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Section A :

Physique théorique.

Inverse scattering for the one-dimensional Stark effect and application to the cylindrical KdV equation

by

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ABSTRACT. — We develop the inverse spectral and scattering theory for one-dimensional Stark operators, i. e.

$$-\frac{d^2}{dx^2} + fx + u(x) \qquad \text{on} \qquad L^2(\mathbb{R}) .$$

The potential u is determined in a special case; this allows existence and approximate solitary-wave behavior to be proved for solutions of a non-linear evolution equation corresponding to the Stark Hamiltonian. Connection is made to the solitary solution discovered by Calogero and Degasperis. This behavior is associated with resonances in much the same way as solitons are associated with bound states in the theory without the linear term fx.

RÉSUMÉ. — Nous développons la théorie inverse du spectre et de la diffusion pour les opérateurs de Stark en une dimension, c'est-à-dire

$$-\frac{d^2}{dx^2} + fx + u(x) ,$$

agissants sur $L^2(\mathbb{R})$. Le potentiel u est déterminé dans un cas spécial, ce

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qui permet la démonstration de l'existence et du comportement quasi solitaire de solutions ondoyantes d'une équation d'évolution non-linéaire correspondante à l'hamiltonien de Stark. On fait connexion à la solution solitaire découverte par Calogero et Degasperis. L'association entre ces solutions et les résonances ressemble à celle entre les solitons et les états liés dans la théorie ordinaire sans terme linéaire fx.

INTRODUCTION

The spectral and scattering theory of Stark operators

$$-\frac{d^2}{dx^2} + fx + u(x) \qquad \text{on} \qquad L^2(\mathbb{R}) \tag{0.1}$$

has been studied intensively in recent years [1] [2] and is now on a very satisfactory footing. The adjustable, conventionally positive parameter f is proportional to the electric field, and u represents an atomic potential in the physical situation for which (0.1) provides a simple model; f will be fixed but not set to 1, because the limit $f \to 0$ is of independent interest in the Stark effect. The model dates from Titchmarsh [3], and if

$$u(x) = 1/x \qquad \text{and} \qquad x \in \mathbb{R}^+,$$

then (0.1) is realistic for the Stark effect in hydrogen after separation of variables.

In this paper we address three questions that deserve analysis now that the direct spectral and scattering theory of (0.1) is well understood:

1. What kind of inverse spectral theory for (0.1) can be rigorously founded? More specifically:

2. Since (0.1) has resonances rather than eigenvalues with reasonable assumptions on u—for instance $u \in L^2 + L^\infty$ with a certain bound on u' [4] [5]—how are resonances reflected in the inverse spectral and scattering theory? Since complex scaling establishes a connection between self-adjoint Hamiltonians with continuous spectrum and non-self-adjoint operators with discrete eigenvalues corresponding to resonances, this is an important question in the development of non-self-adjoint inverse scattering theory.

3. Granting a reasonable answer to question 2, are resonances associated with nonlinear wave equations and their solitary-wave solutions?

Our attention was brought to this problem by a series of articles [*6*] by Calogero and Degasperis on inverse scattering for (0.1). They transcribe the ordinary inverse-scattering formalism to the case of (0.1), allowing Airy functions, which solve

$$\left(-\frac{d^2}{dx^2}+x\right)y(x)=0\,,$$

to play the role of the usual exp $(\pm\, ikx)$ or sines and cosines. Then they assume the existence of a single bound state and solve the Gel'fand-Levitan equation, finding a potential that can be turned into a solution of a non-linear wave equation, the « cylindrical K*d*V equation » for $q(x, t)$,

$$q_t + qxxx - 6qxq + q/2t = 0\,. \tag{0.2}$$

Their solution preserves its shape, though it gets scaled as a function of the introduced parameter t. Since the solution is an explicit expression involving Airy functions, it is a matter of straight-forward though tedious computation to verify that it is a solution of (0.2) in the classical sense. However, it fails—marginally—to be L^2, and its derivatives do not fall off as $y \to 0$, owing to the wild oscillations of the Airy functions. Consequently, as pointed out by Calogero and Degasperis, the assumptions they make to derive inverse scattering are actually violated. Indeed, since their u' is bounded and u'' increases as $x \to -\infty$ at worst as $|x|^{1/2}$, the Liouville transformation of (0.1) shows that its solutions have the same asymptotics as the Airy functions [*7*] [*8*]. It follows that there are no bound states in the spectrum of (0.1); the spectrum is absolutely continuous from $-\infty$ to ∞ (cf. [*5*]). Therefore, the violations of the assumptions for inverse scattering are not merely technical—the spectral function for their u is not the one hypothesized (except perhaps with some peculiar, non-self-adjoint boundary conditions at ∞).

In the light of this and the recent progress in direct scattering and spectral theory, it is possible and timely to put the inverse scattering theory of (0.1) on a rigorous basis. This is consequently our first task. (Some general work on inverse scattering for potentials not vanishing at infinity has been done by Kulish [*9*].) Then, in order to clarify the nature and significance of the solitary solution discovered by Calogero and Degasperis, we examine the analogous resonance problem. It has a unique solution within the class of potentials for which the Gel'fand-Levitan equations are justified. Its transformation into a solution of (0.2) is continuously differentiable and square-integrable, but it is not quite form-preserving. The form-preserving solution turns out to be very special, for not only is it the limit of the solutions we find as the imaginary part of the resonance goes to zero, but it is also the asymptotic form of our solutions as $t \to \infty$.

I. INVERSE SCATTERING AND SPECTRAL THEORY FOR STARK OPERATORS

This section discusses the inverse scattering theory for (0.1) using as a comparison the operator

$$-\frac{d^2}{dx^2} + fx. \tag{1.1}$$

Our method, in a word, is to translate the relevant parts of the article by Faddeev [*10*] from the language of sines to that of Airy functions. We shall not reiterate all the details of [*10*], of course, but to facilitate comparison we refer to equations of that article by their numbers, preceded by F. Faddeev's notation will also be followed as far as possible.

All that is required of u is that it be continuous and either bounded by a constant times $(1 + |x|)^{-1}$ or absolutely integrable on $\mathbb{R}$. This clearly implies that the eigenfunctions of (0.1) are twice differentiable in x. Moreover, the solution of

$$\left(-\frac{d^2}{dx^2} + fx - fz\right)y = 0$$

can then be asymptotically integrated with the methods of [*7*] [*8*]—in fact this only requires that $u/\sqrt{|x|}$ be absolutely integrable for $|x| \geq$ some constant. One discovers

a) the asymptotics of the eigenfunctions and their first derivatives are the same as those of the Airy functions and their derivatives as $x \to \pm\infty$;

b) the subdominants ϕ and $f_{\mathrm{I,II}}$ defined below depend analytically on z.

The restriction of boundedness for finite x could easily be relaxed. In terms of the Airy function $\mathrm{A}i(x)$, the standard solutions of

$$\left(-\frac{d^2}{dx^2} + fx - fz\right)y(x-z) = 0 \tag{1.2}$$

are

$$\begin{aligned} a(x-z) &= f^{1/3}\mathrm{A}i(f^{1/3}(x-z)), \\ e_{\mathrm{I}}(x-z) &= 2\pi f^{-2/3}e^{\pi i/6}\mathrm{A}i(e^{2\pi i/3}f^{1/3}(x-z)), \\ e_{\mathrm{II}}(x-z) &= 2\pi f^{-2/3}e^{-\pi i/6}\mathrm{A}i(e^{-2\pi i/3}f^{1/3}(x-z)) \\ &= \overline{e_{\mathrm{I}}(\overline{x-z})}. \end{aligned} \tag{1.3}$$

The important properties of these functions [*11*] are that $a(x)e_{\mathrm{I,II}}(x)$ is uniformly bounded and $\to 0$ as $|x|^{-1/4}$ as $|x| \to \infty$ in the lower (resp. upper) half plane, and:

$$a = \frac{f}{2\pi i}(e_{\mathrm{I}} - e_{\mathrm{II}}) \qquad \text{is real on the real axis;}$$

$$\mathrm{W}\{a, e_{\mathrm{I,II}}\} \equiv a\frac{d}{dx}e_{\mathrm{I,II}} - e_{\mathrm{I,II}}\frac{d}{dx}a = 1;$$

$$\mathrm{W}\{e_{\mathrm{I}}, e_{\mathrm{II}}\} = \frac{2\pi i}{f}. \tag{1.4}$$

(Note that Faddeev uses the notation $[f; g] \equiv -\mathrm{W}\{f, g\}$, the only point at which we shall not attempt to make our notation conform to his.) The Airy functions divide the complex plane into three sectors: the function $a(z)$ is subdominant for $|\arg z| < \pi/3$, $e_{\mathrm{I}}(z)$ for $-\pi < \arg z < -\pi/3$, and $e_{\mathrm{II}}(z)$ for $\pi/3 < \arg z < \pi$ (all assuming as usual $f > 0$). More precisely, the asymptotic formulae for the Airy functions [*8*] [*11*] imply the following estimates for fixed f (and they remain valid when differentiated by x term by term): as $z \to \infty$,

$$a(z) = \begin{cases} \dfrac{f^{1/4}}{2\sqrt{\pi}} z^{-1/4} \exp\left(-\dfrac{2f^{1/2}z^{3/2}}{3}\right)\Sigma_1, & |\arg z| < \pi \\[2ex] \dfrac{f^{1/4}}{\sqrt{\pi}} z^{-1/4}\left[\sin\left(\dfrac{2}{3}f^{1/2}(-z)^{3/2} + \pi/4\right)\Sigma_2 \right. & \\[2ex] \left. - \dfrac{\cos(2f^{1/2}(-z)^{3/2}/3 + \pi/4)}{(2f^{1/2}(-z)^{3/2}/3)}\Sigma_3\right], & \pi/3 < \arg z < 5\pi/3, \end{cases} \tag{1.5}$$

where $\Sigma_i \to 1$ as $z \to \infty$ and has an explicit asymptotic power series in $1/(2f^{1/2}z^{3/2}/3)$. We shall not be excessively concerned with these expressions, but need only the qualitative exponential and power law fall-off in the complex plane. We do note, however, that e_{I} and e_{II} fall off exponentially in $|z|$ as $z \to -\infty$ with fixed negative and respectively positive imaginary part and so do their derivatives. (This is not obvious, but is straightforward to show with the methods of [*7*] [*8*].)

Let y solve

$$y''(x) + (fx - fz + u(x))y = 0, \tag{1.6}$$

and define particular solutions $y = \phi, f_{\pm}$, such that

$$\lim_{x\to+\infty} \phi(x, z)/a(x - z) = 1$$

$$\lim_{x\to-\infty} f_{\mathrm{I}}(x, z)/e_{\mathrm{I}}(x - z) = 1$$

$$\lim_{x\to-\infty} f_{\mathrm{II}}(x, z)/e_{\mathrm{II}}(x - z) = 1. \tag{1.7}$$

These are well-defined according to a theorem of [*7*]; moreover, variation of parameters yields something analogous to the Lippmann-Schwinger equation (F 1.5-1.6),

$$\phi(x, z) = a(x - z) + \int_x^\infty dy(a(x - z)e_{\mathrm{I,II}}(y - z) - a(y - z)e_{\mathrm{I,II}}(x - z))u(y)\phi(y, z), \quad (1.8)$$

and

$$f_{\mathrm{I,II}}(x, z) = e_{\mathrm{I,II}}(x - z) + \int_{-\infty}^x dy(a(y - z)e_{\mathrm{I,II}}(x - z) - a(x - z)e_{\mathrm{I,II}}(y - z))u(y)f_{\mathrm{I,II}}(y, z). \quad (1.9)$$

The first of these, (1.8), represents two distinct equations for the function ϕ, while (1.9) represents two equations, one for each of the functions f_{I} and f_{II}. We next prove some technical facts about the solution ϕ and its related Airy function:

LEMMA I.1. — Let

$$\tilde{a}(x) = \begin{cases} a(x), & x > 0 \\ \sup\limits_{y \le x} | a(x) |, & x < 0. \end{cases}$$

Then

$$| \phi(x, z) - a(x, z) | \le C(z)\tilde{a}(x - z),$$

where $C(z) \downarrow 0$ as $z \to \pm \infty$ (independently of x).

Remark. — The function $\tilde{a}$ is the positive envelope of a, which is oscillatory for $x < 0$. The lemma implies that for large $| z |$, the ratio of ϕ to a goes to 1 uniformly outside small intervals containing the zeroes of a.

Proof. — Fix z temporarily and regard

$$\Phi \equiv \phi(x, z)/\tilde{a}(x - z)$$

as an element of $C(\mathbb{R})$, the continuous functions with the supremum norm. From (1.8),

$$\Phi = a/\tilde{a} + T\Phi,$$

where

$$T\Phi \equiv \frac{a(x - z)}{\tilde{a}(x - z)} \int_x^\infty e_{\mathrm{I}}(y - z)\tilde{a}(y - z)u(y)\Phi(y, z)dy + \frac{e_{\mathrm{I}}(x - z)}{\tilde{a}(x - z)} \int_x^\infty a(y - z)\tilde{a}(y - z)u(y)\Phi(y, z)dy$$

Since $\Phi = (1 - T)^{-1}(a/\tilde{a}) = \sum\limits_{k=0}^\infty T^k(a/\tilde{a})$ if $|| T ||_{\mathrm{op}} < 1$, it suffices to bound the operator norm of T. It is at most:

$$\int_{-\infty}^\infty dy \,| e_{\mathrm{I}}(y - z)\tilde{a}(y - z)u(y) | + \sup_x \left| \frac{e_{\mathrm{I}}(x - z)}{\tilde{a}(x - z)} \right| \int_x^\infty | a(y - z)\tilde{a}(y - z)u(y) | \, dy$$

$$\equiv \int_{-\infty}^\infty dy \,| e_{\mathrm{I}}(y - z)\tilde{a}(y - z)u(y) | + \sup_x B(x, z). \quad (1.10)$$

For x real the functions $|e_1(x)\tilde{a}(x)|$ and $|a(x)\tilde{a}(x)|$ are bounded pointwise by constant times $(1+|x|)^{-1/2}$, so

$$\int_{-\infty}^{\infty} |e_1(y-z)\tilde{a}(y-z)u(y)|\,dy$$

and

$$\int_{-\infty}^{\infty} |a(y-z)\tilde{a}(y-z)u(y)|\,dy \to 0 \qquad \text{as} \qquad z \to \pm\infty$$

by the Lebesgue dominated convergence theorem. (If u is bounded by $(1+|x|)^{-1}$ but is not absolutely integrable, a partition of the interval of integration into $|x-z| < |z|^{1/2}$ and $|x-z| > |z|^{1/2}$ is the easiest way to show that $\int (1+|y-z|)^{-1/2}(1+|y|)^{-1}dy \to 0$.) This shows that the first integral of (1.10) approaches 0, and, since

$$\sup_{x \le z+c} \left| \frac{e_1(x-z)}{\tilde{a}(x-z)} \right| = \sup_{x \le c} \left| \frac{e_1(x)}{\tilde{a}(x)} \right| < \infty\,,$$

that $\sup\limits_{x \le z+c} \mathrm{B}(x, z) \to 0$ as $z \to \infty$ for any fixed c. We take c large enough that if $x \geq c$, $|e_1(x)|$ increases and $a(x)$ decreases monotonically in x. Then

$$\begin{aligned} \sup_{x \ge z+c} \mathrm{B}(x, z) &\le \sup_{x \ge z+c} \int_{x}^{\infty} |e_1(y-z)a(y-z)u(y)|\,dy \\ &\le \int_{-\infty}^{\infty} |e_1(y-z)a(y-z)u(y)|\,dy \to 0 \end{aligned}$$

as before. □

Corollary I.2. — For any fixed x,

$$\phi(x, z) - a(x-z) \in \mathrm{L}^2(\mathbb{R}, dz)\,.$$

Remark. — Actually, we get $\mathrm{L}^{4/3+\varepsilon}$ for all $\varepsilon > 0$.

Proof. — $\phi - a$ is bounded, so we consider its behavior for large $|z-x|$. By (1.8) and Lemma I.1,

$$\begin{aligned} |\phi(x, z) - a(x-z)| \le |a(x-z)| &\int_{-\infty}^{\infty} |u(y)e_1(y-z)\tilde{a}(y-z)|\,dy \\ &+ |e_1(x-z)| \int_{x}^{\infty} |u(y)\tilde{a}^2(y-z)|\,dy\,. \end{aligned}$$

The first term is $0(|x-z|^{-1/4})$ times a convolution of $u \in \mathrm{L}^1$ (or $\mathrm{L}^{1+\varepsilon}$) with $\tilde{a}e_1 \in \mathrm{L}^{2+\epsilon}$, for arbitrarily small ε, so it is $0(|x-z|^{-1/4})$ times a function $\in \mathrm{L}^{2+\varepsilon}(\mathbb{R}, dz)$ by Young's inequality [*13*]. The Hölder inequality then makes the product in L^2 (in fact, $\mathrm{L}^{4/3+\varepsilon}$). A similar estimate takes care of the second term for $z \gg x$. For $z \ll x$, the second term is instead bounded by $(\sup_x |e_{\mathrm{I}}(x)a(x)|)\int_x^{\infty} |u(y)a(y-z)|\,dy$, which goes rapidly to 0 because $a(x)$ falls off exponentially fast for large, positive x. $\square$

LEMMA I.3. — As $|z| \to \infty$, arg z fixed different from 0 or π, if $u \in \mathrm{L}^1(\mathbb{R})$, then

$$|\phi(x, z) - a(x, z)| \leq \text{const.}\, |\operatorname{Im} z|^{-1/2}\tilde{a}(x, z),$$

where now

$$\tilde{a}(x, z) \equiv \sup_{y \geq x} |a(y-z)|.$$

Proof. — We argue as in Lemma I.1 (only use e_{II} in place of e_{I} in case $z \to \infty$ in the lower half-plane), and need to bound (1.10). In this case, since z is complex but x and y are real, it is not difficult to see that (1.10) is essentially bounded by

$$\|u\|_1 \sup_{y \in \mathbb{R}} |a(y-z)e_{\mathrm{I}}(y-z)| = 0(|\operatorname{Im} z|^{-1/2}). \qquad \square$$

If we now define

$$\mathrm{M}_{\mathrm{I,II}}(z) = \mathrm{W}\{\phi, f_{\mathrm{I,II}}\},$$
$$\mathrm{M}_{\mathrm{I}}(z) = \overline{\mathrm{M}_{\mathrm{II}}(\overline{z})},$$

then substitution from (1.8) and (1.9) yields (F 1.18)

$$\begin{aligned}\mathrm{M}_{\mathrm{I,II}}(z) &= 1 + \int_{-\infty}^{\infty} dyu(y)a(y-z)f_{\mathrm{I,II}}(y, z)\\ &= 1 + \int_{-\infty}^{\infty} dyu(y)e_{\mathrm{I,II}}(y-z)\phi(y, z).\end{aligned} \tag{1.11}$$

The functions $\mathrm{M}_{\mathrm{I,II}}(z)$ are entire, and M_{I} has zeroes only in the upper half plane. Accordingly, M_{II} has zeroes in the lower half plane, and we take them as our definition of resonances; this agrees with the definition by complex scaling.

The kernel $\mathrm{R}_z(x, y)$ of the inverse of the operator $-\dfrac{d^2}{dx^2} + f(x-z) + u(x)$ has a different functional form in the two half planes. The analytic structure of R in the variable z is that it has simple poles at the zeroes of $\mathrm{M}_{\mathrm{I,II}}$ and a cut on the whole real axis. Specifically (F 2.1),

$$\begin{aligned}\mathrm{R}_z^{\mathrm{I,II}}(x, y) &= \frac{f_{\mathrm{I,II}}(x_<, z)\phi(x_>, z)}{\mathrm{M}_{\mathrm{I,II}}(z)},\\ x_< &= \min(x, y)\\ x_> &= \max(x, y).\end{aligned} \tag{1.12}$$

The appropriate Green function for Im $z > 0$ is R_z^{I} and that for Im $z < 0$ is R_z^{II}. From Stone's Theorem, if P_i is the spectral projection for

$$-\frac{d^2}{dx^2} + fx + u$$

onto a real interval i, and if we denote integral operators with the same symbols as their kernels, then (with (1.4)),

$$\frac{1}{2}(P_{[a,b]} + P_{(a,b)}) = \frac{1}{2\pi i}\int_a^b (R_z^{\mathrm{I}} - R_z^{\mathrm{II}})d(fz) = \int_a^b dz \frac{\phi(x, z)\phi(y, z)}{M_{\mathrm{I}}(z)M_{\mathrm{II}}(z)}. \quad (1.13)$$

Thus the spectral function is

$$w(z) = \frac{1}{M_{\mathrm{I}}(z)M_{\mathrm{II}}(z)} = \frac{1}{|M_{\mathrm{I}}(z)|^2} \qquad \text{for } z \text{ real}, \quad (1.14)$$

and the continuum eigenfunctions are $\phi(x, z)/M_{\mathrm{I}}(z)$.

Since asymptotic integration of the Schrödinger equation shows that there are no square-integrable solutions as $x \to -\infty$, we conclude from the discussion above that the spectrum of (0.1) is purely absolutely continuous. (Non-one-dimensional versions of this fact have been proved in [*4*] [*5*].)

As a final part of the preparation for the equations of inverse Stark scattering, we collect some facts about the unitary transformation that performs the spectral resolution of $-\frac{d^2}{dx^2} + fx$, which has been dubbed the Airy transform by Widder [*13*]. It is defined by

$$A = \mathcal{F} \exp(-ik^3/3f)\mathcal{F}^{-1},$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform to a variable k, and $\exp(-ik^3/3f)$ is a multiplication operator. It is equivalent to convolution by a:

$$[Af](z) = \int_{-\infty}^{\infty} f(y)a(y - z)dy, \qquad \text{a. e.}$$

The properties of A follow readily from those of $\mathcal{F}$. For instance, with Sobolev's Theorem, if $f \in D(-d^2/dx^2) = W_2$, then $Af \in W_2 \subset C^1(\mathbb{R})$.

These facts can now be used to derive a Gel'fand-Levitan equation. Define $K(x, y)$ so that (F 4.3)

$$\phi(x, z) = a(x - z) + \int_x^{\infty} K(x, y)a(y - z)dy. \quad (1.15)$$

The existence and functional properties of $K(x, y)$, $y \geq x$, follow from the observation that the integral in (1.15) is the Airy transform in y of $K(x, y)$, interpreted as 0 for $y < x$.

THEOREM I.4. — For almost every $y \geq x$,

$$K(x, y) = \int_{-\infty}^{\infty} [\phi(x, z) - a(x - z)]a(y - z)dz . \qquad (1.16)$$

Proof. — This is just the inverse Airy transform of (1.15). By Corollary I.2, $\phi - a \in L^2(\mathbb{R}, dz)$, so for fixed x, $K \in L^2(\mathbb{R}, dy)$, since it is the result of three unitary operations on the function $\phi - a$. □

By substitution into the defining equations for ϕ and K, one finds in the standard way that u and K are related by

$$K(x, x) = \frac{1}{2}\int_{x}^{\infty} dyu(y) , \qquad (1.17)$$

and that K(x, y) satisfies the differential equation

$$K_{xx} = K_{yy} + (u(x) + fx - fy)K .$$

Define the operator U by

$$[Uf](x) = f(x) + \int_{x}^{\infty} K(x, y)f(y)dy .$$

Observe that U converts the generalized eigenfunctions a into ϕ. From (1.14) we obtain the operator equations (F 8.2)

$$I = (U^{-1})^*wU^{-1} = UwU^* ,$$
$$Uw = (U^*)^{-1} . \qquad (1.18)$$

Of course, U* is the operator

$$[U^*g](x) = g(x) + \int_{-\infty}^{x} K(y, x)g(y)dy .$$

LEMMA I.5. — The kernel of $(U^*)^{-1}$ is simply $\delta(x - y)$ when $y > x$.

Proof. — The kernel of $(U^*)^{-1}$—call it G(x, y)—solves

$$G(x, y) + \int_{-\infty}^{x} K(z, x)G(z, y)dz = \delta(x - y) ,$$

from which it is easy to see that G$(x, y) = \delta(x - y)$ when $y > x$. □

The Gel'fand-Levitan equation (F 8.5) follows from writing (1.18) out explicitly, using Lemma I.5. For $y > x$,

$$0 = K(x, y) + \Omega(x, y) + \int_{x}^{\infty} dtK(x, t)\Omega(t, y) , \qquad (1.19)$$

where

$$\Omega(x, y) = \int_{-\infty}^{\infty} dza(x - z)a(y - z)(w(z) - 1) . \qquad (1.20)$$

To summarize: given a spectral function $w(z)$, corresponding to a potential with our assumptions (guaranteeing that $w(z)$ is absolutely continuous), the prescription for recovering u is to solve (1.19) and (1.20) for K, then to differentiate according to (1.17).

We next discuss the asymptotics of the spectral function for z complex. For simplicity suppose that u is absolutely integrable and falls off faster than $1/x$ as $x \to \pm\infty$. In the example, where this fails, we have explicit control on $w(z)$. From (1.11) and Lemma 1.3, if $\operatorname{Im} z > 0$, then

$$| \mathrm{M}_{\mathrm{I}}(z) - 1 | \leq (1 + 0(|\operatorname{Im} z|^{-1/2}) \| u \|_1 \sup_y | e_{\mathrm{I}}(y - z) a(y - z) |$$
$$= 0(|\operatorname{Im} z|^{-1/2}) \quad \text{as} \quad \operatorname{Im} z \to \infty .$$

Moreover, for $\operatorname{Re} z \to \pm\infty$,

$$| \mathrm{M}_{\mathrm{I}}(z) - 1 | \leq \int_{|y-z| \leq \operatorname{Re} z/2} | u(y) e_{\mathrm{I}}(y - z) a(y - z) | \, dy$$
$$+ \int_{|y-z| \geq \operatorname{Re} z/2} | u(y) e_{\mathrm{I}}(y - z) a(y - z) | \, dy = 0(|\operatorname{Re} z|^{-1/2}) .$$

The net result is that $\mathrm{M}_{\mathrm{I}}(z) - 1 = 0(|z|^{-1/2})$ throughout the upper half plane. From Young's inequality, since $u \in \mathrm{L}^1$ and $ae_{\mathrm{I}} \in \mathrm{L}^{2+\varepsilon}$, it follows that $\mathrm{M}_{\mathrm{I}}(z) - 1 \in \mathrm{L}^{2+\varepsilon}$. It is not hard to see from this fact, the fall-off of $\mathrm{M}_{\mathrm{I}} - 1$, and (1.14) that also $w(z) - 1 = 0(|z|^{-1/2})$ and $w(z) - 1 \in \mathrm{L}^{2+\varepsilon}$. □

Nothing prevents $w(z) - 1$ from falling off faster than $|z|^{-1/2}$, and our experience seems to indicate that ordinarily it does. This is the case in the example to be discussed in section 2.

This section will conclude with a discussion of the properties of the kernel $\Omega(x, y)$ and the existence of a solution to the Gel'fand-Levitan equation (1.19). By (1.20), Ω is an Airy transform of alternatively

$$a(y - z)(w(z) - 1) \in \mathrm{L}^2(\mathbb{R}, dz) \qquad \text{or} \qquad a(x - z)(w(z) - 1) \in \mathrm{L}^2(\mathbb{R}, dz) ,$$

so $\Omega(x, y) \in \mathrm{L}^2(\mathbb{R}, dx)$ for each fixed y and $\in \mathrm{L}^2(\mathbb{R}, dy)$ for each fixed x. As the kernel of the operator $w - 1$, Ω represents a bounded linear transformation of $\mathrm{L}^2(\mathbb{R})$ to itself provided only that the function $w(z)$ is bounded. (This is automatic, of course, since the spectrum is absolutely continuous and w falls off at $\pm\infty$.) If for real z, $w(z) < 2$, then $\| w - 1 \|_{\mathrm{op}} < 1$, and the Gel'fand-Levitan equation can be solved uniquely for $\mathrm{K}(x, y)$ by the Neumann series; (1.19) should be thought of as an integral equation for a square-integrable function of y with x as a parameter. Alternatively, if $w(z) - 1$ falls off as $|z|^{-3/2-\varepsilon}$, $\varepsilon > 0$, then $\Omega(t, y)$ becomes a Hilbert-Schmidt kernel on $[x, \infty)$ for each x, and the nonexistence of a homogeneous solution can be shown as in Faddeev [*10*] section 9, with minor modifications. The analytic Fredholm theorem then guarantees the existence of $\mathrm{K}(x, y) \in \mathrm{L}^2(\mathbb{R}, dy)$ for each x. We shall not attempt to find optimal conditions for the existence and uniqueness of $\mathrm{K}(x, y)$.

Some additional technical conditions on $w(z)$ are necessary to ensure that K be differentiable. For instance, suppose that

$$\frac{d^2}{dz^2} a(x - z)(w(z) - 1) \in L^2(\mathbb{R}, dz)$$

for all x—owing to the rapid oscillations of a this means essentially that $w(z) - 1$ falls off faster than $z^{-5/4}$ as $z \to +\infty$ and that the derivatives fall off by additional powers of z. Then with Sobolev's lemma and the fact that A^{-1} transforms W_2 into itself, one finds that

a) $\Omega(x, y) \in C^1 \cap L^2$ in either variable with the other fixed, and
b) $\Omega_x(x, y) \subset C^1 \cap L^2(\mathbb{R}, dy)$.

Then from the Gel'fand-Levitan equation (1.19) and its differentiated versions, we find that $K \in C^1$ in both x and y, so $u(x)$ is well-defined.

In the following section we discuss an example where we have fairly explicit control on Ω, and will thus not have to check such general conditions.

II. AN EXAMPLE WITH AN APPROXIMATE SOLITARY-WAVE SOLUTION OF CYLINDRICAL KdV

We now make an ansatz: suppose that $w(z) - 1$ is an analytic function the only singularities of which are a pair of complex-conjugate simple poles. For definiteness, for $\operatorname{Im} z_0 > 0$,

$$w(z) = 1 + \frac{\alpha}{z - z_0} + \frac{\bar{\alpha}}{z - \bar{z}_0}, \tag{2.1}$$

which is positive for $|\alpha|$ small enough and z real. We can then solve for Ω with the aid of contour integration, and for consistency must verify that everything falls off properly at ∞. We also set $f = 1$ from now on.

From (1.20) with $y > x$,

$$\begin{aligned}\Omega(x, y) &= \int_{-\infty}^{\infty} a(x - z)a(y - z)\left(\frac{\alpha}{z - z_0} + \frac{\bar{\alpha}}{z - \bar{z}_0}\right)dz \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} (e_{\mathrm{I}}(x - z) - e_{\mathrm{II}}(x - z))a(y - z)\left(\frac{\alpha}{z - z_0} + \frac{\bar{\alpha}}{z - \bar{z}_0}\right)dz .\end{aligned}$$

The contour of the integral containing e_{I} can be closed in the upper half plane of z, and the contour containing e_{II} in the lower half-plane. By the residue theorem,

$$\begin{aligned}\Omega(x, y) &= \alpha a(y - z_0)e_{\mathrm{I}}(x - z_0) + \bar{\alpha} a(y - \bar{z}_0)e_{\mathrm{II}}(x - \bar{z}_0) \\ &= 2 \operatorname{Re}(\alpha a(y - z_0)e_{\mathrm{I}}(x - z_0))\end{aligned}$$

since x and y are real. If $y < x$, then similarly

$$\Omega(x, y) = 2 \operatorname{Re} (\alpha a(x - z_0)e_I(y - z_0)),$$

so in all

$$\Omega(x, y) = 2 \operatorname{Re} (\alpha a(x_> - z_0)e_I(x_< - z_0)). \tag{2.2}$$

Observe that $\Omega(x, y)$ is exponentially decreasing in each variable at ∞, bounded by a constant, and smooth except at $x = y$. It is just the real part of something like a Green function, and is in fact Hilbert-Schmidt:

PROPOSITION II.1. — For any $x > -\infty$,

$$\int_x^\infty dt \int_x^\infty dy \, | \Omega(t, y) |^2 < \infty .$$

Proof. — This integral is bounded essentially by a constant times

$$\begin{aligned}
&\int_x^\infty dt \int_t^\infty dy \, | a^2(y - z_0)e_I^2(t - z_0) | \\
&\qquad \leq \text{const.} \int_x^\infty \left| e_I^2(t - z_0)(t - z_0)^{-1} \exp\left(-\frac{4}{3}(t - z_0)^{3/2} \right) \right| dt \\
&\qquad \leq \text{const.} \int_x^\infty (1 + | t - z_0 |)^{-3/2} dt < \infty . \qquad \square
\end{aligned}$$

COROLLARY II.2. — For any $x > -\infty$, the function $K(x, y)$ exists and is unique.

Proof. — This follows automatically from II.1 as in Faddeev's article [*10*]. $\square$

Now consider what happens if the imaginary part of z_0 is decreased to zero with α imaginary. We shall see that the kernel Ω approaches the one of Calogero and Degasperis [*6*] pointwise, i. e., a constant times

$$a(x - z_0)a(y - z_0),$$

which, however, is no longer exponentially decreasing or even square-integrable in y for fixed x; it falls off as $| y - z_0 |^{-1/4}$ times an oscillatory factor, and the corresponding $u(x)$ fails to satisfy the assumptions guaranteeing the validity of the inverse scattering formalism.

If $| \alpha |$ is taken small enough and z_0 is kept nonreal, then, since

$$\int_{-\infty}^\infty dt \, | \Omega(t, y) | \qquad \text{and} \qquad \int_{-\infty}^\infty dt \left| \frac{\partial}{\partial y} \Omega(t, y) \right|$$

are each bounded by a constant times $| \alpha |$ independently of y, the operator norm of Ω acting in the variable y on $C^1 [x, \infty)$ (with norm $\| f \|_\infty + \| f' \|_\infty$)

is less than 1. Therefore $K(x, y)$ is continuously differentiable in the variable $y \geq x$, uniformly in x, and equals the convergent Neumann series

$$K(x, y) = -\Omega(x, y) + \int_x^\infty \Omega(x, t)\Omega(t, y)dt - \int_x^\infty dt_1 \int_x^\infty dt_2 \Omega(x, t_1)\Omega(t_1, t_2)\Omega(t_2, y) + \dots \quad (2.3)$$

The series on the right when differentiated by x also converges to $\frac{\partial K(x, y)}{\partial x}$, and thus $u(x) = \frac{1}{2}\frac{d}{dx}K(x, x)$ is well-defined as an element of $C^1(\mathbb{R})$. □

LEMMA II.3. — With Ω as above and $|\alpha|$ small enough,

$$|K(x, x)| < \text{const.}\ (1 + |x|)^{-1/2}.$$

Proof. — The asymptotic formulae (1.5) proved a bound of this form for $|\Omega(x, x)|$, and when $y = x$ the other terms of (2.3) are bounded by terms such as

$$|e_1(x - z_0)|^2 (\sup_{y \geq x} |a(y - z_0)|)^2 \left[\sup_{t_1} \int_{-\infty}^\infty |\Omega(t_1, t_2)| dt_2\right]^{n-1} = |e_1(x - z_0)|^2 (\sup_{y \geq x} |a(y - z_0)|)^2 \cdot 0(|\alpha|^{n-1}).$$

Thus the sum of the terms other than $\Omega(x, x)$ is bounded pointwise by a constant times

$$|e_1(x - z_0)|^2 \sup_{y \geq x} |a(y - z_0)|^2 \leq \text{const.}\ |x - z_0|^{-1}. \quad \square$$

PROPOSITION II.4. — The potential $u(x)$ belongs to the class of potentials for which the inverse-scattering formalism has been justified; in particular, it is continuously differentiable, and

$$|u(x)| \leq \text{const.}\ (1 + |x|)^{-1}.$$

Proof. —

$$|u(x)| = \left|\frac{1}{2}\frac{d}{dx}K(x, x)\right| \leq \frac{1}{2}\left|\frac{d}{dx}\Omega(x, x)\right| + \frac{1}{2}|K(x, x)\Omega(x, x)| + \frac{1}{2}\int_x^\infty \left(\left|\frac{\partial K}{\partial x}(x, t)\Omega(t, x)\right| + \left|K(x, t)\frac{\partial}{\partial x}\Omega(t, x)\right|\right)dt$$

by (1.19). The first two terms are $0(|x - z_0|^{-3/2})$ and $0(|x|^{-1})$ respectively because of Lemma II.3 and the Airy-function estimates (1.5). From (2.3) the integrated terms are bounded by sums of terms such as

$$\Omega(x, x)\Omega(x, t_1) \dots \Omega(t_n, x)$$

and

$$\int_x^{\infty} dt_1 \ldots \int_x^{\infty} dt_n \frac{\partial \Omega(x, t_1)}{\partial x} \Omega(t_1, t_2) \ldots \Omega(t_{n-1}, t_n)\Omega(t_n, x)$$
$$= \int_x^{\infty} dt_1 \ldots \int_x^{\infty} dt_n \Omega(x, t_1)\Omega(t_1, t_2) \ldots \Omega(t_{n-1}, t_n) \frac{\partial \Omega(t_n, x)}{\partial x}.$$

From (1.5) we glean the estimate

$$e_I(x - z_0)e_I'(x - z_0)\int_x^{\infty} a^2(y - z_0)dy = 0(|x - z_0|^{-1}),$$

with which the integrated terms can be bounded by

$$0(|x - z_0|^{-1}) \cdot \sum_n (\text{const.}\,|\alpha|)^n = 0(|x - z_0|^{-1}).$$

The fact that $u(x) \in C^1$ follows from insertion of the differentiated Neumann series for K into the formula for u; the gravest effect of differentiation is to produce additional factors of $|x - z_0|^{1/2}$ from the exponential factors in the Airy functions. As a consequence of the $|x - z_0|^{-1}$ fall-off just demonstrated, the series for u' and even u'' remain convergent to bounded functions. □

At this point we can take over the computations of Calogero and Degasperis [6] wholesale. They have shown that if the validity of the inverse scattering method is granted, then some of the solutions of the cylindrical KdV equation (0.2) are given by the form-preserving expression

$$q(x, t) = (12t)^{-1/3}u((12t)^{-1/3}x\,;\,t), \tag{2.4}$$

where $u(x\,;\,t)$ is the solution to the inverse-scattering problem for a parametrized spectral function $w(z\,;\,t) = w(z(t/t_0)^{1/3})$. For convenience we set the arbitrary constant t_0 to 1. If $w(z)$ has the specific form (2.1), then

$$\begin{aligned} w(z\,;\,t) &= 1 + \frac{\alpha}{zt^{1/3} - z_0} + \frac{\bar{\alpha}}{zt^{1/3} - \bar{z}_0} \\ &= 1 + \frac{\alpha t^{-1/3}}{z - z_0 t^{-1/3}} + \frac{\bar{\alpha} t^{-1/3}}{z - \bar{z}_0 t^{-1/3}}. \end{aligned} \tag{2.5}$$

In other words, $w(z\,;\,t)$ has the same functional form as $w(z\,;\,1) = w(z)$, but with $\alpha \to \alpha t^{-1/3}$ and $z_0 \to z_0 t^{-1/3}$.

The kernel function at time t is thus

$$\Omega(x, y\,;\,t) = 2\,\mathrm{Re}\,(\alpha t^{-1/3} a(x_> - z_0 t^{-1/3}) e_I(x_< - z_0 t^{-1/3})).$$

Suppose α is imaginary, $\alpha = i\rho$, $\rho \in \mathbb{R}$. Then for each x,

$$\lim_{t \to \infty} t^{1/3}(\Omega(x, y\,;\,t) - \Omega_{CD}(x, y\,;\,t)) = 0,$$

in the sense of functions of y belonging to $C^0(\mathbb{R})$, $C^1[x, \infty)$, or $L^1[x, \infty)$, where

$$\Omega_{CD} = -2\pi\rho t^{-1/3} a(x - \operatorname{Re} z_0 t^{-1/3}) a(y - \operatorname{Re} z_0 t^{-1/3})$$

(cf. (1.4)), is the kernel function of Calogero and Degasperis. The limits in $C^1[x, \infty)$ and $L^1[x, \infty)$ are not uniform in x as $x \to -\infty$, since $\frac{\partial}{\partial y}\Omega_{CD}$ fails to have a limit as $y \to -\infty$, and $\Omega_{CD} \notin L^1(\mathbb{R}, dy)$, whereas $\frac{\partial}{\partial y}\Omega \to 0$ and $\Omega \in L^1(\mathbb{R}, dy)$. The L^1 convergence, however, ensures the convergence of the series expansion of the operator

$$\left(1 + \int_x^\infty dt\Omega(t, y)\cdot\right)^{-1}$$

on $\dot{C}^1[x, \infty)$ in such a way that for each x,

$$K(x, y; t) = -\left(1 + \int_x^\infty dt\Omega(t, y)\cdot\right)^{-1}\Omega(x, y)$$

$$\to K_{CD}(x; y; t) = 2\pi\rho t^{-1/3} \frac{a(x - \operatorname{Re} z_0 t^{-1/3}) a(y - \operatorname{Re} z_0 t^{-1/3})}{1 + \int_x^\infty a^2(r - \operatorname{Re} z_0 t^{-1/3}) dr}$$

in the sense that

$$\| K(x, y; t) - K_{CD}(x, y; t) \|_{op, C^1[x,\infty)} = o(t^{-1/3}).$$

It happens that the convergence of the potentials is uniform on $-\infty < x < \infty$, because a cancellation in the derivatives ensures that

$$u_{CD}(x; t) = -\pi\rho t^{-1/3} \frac{d}{dx}\left[\frac{a^2(x - \operatorname{Re} z_0 t^{-1/3})}{1 + \int_x^\infty a^2(r - \operatorname{Re} z_0 t^{-1/3}) dr}\right]$$

falls off as $|x|^{-1/2}$ as $x \to -\infty$ (cf. [*6*]). The solution to (0.2) found by Calogero and Degasperis,

$$q_{CD}(x, t) = (12t)^{-1/3} u_{CD}((12t)^{-1/3} x; t)$$

is consequently the limit of our

$$q(x, t) = (12t)^{-1/3} u((12t)^{-1/3} x; t)$$

in the sense that

$$\sup_x | q(x, t) - q_{CD}(x, t) | = o(t^{-1/3}),$$

whereas separately $\sup_x | q(x, t) |$ and $\sup_x | q_{CD}(x, t) |$ are at most $o(t^{-1/3})$.

The behaviour of the solutions of the nonlinear wave equation (0.2) is as follows. The solution q_{CD} is soliton-like in that it preserves its shape as $t \to \infty$, though it broadens and has a decreasing amplitude. Clearly a logarithmic change of variables, $\xi = \ln x$, $\tau = \ln (12t)^{-1/3}$, converts it into a traveling wave solution, depending only on $\xi - \tau$, of another, somewhat less reasonable nonlinear equation. The amplitudes of some of the new quasisolitary waves constructed in this article as solutions of (0.2) decrease at the same rate as that of q_{CD}, and their shapes become that of q_{CD} asymptotically in some sense (e. g., uniform convergence of $(12t)^{1/3}q$ on compacts in a variable $x' = (12t^{-1/3}x)$ as $t \to \infty$). Of course, $q \in L^{1+\varepsilon}$ whereas q_{CD} belongs only to $L^{2+\varepsilon}$, so the tails are different. (For general information on nonlinear wave equations, solitary waves, etc., see [*14*].)

There is a second connection between the new quasisolitary solutions and q_{CD}, viz., that for fixed t, as $\operatorname{Im} z_0 \downarrow 0$, $\operatorname{Re} z_0$ fixed, $\Omega(x, y)$, $K(x, y)$, and $q(x)$ approach the related quantities of Calogero and Degasperis pointwise. Hence, although their solution of (0.2) makes use of a technically invalid inverse-scattering formalism, it can be regarded as a limiting case of solutions connected with Stark resonances. It is natural to conjecture that there is a one-to-one correspondence between sets of resonances with positions and residues scaling according to (2.5) and square-integrable solutions of (0.2) that approach form-preserving solutions, which fail to be square-integrable. The connection should not be special to the problem of the Stark effect, but may also hold for ordinary resonances in other situations, such as the Auger effect.

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