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Section A :
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Superstable interactions in quantum statistical mechanics: Maxwell-Boltzmann statistics

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ABSTRACT. — We consider a system of quantum particles in $\mathbb{R}^\nu (\nu \leqslant 3)$ obeying Maxwell-Boltzmann statistics and interacting via a superstable and lower regolar potential. The following bounds for the Reduced Density Matrices are proved for any value of β and x :

$$\rho_\Lambda(x_1 \ldots x_n ; y_1 \ldots y_n) \leqslant \xi^n$$

These inequalities are a consequence of the Ginibre representation [*1*] [*2*] and estimates on the classical correlation functions defined on the space of paths, in analogy with the Ruelle superstable estimates [*3*]. These bounds allow us to obtain the existence of the pressure, its independence of boundary conditions, and the existence of the thermodynamic limit, extending previons results of Ginibre.

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1. INTRODUCTION

In 1965, Ginibre proposed an approach to Quantum Statistical Mechanics based on functional integration [1] [2].

Consider a system of interacting quantum particles in a box Λ ($\Lambda \subset \mathbb{R}^\nu$ is assumed to be open and sufficiently regular). The (formal) Hamiltonian is :

$$H_\Lambda^n = -\frac{\hbar^2}{2m}\sum_{i=1}^{n}\Delta_{x_i} + \mathcal{U}(x_1 \dots x_n) \qquad (1.1)$$

where m is the mass of the particles, $\hbar = h/2\pi$, h is the Planck constant, and $\mathcal{U} : \Lambda^n \to \mathbb{R}$ the potential energy.

We introduce the spaces $\mathcal{G}_n(\Lambda) = L_2(\Lambda^n)$ and $\mathcal{G}(\Lambda) = \bigoplus_{n\geq 0} \mathcal{G}_n(\Lambda)$. If $\mathcal{U}$ is regular enough, a semigroup of operators is given in $\mathcal{G}$ by the following kernel :

$$W_\beta^{n,\Lambda}(X, Y) = [\exp(-\beta H_\Lambda^n)](X, Y) = \int P_{X,Y}^\theta(d\underline{\omega})\alpha_\Lambda(\underline{\omega}) \exp[-\mathcal{U}^\theta(\underline{\omega})]. \qquad (1.2)$$

Here $P_{X,Y}^\theta(d\underline{\omega}) = \prod_{i=1}^{n} P_{x_i y_i}^\theta(d\omega_i)$, $X = \{x_i\}_{i=1}^n$, $Y = \{y_i\}_{i=1}^n$, $x_i, y_i \in \Lambda$ and $P_{x_i y_i}^\theta(d\omega_i)$ is the conditional Wiener measure given by the Green function $\exp -\frac{|x-y|^2}{2\theta}\left(\frac{1}{\sqrt{2\pi\theta}}\right)$, $\theta = \frac{\beta\hbar}{m}$, $\beta > 0$. $P_{xy}^\theta(d\omega)$ is a measure on the space of continuous functions $\omega = [0, \theta] \to \mathbb{R}^\nu$, $\underline{\omega} = \{\omega_1 \dots \omega_n\}$. The function α_Λ is defined as $\alpha_\Lambda(\underline{\omega}) = 1$ if the range of all the ω_i's is contained in Λ and $\alpha_\Lambda(\underline{\omega}) = 0$ otherwise. Finally $\mathcal{U}^\theta(\underline{\omega}) = \int_0^\theta d\tau \mathcal{U}^\theta(\omega_1(\tau)\dots\omega_n(\tau))$, where $\mathcal{U}^\theta = \frac{\beta}{\theta}\mathcal{U}$.

We denote by H_Λ^n the generator of $W_t^{n,\Lambda}$, because it is the selfadjoint version of the formal hamiltonian with Dirichelet boundary conditions [2].

For $\varepsilon = 0, \pm 1$, we define the grand-canonical particles density matrices

$$\sigma_\Lambda^{\varepsilon,n} = Z^n S_n^\varepsilon(\exp - \beta H_\Lambda^n)(Z_\Lambda^\varepsilon)^{-1}, \qquad (1.3)$$

as positive trace class operators on $\mathcal{G}_n(\Lambda)$. Here $S_n^\varepsilon = 1/n!$ in the case of Maxwell-Boltzmann (M. B.) statistics ($\varepsilon = 0$) and S_n^ε is the canonical projection on the symmetric or antisymmetric functions of $\mathcal{G}_n(\Lambda)$ for Bose-Einstein (B. E.) Statistics ($\varepsilon = 1$) and for Fermi-Dirac (F. D.) statistics

($\varepsilon = -1$) respectively. Z_Λ^ε denotes the partition function for activity z and inverse temperature β and is defined as :

$$Z_\Lambda^\varepsilon = \sum_{n \geqslant 0} \mathrm{Tr}_n\, z^n S_n^\varepsilon \exp - \beta H_\Lambda^n \tag{1.4}$$

where Tr_n means trace on $\mathscr{G}_n(\Lambda)$.

The m-particle reduced density matrices (RDM) are defined by:

$$\rho_\Lambda^{\varepsilon,m} = \sum_{n=0}^{\infty} \frac{(n+m)\,!}{n\,!} \mathrm{Tr}_n\, \sigma_\Lambda^{\varepsilon,n+m} \tag{1.5}$$

where the above Tr_n means the n-particle partial trace on $\mathscr{G}_{n+m}(\Lambda)$.

In terms of kernels this becomes

$$\rho_\Lambda^{\varepsilon,m}(x_1 \dots x_m ; y_1 \dots y_m) = \sum_{n \geqslant 0} \frac{(n+m)\,!}{n\,!} \int_{\Lambda^n} du_1 \dots du_n \sigma_\Lambda^{\varepsilon,n+m}(x_1 \dots x_m, u_1 \dots u_n ; y_1 \dots y_m, u_1 \dots u_n) \tag{1.6}$$

The integral representation (1.2) allows to write the RDM's in the following way when $\varepsilon = 0$:

$$\rho_\Lambda(X, Y) = \int P_{X,Y}^\theta(d\underline{\omega})\rho_\Lambda(\underline{\omega}) \tag{1.7}$$

where

$$\rho_\Lambda(\underline{\omega}) = Z_\Lambda^{-1} z^{|\underline{\omega}|} \int d\underline{\omega}' \alpha_\Lambda(\underline{\omega} \cup \underline{\omega}') \cdot \exp\{ - \mathscr{U}^\theta(\underline{\omega} \cup \underline{\omega}') \} \tag{1.8}$$

$$Z_\Lambda = \int d\underline{\omega} \alpha_\Lambda(\underline{\omega}) \exp\{ - \mathscr{U}^\theta(\underline{\omega}) \} \tag{1.9}$$

Here $X = \{ x_i \}_{i=1}^n$ and we have dropped the indices m and ε for notational simplicity. In this formula $\underline{\omega} \in \Omega = \bigcup_{n \geqslant 0} \Omega_n$ and Ω_n is the symmetrized space of all sets of continuous trajectories $\{ \omega_i \}_{i=1}^n$, $\omega_i : [0, \theta] \to \mathbb{R}^\nu$. The union $\underline{\omega} \cup \underline{\omega}'$ denotes the joint set of the trajectories in $\underline{\omega}$ and $\underline{\omega}'$, $|\underline{\omega}|$ denotes the number of components of $\underline{\omega} \in \Omega$ and $Z_\Lambda = Z_\Lambda^0$; finally $d\underline{\omega}$ is a measure on Ω defined as :

$$d\underline{\omega} = \sum_{n \geqslant 0} \frac{1}{n\,!} d\omega_1 \dots d\omega_n , \tag{1.10}$$

where

$$\int d\omega = z \int dx P_{xx}^\beta(d\omega) . \tag{1.11}$$

The above representation holds for M. B. statistics and is a direct consequence of 1.2 and the Feynman-Kac formula.

Analogous representations hold for the B. E. and F. D. statistics, but with more complicated spaces of trajectories due to the combinatorics arising from the statistics.

The RDM's are expressed in term of the $\rho_\Lambda(\underline{\omega})$'s. They have the same structure of classical correlation functions in which points are replaced by trajectories. Thus it is tempting to use the technology of Classical Statistical Mechanics to obtain results for the quantum case. This has been done by Ginibre, who proved the existence and uniqueness of infinite volume limit RDM's (for M. B., B. E., F. D. statistics) by means of a low activity expansion, similar to that used in Classical Statistical Mechanics (see ref. [*1*] [*2*]).

Existence and properties of thermodynamic functions were also obtained by Classical Statistical Mechanics techniques.

In this paper we want to apply to Quantum Statistical Mechanics a classical idea due to Ruelle called superstability [*3*]. Roughly speaking the problem solved in [*3*] is the following. In several problems one needs to control large fluctuations of the number of particles in a small region Γ of the physical space. Such fluctuations are prevented by the Gibbs factor in a trivial way if the potential is non negative and positive at the origin. But in the presence even of an arbitrarily small negative part, one needs to solve a genuine many body problem, because the interaction between the particles in Γ and the external particles, has to be controlled. Ruelle solves this technical difficulty for a very large class of interactions, called superstable interactions (see definition below). More precisely, for such a class of interactions he proves that the grand canonical probability of finding more than n particles in a box Γ is bounded by $\exp\left(-\frac{k_1 n^2}{\text{vol}\,\Gamma}+k_2 n\right)$ where k_1 and k_2 are constants depending only on the temperature β^{-1}, the activity z and the interaction.

This probability estimate is a rather simple consequence of a highly non-trivial estimate on the correlation functions.

It has to be remarked that this approach is not perturbative (a free gas does not satisfy the above probability estimate), so it works for all values of z and β. Among the consequences of this probability estimate is the existence of the infinite volume correlation functions. These have already been obtained (together with their uniqueness) by means of perturbative techniques working only in some region of z and β (see [*3*] and [*4*]).

The plan of this paper is the following. We use the Ginibre representation and consider the $\rho_\Lambda(\underline{\omega})$'s rather then the RDM's as the main object to investigate. They are correlation functions of a Classical Statistical Mechanical system of interacting trajectories. Such objects are expected to satisfy bounds of the same kind as those obtained in [3] for classical particles.

Let's now outline the content of the paper. In section 2 we prove the basic estimates on the $\rho_\Lambda(\underline{\omega})$'s for a large class of interactions in the case of M. B. statistics for $\nu \leqslant 3$. These estimates allow us to control the fluctuations on the number of particles in a box and hence the thermodynamic limit for any value of z and β (section 5). Moreover, for the case of M. B. statistics, we obtain the results discussed in [3] thus extending to a wider class of interactions all the results already proven, controlling fluctuations by positivity or by presence of hard-cores. In particular in Section 4, we obtain the existence of the pressure and its independence of the quantum boundary conditions. Section 3 is devoted to a probability estimate on the number of particles in a bounded region Γ, and we discuss its classical limit. Finally, in Appendix A we deduce a bound concerning Brownian motion that will be used systematically, throughout the paper in combination with the estimates in Section 2. In Appendix B, we prove some lemmas that are technical devices in deducing the main estimate of Section 2.

Although our results are obtained only for M. B. statistics, we believe that this work might provide a conceptual framework for the physically more interesting B. E. statistics. Furthermore some of our results may be applied also to bosons and fermions in one dimension, since, in this case, the partition function is the same for all statistics if the potential is sufficiently repulsive [5].

The main difficulty arising in dealing with B. E. statistics using this approach is the arbitrary « length » of the trajectories. The correlation bounds that one would hope to prove, at least in a small activity region, thus become more difficult, and might require a non trivial modification of the method. Furthermore if one has such estimates (even for any activity), one cannot automatically deduce uniform bounds on the RDM's because of the divergence of the free measure. Thus the statistical mechanics of interacting bosons at high activity has still to be understood for such simple interactions as pure hard-cores or positive potentials.

It should be remarked that quantum systems of charged particles interacting via a positive definite potential have been considered in [6] by means of the Ginibre representation and the sine — Gordon transformation — Furthermore some properties of superstability in a Quantum mechanical context have already been applied to deduce the barometric formula [7].

We conclude this section by stating our assumptions on the potential energy.

We assume that our particles system interact via a two-body potential $\phi, \phi : (0, +\infty) \to \mathbb{R}^1$, such that $\phi = \phi_1 + \phi_2$, and where ϕ_1 is continuous, positive, and strictly positive in a neighbour of the origin, and ϕ_2 is continuous and stable, i. e.

$$\sum_{i<j}^{(1,n)} \phi_2(x_i - x_j) \geq -\mathrm{B}n, \qquad \mathrm{B} > 0. \tag{1.12}$$

Let us consider a partition $\mathcal{Q}$ of $\mathbb{R}^\nu$ into half open cubes with side 1 :

$$\mathcal{Q} = \left\{ \Delta \mid \Delta = \prod_{i=1}^{\nu} \left(r_i - \frac{1}{2}, r_i + \frac{1}{2} \right], \quad r_i \in \mathbb{Z} \right\} \tag{1.13}$$

We assume that there exist positive constants A, B such that

$$\mathcal{U}(\mathrm{X}) \geq \mathrm{A} \sum_{\Delta \in \mathcal{Q}} n(\mathrm{X}, \Delta)^2 - \mathrm{B} \sum_{\Delta \in \mathcal{Q}} n(\mathrm{X}, \Delta) \tag{1.14}$$

Here

$$\mathcal{U}(\mathrm{X}) = \sum_{i<j} \phi(|x_i - x_j|), \qquad \mathrm{X} = \{ x_1 \dots x_m \} \in \mathbb{R}^{\nu m} \tag{1.15}$$

and n (X, Δ) denotes the number of particles of the configuration X in the element $\Delta \in \mathcal{Q}$.

The following decay property, called lower-regularity, is also required. Put

$$\mathrm{W}(\mathrm{X} \mid \mathrm{Y}) = \sum_{\substack{x_i \in \mathrm{X} \\ y_i \in \mathrm{Y}}} \phi(|x_i - y_i|), \tag{1.16}$$

where $\mathrm{X} = \{ x_1 \dots x_n \}$, $\mathrm{Y} = \{ y_1 \dots y_n \}$, $x_i \neq y_i \ \forall\ i, j$, then

$$-\mathrm{W}(\mathrm{X} \mid \mathrm{Y}) \leq \sum_{\Delta, \Delta^1 \in \mathcal{Q}} \Psi(\Delta, \Delta')n(\mathrm{X}, \Delta)n(\mathrm{Y}, \Delta') \tag{1.17}$$

where

$$\Psi(\Delta, \Delta') = \sup_{\substack{x \in \Delta \\ y \in \Delta'}} \phi(|x - y|) \tag{1.18}$$

and ϕ is the negative part of ϕ.

We require the existence of a positive, decreasing function Ψ, defined on the positive, integers, such that

$$\Psi(k) \geq \sup \{ \Psi(\Delta, \Delta') \mid d(\Delta, \Delta') = k \} \tag{1.19}$$

and

$$\sum_{k=0}^{\infty} \Psi(k) k^{\nu+\mu-1} = \mathrm{F} < +\infty \tag{1.20}$$

where $\mu > \frac{1}{2}$ and

$$d(\Delta, \Delta') = \max_{1 \leqslant i \leqslant \nu} \inf \{ | x_i - y_i | \mid x \in \Delta, y \in \Delta' \} \tag{1.21}$$

2. MAIN ESTIMATE

We introduce the functions:

$$\bar{\rho}_\Lambda(\underline{\eta}) = \frac{1}{Z_\Lambda} \int_\Omega d\underline{\omega} \exp \{ - \mathscr{U}^\theta(\underline{\omega} \cup \underline{\eta}) \} \alpha_\Lambda(\underline{\omega}) \tag{2.1}$$

which are the correlation functions up to the factor $z^{|\eta|}$.

Let $\mathscr{B}([0, \theta])$ be the σ-algebra of Borel sets in $[0, \theta]$; Ω_B the set of all measurable functions

$$\tilde{\omega} : B \to \mathbb{R}^\nu, \qquad B \in \mathscr{B}([0, \theta]).$$

We define the following large space of trajectories:

$$\tilde{\Omega} = \left[\bigcup_{n \geqslant 1} \tilde{\Omega}_1^{\otimes n\text{-symm}} \right] \cup \tilde{\Omega}_0 \tag{2.2}$$

where $\tilde{\Omega}_0$ is the vacuum element and

$$\tilde{\Omega}_1 = \bigcup_{B \in \mathscr{B}([0,\theta])} \Omega_B \tag{2.3}$$

Clearly Ω is a subset of $\tilde{\Omega}$ and it is useful to consider the following extension of $\mathscr{U}^\theta$, defined in sect. 1, from Ω to $\tilde{\Omega}$: if $\underline{\tilde{\omega}} \in \tilde{\Omega}$ we put

$$\mathscr{U}^\theta(\underline{\tilde{\omega}}) = \frac{1}{2} \sum_{i \neq j} \frac{\beta}{\theta} \int_0^\theta d\tau \chi(\{ \tau \in \mathscr{D}(\tilde{\omega}_i) \cap \mathscr{D}(\tilde{\omega}_j) \}) \phi(| \tilde{\omega}_i(\tau) - \tilde{\omega}_j(\tau) |) \tag{2.4}$$

where $\mathscr{D}(\tilde{\omega})$ is the domain of $\tilde{\omega}$.

We notice that for a fixed $\tilde{\eta} \in \tilde{\Omega}$ ($\underline{\tilde{\eta}} \cup \underline{\omega}$ still denotes the joint set of $\underline{\omega}$ and $\underline{\tilde{\eta}}$ i. e. $\underline{\tilde{\eta}} \cup \underline{\omega} \in \tilde{\Omega}_1^{\otimes p\text{-symm}}$ where $p = | \underline{\tilde{\eta}} | + | \underline{\omega} |$, and $| \underline{\tilde{\eta}} |$ is the number of components of $\underline{\tilde{\eta}}$)

$$\mathscr{U}^\theta(\underline{\tilde{\eta}} \cup \cdot) : \Omega \to \mathbb{R} \tag{2.5}$$

is measurable on Ω w. r. t. the σ-algebra of the Borel sets corresponding to the pointwise convergence topology. In fact, for each positive integer l, let $\phi_l(x) = \min \{ \phi(x), l \}$ and $\mathscr{U}_l^\theta$ the analogous of $\mathscr{U}^\theta$ with ϕ replaced by ϕ_l. Then $\mathscr{U}_l^\theta(\underline{\tilde{\eta}} \cup \cdot)$ is continuous in the pointwise topology on Ω for fixed $\underline{\tilde{\eta}}$, by Lebesgue dominated convergence theorem. The limit

$$\lim_{l \to \infty} \mathscr{U}_l^\theta(\underline{\tilde{\eta}} \cup \cdot) \tag{2.6}$$

exists a. e. $d\underline{\omega}$ by monotone convergence theorem and coincides with $\mathscr{U}^\theta(\underline{\tilde{\eta}} \cup \cdot)$ which is then measurable.

The above remark allows us to put, for each $\underline{\tilde{\eta}} \in \tilde{\Omega}$

$$\tilde{\rho}_\Lambda(\underline{\tilde{\eta}}) = \frac{1}{Z_\Lambda} \int d\underline{\omega} \alpha_\Lambda(\underline{\omega}) \exp\{-\mathscr{U}^\theta(\underline{\tilde{\eta}} \cup \underline{\omega})\} \tag{2.7}$$

which is an useful extension of definition (2.1).

Let now Γ be a bounded measurable region in $\mathbb{R}^\nu$ and define the map

$$\Pi_\Gamma : \Omega \to \tilde{\Omega} \tag{2.8}$$

as follows: if $\underline{\omega} \equiv \{\omega_1 \dots \omega_n\}$, then

$$\Pi_\Gamma \underline{\omega} = (\Pi_\Gamma \omega_1 \dots \Pi_\Gamma \omega_n). \tag{2.9}$$

Here $\Pi_\Gamma \omega$ is the trajectory of $\tilde{\Omega}_1$ whose domain is the measurable set

$$B = \{\tau \in [0, \theta] \mid \omega(\tau) \in \Gamma\} \tag{2.10}$$

and, if it has non zero Lebesgue measure,

$$(\Pi_\Gamma \omega)(\tau) = \omega(\tau), \qquad \tau \in B \tag{2.11}$$

Otherwise, if B has zero measure,

$$\Pi_\Gamma \omega \in \Omega_0 \tag{2.12}$$

We introduce also the map

$$s : \Omega \to \tilde{\Omega} \qquad \underline{\omega} \to s(\underline{\omega}) = \bigcup_\Delta \Pi_\Delta \underline{\omega} \tag{2.13}$$

The union is made on all $\Delta \in \mathscr{Q}$ such that $\Pi_\Delta(\underline{\omega})$ has domain of non zero Lebesgue measure.

Notice that for all $\underline{\eta} \in \Omega$

$$\bar{\rho}_\Lambda(\underline{\eta}) = \tilde{\rho}_\Lambda(s(\underline{\eta})) \tag{2.14}$$

Let $\underline{\tilde{\eta}} \in \tilde{\Omega}$ be such that each trajectory in it is completely contained in a tessera $\Delta \in \mathscr{Q}$; we also assume that the components of $\underline{\tilde{\eta}}$ have domain with positive measure. Let also Δ_1 be the first lexicographic element in the set of tesserae covering $\underline{\tilde{\eta}}$. We fix the origin in the center of Δ_1 and consider, for each integer q, the cubic region

$$\Lambda_q = \left[-l_q - \frac{1}{2}, l_q + \frac{1}{2} \right]^\nu \tag{2.15}$$

where

$$l_q = \mathscr{I}(e^{\alpha q}) \tag{2.16}$$

$\alpha > 0$ to be fixed later, and for $x \in \mathbb{R}^+$, $\mathscr{I}(x)$ is the integer part of x.

If Γ is a region paved by $\mathcal{Q}$, we denote:

$$|\Gamma| = \text{Card}\{\Delta \in \mathcal{Q} \mid \Delta \subset \Gamma\} \tag{2.17}$$

and, $\tilde{\eta}_\Gamma$ the set of trajectories of $\underline{\tilde{\eta}}$ contained in tesserae of Γ.

We define, for $\underline{\xi} \in \tilde{\Omega}$

$$E^\theta(\underline{\xi}) = \frac{1}{\theta} \sum_{\Delta \in \mathcal{Q}} \int n^2(\underline{\xi}(\tau), \Delta) d\tau \tag{2.18}$$

For notational simplicity we systematically omit the effective integration domain.

PROPOSITION 2.1. — There exist an integer q_0, an α small enough and a constant $h(q_0)$ (depending only on q_0), such that:

$$\tilde{\rho}_\Lambda(\underline{\tilde{\eta}}) \leqslant \sum_{q \geqslant q_0} c_q \exp\{-\gamma E^\theta(\underline{\tilde{\eta}}_{\Lambda_q})\} \tilde{\rho}_\Lambda(\underline{\tilde{\eta}}_{\Lambda_q^c}) + c_0 \exp\{-\gamma E^\theta(\underline{\tilde{\eta}}_{\Lambda_{q_0}})\} \tilde{\rho}_\Lambda(\underline{\tilde{\eta}}_{\Lambda_{q_0}^c}) \tag{2.19}$$

where

$$\gamma = \frac{\beta A}{2}, \qquad q > q_0$$

$$c_0 = \exp\left\{\left[\beta B + \frac{4\beta B^2}{A} + \frac{Z}{(2\pi\theta)^{\nu/2}}\left(1 + (1 + 2\alpha)^{2(\nu+1)} e^{\beta B} + \frac{e^{\beta B} f(\theta)}{|\Lambda_{q_0}|}\right)\right] |\Lambda_{q_0}| + h(q_0)\right\}$$

$$c_q = \exp\left\{-\frac{\beta A}{4}(q-1)|\Lambda_{q-1}| - \left[\left(\frac{\beta A}{8} q + \frac{4\beta B^2}{A} + \frac{Z}{\sqrt{2\pi\theta^\nu}}(1 + (1 + 2\alpha)^{2(\nu+1)} e^{\beta B}) + \frac{e^{\beta B} f(\theta)}{|\Lambda_q|}\right)\right] |\Lambda_q|\right\} \tag{2.20}$$

and

$$f(\theta) = \sum_{s > q_0 + 2} (2l_{s+1} + 1)^\nu \exp\left\{-c_1 \frac{l_s^2}{2\theta} \frac{\alpha^2}{(1 + 2\alpha)^4} + v F s^{1/2} (2l_s + 1)^{\nu/2}\right\} \tag{2.21}$$

c_1 and v being positive constants (see A.15 and Lemma 2.3 below). c_1 depends on ν, and v on ν and α.

REMARK 1. — $f(\theta)$ is bounded if $\nu \leqslant 3$ and is such that

$$\lim_{\theta \to 0^+} f(\theta) = 0 \tag{2.22}$$

REMARK 2. — The sequence c_q is fastly convergent, provided $\exp \alpha\nu < 2$, and we denote by D its sum.

Proposition 2.1 will be proven below. It implies the following.

PROPOSITION 2.2. — Under the same hypotheses of Proposition 1, if $\nu \leqslant 3$, $\tilde{\rho}_\Lambda(\underline{\tilde{\eta}})$ verifies the following uniform bound in Λ:

$$\tilde{\rho}_\Lambda(\underline{\tilde{\eta}}) \leqslant \exp\{ -\gamma E^\theta(\underline{\tilde{\eta}}) + \delta \mid \underline{\tilde{\eta}} \mid \} \tag{2.23}$$

where δ is such that

$$\exp \delta > c_0 + D \tag{2.24}$$

Proof. — Let $\underline{\tilde{\eta}}'$ be a proper subset of components of $\underline{\tilde{\eta}}$; then we assume that (2.23) is true for any such $\underline{\tilde{\eta}}'$.

Therefore:

$$\begin{aligned} \tilde{\rho}_\Lambda(\underline{\tilde{\eta}}) &\leqslant c_0 \exp\{ -\gamma E^\theta(\underline{\tilde{\eta}}_{\Lambda_{q_0}}) - \gamma E^\theta(\underline{\tilde{\eta}}_{\Lambda^c_{q_0}}) + \delta \mid \underline{\tilde{\eta}}_{\Lambda^c_{q_0}} \mid \} \\ &\quad + \sum_{q \geqslant q_0} c_q \exp\{ -\gamma E^\theta(\underline{\tilde{\eta}}_{\Lambda_q}) - \gamma E^\theta(\tilde{\eta}_{\Lambda^c_q}) + \delta \mid \tilde{\eta}_{\Lambda^c_q} \mid \} \\ &\leqslant \exp\{ -\gamma E^\theta(\underline{\tilde{\eta}}) + \delta \mid \underline{\tilde{\eta}} \mid \} \end{aligned} \tag{2.25}$$

since $\mid \underline{\tilde{\eta}}_{\Lambda^c_q} \mid \leqslant \mid \underline{\tilde{\eta}} \mid - 1$. But (2.23) is true when $\underline{\tilde{\eta}} \in \tilde{\Omega}_0$ and proposition 2.2 follows by induction □.

We consider now $\underline{\eta} \in \Omega$. It is true that $s(\underline{\eta})$ verifies the hypotheses of Proposition 2.1 and 2.2, and therefore, by (2.14) we get the following:

THEOREM 2.1. — If $\nu \leqslant 3$

$$\rho_\Lambda(\underline{\eta}) \leqslant \exp\{ -\gamma E^\theta(\underline{\eta}) + \delta \mid s(\underline{\eta}) \mid \} \tag{2.26}$$

We notice that $\mid s(\underline{\eta}) \mid$ has a simple geometric interpretation as volume of the region (shadow of $\underline{\eta}$)

$$S(\underline{\eta}) = \{ \Delta \in \mathscr{Q} \mid \exists t \in [0, \theta] \quad \text{and} \quad i \in \{ 1 \dots \mid \underline{\eta} \mid \} \text{ s. t. } \eta_i(t) \in \Delta \}. \tag{2.27}$$

We now prove Proposition 2.1.

Fixed $\underline{\tilde{\eta}}$ and some integer q_0 (to be fixed later) denoting $\underline{\omega} \cup \underline{\tilde{\eta}} = \underline{\xi}$ and

$$q \mid \Lambda_q \mid = \varphi(q) \tag{2.28}$$

we put the following decomposition:

$$1 = \chi_{q_0 - 1}(\underline{\xi}) + \sum_{q \geqslant q_0} \tilde{\chi}_q(\underline{\xi}) \tag{2.29}$$

where, denoting as usual by $\chi(S)$ the indicator of the set S, for each $q \geqslant q_0 - 1$

$$\chi_q = \chi(\{ \underline{\omega} \in \Omega \mid E^\theta_m(\underline{\xi}) \leqslant \varphi(m) \ \forall\ m \geqslant q \}) \tag{2.30}$$

and for $q \geqslant q_0$

$$\tilde{\chi}_q = \chi_q \chi(\{ \underline{\omega} \in \Omega \mid E^\theta_{q-1}(\underline{\xi}) > \varphi(q-1) \} \tag{2.31}$$

having defined

$$E^\theta_k(\underline{\xi}) = \sum_{\Delta \in \Lambda_k} \frac{1}{\theta} \int d\tau n^2(\underline{\xi}(\tau), \Delta) \tag{2.32}$$

By (2.29),

$$Z_\Lambda \tilde{\rho}_\Lambda(\underline{\tilde{\eta}}) = I_0 + \sum_{q \geq q_0} I_q \tag{2.33}$$

where

$$I_0 = \int \exp\{-\mathcal{U}^\theta(\underline{\xi})\}\alpha_\Lambda(\underline{\omega})\chi_{q_0-1}(\xi)d\underline{\omega} \tag{2.34}$$

$$I_q = \int \exp\{-\mathcal{U}^\theta(\underline{\xi})\}\alpha_\Lambda(\underline{\omega})\tilde{\chi}_q(\xi)d\underline{\omega} \tag{2.35}$$

We now bound the r. h. s. of (2.33) term by term.

It will be useful to consider different contributions to I_q and I_0 arising from the different behaviour of the trajectories $\underline{\omega}$: we divide them in three classes: $\underline{\omega}_1$ are trajectories completely contained in the interior of Λ_q, $\underline{\omega}_2$ are trajectories completely outside Λ_q and $\underline{\zeta}$ are trajectories which cross the boundary of Λ_q. Furthermore the trajectories $\underline{\zeta}$ are divided in two classes: $\underline{\zeta}_1$ are trajectories which do not go out of Λ_{q+2} while go $\underline{\zeta}_2$ outside of this region. To be precise, we decompose the total energy as follows:

$$\begin{aligned}
\mathcal{U}^\theta(\underline{\omega}_1 \cup \underline{\omega}_2 \cup \underline{\zeta}_1 \cup \underline{\zeta}_2 \cup \underline{\tilde{\eta}}) &= \mathcal{U}^\theta(\underline{\omega}_1 \cup \underline{\tilde{\eta}}_{\Lambda_q} \cup \pi_{\Lambda_q}\underline{\zeta}_1 \cup \pi_{\Lambda_q}\underline{\zeta}_2) \\
&+ \mathcal{U}^\theta(\underline{\omega}_2 \cup \underline{\tilde{\eta}}_{\Lambda_q^c}) + \mathcal{U}^\theta(\pi_{\Lambda_q^c}\underline{\zeta}_1 \cup \pi_{\Lambda_q^c}\underline{\zeta}_2) \\
&+ W^\theta(\underline{\omega}_1 \cup \underline{\tilde{\eta}}_{\Lambda_q} \cup \pi_{\Lambda_q}\underline{\zeta}_1 \cup \pi_{\Lambda_q}\underline{\zeta}_2 \mid \underline{\omega}_2 \cup \underline{\tilde{\eta}}_{\Lambda_q^c} \cup \pi_{\Lambda_q^c}\underline{\zeta}_1 \cup \pi_{\Lambda_q^c}\underline{\zeta}_2) \\
&+ W^\theta(\pi_{\Lambda_q^c}\underline{\zeta}_1 \mid \underline{\omega}_2 \cup \underline{\tilde{\eta}}_{\Lambda_q^c}) \\
&+ W^\theta(\pi_{\Lambda_q^c}\underline{\zeta}_2 \mid \underline{\omega}_2 \cup \underline{\tilde{\eta}}_{\Lambda_q^c})
\end{aligned} \tag{2.36}$$

where, if $\underline{\tilde{\omega}}$ and $\underline{\tilde{\omega}}^1$ are in $\tilde{\Omega}$, we put

$$W^\theta(\underline{\tilde{\omega}} \mid \underline{\tilde{\omega}}^1) = \frac{1}{2}\frac{\beta}{\theta}\sum_{j \neq j^1}\int d\tau\chi(\{\mathcal{D}(\tilde{\omega}_j) \cap \mathcal{D}(\tilde{\omega}_{j^1}^1)\})\phi(|\tilde{\omega}_j(\tau) - \tilde{\omega}_{j^1}^1(\tau)|). \tag{2.37}$$

We also perform the integration according to the behaviour of trajectories. In fact we use the following identity:

$$1 = \alpha_{\Lambda_q}(\omega) + \alpha_{\Lambda_q^c}(\omega) + \alpha'_{\partial\Lambda_q}(\omega) \tag{2.38}$$

where α_Γ is the indicator of the event ω is always in the interior of Γ, while $\alpha'_{\partial\Gamma}$ is the indicator of the set $\{\omega \mid \exists\tau \in [0, \theta] \text{ s. t. } \omega(\tau) \in \partial\Gamma\}$. Furthermore if α is any indicator on ω, and $\omega = (\omega_1 \ldots \omega_n)$ we denote

$$\alpha(\underline{\omega}) = \prod_{i=1}^{n} \alpha(\omega_i) \tag{2.39}$$

We have:

$$I_q = \int d\underline{\omega}_1\alpha_{\Lambda_q}(\underline{\omega}_1)\int d\underline{\omega}_2\alpha_{\Lambda_q^c}(\underline{\omega}_2)\int d\underline{\zeta}\alpha'_{\partial\Lambda_q}(\underline{\zeta})\tilde{\chi}_q(\underline{\xi}) \exp\{-\mathcal{U}^\theta(\underline{\omega}_1 \cup \underline{\omega}_2 \cup \underline{\zeta} \cup \underline{\tilde{\eta}})\} \tag{2.40}$$

To get (2.40) we used twice the identity:

$$\int d\underline{\omega} f(\underline{\omega}) \sum_{\omega' \subset \omega} \alpha(\underline{\omega}')\alpha^c(\underline{\omega} \setminus \underline{\omega}') = \int d\underline{\omega}_1 \alpha(\underline{\omega}_1) \int d\underline{\omega}_2 \alpha^c(\underline{\omega}_2) f(\underline{\omega}_1 \cup \underline{\omega}_2) \quad (2.41)$$

where $\alpha^c = 1 - \alpha$, f is a symmetric function on trajectories and the other notations have an obvious meaning.

Let now $\hat{\chi}_s$ denote the indicator of the set

$$\{ \underline{\omega} \mid \omega(\tau) \in \text{Int}\, \Lambda_s \ \forall\ \tau \in [0, \theta]\, ;\ \exists\ \tau' \in [0, \theta] \text{ s. t. } \omega(\tau') \in \partial\Lambda_{s-1} \} \quad (2.42)$$

Then

$$\alpha'_{\partial\Lambda_q} = \alpha'_{\partial\Lambda_q}\alpha_{\Lambda_{q+2}} + \sum_{s > q+2} \alpha'_{\partial\Lambda_q}\hat{\chi}_s = \alpha'_{\partial\Lambda_q}(\alpha_{\Lambda_{q+2}} + \bar{\chi}_q) \quad (2.43)$$

where $\bar{\chi}_q$ is defined by the last steep of (2.43). Using still (2.41) we get finally:

$$\text{I}_q = \int d\underline{\omega}_1 \alpha_{\Lambda_q}(\underline{\omega}_1) \int d\underline{\omega}_2 \alpha_{\Lambda_q^c}(\omega_2) \int d\underline{\zeta}_1 \alpha'_{\partial\Lambda_q}(\underline{\zeta}_1)\alpha_{\Lambda_{q+2}}(\underline{\zeta}_1) \int d\underline{\zeta}_2 \alpha'_{\partial\Lambda_q}(\underline{\zeta}_2)\bar{\chi}_q(\underline{\zeta}_2)\tilde{\chi}_q(\underline{\xi})$$
$$\exp\{ -\mathscr{U}^\theta(\underline{\xi}) \}\, \alpha_\Lambda(\underline{\omega}) \quad (2.44)$$

We now estimate the various terms in the decomposition of energy and use such bounds to evaluate (2.44). We need some lemmas.

LEMMA 2.1. — For each positive integer q

$$\mathscr{U}^\theta(\tilde{\underline{\eta}}_{\Lambda_q} \cup \pi_{\Lambda_q}(\underline{\omega})) \geqslant \frac{\beta \text{A}}{4} \text{E}_q(\underline{\xi}) + \frac{\beta \text{A}}{2} \text{E}_q(\tilde{\underline{\eta}}_{\Lambda_q}) - \frac{4\beta \text{B}^2}{\text{A}} |\Lambda_q| \quad (2.45)$$

The proof of this lemma is a slight modification of the one of [3] and is given in Appendix B.

Now we have to estimate the interaction among the trajectories according to the decomposition (2.36). The first two W-terms in (2.36) are essentially classical in the sense that they contain localized pieces of trajectories. To treat them we use Lemma 2.2 below. The contribution of the last W-term to the integration will be controlled with probabilistic arguments.

LEMMA 2.2. — If $\underline{\xi}_1$ and $\underline{\xi}_2$ are sets of trajectories of $\underline{\xi}$ contained in $\Lambda_{q+a}(a \geqslant 0)$ and Λ_q^c respectively and if $\tilde{\chi}_q(\underline{\xi}) = 1$, then there exist an α small enough and a $q_0^{(1)}$ large enough, s. t. for each $q \geqslant q_0^{(1)}$,

$$\text{W}^\theta(\underline{\xi}_1 \mid \underline{\xi}_2) \geqslant -\frac{\beta \text{A}}{16} \varphi(q) \quad (2.46)$$

Lemma 2.2 will be used with $\underline{\xi}_1 = \tilde{\underline{\eta}}_{\Lambda_q} \cup \pi_{\Lambda_q}\underline{\omega}$, $\underline{\xi}_2 = \tilde{\underline{\eta}}_{\Lambda_q^c} \cup \pi_{\Lambda_q^c}\underline{\omega}$ and $a = 0$, or with $\underline{\xi}_1 = \pi_{\Lambda_q^c}\underline{\zeta}_1$, $\underline{\xi}_2 = \tilde{\underline{\eta}}_{\Lambda_q^c} \cup \underline{\omega}_2$ and $a = 2$. Also this lemma is proven following Ruelle [3] with minor modifications and the proof is given in Appendix B for sake of completness.

The interaction of trajectories $\underline{\zeta}_2$, which could be very long, cannot be treaten by (2.46).

Lemma 2.3. — Let $\underline{\zeta}_2$ be trajectories in $\underline{\xi}$ contained in Λ_s, $s > q$ and $\chi_q(\underline{\xi}) = 1$. Then there is a $q_0^{(2)}$ large enough such that for each $q \geqslant q_0^{(2)}$

$$W^\theta(\pi_{\Lambda_q^c}\underline{\zeta}_2 \mid \underline{\tilde{\eta}}_{\Lambda_q^c} \cup \underline{\omega}_2) \geqslant -\beta v \mid \underline{\zeta}_2 \mid F\varphi(s)^{1/2} \tag{2.47}$$

where v is a constant depending only on dimensions and α.

Proof. — We denote $\underline{\xi}_3 = \pi_{\Lambda_q^c}\underline{\zeta}_2$ and $\underline{\xi}_2 = \underline{\tilde{\eta}}_{\Lambda_q^c} \cup \underline{\omega}_2$. By lower regularity, fixed $\tau \in [0, \theta]$

$$-W^\theta(\underline{\xi}_3(\tau) \mid \underline{\xi}_2(\tau)) \leqslant \frac{\beta}{\theta} \sum_{\Delta \subset \Lambda_s} \sum_{\Delta' \subset \Lambda_q^c} \psi(\Delta, \Delta') n(\underline{\xi}_3(\tau), \Delta) n(\underline{\xi}_2(\tau), \Delta'). \tag{2.48}$$

Then

$$-W^\theta(\underline{\xi}_3(\tau) \mid \underline{\xi}_2(\tau)) \leqslant \frac{\beta}{\theta} \sum_{k=0}^{\infty} \psi(k) \sum_{\Delta \subset \Lambda_s} n(\underline{\xi}_3(\tau), \Delta) \sum_{\substack{\Delta' \subset \Lambda_q^c \\ d(\Delta', \Delta) = k}} n(\underline{\xi}_2(\tau), \Delta')$$

$$\leqslant \frac{\beta}{\theta} v_1 \left[\sum_{k=0}^{\infty} \psi(k)(k+1)^{\frac{v-1}{2}} \sum_{\Delta \subset \Lambda_s} n(\underline{\xi}_3(\tau), \Delta) \left(\sum_{\substack{\Delta' \subset \Lambda_q^c \\ d(\Delta, \Delta') = k}} n^2(\underline{\xi}_2(\tau), \Delta') \right)^{1/2} \right], \tag{2.49}$$

by a convexity inequality, for some v_1, depending on v. But:

$$\sup_{\Delta \subset \Lambda_s} \sum_{\substack{\Delta' \subset \Lambda_q^c \\ d(\Delta, \Delta') = k}} n^2(\underline{\xi}_2(\tau), \Delta') \leqslant \sum_{\Delta' \subset \Lambda_{s+r(k)}} n^2(\underline{\xi}_2(\tau), \Delta') \tag{2.50}$$

where $r(n)$ is the smallest of the integers r such that the set

$$\{ \Delta' \in \mathcal{Q} \mid d(\Delta, \Delta') = k \ \forall\ \Delta \subset \Lambda_s \} \tag{2.51}$$

is contained in Λ_{s+r}.

Obviously

$$\sum_{\Delta \subset \Lambda_s} n(\underline{\xi}_3(\tau), \Delta) \leqslant \mid \underline{\xi}_3 \mid \tag{2.52}$$

Then, integrating (2.49), still by convexity we get

$$-W^\theta(\underline{\xi}_3 \mid \underline{\xi}_2) \leqslant \beta v_1 \mid \underline{\xi}_3 \mid \sum_{k=0}^{\infty} \psi(k)(k+1)^{\frac{v-1}{2}} (E^\theta_{s+r(k)}(\underline{\xi}))^{1/2}$$

since $\underline{\xi}_2$ is a subset of $\underline{\xi}$; using the condition $\chi_q(\underline{\xi}) = 1$ we have then

$$-W^\theta(\underline{\xi}_3 \mid \underline{\xi}_2) \leqslant \beta v_1 \mid \underline{\xi}_3 \mid \sum_{k=0}^{\infty} \psi(k)(k+1)^{\frac{v-1}{2}} [\varphi(s + r(k))]^{1/2} \tag{2.53}$$

We now use Lemmas B.1) and B.2); we fix

$$\varepsilon = \min\left\{\varepsilon_0, \frac{2\left(\mu - \frac{1}{2}\right)}{\nu + 1}\right\}, \qquad q_0^{(2)} = s_0$$

(see Lemma B.1). By Lemmas B.1 and B.2:

$$r(k) < \frac{\log(k+2)}{\alpha} + 1$$

$$\frac{\varphi(s+r)}{\varphi(s)} < (f(\alpha, \varepsilon))^r \tag{2.54}$$

for each $r > 0$. Then

$$\varphi(s + r(k)) \leqslant \varphi(s)(k+2)^{(\nu-1)+2\mu} f(\alpha, \varepsilon) \tag{2.55}$$

Therefore

$$-\mathrm{W}(\underline{\xi}_3 \mid \underline{\xi}_2) \leqslant \beta v \mid \underline{\zeta}^1 \mid \varphi(s)^{1/2}\mathrm{F} \tag{2.56}$$

where

$$v = v_1 \sqrt{f(\alpha, \varepsilon)}\, 3^{(\nu - 1 + \mu)} \qquad \square \tag{2.57}$$

Obviously the bound (2.47) is useful only combined with a phase space bound, based on the fact that very long trajectories have small $d\omega$-measure.

LEMMA 2.4. — For any $s > q + 2$, $q \geqslant q_0^{(2)}$

$$\int d\omega \alpha'_{\zeta\Lambda_q}(\omega)\tilde{\chi}_s(\omega) \leqslant \frac{z \mid \Lambda_s \mid}{(2\pi\theta)^{\nu/2}} \cdot \exp\left\{ - c_1 \frac{l_s^2}{2\theta} \frac{\alpha^2}{(1+2\alpha)^4} \right\} \tag{2.58}$$

where c_1 has been introduced in A.15.

Proof. — The lemma follows from the probability estimate A.15 and the inequalities

$$l_{s-1} - l_q > l_{s-1} - l_{s-2} > \frac{\alpha l_s}{(1+2\alpha)^2} \tag{2.59}$$

where we used Lemma B.1 and the fact that

$$s - 1 > q \geqslant q_0^{(2)} = s_0 \tag{2.60}$$

Volume integration gives the extra factor $\mid \Lambda_s \mid$ $\square$.

Above lemmas allow to estimate I_q, $q \geqslant q_0$. In fact we fix

$$q_0 = \max\{ q_0^{(1)}, q_0^{(2)} \} \tag{2.61}$$

and in (2.44), decomposing the energy according to (2.36). We bound $\mathcal{U}^\theta(\underline{\omega}_1 \cup \underline{\tilde{\eta}}_{\Lambda_q} \cup \pi_{\Lambda_q}\underline{\zeta}_1 \cup \pi_{\Lambda_q}\underline{\zeta}_2)$ by lemma 2.1, and $\mathcal{U}^\theta(\pi_{\Lambda_q^c}\underline{\zeta}_1 \cup \pi_{\Lambda_q^c}\underline{\zeta}_2)$ by stability:

$$\mathcal{U}^\theta(\pi_{\Lambda_q^c}\underline{\zeta}_1 \cup \pi_{\Lambda_q^c}\underline{\zeta}_2) \geqslant -\mathrm{B}(\mid \underline{\zeta}_1 \mid + \mid \underline{\zeta}_2 \mid) \tag{2.62}$$

Also we bound interaction energy by lemmas 2.2 and 2.3 since $\tilde{\chi}_q(\underline{\xi}) = 1$;

we also decompose the integration on $\underline{\zeta}_2$ according to the partition (2.43):

$$\mathrm{I}_q \leqslant \exp\left\{ -\frac{\beta A}{2} E^\theta(\underline{\tilde\eta}_{\Lambda_q}) \right\} \exp\left\{ -\frac{\beta A}{4}\varphi(q-1) + \frac{\beta A}{8}\varphi(q) + \frac{4\beta B^2}{A}|\Lambda_q| \right\}$$
$$\int d\underline{\omega}_1 \alpha_{\Lambda_q}(\underline{\omega}_1) \int d\underline{\omega}_2 \alpha_\Lambda(\underline{\omega}_2)\alpha_{\Lambda_q^c}(\underline{\omega}_2) \exp\{ -\mathcal{U}^\theta(\underline{\omega}_2 \cup \underline{\tilde\eta}_{\Lambda_q^c}) \} \quad (2.63)$$
$$\int d\underline{\zeta}_1 \alpha_{\Lambda_{q+2}}(\underline{\zeta}_1) e^{\beta B|\zeta_1|} \sum_{k \geqslant 0} \frac{1}{k!} \sum_{\substack{s_1 \dots s_k \\ s_i > q+2}} \int d\zeta_2^{(1)} \dots d\zeta_2^{(k)}$$
$$\prod_{i=1}^k (\hat\chi_{s_i}(\zeta_2^{(i)})\alpha'_{\partial\Lambda_q}(\zeta_2^{(i)})) \exp\{ \beta B + \beta v F \varphi(s_i)^{1/2} \}$$

Then, using lemma 2.4 we get the estimate for I_q, $q \geqslant q_0$

$$\mathrm{I}_q \leqslant \exp\left\{ -\frac{\beta A}{2} E^\theta(\underline{\tilde\eta}_{\Lambda_q}) c_q \int d\underline{\omega}_2 \alpha_\Lambda(\underline{\omega}_2)\alpha_{\Lambda_q^c}(\underline{\omega}_2) \exp\{ -\mathcal{U}^\theta(\underline{\omega}_2 \cup \underline{\tilde\eta}_{\Lambda_q^c}) \} \right.$$
$$\leqslant \exp\left\{ -\frac{\beta A}{2} E^\theta(\underline{\tilde\eta}_{\Lambda_q}) c_q \int d\underline{\omega} \alpha_\Lambda(\underline{\omega}) \exp\{ -\mathcal{U}^\theta(\underline{\omega} \cup \underline{\tilde\eta}_{\Lambda_q^c}) \} \right. \quad (2.64)$$

where c_q is the coefficient of proposition 2.1.

To estimate I_0, which has the same structure of I_{q_0} but with χ_{q_0-1} in place of $\tilde\chi_{q_0}$, we cannot use lemma 2.2. In place of it we have, with the same meaning of notation as in Lemma 2.2:

LEMMA 2.5. — If $\underline{\xi}$ is such that $\chi_{q_0-1}(\underline{\xi}) = 1$ and q_0 is fixed as above, there exists a function $h(q_0)$ which does not depend on β, z, θ, such that

$$W^\theta(\underline{\xi}_1 \mid \underline{\xi}_2) \geqslant -\frac{1}{2}\beta h(q_0) \quad (2.65)$$

Also Lemma 2.5 is proven in Appendix B.

Then, since $\chi_{q_0-1}(\underline{\xi}) = 1$, using Lemmas 2.1, 2.3, 2.4 and 2.5, eq. (2.36) and performing integration as for I_q, we get the estimate

$$\mathrm{I}_0 \leqslant \exp\left\{ -\frac{\beta A}{2} E^\theta(\tilde\eta_{\Lambda_{q_0}}) \right\} c_0 \int d\underline{\omega} \alpha_1(\underline{\omega}) \exp\{ -\mathcal{U}^\theta(\underline{\omega} \cup \underline{\tilde\eta}_{\Lambda_{q_0}^c}) \} \quad (2.66)$$

where c_0 is given in Proposition 2.1. Therefore (2.19) is proven. □

3. A PROBABILITY ESTIMATE

In this section we obtain a probability estimate on the number of particles in a given bounded region Γ. We shall also compare this estimate with the classical one, already deduced in [3].

Let Γ and Λ be two bounded regions, exactly paved by $\mathcal{Q}$, $\Lambda \supset \Gamma$, and a, a positive integer. We want to obtain an estimate on $Pr_\Lambda(N_\Gamma \geqslant a)$, not depending on Λ, where $Pr_\Lambda(N_\Gamma \geqslant a)$ is the Λ grand-canonical probability of finding more than a particles in Γ. Such a probability is defined as:

$$Pr_\Lambda(N_\Gamma \geqslant a) = \mathrm{Tr}\, \sigma_\Lambda E_\Gamma(a) \tag{3.1}$$

where $E_\Gamma(a) = \int_a^\infty dE_\lambda$, and dE_λ is the spectral measure of the selfadjoint operator N_Γ, number of particles in Γ.

Straightforward calculations on the Fock space $\mathcal{G}(\Lambda)$, give:

$$Pr_\Lambda(N_\Gamma \geqslant a) = \sum_{n\geqslant a}\sum_{m\geqslant 0} \frac{(n+m)!}{m!\,n!}$$
$$\int_{\Gamma^n} dx_1 \dots dx_n \int_{(\Lambda\setminus\Gamma)^m} dy_1 \dots dy_m \sigma_\Lambda(x_1 \dots x_n\, y_1 \dots y_m, x_1 \dots x_n\, y_1 \dots y_m) \tag{3.2}$$

where $\sigma_\Lambda(\cdot,\cdot)$ is the kernel of the density matrix. Eq. (3.2) may also be assumed as a definition for $Pr_\Lambda(N_\Gamma \geqslant a)$ in analogy with the classical case. Thus

$$Pr_\Lambda(N_\Gamma \geqslant a) \leqslant \sum_{n\geqslant a} \int_{\Gamma^n} \frac{dX_n}{n!} z^n \int P^\theta_{X_nX_n}(d\underline{\omega})\rho_\Lambda(\underline{\omega}) \leqslant C + R \qquad (X_n = x_1 \dots x_n) \tag{3.3}$$

where

$$C = \sum_{n\geqslant a} \int_{\Gamma^n} \frac{dX_n}{n!} z^n \int P^\theta_{X_nX_n}(d\underline{\omega})\rho_\Lambda(\underline{\omega})\alpha_{\overline{\Gamma}}(\underline{\omega}), \tag{3.4}$$

$$\overline{\Gamma} = \{\Delta \in \alpha \mid d(\Delta, \Gamma) \leqslant d\}, \qquad d > 0 \tag{3.5}$$

and

$$R = \sum_{\substack{h,k\\ h+k\geqslant a\\ h\geqslant 1}} \frac{z^h}{h!}\frac{z^k}{k!} \int_{\Gamma^k} dX_k \int P^\theta_{X_kX_k}(d\underline{\omega}_1)\alpha_{\overline{\Gamma}}(\underline{\omega}_1) \int_{\Gamma^h} dX_h \int P^\theta_{X_hX_h}(d\underline{\omega}_2)\chi(\underline{\omega}_2 \cap \overline{\Gamma}^c \neq \emptyset)\rho_\Lambda(\underline{\omega}_1 \cup \underline{\omega}_2) \tag{3.6}$$

$$C \leqslant \sum_{n\geqslant a} \frac{|\Gamma|^n}{n!} \frac{z^n}{(2\pi\theta)^{\frac{n\nu}{2}}} \exp\left\{ -\gamma\frac{n^2}{|\overline{\Gamma}|} + \delta|\overline{\Gamma}| \right\} \tag{3.7}$$

Hence:

$$C \leqslant \exp\left\{ -\gamma\frac{a^2}{|\overline{\Gamma}|} + \left(\frac{z}{(2\pi\theta)^{\nu/2}} + \delta\right)|\overline{\Gamma}| \right\} \leqslant \exp\left\{ -\gamma'\frac{a^2}{|\Gamma|} + ga \right\} \tag{3.8}$$

for some $\gamma' > 0$ depending on d and such $\gamma' \to \gamma$ when $d \to 0$ and $g > 0$. The last step is due to the fact that $c \leqslant 1$.

Furthermore:

$$\mathrm{R} \leqslant \sum_{\substack{h,k \\ h+k \geqslant a \\ h \geqslant 1}} \frac{z^k}{k!} \int_{\Gamma^k} d\mathrm{X}_k \int \mathrm{P}^{\theta}_{\mathrm{X}_k \mathrm{X}_k}(d\underline{\omega}_1) \alpha_{\overline{\Gamma}}(\omega_1) \exp \{ - \gamma \mathrm{E}^{\theta}(\underline{\omega}_1) + \delta \mid s(\underline{\omega}_1) \mid \} \frac{z^h}{h!} \int_{\Gamma^h} d\mathrm{X}_h \int \mathrm{P}^{\theta}_{\mathrm{X}_h \mathrm{X}_h}(d\underline{\omega}_2) \chi(\underline{\omega}_2 \cup \overline{\Gamma}^c \neq \varnothing) \exp \delta \mid s(\underline{\omega}_2) \mid \quad (3.9)$$

Using Schwartz inequality in the last integral and A.15 and A.21:

$$\mathrm{R} \leqslant \sum_{h \geqslant 1} \sum_{\substack{k \geqslant 0 \\ k+h \geqslant a}} \left(\frac{z}{\sqrt{2\pi\theta^{\nu}}} \right)^{h+k} \frac{|\Gamma|^{h+k}}{k!\,h!} \exp \{ -\gamma k + \delta |\overline{\Gamma}| \} \, \mathrm{I}(z\delta)^{h/2} \exp \left(- c_1 \frac{hd^2}{2\theta} \right) \quad (3.10)$$

$$\leqslant \exp \left\{ - \frac{c_1 d^2}{2\theta} - g_1(a-1) + \frac{2 |\Gamma| g_2 z}{\sqrt{2\pi\theta^{\nu}}} \right\} \quad (3.11)$$

where $g_1 = \min \left(\gamma, \frac{c_1 d^2}{2\theta} \right)$ and g_2 some positive constant.

Thus the estimate we obtains is rather different from the classical one that is (3.8). This difference is due to the delocalization of the quantum particles that is responsible for the extra addendum (3.11). However C and R have different order of magnitude. In fact, as is expected to be true, $\mathrm{R} \to 0$ as $\hbar \to 0$ and $z \to 0$ in such a way that $\frac{z}{\sqrt{2\pi\theta^{\nu}}} \to z_0$, that is the classical activity. This may be seen by realizing that in all coefficients the activity appears only in the form $z/\sqrt{2\pi\theta^{\nu}}$, so that in the above classical limit, $\mathrm{R} \to 0$ and C tends to the classical estimate, with the same coefficients obtained in [*3*] in which z is replaced by z_0.

To get the right scaling see Ref. [*8*], Th. 10.1, p. 106.

Of course other scaling are possible to obtain the classical limit, all of that describing different physical situations.

4. PRESSURE AND EQUIVALENCE OF THE BOUNDARY CONDITIONS

The existence of the pressure may be obtained, in our context, by a combination of the methods employed in the previous sections and the classical ones in Statistical Mechanics.

Let's consider the following rectangular regions:

$$\begin{aligned} \Lambda &= [-L_1, L_1] \times S \\ \Lambda_1 &= [-L_1, 0] \times S \\ \Lambda_2 &= [0, L_1] \times S \\ S &= \prod_{i=2}^{\nu} [-L_i, L_i], \qquad L_i \in \mathbb{N}, \qquad i = 1 - \nu. \end{aligned} \tag{4.1}$$

We prove the following inequality, for L_1 large enough:

$$Z_\Lambda \leqslant Z_{\Lambda_1} \cdot Z_{\Lambda_2} \exp \{ m_0 S L_1^b \} \tag{4.2}$$

where $m_0 > 0$ and $0 < b < 1$.

To do this let $\Lambda(l) = [-l, l] \times S$, $l < L_1$ integer and:

$$\begin{aligned} \chi_k(\underline{\omega}) &= \chi(\{ E^\theta_{\Lambda(l)}(\underline{\omega}) > k \, | \Lambda(l) | \text{ for some } l \}) \\ \chi^c_k(\underline{\omega}) &= \chi(\{ E^\theta_{\Lambda(l)}(\underline{\omega}) \leqslant k \, | \Lambda(l) | \text{ for all } l \}) \\ \chi^l_k(\underline{\omega}) &= \chi(\{ E^\theta_{\Lambda(l)}(\underline{\omega}) > k \, | \Lambda(l) | \}) \end{aligned} \tag{4.3}$$

We claim that, for sufficiently large k:

$$\int d\underline{\omega} \exp \{ -\mathscr{U}^\theta(\underline{\omega}) \} \alpha_\Lambda(\underline{\omega}) \chi_k(\underline{\omega}) \leqslant \frac{1}{2} Z_\Lambda \tag{4.4}$$

Proof.

$$\begin{aligned} \int d\underline{\omega} e^{-\mathscr{U}^\theta(\underline{\omega})} \alpha_\Lambda(\underline{\omega}) \chi_k(\underline{\omega}) &\leqslant \sum_{l \geqslant 1} \int d\underline{\omega} \exp \{ -\mathscr{U}^\theta(\underline{\omega}) \} \chi^l_k(\underline{\omega}) \alpha_\Lambda(\underline{\omega}) \\ &\leqslant Z_\Lambda \sum_{l \geqslant 1} \frac{1}{Z_\Lambda} \int d\underline{\omega}' \chi(\underline{\omega}' \cap \Lambda(l) \neq \emptyset) \chi^l_k(\underline{\omega}') \\ &\quad \int d\underline{\omega} \chi(\underline{\omega} \cap \Lambda(l) = \emptyset) \exp \{ -\mathscr{U}^\theta(\underline{\omega} \cup \underline{\omega}') \} \alpha_\Lambda(\underline{\omega} \cup \underline{\omega}') \\ &\leqslant Z_\Lambda \sum_{l \geqslant 1} \int d\underline{\omega}' \chi(\underline{\omega}' \cap \Lambda(l) \neq \emptyset) \chi^l_k(\underline{\omega}') \rho_\Lambda(\underline{\omega}') \alpha_\Lambda(\underline{\omega}') \\ &\leqslant Z_\Lambda \sum_{l \geqslant 1} \int d\underline{\omega}' \chi(\underline{\omega}' \cap \Lambda(l) \neq \emptyset) \chi^l_k(\underline{\omega}') \\ &\quad \alpha_\Lambda(\underline{\omega}') \exp \{ -\gamma E^\theta_{\Lambda(l)}(\underline{\omega}') + \delta \, | s(\underline{\omega}') | \} \end{aligned} \tag{4.5}$$

We shall prove:

$$\int d\underline{\omega}' \chi(\omega' \cap \Lambda(l) \neq \emptyset) e^{\delta |s(\omega')|} \leqslant e^{m_1 |\Lambda(l)|} \tag{4.6}$$

for some positive m_1, and hence (4.4) follows by (4.5), (4.6), (4.3).

Proof of (4.6):

$$\sum_{n\geq 0}\frac{z^n}{n!}\left(\int dx\int P^{\theta}_{xx}(d\omega)e^{\delta|s(\omega)|}\chi(\omega\cap\Lambda(l)\neq\emptyset)\right)^n$$

$$\leqslant\sum_{n\geq 0}\frac{z^n}{n!}\left[\left(\int dx\int P^{\theta}_{xx}(d\omega)e^{2\delta|s(\omega)|}\right)^{1/2}\left(\int P^{\theta}_{xx}(d\omega)\chi(\omega\cap\Lambda(l)\neq\emptyset)\right)^{1/2}\right]^n$$

$$\leqslant\sum_{n\geq 0}\frac{z^n}{n!}\left[\frac{I(2\delta)^{1/2}}{(2\pi\theta)^{\nu/2}}\int dx\exp\left\{-c_1\frac{d(x,\Lambda(l))^2}{2\theta}\right\}\right]^n \tag{4.7}$$

as follows by (A.15) and (A.21).

Thus:

$$Z_\Lambda\leqslant 2\int d\underline{\omega}\exp\{-\mathscr{U}^\theta(\underline{\omega})\}\chi^c_k(\underline{\omega})$$

$$=2\int d\underline{\omega}_1\alpha_{R_1}(\underline{\omega}_1)\exp\{-\mathscr{U}^\theta(\underline{\omega}_1)\}\int d\underline{\omega}_2\alpha_{R_2}(\underline{\omega}_2)\exp\{-\mathscr{U}^\theta(\underline{\omega}_2)-W^\theta(\underline{\omega}_1\,|\,\underline{\omega}_2)\}$$

$$\int d\underline{\zeta}\chi(\underline{\zeta}\cap\Lambda(l_0)\neq\emptyset)\chi^c_k(\underline{\zeta}\cup\underline{\omega}_1\cup\underline{\omega}_2)\exp\{-W^\theta(\underline{\zeta}\,|\,\underline{\omega}_1\cup\underline{\omega}_2)-\mathscr{U}^\theta(\underline{\zeta})\} \tag{4.8}$$

where: $R_1=\Lambda_1\setminus\Lambda(l_0)$, $R_2=\Lambda_2\setminus\Lambda(l_0)$ and l_0 will be fixed later.

The last integral in r. h. s. of (4.8) may be treated as in Section 2. In fact the contribution of the trajectories $\underline{\zeta}_1\cup\underline{\zeta}_2$ (see lemma 2.3, 2.4 and 2.5) give $\exp m_2\,|\,\Lambda(l_0)\,|$ for some $m_2>0$ and for sufficiently large l_0.

Moreover if $\underline{\omega}_1$ and $\underline{\omega}_2$ are such that $\chi^c_k(\underline{\omega}_1\cup\underline{\omega}_2)=1$ then:

$$-W^\theta(\underline{\omega}_1\,|\,\underline{\omega}_2)\leqslant km_3SL_1\cdot\sup_{\Delta\in R_1}\sum_{\Delta'\in R_2}\psi(\Delta,\Delta') \tag{4.9}$$

Thus we have, for some positive a:

$$Z_\Lambda\leqslant Z_{\Lambda_1}Z_{\Lambda_2}2\exp\{m_2Sl_0+km_3SL_1l_0^{-a}\} \tag{4.10}$$

because of (1.20). Hence the inequality (4.2) follows after a suitable choice of l_0 as function of L_1.

As consequence of inequality (4.2), one can get the existence of the limit $\dfrac{1}{\beta\,|\,\Lambda\,|}\log Z_\Lambda$ for a suitable sequence of increasing regions. We shall denote such limit P.

It has to be remarked that for short range potential [*2*] or for bounded potentials, the proof of the existence of the pressure may be considerably shortned.

An interesting problem arising in Quantum Statistical Mechanics is how the thermodynamical functions, as the pressure, depend on the different boundary conditions, that, in principle, may be chosen for the Laplacian operator.

It may be proven [9], [10], [2], that if we denote by Z_Λ^σ the partition function constructed via Δ^σ, the Laplacian operator with boundary conditions $\frac{\partial\psi}{\partial n} - \sigma\psi$, where $\frac{\partial}{\partial n}$ is the inward normal derivative on $\partial\Lambda$, then $Z_\Lambda \equiv Z_\Lambda^\infty \leqslant Z_\Lambda^\sigma \leqslant Z_\Lambda^0, \sigma > 0$.

Thus if one prove that P^0, the pressure obtained by the Neumann boundary conditions $\sigma = 0$, is equal to $P^\infty \equiv P$, one can get the equivalence of the pressure for all other intermediate σ boundary conditions.

The estimate $\rho_\Lambda(\underline{\eta}) \leqslant \exp \delta \,|\, s(\underline{\eta}) \,|$ allows to do this, so that previous results obtained by Novikov (see [7] [2]) for hard spheres or positive potentials, may be generalized to our situation.

LEMMA 4.1. — Let μ_Λ be any Borel measure on Ω with the following properties: $(\Lambda = [-L, L]^\nu)$

$$\int \mu_\Lambda(d\underline{\omega})\chi(\underline{\omega} \cap \Lambda^c \neq \emptyset) = 0 \tag{4.11}$$

$$\int \mu_\Lambda(d\underline{\omega}) f(\underline{\omega})\alpha_\Lambda(\underline{\omega}) = \int d\underline{\omega} f(\underline{\omega})\alpha_\Lambda(\underline{\omega}) \tag{4.12}$$

for all positive measurable functions f.

$$\int \mu_\Lambda(d\underline{\omega})\chi(\underline{\omega} \cap \partial\Lambda \neq \emptyset) e^{\delta|s(\omega)|} \leqslant e^{m_4 L^2} \tag{4.13}$$

for some $m_4 > 0$. Then defining:

$$Z_\Lambda^\mu = \int \mu_\Lambda(d\underline{\omega}) \exp\{-\mathcal{U}^\theta(\underline{\omega})\} \tag{4.14}$$

the following inequalities hold:

$$Z_\Lambda \leqslant Z_\Lambda^\mu \leqslant Z_\Lambda \exp m_4 L^2 \tag{4.15}$$

Proof.

$$Z_\Lambda^\mu = \int \mu_\Lambda(d\underline{\omega}) \exp\{-\mathcal{U}^\theta(\underline{\omega})\} = \int \mu_\Lambda(d\underline{\omega})\chi(\underline{\omega} \cup \partial\Lambda \neq \emptyset)$$
$$\int \exp\{-\mathcal{U}^\theta(\underline{\eta} \cup \underline{\omega})\}\alpha_\Lambda(\underline{\eta})d\underline{\eta} \leqslant \int \mu_\Lambda(d\underline{\omega})\chi(\underline{\omega} \cap \partial\Lambda \neq \emptyset)\rho_\Lambda(\underline{\omega}) \cdot Z_\Lambda \tag{4.16}$$

The thesis follows from (4.13) □.

Now, using the images methods, combining the arguments given by Novikov [9] [2] and (A.21), one can easily see that $Z_\Lambda^0 = Z_\Lambda^\mu$ for some suitable μ satisfying (4.11), (4.12), (4.13). So the desired result may be obtained by the use of (4.15).

We summarize the content of this section in the following.

THEOREM 4.1. — Let $\Lambda_n = [-n, n]^\nu$ and $\sigma \geqslant 0$. Then the following limit: $(\sigma \geqslant 0)$

$$P_\sigma = \lim_{n \to +\infty} \beta^{-1} n^{-\nu} \log Z^\sigma_{\Lambda_n} \tag{4.17}$$

exists. Moreover:

$$P_\sigma = P_\infty = P \tag{4.18}$$

As a final remark, we mention that it is possible to prove the continuity of the pressure as function of the density with an appropriate (but straightforward) use of the arguments in [3] and those of the next section.

5. THERMODYNAMIC LIMIT

In this section we discuss the thermodynamic limit and prove the existence of the infinite volume correlation functions and RDM. It extends, for arbitrary z, previous results obtained by Ginibre [1] [2] via low-activity expansions.

To this purpose it is convenient to introduce a family of seminorms on the real valued functions defined on Ω.

We define for each $m > 0$ and Γ bounded:

$$\| f \|^m_\Gamma = \sup_{\substack{\alpha\,(\underline{\omega}) = 1 \\ |\underline{\omega}| = m}} | f(\underline{\omega}) | \tag{5.1}$$

for any $f : \Omega \to \mathbb{R}$.

As a consequence of the estimate

$$\bar{\rho}_\Lambda(\underline{\omega}) \leqslant \exp \{ \delta \, | s(\underline{\omega}) | \} \tag{5.2}$$

we are able to prove the following Proposition.

PROPOSITION 5.1. — Let $\Lambda \nearrow \mathbb{R}^\nu$ be a sequence of bounded open regions. One can extract a subsequence $\{ \Lambda_n \}_{n=1}^\infty$ such that there exists

$$\lim_{\Lambda_n \to \infty} \bar{\rho}_{\Lambda_n}(\underline{\omega}) = \bar{\rho}(\underline{\omega}) \qquad \underline{\omega} \in \Omega \tag{5.3}$$

moreover

$$\sup_m \| \bar{\rho} \|^m_\Gamma \leqslant \exp \{ \delta \, | \Gamma | \} \tag{5.4}$$

and for all m and Γ bounded

$$\lim_{n \to \infty} \| \bar{\rho} - \bar{\rho}_{\Lambda_n} \|^m_\Gamma = 0 \tag{5.5}$$

Proof. — By the estimate

$$\bar{\rho}_{\Lambda_n}(\underline{\omega}) \exp \{ -\delta \, | s(\underline{\omega}) | \} \leqslant 1 \tag{5.6}$$

and the Banach-Alaoglu Theorem, there exists $\bar{\rho}$ such

$$(\bar{\rho}_{\Lambda_n} - \bar{\rho}) \exp \{ -\delta | s(\cdot) | \} \to 0 \quad (5.7)$$

as $n \to \infty$ for some subsequence $\{ \Lambda_n \}_{n=1}^{\infty}$ in the topology of L^{∞} as dual of $L^2(\Omega, d\underline{\omega})$.

On the other hand the $\bar{\rho}_{\Lambda}'s$ satisfy the Mayer-Montroll equations: (see (2), p. 371)

$$\bar{\rho}_{\Lambda}(\underline{\omega}) = \alpha_{\Lambda}(\underline{\omega}) \exp \{ -\mathscr{U}^{\theta}(\underline{\omega}) \} \int d\underline{\eta} K(\underline{\omega})\underline{\eta}\bar{\rho}_{\Lambda}(\underline{\eta}) \quad (5.8)$$

where

$$K(\underline{\omega}) | \underline{\eta}) = \prod_{i=1}^{n} (\exp \{ -W^{\theta}(\underline{\omega} | \eta_i) \} - 1) \quad (5.9)$$

Thus, because of the estimate

$$\int d\underline{\eta} K(\underline{\omega} | \underline{\eta}) \exp \{ \delta | s(\underline{\eta}) | \} < +\infty \quad (5.10)$$

that will be proven later, also (5.3) holds.

Now, using the pointwise convergence of $\bar{\rho}_{\Lambda_n}(\underline{\omega})$, we can obtain also the uniform convergence on the set $\Omega_{\Gamma}^m \subset \Omega$ of all trajectories $\underline{\omega}$ such that $\alpha_{\Gamma}(\underline{\omega}) = 1$, $| \underline{\omega} | = m$. In fact we prove:

$$\int K(\underline{\omega} | \underline{\eta} | \bar{\rho}_{\Lambda_n}(\underline{\eta}) - \bar{\rho}(\underline{\eta}) | d\underline{\eta} \underset{n\to\infty}{\longrightarrow} 0 \quad (5.11)$$

uniformly in $\underline{\omega} \in \Omega_{\Gamma}^m$ simultaneously obtaining the bound (5.10). Let us put

$$F_n(\underline{\eta}) = | \bar{\rho}_{\Lambda_n}(\underline{\eta}) - \bar{\rho}(\underline{\eta}) | \quad (5.12)$$

then

$$\int K(\underline{\omega} | \underline{\eta}) F_n(\underline{\eta}) d\underline{\eta} = \int d\underline{\gamma} K(\underline{\omega} | \underline{\gamma}) \chi_{\Gamma}(\underline{\gamma}) \int d\underline{\zeta} K(\underline{\omega} | \underline{\zeta}) \chi_{\Gamma}^c(\underline{\zeta}) F_n(\underline{\gamma} \cup \underline{\zeta}) \quad (5.13)$$

where

$$\begin{aligned} \chi_{\Gamma}(\underline{\gamma}) &= \chi(\{ \underline{\gamma} | d(\gamma_i(\tau), \Gamma) \leqslant d \ \forall\, i \text{ and some } \tau \in [0, \theta] \}) \\ \chi_{\Gamma}^c(\underline{\zeta}) &= \chi(\{ \underline{\zeta} | d(\zeta_i(\tau), \Gamma) > d \ \forall\, i \text{ and some } \tau \in [0, \theta] \}) \end{aligned} \quad (5.14)$$

Now we estimate $\exp \{ \delta | s(\underline{\gamma}) | \} K(\underline{\omega} | \underline{\gamma}) \chi_{\Gamma}(\underline{\gamma})$ and $\exp \{ \delta | s(\underline{\zeta}) | \} K(\underline{\omega} | \underline{\zeta}) \chi_{\Gamma}^c(\underline{\zeta})$ in terms of two integrable functions $g_1(\underline{\gamma})$ and $g_2(\underline{\gamma})$ not depending on $\underline{\omega}$ but only on $m = | \underline{\omega} |$. So we obtain (5.10) and, by the use of dominated convergence theorem also (5.11).

We have: $(n = | \underline{\gamma} |)$

$$(\exp \delta | s(\underline{\gamma}) |) K(\underline{\omega} | \underline{\gamma}) \chi_{\Gamma}(\underline{\gamma}) \leqslant (1 + e^{\beta \bar{B} m})^n \chi_{\Gamma}(\underline{\gamma}) \exp \delta | s(\underline{\gamma}) | \quad (5.15)$$

where $\bar{B}$ is a minimum of ϕ. The integrability of the r. h. s. of (5.15) follows easily by the Schwartz inequality, (A.15) and the $d\underline{\gamma}$ integrability of χ_{Γ} (gaussian decay of $d\gamma$).

Moreover if $\underline{\zeta}$ is such that $\chi_{\Gamma}^{c}(\underline{\zeta}) = 1$,

$$\mathrm{K}(\underline{\omega} \mid \underline{\zeta}) \leqslant \prod_{j=1}^{|\zeta|} [(e^{-\mathrm{W}^{\theta}(\omega|\zeta_j)} + 1)\mathrm{W}^{\theta}(\underline{\omega} \mid \zeta_j)]$$
$$\leqslant \prod_{j=1}^{|\zeta|}\left[(1 + e^{\beta\bar{\mathrm{B}}m})\prod_{k=1}^{m}\frac{\beta}{\theta}\int d\tau \mid \phi(\mid \omega_k(\tau) - \zeta_j(\tau) \mid) \mid\right.$$
$$\leqslant \prod_{j=1}^{|\zeta|}\left[(1 + e^{\beta\bar{\mathrm{B}}m})\sum_{k=1}^{m}\int d\tau\tilde{\phi}(d(\partial\Gamma, \zeta_j(\tau)))\frac{\beta}{\theta}\right] \quad (5.16)$$

where $\tilde{\phi}$ is some positive decreasing function not depending on $\underline{\omega}$, such that:

$$\int_{|r|>d} \tilde{\phi}(r)d^{\nu}r < +\infty \quad (5.17)$$

$$\int d\underline{\zeta} \exp\{\delta \mid s(\underline{\zeta}) \mid\}\mathrm{K}(\underline{\omega} \mid \underline{\zeta})\chi_{\Gamma}^{c}(\underline{\zeta})$$
$$\leqslant \sum_{n\geqslant 0} \frac{[(1 + e^{\beta\bar{\mathrm{B}}m})]^n}{n!}\left\{\frac{m\beta}{\theta}\int d\zeta\int \tilde{\phi}(d(\partial\Gamma, \zeta(\tau))d\tau \exp\{\delta \mid s(\zeta) \mid\} \chi_{\Gamma}^{c}(\zeta)\right\}^n \quad (5.18)$$

and finally, for some $c_\nu > 0$

$$\int d\zeta\int \mid \tilde{\phi}(d(\partial\Gamma, \zeta(\tau))) \mid d\tau \exp\{\delta \mid s(\zeta) \mid\}\chi_{\Gamma}^{c}(\zeta)$$
$$\leqslant \frac{\beta}{\theta}\int_0^\theta d\tau\int \mathrm{P}_{00}^{\theta}(d\zeta)\exp\{\delta c_\nu \mid s(\zeta) \mid\}\int_{d(\partial\Gamma,\zeta(t)+r)\geqslant d} d^{\nu}r\tilde{\phi}(d(\partial\Gamma, \zeta(t) + r))$$
$$\leqslant \beta\frac{\mathrm{I}(c_\nu\delta)}{(2\pi\theta)^{\nu/2}}\int_{|r|>d}\tilde{\phi}(r)d^{\nu}\underline{r}. \quad \square \quad (5.19)$$

Let us put

$$\rho(\underline{\omega}) = z^{|\omega|}\bar{\rho}(\underline{\omega}) \quad (5.20)$$

and define the infinite volume RDM by

$$\rho(\mathrm{X}, \mathrm{Y}) = \int \mathrm{P}_{\mathrm{XY}}^{\theta}(d\underline{\omega})\rho(\underline{\omega}) \quad (5.21)$$

First we observe that the following bound

$$\rho(\mathrm{X}, \mathrm{Y}) \leqslant \xi^{|\mathrm{X}|} \quad (5.22)$$

holds in virtue of (A.15), where

$$\xi = \left(\frac{z\mathrm{I}(\delta)}{(2\pi\theta)^{\nu/2}}\right) \quad (5.23)$$

(The same bound obviously holds for the $\zeta_\Lambda(\mathrm{X}, \mathrm{Y})$'s). It is not hard to prove that $\rho_{\Lambda_n}(X, Y) \to \rho(X, Y)$ uniformly on compact sets for X and Y. Infact (see [2], p. 379 for details)

$$|\rho(X, Y) - \rho_{\Lambda_n}(X, Y)| \leqslant \int P^{\theta}_{XY}(d\underline{\omega})\, |\rho(\underline{\omega}) - \alpha_{\Lambda_n}(\underline{\omega})\rho_{\Lambda_n}(\underline{\omega})|$$

and one can split the above integration in two parts: trajectories « near » $X \cup Y$ give rise to a small contribution for large n in virtue of Proposition 5.1. The others, for which at least one goes far enough from $X \cup Y$ give a small contribution for the gaussian decay of P^{θ}_{XY}. To summarize:

PROPOSITION 5.2. — The RDM's have a limit for $n \to +\infty$ uniformly on compacts sets. Moreover such limit is given by (5.20) and satisfy the bound (5.21).

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APPENDIX A

Let P_x the Wiener measure associated with a ν-dimensional Brownian motion starting at the point $x \in \mathbb{R}^\nu$. P_x lives on the Borel sets of the space $M = \prod_{t \in [0, +\infty)} (\mathbb{R}^\nu)_t$, where $(\mathbb{R}^\nu)_t = \mathbb{R}^\nu$.

More precisely, P_x is concentrated in the small subset of M of all Hölder continuous trajectories $\left(\text{with exponent } \alpha < \frac{1}{2}\right)$ as consequence of the following Lemma 2 :

LEMMA A.1. — Defining:

$$E(s ; \delta, l) = \{ \omega \in M \mid \exists\, t, t' \in [0, s] \text{ s. t. } |t - t'| < \delta, |\omega(t) - \omega(t')| > l \} \qquad (A.1)$$

then:

$$P_x(E(s ; \delta, l)) \leqslant C(s, \delta) \int_{|x| > l/\delta} g_\delta(x) dx = C(s, \delta) G(l, \delta) \qquad (A.2)$$

where $g_t(x) = (2\pi t)^{-\nu/2} \exp(- x^2 \setminus 2t)$, and (A.2) defines G.

$$C(s, \delta) = 4\left[\mathscr{I}\left(\frac{s}{\delta}\right) + 1\right] \qquad (A.3)$$

The above lemma has as corollary the estimate (A.10) (see below) that is the main tool in proving the result of Section 2.

The following lemma is the core of Proposition A.1 that plays a central role in Section 3, 4, 5.

LEMMA A.2. — Let $\{ \Delta_1 \ldots \Delta_N \}$ be a finite sequence of elements of $\mathscr{Q}$ such that $d(\Delta_i, \Delta_j) \geqslant 1$ for all $i \neq j$, $i, j = 1 \ldots N$. Suppose that $d(x, \Delta_i) \geqslant 1$ for all i. (Here $d(x, \Delta_i)$ denotes the Euclidean distance between x and Δ_i). Denoting:

$$E(\alpha) = \{ \omega \mid \exists\, t_1 \ldots t_N \in [0, \alpha] \text{ s. t. } \omega(t_i) \in \Delta_i \} \qquad (A.4)$$

then:

$$P_x(E(\alpha)) \leqslant 4^N (2N + 1)^{3N} G\left(1, \frac{2\alpha}{N}\right)^N \qquad (A.5)$$

Proof. — Let $\bar{t}_i = \bar{t}_i(\omega)$ be the family of the random variables time of the first entrance in the set Δ_i and consider the event:

$$F = \{ \omega \mid \bar{t}_i \leqslant \bar{t}_{i+1} \quad i = 1 \ldots N \} \qquad (A.6)$$

Then:

$$\chi_F \leqslant \sum_{\substack{s_1 = 0 \\ s_1 \leqslant s_2 \ldots s_N}}^{N-1} \cdots \sum_{s_N = 0}^{N-1} \chi\left(\frac{s_1 \alpha}{N} < \bar{t}_1 \leqslant \frac{(s_1 + 1)}{N} \alpha\right) \cdots \chi\left(\frac{s_N \alpha}{N} < \bar{t}_N \leqslant \frac{(s_N + 1)}{N} \alpha\right) \qquad (A.7)$$

Defining the new random variables:

$$\tau_i = \bar{t}_i - \bar{t}_{i-1}, \qquad \tau_1 = \bar{t}_1 \qquad (A.8)$$

putting, $r_i = (s_i - s_{i-1}) + 1$, $r_1 = s_1 + 1$

$$\chi_F \leqslant \sum_{r_1 = 1}^{N} \cdots \sum_{r_N = 1}^{N} \chi\left(\sum_{i=1} r_i \leqslant 2N\right) \prod_{i=1}^{N} \chi\left(\tau_i \leqslant r_i \frac{\alpha}{N}\right) \qquad (A.9)$$

We define the new processes:

$$\omega_1 = \omega$$
$$\omega_i(t) = \omega_{i-1}(t + \tau_{i-1}) - \omega_{i-1}(\tau_{i-1}), \qquad i = 2 \ldots \mathrm{N} \qquad (\mathrm{A}.10)$$

In virtue of the strong Markov property $\omega_i(t)$ is independent of $\omega_{i-1}(t)$, $t < \tau_{i-1}$. Thus since:

$$\chi\left(\tau_i \leqslant r_i \frac{\alpha}{\mathrm{N}}\right) \leqslant \chi\left(\,|\,\omega_i(t)\,| \geqslant 1\,;\, t \leqslant \frac{r_i\alpha}{\mathrm{N}}\right) \qquad (\mathrm{A}.11)$$

we have:

$$\mathrm{P}_x(\mathrm{F}) \leqslant \sum_{\substack{r_1 \ldots r_\mathrm{N} \\ r_i = 1 \ldots \mathrm{N}}} \chi\left(\sum_{i=1}^{\mathrm{N}} r_i \leqslant 2\mathrm{N}\right) \prod_{i=1}^{\mathrm{N}} \mathrm{C}\left(\alpha, \frac{r_i\alpha}{\mathrm{N}}\right) \mathrm{G}\left(1, \frac{r_i\alpha}{\mathrm{N}}\right) \leqslant 4^{\mathrm{N}}(\mathrm{N}+1)^{2\mathrm{N}} \mathrm{G}\left(1, \frac{2\alpha}{\mathrm{N}}\right)^{\mathrm{N}} \qquad (\mathrm{A}.13)$$

after maximizing on r_i with the constraint $\Sigma r_i \leqslant 2\mathrm{N}$.

The thesis follows taking into account all possible permutations. □

The above estimates on P_x induce estimates on the conditional Wiener measure P^θ_{xy} that is the only measure used for our purposes.

By the following inequality we have, for all events $\mathrm{H}_{\theta\setminus 2}$ depending only on what happens in $[0, \theta/2]$,

$$\int \mathrm{P}^\theta_{xy}(d\omega)\chi_{\mathrm{H}_{\theta/2}}(\omega) \leqslant \int du \int \mathrm{P}^{\theta/2}_{xu}(d\omega)\mathrm{H}_{\theta/2}(\omega) g_{\theta/2}(u-y) \leqslant (\pi\theta)^{-\nu/2} \int \mathrm{P}_x(d\omega)\mathrm{H}_{\theta/2}(\omega) \qquad (\mathrm{A}.14)$$

Hence:

$$\int \mathrm{P}^\theta_{xy}(d\omega)\chi_{\mathrm{E}(s,\delta,l)}(\omega) \leqslant (2\pi\theta)^{-\nu/2}\left(\exp - c_1 \frac{l^2}{2\delta}\right) \qquad (\mathrm{A}.15)$$

$$\int \mathrm{P}^\theta_{xy}(d\omega)\chi_{\mathrm{E}\left(\frac{\theta}{2}\right)}(\omega) \leqslant (\pi\theta)^{-\nu/2}\left(\exp - c_1 \frac{\mathrm{N}^2}{\theta}\right) \qquad (\mathrm{A}.16)$$

for $l \geqslant 1$, θ smaller then some fixed constant, and c_1, depending only on the dimensions.

(A.15) is a consequence of (A.14) Lemma A.1 and some tricks [*1*].

(A.16) follows easily from(A.14) and Lemma A.2. Finally we prove:

Proposition A.1. — For k large enough:

$$\mathrm{P}^\theta_{xy}(\,|\,s(\omega)\,| = k) \leqslant c_2 \exp - c_3 \frac{k^2}{2\theta} \qquad (\mathrm{A}.17)$$

for some positive constants c_2 and c_3.

Proof. — Given a shadow S, we construct the set T(S) in the following way. We take the first element of S in the lexicographic order $\Delta_1 \in \mathrm{S}$ and consider the set S R_1 where:

$$\mathrm{R}_i = \{\, \Delta \mid d(\Delta, \Delta_1) = 0 \,\} \qquad (\mathrm{A}.18)$$

Let Δ_2 be the first element of $\mathrm{S}\setminus\mathrm{R}_1$. We iterate defining Δ_i as the first element of $(\mathrm{S}\setminus\mathrm{R}_1)\setminus\mathrm{R}_2 \ldots \setminus\mathrm{R}_{i-1}$. Let $\mathrm{T(S)} = \bigcup_i \Delta_i$. Then T(S) contains at least $|\,\mathrm{S}\,|/c_\nu$ elements, where c_ν is a positive constant depending only on the dimensions. Then:

$$\mathrm{P}^\theta_{xy}(\,|\,s(\omega)\,| = k) \leqslant \mathrm{P}^\theta_{xy}(\mathrm{E(T(S))}) \qquad (\mathrm{A}.19)$$

where E(T(S)) is the event in which the Brownian particle starting at x visits all the tesserae of T(S) in the time θ. Hence:

$$\mathrm{E(T(S))} = \bigcup_{\substack{n,n' \\ n+n'=k'}} \mathrm{G}\left(n, \frac{\theta}{2}\right) \cap \tilde{\mathrm{G}}\left(n', \frac{\theta}{2}\right) \qquad (\mathrm{A}.20)$$

where $k' = |T(S)| - 2$ and $G\left(n, \frac{\theta}{2}\right)$ and $\tilde{G}\left(n', \frac{\theta}{2}\right)$ denote respectively the events in which the particles visit at least n tesserae of T(S) in $\left[0, \frac{\theta}{2}\right]$ and visit at least n' tesserae in $\left[\frac{\theta}{2}, \theta\right]$. Since $P^{\theta}_{xy}\left(\tilde{G}\left(n, \frac{\theta}{2}\right)\right) = P^{\theta}_{yx}\left(G\left(n, \frac{\theta}{2}\right)\right)$ we obtain (A.17) by (A.16), the inequality $n^2 + n'^2 \geqslant \frac{1}{2}.(n + n')^2$, Schwartz inequality and a rearrangement of the constants. As consequence of the above proposition we have the following estimate:

$$\int P^{\theta}_{xy}(d\omega)(\exp \delta \,|\, s(\omega) \,|\,) \leqslant (2\pi\theta)^{-\nu/2} I(\delta) \tag{A.21}$$

where $I(\delta)$ depends only on the dimensions if is chosen in some fixed but arbitrary interval $[0, \theta_0]$.

APPENDIX B

We begin with some elementary considerations.

LEMMA B.1. — Given $\varepsilon > 0$, $0 < \alpha < 1$, there exists an integer s_0 large enough, such that for each $s \geqslant s_0$:

$$1 + \alpha < \frac{l_{s+1}}{l_s} < e^{\alpha(1+\varepsilon)} \tag{B.1}$$

$$\frac{\varphi(s+1)}{\varphi(s)} < e^{\alpha(v+\varepsilon(v+1))} \equiv f(\alpha, \varepsilon) \tag{B.2}$$

Proof. —

$$\frac{l_{s+1}}{l_s} > \frac{e^{\alpha(s+1)} - 1}{e^{\alpha s}} = e^{\alpha} - \frac{1}{e^{\alpha s}} > 1 + \alpha \tag{B.3}$$

if

$$s \geqslant s_1 > \frac{1}{\alpha} \log 2/\alpha^2 \tag{B.4}$$

$$\frac{l_{s+1}}{l_s} < \frac{e^{\alpha(s+1)}}{e^{\alpha s} - 1} = e^{\alpha}\left(1 + \frac{1}{e^{\alpha s} - 1}\right) \tag{B.5}$$

and, of course, if $s \geqslant s_2$ where

$$s_2 > \frac{1}{\alpha} \log\left(1 + \frac{1}{e^{\alpha\varepsilon} - 1}\right)$$

(B.1) is verified.

Finally, if $s > s_3$, where

$$s_3 > \frac{1}{e^{\varepsilon\alpha} - 1} \tag{B.6}$$

we have

$$\frac{s+1}{s} < e^{\alpha\varepsilon} \tag{B.7}$$

and (B.2) follows from the inequality

$$\frac{2l_{s+1} + 1}{2l_{s+1}} < e^{\alpha(1+\varepsilon)} \frac{2l_s + \dfrac{l_s}{l_{s+1}}}{2l_{s+1}} < e^{\alpha(1+\varepsilon)}. \qquad \square$$

LEMMA B.2. — Put, for each $k \in \mathbb{N}$, $r(k) = \min\{ r \in \mathbb{N} \mid l_{q+r} > l_q + k \ \forall q \geqslant 1 \}$. Then

$$r(k) \leqslant 1 + \frac{1}{\alpha} \log (k+2) \tag{B.8}$$

Proof. —

$$\{ r \in \mathbb{N} \mid l_{q+r} > l_q + k \ \forall q \geqslant 1 \} \supseteq \{ r \in \mathbb{N} \mid \exp \alpha(q+r) - \exp \alpha q > k + 1 \ \forall q \geqslant 1 \}$$
$$\supseteq \{ r \in \mathbb{N} \mid \exp \alpha r > k + 2 \} \tag{B.9}$$

Then:

$$r(k) \leqslant \min \{ r \in \mathbb{N} \mid e^{\alpha r} > k + 2 \} \leqslant 1 + \frac{1}{\alpha} \log (k+2). \qquad \square$$

Estimates (B.2) and (B.8) are used in Lemma 2.3.

In this appendix we use, instead of (B.1) and (B.2) the estimates

$$1 + \alpha < \frac{l_{s+1}}{l_s} < 1 + 2\alpha \tag{B.10}$$

$$\frac{\varphi(s+1)}{\varphi(s)} < (1 + 2\alpha)^{v+1} \tag{B.11}$$

which follow from (B.1) and (B.2), taking $\varepsilon < \varepsilon_0$, where

$$\varepsilon_0 = 1 - 2\alpha e^{2\alpha} \tag{B.12}$$

is positive, if α is small enough.

Proof of Lemma 2.1. — By superstability, fixed τ in the intersection of the domains of the trajectories in $\tilde{\underline{\eta}}_{\Lambda_q}$ and $\pi_{\Lambda_q}\underline{\omega}$, we have

$$\mathcal{U}^\theta(\tilde{\underline{\eta}}_{\Lambda_q}(\tau) \cup \pi_{\Lambda_q}\underline{\omega}(\tau)) \geqslant \frac{\beta A}{\theta} \sum_{\Delta \subset \Lambda_q} n^2(\tilde{\underline{\eta}}_{\Lambda_q}(\tau) \cup \pi_{\Lambda_q}\underline{\omega}(\tau), \Delta) - \frac{\beta B}{\theta} \sum_{\Delta \subset \Lambda_q} n(\tilde{\underline{\eta}}_{\Lambda_q}(\tau) \cup \pi_{\Lambda_q}\underline{\omega}(\tau), \Delta) \tag{B.13}$$

In fact, for any n such that

$$n < \frac{A}{4B} n^2 + \frac{4B}{A} \tag{B.14}$$

we have

$$\mathcal{U}^\theta(\tilde{\underline{\eta}}_{\Lambda_q}(\tau) \cup \pi_{\Lambda_q}\underline{\omega}(\tau)) \geqslant \frac{\beta A}{4\theta} \sum_{\Delta \subset \Lambda_q} n^2(\tilde{\underline{\eta}}_{\Lambda_q} \cup \pi_{\Lambda_q}\underline{\omega}(\tau), \Delta) - \frac{4\beta B^2}{4\theta} |\Lambda_q| + \frac{\beta A}{2\theta} \sum_{\Delta \subset \Lambda_q} n^2(\tilde{\underline{\eta}}_{\Lambda_q}(\tau), \Delta) \tag{B.15}$$

Therefore, integrating in τ we prove the lemma. □

The proofs of Lemma 2.2 and 2.5 are based on the following decomposition of the interaction energy: fixed τ in the intersection of the domains of the trajectories in $\underline{\xi}_1$ and $\underline{\xi}_2$, by lower regularity (1.17), if $a \geqslant 0$,

$$-W^\theta(\underline{\xi}(\tau) \mid \underline{\xi}_2(\tau)) \leqslant \frac{1}{2}\frac{\beta}{\theta} \sum_{\Delta \subset \Lambda_{q+a}} \sum_{\Delta' \subset \Lambda_q^c} \psi(\Delta, \Delta')[n^2(\underline{\xi}_1(\tau), \Delta) + n^2(\underline{\xi}_2(\tau), \Delta')] \tag{B.16}$$

Omitting the dependance on $\underline{\xi}_1$, $\underline{\xi}_2$ we write the r. h. s. of (B.16) as follows

$$\begin{aligned} \frac{1}{2}\frac{\beta}{\theta}\Bigg\{ &\sum_{\Delta \subset \Lambda_{q+a} \setminus \Lambda_{q-1}} \sum_{\Delta' \subset \Lambda_{q+a+1} \setminus \Lambda_q} \psi(\Delta, \Delta')n^2(\Delta) + \sum_{\Delta \subset \Lambda_{q-1}} \sum_{\Delta' \subset \Lambda_{q+a+1} \setminus \Lambda_q} \psi(\Delta, \Delta')n^2(\Delta) \\ &+ \sum_{\Delta \subset \Lambda_{q+a}} \sum_{\Delta' \subset \Lambda_{q+a+1}^c} \psi(\Delta, \Delta')n^2(\Delta) + \sum_{\Delta \subset \Lambda_{q+a}} \sum_{\Delta' \subset \Lambda_{q+a+1} \setminus \Lambda_q} \psi(\Delta, \Delta')n^2(\Delta') \\ &+ \sum_{\Delta \subset \Lambda_{q+a}} \sum_{\Delta' \subset \Lambda_{q+a+1}^c} \psi(\Delta, \Delta')n^2(\Delta') \Bigg\} \end{aligned} \tag{B.17}$$

The first term in the bracket is a « short range » term and is easily bounded by (see 1.20)

$$T_1 = \frac{1}{2} F \frac{\beta}{\theta} \sum_{\Delta \subset \Lambda_{q+a} \setminus \Lambda_{q-1}} n^2(\Delta) \tag{B.18}$$

The fourth term is also of « short range » type and is bounded as

$$T_4 = \frac{1}{2} F \frac{\beta}{\theta} \sum_{\Delta' \subset \Lambda_{q+a+1} \setminus \Lambda_q} n^2(\Delta') \tag{B.19}$$

The second and third terms are of similar nature and are bounded respectively by

$$T_2 = \frac{1}{2}\frac{\beta}{\theta} F(l_q - l_{q-1}) \sum_{\Delta \subset \Lambda_{q-1}} n^2(\Delta) \tag{B.20}$$

and

$$T_3 = \frac{1}{2}\frac{\beta}{\theta}F(l_{q+a+1} - l_{q+a}) \sum_{\Delta \subset \Lambda_{q+a}} n^2(\Delta) \tag{B.21}$$

where $F(n)$ denotes the rest of order n of the series defining F.

The last term is bounded by

$$T_5 = \sum_{\Delta \subset \Lambda_{q+a}} \sum_{k=1}^{\infty} \psi_k\Big(\sum_{\Delta' \subset \Lambda_{q+a+k+1} \setminus \Lambda_{q+a+k}} n^2(\Delta')\Big) \tag{B.22}$$

where

$$\psi_k = \psi(l_{q+a+k} - l_{q+u}) \tag{B.23}$$

After integration on τ, since $\underline{\xi}_1$ and $\underline{\xi}_2$ are subsets of $\underline{\xi}$, we have

$$- W^\theta(\underline{\xi}_1 \mid \underline{\xi}_2) \leqslant \beta F E^\theta_{\Lambda_{q+a+1}\setminus\Lambda_{q-1}}(\underline{\xi}) + \beta F(l_q - l_{q-1}) E^\theta_{\Lambda_{q+a}}(\underline{\xi})$$
$$+ \frac{\beta}{2} \sum_{\Delta \subset \Lambda_{q+a}} \sum_{k=1}^{\infty} \psi_k\big(E^\theta_{\Lambda_{q+a+k+1}\setminus\Lambda_{q+a+1}}(\underline{\xi}) - E^\theta_{\Lambda_{q+a+k}\setminus\Lambda_{q+a+1}}(\underline{\xi})\big), \tag{B.24}$$

where we have collected T_1 and T_4 to get the first contribution and T_2 and T_3 to get the second one, since $l_{q+a+1} - l_{q+a} > l_q - l_{q-1}$ for q large enough.

We have now:

LEMMA B.3. — Let $p = q + a$. Then:

$$\sum_{k=1}^{\infty} \psi_k\big(E^\theta_{\Lambda_{p+k+1}\setminus\Lambda_{p+1}} - E^\theta_{\Lambda_{p+k}\setminus\Lambda_{p+1}}\big) \leqslant \sum_{k=1}^{\infty} (\psi_k - \psi_{k+1}) E^\theta_{\Lambda_{p+k+1}} \tag{B.25}$$

Proof. — In fact

$$\sum_{k=1}^{\infty} \psi_k\big(E^\theta_{\Lambda_{p+k+1}\setminus\Lambda_{p+1}} - E^\theta_{\Lambda_{p+k}\setminus\Lambda_{p+1}}\big) = \sum_{k=1}^{\infty} \psi_k E^\theta_{\Lambda_{p+k+1}\setminus\Lambda_{p+1}} - \sum_{k=2}^{\infty} \psi_k E^\theta_{\Lambda_{p+k}\setminus\Lambda_{p+1}}$$
$$= \sum_{k=1}^{\infty} \psi_k E^\theta_{\Lambda_{p+k+1}\setminus\Lambda_{p+1}} - \sum_{k'=1}^{\infty} \psi_{k'+1} E^\theta_{\Lambda_{p+k'+1}\setminus\Lambda_{p+1}} \leqslant \sum_{k=1}^{\infty} (\psi_k - \psi_{k+1}) E^\theta_{\Lambda_{p+k+1}}. \quad \square \tag{B.26}$$

LEMMA B.4. — If $q \geqslant q_0^{(1)}$ with q_0 large enough, and α is small enough:

$$\frac{\varphi(q+a+1) - \varphi(q-1)}{\varphi(q)} < \frac{A}{48F} \tag{B.27}$$

$$\frac{\varphi(q+a)F(l_q - l_{q-1})}{\varphi(q)} < \frac{A}{48} \tag{B.28}$$

$$\frac{\varphi(q+a)\sum_{k=1}^{\infty}(\psi_k - \psi_{k+1})\psi(q+a+k+1)}{\varphi(q)} < \frac{A}{24} \tag{B.29}$$

Proof. — Since $\dfrac{\varphi(q-1)}{\varphi(q)} < 1$ and, by (B.11)

$$\frac{\varphi(q+a+1) - \varphi(q-1)}{\varphi(q-1)} < (1 + 2\alpha)^{(a+1)(v+1)} - 1 \tag{B.30}$$

inequality (B.27) is true if we choose α so small that

$$(1 + 2\alpha)^{(a+1)(\nu+1)} - 1 < \frac{A}{48F} \tag{B.31}$$

Furthermore (B.11) and the fact that

$$l_q - l_{q-1} > \alpha l_{q-1} \to \infty \quad \text{as} \quad q \to \infty \tag{B.32}$$

prove (B.28), if q_0 is chosen large enough.

We have, denoting $q + a = p$, for $k \geqslant 1$

$$\frac{\varphi(p + k + 1)}{\frac{1}{\alpha}\log(l_{p+k+1} - l_{p+1} + 3)[2(l_{p+k+1} - l_{p+1}) + 3]^{\nu}} \leqslant \left(\frac{l_{p+k+1}}{l_{p+k} - l_p}\right)^{\nu+1} \tag{B.33}$$

but by (B.10)

$$\frac{l_{p+k+1}}{l_{p+k} - l_p} = \frac{\dfrac{l_{p+k+1}}{l_{p+k}}}{1 - \dfrac{l_p}{l_{p+k}}} < \frac{1 + 2\alpha}{1 - (1 + \alpha)^{-k}} < \frac{1 + 2\alpha}{1 + (1 + \alpha)^{-1}} = \alpha^{-1}(1 + \alpha)(1 + 2\alpha) \tag{B.34}$$

Then the l. h. s. of (B.29) is smaller then

$$[\alpha^{-1}(1 + \alpha)(1 + 2\alpha)]^{\nu+1} \sum_{k=1}^{\infty} [\psi(l_{q+a+k} - l_{q+a}) - \psi(l_{q+a+k+1} - l_{q+a})]$$
$$\times \alpha^{-1}\log(l_{q+a+k+1} - l_{q+a+1} + 3)[2(l_{q+a+k+1} - l_{q+a+1}) + 3]^{\nu}$$
$$\leqslant \alpha^{-1}[\alpha^{-1}(1 + \alpha)(1 + 2\alpha)]^{\nu+1} \sum_{s=l_{q+q+1}-l_{q+a+1}}^{\infty} [\psi(s) - \psi(s + 1)](2s + 3)^{\nu}\log(s + 3) \tag{B.35}$$

but the last series converges by (1.20), and then (B.35) is bounded by an infinitesimal as $q \to \infty$; therefore, if q_0 is large enough, by (B.11), (B.29) is proven. □

Lemmas B.3 and B.4 and (B.24) imply Lemma 2.2 since $\underline{\xi}$ is such that $\tilde{\chi}_q(\underline{\xi}) = 1$ for $q \geqslant q_0$ and q_0 is chosen large enough.

Lemma 2.5 is easily proven using still (B.24), Lemma B.3, (B.28) and (B.29), since $\chi_{q_0-1}(\underline{\xi}) = 1$. Thus:

$$- W^{\theta}(\underline{\xi}_1 \mid \underline{\xi}_2) \leqslant \beta\left[F\varphi(q_0 + a + 1) + \frac{A}{24}\varphi(q_0)\right] \equiv \frac{1}{2}\beta h(q_0) \tag{B.36}$$

with $h(q_0)$ defined by the last step and then independent of β, z, θ.

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