta(\theta_\phi, \mathrm{G}, \pi(p))(\overline{x}))\mu(\phi, \mathrm{G}, \pi(p))(\overline{x})$$

for $\overline{x} \in \mathrm{C}(\theta_\phi, m; \mathrm{G}, e)$, $p \in \mathrm{P}$. Hence, for G-maximal θ_ϕ, θ_ϕ^η is G-maximal iff

$$a^\alpha_m(\delta(\theta_\phi, \mathrm{G}, m)(\overline{x})) = \mu^{-1}(\phi, \mathrm{G}, m)(\overline{x})$$

for $\overline{x} \in \mathrm{C}(\theta_\phi, m; \mathrm{G}, e)$, $m \in \mathrm{M}$. For transitive and G-maximal θ_ϕ it follows from Proposition 6 that θ_ϕ^η is G-maximal iff

$$a^\alpha_{m_0}(\delta(\theta_\phi, \mathrm{G}, m_0)(\overline{x})) = \mu^{-1}(\phi, \mathrm{G}, m_0)(\overline{x}).$$

By (*), we have the following result.

PROPOSITION 8. — Suppose that θ_ϕ is transitive and that θ_ϕ^η is G-maximal. Take $\chi \in \pi_1^\wedge(\mathrm{M}, m_0)$. Then $\theta_\phi^{\eta\chi}$ is G-maximal iff

$$\mathrm{Im}\ \varepsilon(\theta_\phi, \mathrm{G}, m_0) \subset \mathrm{Ker}\ \chi.$$

7. AN EXAMPLE FOR THE HEISENBERG ALGEBRA

Let $\mathrm{H} = (\mathbb{R}^3, .)$ be the Heisenberg group with multiplication

$$(x_1, x_2, x_3)\cdot(y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2).$$

Denote by $\mathfrak{h}$ the Lie algebra of left invariant vectorfields on H. We shall now construct a Lie algebra homomorphism ($\dot{\mathbb{R}}^2 = \mathbb{R}^2 - (0, 0)$)

$$\phi : \mathfrak{h} \to \mathscr{F}(\dot{\mathbb{R}}^2, dx \wedge dy).$$

Consider the basis

$$\{ \partial/\partial x_1, \partial/\partial x_2 + x_1 \partial/\partial x_3, \partial/\partial x_3 \}$$

of $\mathfrak{h}$ and define

$$\phi(\partial/\partial x_1) = y, \qquad \phi(\partial/\partial x_2 + x_1\partial/\partial x_3) = x, \qquad \phi(\partial/\partial x_3) = 1.$$

It is not hard to conclude that ϕ is a Lie algebra homomorphism and that

$$\theta_\phi : \mathfrak{h} \to \mathscr{M}(\dot{\mathbb{R}}^2)$$

is given by

$$\theta_\phi(\partial/\partial \mathrm{x}_1) = \partial/\partial \mathrm{x}, \qquad \theta_\phi(\partial/\partial \mathrm{x}_2 + \mathrm{x}_1\partial/\partial \mathrm{x}_3) = -\partial/\partial \mathrm{y}, \qquad \phi(\partial/\partial \mathrm{x}_3) = 0.$$

By Proposition 4, θ_ϕ is H-maximal. Furthermore,

$$\varepsilon(\theta_\phi, \mathrm{H}, *) : \mathrm{C}(\theta_\phi, *; \mathrm{H}, e) \rightarrow \pi_1(\dot{\mathbb{R}}^2, *)$$

is surjective for any $* \in \dot{\mathbb{R}}^2$.

Since $dx \wedge dy$ is exact, $(\dot{\mathrm{R}}^2, dx \wedge dy)$ is quantizable; denote by

$$\varepsilon = (\dot{\mathbb{R}}^2 \times \mathrm{A}, \alpha, \mathrm{A}, \lambda, \dot{\mathbb{R}}^2, dx \wedge dy)$$

the trivial (A, λ)-bundle associated with the system $\{ f_{ij}, \alpha_i; i, j \in \mathrm{J} \}$ of quantizing functions [8] given by

$$f_{ij} = e, \qquad \alpha_i = xdy \mid \mathrm{U}_i .$$

Hence, in view of Proposition 4, θ_ϕ^ε is a H-maximal $\mathfrak{h}$-action on $\dot{\mathbb{R}}^2 \times \mathrm{A}$. The following result now follows from Proposition 8.

PROPOSITION 9. — $\theta_\phi^{\varepsilon\chi}$ is H-maximal if and only if

$$\chi \in \pi_1^{\mathrm{A}}(\dot{\mathbb{R}}^2, *)$$

is trivial.

By using Proposition 7, we can conclude that for unitary and faithful ρ the skew-adjoint representation $\mathrm{D}_\rho(\theta_\phi^{\varepsilon\chi}, \theta_\phi)$ integrates up to a unitary representation of the Heisenberg group if and only if

$$\chi \in \pi_1^{\mathrm{A}}(\dot{\mathbb{R}}^2, *)$$

is trivial.

8. ACTIONS ON BUNDLES OVER MOMENTUM PHASE SPACE

We shall now apply the results of § 6 to actions on cotangent bundles. Let $\delta : \mathfrak{g} \rightarrow \mathscr{M}(\mathrm{X})$ be a $\mathfrak{g}$-action on X. Consider the cotangent bundle $\mathrm{T}^*\mathrm{X}$ with projection $\nu : \mathrm{T}^*\mathrm{X} \rightarrow \mathrm{X}$. Then δ induces a $\mathfrak{g}$-action θ_δ on $\mathrm{T}^*\mathrm{X}$ via $(x \in \mathfrak{g})$

$$\varphi_t^{\theta_\delta(x)}(u_q) = (\varphi_{-t}^{\delta(x)})^* u_q ,$$

$u_q \in \mathrm{T}^*\mathrm{X}$, $\nu(u_q) = q$. Observe that θ_δ is G-maximal iff δ is. Let Ω denote the canonical 1-form on $\mathrm{T}^*\mathrm{X}$ given by

$$\Omega(\xi_{u_q}) = u_q(\nu_* \xi_{u_q}) .$$

The exterior derivative of Ω is the canonical symplectic structure on $\mathrm{T}^*\mathrm{X}$. Since

$$\mathrm{L}_{\theta_\delta(x)}\Omega = 0$$

for $x \in \mathfrak{g}$ [1],

$$x \in \mathfrak{g} \overset{\phi}{\rightarrow} -\Omega(\theta_\delta(x)) \in \mathscr{F}(\mathrm{T}^*\mathrm{X}, d\Omega)$$

defines a Lie algebra homomorphism such that $\theta_\phi = \theta_\delta$ [6]. The corresponding prequantization action $\theta_\delta^\varepsilon$ on the trivial quantizing bundle $\varepsilon = (T^*X \times A, \alpha, A, \lambda, T^*X, d\Omega)$ is the trivial lift of θ_δ [9]. Hence $\theta_\delta^\varepsilon$ is G-maximal iff δ is G-maximal. Observe that $\nu : T^*X \to X$ induces an isomorphism [9]

$$\pi_1(T^*X, u_q) \overset{\cong}{\to} \pi_1(X, q).$$

Thus

$$\pi_1^A(T^*X, u_q) \cong \pi_1^A(X, q)$$

and we have the following result.

PROPOSITION 10. — Let $\delta : \mathfrak{g} \to \mathcal{M}(X)$ be a quasi-complete transitive G-maximal $\mathfrak{g}$-action on X. Denote by $(\theta_\delta^\varepsilon, \theta_\delta)$ the induced prequantization on $\varepsilon = (T^*X \times A, \alpha, A, \lambda, T^*X, d\Omega)$. Take $\chi \in \pi_1^A(X, q_0)$ and denote by χ^* the corresponding element of $\pi_1^A(T^*X, u_{q_0})$. Then $\theta_\delta^{\varepsilon\chi^*}$ is G-maximal if and only if

$$\operatorname{Im} \varepsilon(\delta, G, q_0) \subset \operatorname{Ker} \chi.$$

Hence, in view of Proposition 7, we conclude that for unitary and faithful ρ, $D_\rho(\theta_\delta^{\varepsilon\chi^*}, \theta_\rho)$ is a skew-adjoint representation of $\mathfrak{g}$ which integrates up to a unitary representation of G if and only if

$$\operatorname{Im} \varepsilon(\delta, G, q_0) \subset \operatorname{Ker} \chi.$$

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