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Section A :
Physique théorique.

Asymptotic completeness for the impact parameter approximation to three particle scattering

by

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ABSTRACT. — We prove a theorem of Yajima on the asymptotic completeness of the impact parameter model in three body scattering. The proof depends on developing suitable analogs of the Faddeev equations which allow one to study time dependent Hamiltonians directly. With these equations for the propagator, the physics of the problem is more transparent than it is in Yajima's proof. As a corollary of our methods we show that for high impact velocities, the Faddeev-Watson multiple scattering series is convergent for the impact parameter approximation.

1. INTRODUCTION

The impact parameter model in three body quantum scattering is supposed to approximate a system in which the masses of two of the three particles are very much greater than the mass of the third. The extremely massive particles are assumed to move classically in straight lines with constant velocities. The motion of the third particle is then determined by the Schrödinger equation with the time dependent effective potential which is generated by the very massive particles.

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This model is of interest because calculations involving the full three body system are prohibitively complicated [*12*] [*13*], and because charge transfer reactions such as $He^{++} + H \to He^{+} + H^{+}$ should be fairly accurately described by the approximate treatment (especially at high impact velocities).

The time dependent Hamiltonian for the particle of small mass in the impact parameter model is

$$H(t) = H_0 + V_1(x - a_1 - v_1 t) + V_2(x - a_2 - v_2 t),$$

where $H_0 = -\dfrac{1}{2m}\Delta$. The positions of the very massive particles 1 and 2 at time t are given by $a_1 + v_1 t$ and $a_2 + v_2 t$, respectively, and particle 3 interacts with particles 1 and 2 via the two-body potentials V_1 and V_2, respectively. To avoid trivialities, we assume $v_1 \neq v_2$.

For reasonable choices of V_1 and V_2 which decay sufficiently rapidly at infinity, physical arguments suggest that there are only three possible types of motion which particle 3 can exhibit as $t \to \pm\infty$. It can behave like a free particle; it can be bound to particle 1; or it can be bound to particle 2. If this is indeed correct, the scattering is called asymptotically complete.

Yajima [*18*] has proved asymptotic completeness of this model for a large class of potentials V_1 and V_2 in three or more dimensions. His proof relies on a very clever method of Howland [*7*] which involves the study of a time independent Hamiltonian in a higher number of dimensions. Although this method yields a proof of asymptotic completeness, we feel that the method completely obscures the physics of the problem. The purpose of this paper is to provide a proof of Yajima's theorem which is based on a method which more clearly illustrates the physics.

The two conditions we impose on V_1 and V_2 coincide with those used by Yajima. The first hypothesis is:

$$\text{(H1)}\quad \begin{cases} V_j(x) = (1 + x^2)^{-\delta}(W_{j,1}(x) + W_{j,2}(x)), \text{ where} \\ W_{j,1} \text{ and } W_{j,2} \text{ are real valued functions on} \\ \mathbb{R}^n \text{ for some } n \geqslant 3, \text{ such that } 1 < \delta < 3/2, \\ W_{j,1} \in W^{1,s}(\mathbb{R}^n) \text{ for some } s \in (n/2, n), \text{ and} \\ W_{j,2} \in W^{1,\infty}(\mathbb{R}^n). \end{cases}$$

Here $W^{1,q}(\mathbb{R}^n)$ denotes the Sobolev space [*19*] of $L^q(\mathbb{R}^n)$ functions whose weak first partial derivatives belong to L^q.

If V_1 and V_2 satisfy (H1), then $W_{j,1} \in L^p(\mathbb{R}^n)$ for $p = (1/s - 1/n)^{-1} > n$ and $V_j \in L^q(\mathbb{R}^n)$ for $n/2\delta < q \le p$. Furthermore, $A_j(x) = (1+x^2)^{-\delta/2} \cdot V_j(x)$ and $B_j(x) = (1 + x^2)^{-\delta/2}$ both belong to L^q for $n/\delta < q \le p$. These results imply that V_1 and V_2 are H_0-compact operators, from which it follows that $H_1(t)=H_0+V_1(x-a_1-v_1t)$, $H_2(t)=H_0+V_2(x-a_2-v_2t)$, and $H(t) = H_0 + V_1(x - a_1 - v_1t) + V_2(x - a_2 - v_2t)$ are self-adjoint on the domain of H_0. Furthermore ([14], Theorem II, 27; [18]), these hypotheses guarantee that $H_1(t)$, $H_2(t)$ and $H(t)$ generate strongly continuous unitary propagators $U_1(t, s)$, $U_2(t, s)$ and $U(t, s)$, respectively. For $H(t)$ this means

1) $U(t, s)$ is unitary on $L^2(\mathbb{R}^n)$ and is strongly continuous in (t, s);
2) $U(t, s) = U(t, r)U(r, s)$ for $-\infty < t, r, s < \infty$;
3) if $\psi \in D(H_0^{\frac{1}{2}})$, then $U(t, s)\psi \in D(H_0^{\frac{1}{2}})$ and $i(\partial/\partial t)U(t, s)\psi = H(t)U(t, s)\psi$, where the derivative is understood as the strong derivative in the space $\mathcal{H}_{-1} = \mathcal{H}^*_{+1}$ = dual space of $W^{1,2}(\mathbb{R}^n)$.

The analogous statements hold for $H_1(t)$ and $H_2(t)$.

Our second condition on V_1 and V_2 is that $H_1(t)$ and $H_2(t)$ have no non-negative eigenvalues or resonances. To state this more precisely we first note ([3], Proposition 3.1) that $M_j(z) = A_j(z - H_0)^{-1}B_j$ is an analytic compact operator valued function for $z \in \mathbb{C}\setminus[0, \infty)$ which is norm continuous in the closed cut plane. As $|z| \to \infty$, $\|M_j(z)\| \to 0$, and we denote $M_j(E \pm i0) = \lim_{\varepsilon\to 0} M_j(E \pm i\varepsilon)$ for $E \in [0, \infty)$. Our second hypothesis is

$$\text{(H2)}\begin{cases} 1 \notin \sigma(M_j(E \pm i0)) \quad \text{for} \quad E \in [0, \infty) \\ \text{and} \quad j = 1, 2\,. \end{cases}$$

Using methods of Agmon [1], one can show that if (H1) is satisfied, then $1 \in \sigma(M_j(E \pm i0))$ for some $E > 0$ implies that $H_j(t)$ has a positive eigenvalue E (for all t). There are regularity conditions ([11], Section XIII, 13) which may be imposed on the potentials to guarantee that these eigenvalues are absent. However, (H2) will fail at $E = 0$ for some elements of any reasonable vector space of potentials, but this failure of (H2) is non-generic if (H1) is satisfied. For example, if V_j is replaced by λV_j and (H1) is satisfied, then $1 \in \sigma(M_j(0))$ only for a discrete set of $\lambda \in \mathbb{R}$.

If (H1) and (H2) are satisfied, then $H_1(t)$ and $H_2(t)$ have only finitely many eigenvalues. These eigenvalues are isolated, negative, independent of t, and have finite multiplicities. The projection $P_j^{(0)}(t)$ onto the span of the eigenfunctions of $H_j(t)$ satisfies $\|(1 + x^2)^\gamma P_j^{(0)}(t)(1 + x^2)^\gamma\| < \infty$ for any $\gamma \in \mathbb{R}$. For proofs of these facts, see [11]. We define

$$P_j(t) = e^{im\, v_j x} P_j^{(0)}(t) e^{-im\, v_j x}\,.$$

Our main result is the following:

THEOREM 1.1. — If (H1) is satisfied then the following strong limits exist:

$$\Omega_0^\pm(s) = s\text{-}\lim_{t\to\mp\infty} U(s, t)e^{-i(t-s)H_0}$$

$$\Omega_1^\pm(s) = s\text{-}\lim_{t\to\mp\infty} U(s, t)U_1(t, s)P_1(s)$$

$$\Omega_2^\pm(s) = s\text{-}\lim_{t\to\mp\infty} U(s, t)U_2(t, s)P_2(s)\,.$$

The ranges of $\Omega_0^+(s)$, $\Omega_1^+(s)$ and $\Omega_2^+(s)$, are mutually orthogonal, as are the ranges of $\Omega_0^-(s)$, $\Omega_1^-(s)$ and $\Omega_2^-(s)$. If in addition (H2) is satisfied, then asymptotic completeness holds in the strong sense that

$$L^2(\mathbb{R}^n) = \mathrm{Ran}\ \Omega_0^\pm(s) \oplus \mathrm{Ran}\ \Omega_1^\pm(s) \oplus \mathrm{Ran}\ \Omega_2^\pm(s)$$

for all values of s.

Remarks 1. — Physical arguments indicate that hypothesis (H2) should not be needed, but we have not yet been able to prove the theorem without it.

2. The only difficult part of the theorem is the last statement, which is the only result we will prove. Yajima's proofs of the other statements are straight forward and very simple ([*18*], p. 158-159), and we have no improvements to suggest.

3. Yajima states his results in a somewhat more general form. First of all, he allows N extremely massive particles where we have taken only 2. Our proof easily generalizes. Second, he requires that the extremely massive particles move in straight lines only in the distant past and far future. The trajectories may be curved for some finite period of time. This generalization is a trivial one (see Lemma 1 of [*18*]).

4. To prove asymptotic completeness we will develop time dependent analogs of the three body Faddeev equations [*2*] [*3*] [*5*] [*7*] [*16*] by summing graphs as in [*5*]. These equations allow us to view the motion of particle 3 generated by $H(t)$ as a series in which each term corresponds to a sequence of collisions of particle 3 alternately with particles 1 and 2. Furthermore, after a large time we show that it is unlikely for particle 3 to encounter V_1, then later encounter V_2, and then later encounter V_1 again. Similarly, the sequence V_2 followed by V_1, followed by V_2 is unlikely for large times. Since these multiple scatterings are unlikely the Faddeev series for the adjoints of wave operators $\Omega_i^-(s)^*\psi$, $i = 0, 1, 2$, converges if s is sufficiently large and ψ belongs to a dense set. From this we can conclude the asymptotic completeness.

As a corollary of the proof of Theorem 1.1 we obtain the following theorem, which may very well be useful for answering some questions raised in [*12*].

THEOREM 1.2. — Choose V_1 and V_2 so that hypotheses (H1) and (H2) are satisfied. Then for sufficiently large values of $|v_1 - v_2|$, the Faddeev series for $\langle \psi, \Omega_i^\pm(0)\phi \rangle$ is convergent for all $\phi, \psi \in L^1 \cap L^2$.

Remarks 1. — The Faddeev series for $\langle \psi, \Omega_i^\pm(0)\phi \rangle$ is quite complicated (see Section 5 for formulas), but we hope that the first few terms can be computed numerically. The terms involved are far less complicated in the impact parameter model than in the full three body problem [*13*].

2. Theorem 1.2 is not surprising. It is physically very reasonable, and its analog in the full three body problem is true [*6*].

3. Of course Theorem 1.2 remains true if we replace $\langle \psi, \Omega_i^{\pm}(0)\phi \rangle$ by $\langle \psi, \Omega_i^{+}(s)\phi \rangle$ for any s.

The paper is organized as follows. In Section 2 we derive the analogs of the Faddeev equations. In Section 3 we prove the estimates which show that the Faddeev series converges for suitable large times and that multiple scatterings are unlikely. Asymptotic completeness is then proved in Section 4. Theorem 1.2 is proved in Section 5.

2. TIME DEPENDENT FADDEEV EQUATIONS

In this section we will formally derive an infinite series expression for $U(t, s)\psi$. Given the estimates from Section 3, an argument presented at the end of this section establishes the convergence of the series to $U(t, s)\psi$ for all $\psi \in \mathcal{S}$ if $t > s$ and $s > 0$ is sufficiently large or if $t < s$ and $-s > 0$ is sufficiently large.

This series is analogous to the series for $e^{-itH}\psi$ in the full three body problem which is obtained by formally Fourier transforming the iterates of the Faddeev equations for $(z - H)^{-1}\psi$ with respect to the variable z.

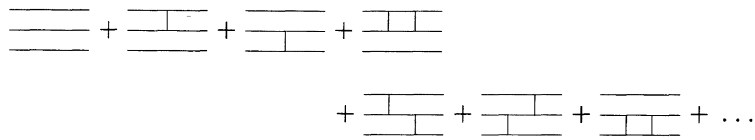
The simplest way to formally derive our series is to use a graphical symbolism similar to that introduced by Weinberg [*17*] (see also [*5*] [*14*]). The formal Dyson expansion for $U(t, s)$ is

$$\begin{aligned} U(t, s) = U_0(t, s) &- i\int_s^t U_0(t, r)V_1(r)U_0(r, s)dr - i\int_s^t U_0(t, r)V_2(r)U_0(r, s)dr \\ &- \int_s^t dr_1 \int_s^{r_1} dr_2 U_0(t, r_1)V_1(r_1)U_0(r_1, r_2)V_1(r_2)U_0(r_2, s) \\ &- \int_s^t dr_1 \int_s^{r_1} dr_2 U_0(t, r_1)V_1(r_1)U_0(r_1, r_2)V_2(r_2)U_0(r_2, s) \\ &- \int_s^t dr_1 \int_s^{r_1} dr_2 U_0(t, r_1)V_2(r_1)U_0(r_1, r_2)V_1(r_2)U_0(r_2, s) \\ &- \int_s^t dr_1 \int_s^{r_1} dr_2 U_0(t, r_1)V_2(r_1)U_0(r_1, r_2)V_2(r_2)U_0(r_2, s) \\ &+ \dots, \end{aligned} \tag{2.1}$$

where $U_0(t, s) = e^{-i(t-s)H_0}$.

If we identify each term in this series with a graph in which the vertical links represent V_j's and the horizontal lines represent free propagation of particles, then we can identify the above series with the sum of all graphs:

$U(t, s) =$



Here the top horizontal line represents particle 1 (infinitely massive); the bottom horizontal line represents particle 2 (infinitely massive); and the middle line represents the particle whose evolution is governed by $U(t, s)$.

To each non-trivial graph G whose associated operator is

$$(-i)^n \int_s^t dr_1 \int_s^{r_1} dr_2 \ldots \int_s^{r_{n-1}} dr_n U_0(t, r_1) V_{j_1}(r_1) U_0(r_1, r_2) \ldots U_0(r_{n-1}, r_n) V_{j_n}(r_n) U_0(r_n, s),$$

we can associate a number $N(G) = \sum_{k=1}^{n-1} (1 - \delta(j_k, j_{k+1}))$. In some sense this number represents the degree of connectivity of the graph G. We can resum (2.1) by first summing all graphs G with the same N(G), then summing over all possible N(G)'s, and then adding on the trivial graph $U_0(t, s)$.

The sum of all non-trivial graphs with $N(G) = 0$ is

$$-i \int_s^t dr U_1(t, r) V_1(r) U_0(r, s) - i \int_s^t dr U_2(t, r) V_2(r) U_0(r, s).$$

The sum of all non-trivial graphs with $N(G) = 1$ is

$$- \int_s^t dr_1 \int_s^{r_1} dr_2 U_1(t, r_1) V_1(r_1) U_2(r_1, r_2) V_2(r_2) U_0(r_2, s)$$
$$- \int_s^t dr_1 \int_s^{r_1} dr_2 U_2(t, r_1) V_2(r_1) U_1(r_1, r_2) V_1(r_2) U_0(r_2, s).$$

For general n, the sum of all graphs with $N(G) = n$ is

$$(-i)^{n+1} \int_s^t dr_1 \int_s^{r_1} dr_2 \ldots \int_s^{r_n} dr_{n+1} U_1(t, r_1) V_1(r_1) U_2(r_1, r_2) V_2(r_2) \ldots V_j(r_{n+1}) U_0(r_{n+1}, s)$$
$$+ (-i)^{n+1} \int_s^t dr_1 \int_s^{r_1} dr_2 \ldots \int_s^{r_n} dr_{n+1} U_2(t, r_1) V_2(r_1) U_1(r_1, r_2) V_1(r_2) \ldots V_k(r_{n+1}) U_0(r_{n+1}, s),$$

where $j = 2$ and $k = 1$ if n is odd, and $j = 1$ and $k = 2$ if n is even.

So, by summing over the N(G) and adding $U_0(t, s)$, we have

$$\begin{aligned} U(t, s)\psi = U_0(t, s)\psi - i\int_s^t dr U_1(t, r)V_1(r)U_0(r, s)\psi - i\int_s^t dr U_2(t, r)V_2(r)U_0(r, s)\psi \\ - \int_s^t dr_1 \int_s^{r_1} dr_2 U_1(t, r_1)V_1(r_1)U_2(r_1, r_2)V_2(r_2)U_0(r_2, s)\psi \\ - \int_s^t dr_1 \int_s^{r_1} dr_2 U_2(t, r_1)V_2(r_1)U_1(r_1, r_2)V_1(r_2)U_0(r_2, s)\psi \\ + \dots \qquad (2.2) \end{aligned}$$

This is the formula which we will use to prove asymptotic completeness in Section 4. To rigorously justify equation (2.2) we must show that the series converges and that it converges to the correct result. We will establish that this is the case when $\psi \in \mathscr{S}$ and either $0 < s < t$ and s is large or $t < s < 0$ and $-s$ is large.

If $\psi \in \mathscr{S}$, then $U_0(r, s)\psi \in D(H_0) \subset D(V_1(r)) \cap D(V_2(r))$, and it is not hard to show that $\int_s^t U(s, r)[V_1(r) + V_2(r)]U_0(r, s)\psi dr$ converges for each t and s. Furthermore, if $\phi \in \mathscr{S}$, then by the fundamental theorem of calculus, we have

$$\langle \phi, (1 - U(s, t)U_0(t, s))\psi \rangle = -i\int_s^t \langle U(r, s)\phi, [V_1(r) + V_2(r)]U_0(r, s)\psi \rangle dr .$$

Therefore, by a density argument we have

$$\psi = U(s, t)U_0(t, s)\psi - i\int_s^t U(s, r[V_1(r) + V_2(r)]U_0(r, s)\psi dr .$$

So, by applying $U(t, s)$ to both sides we obtain

$$U(t, s)\psi = U_0(t, s)\psi - i\int_s^t U(t, r)[V_1(r) + V_2(r)]U_0(r, s)\psi dr . \qquad (2.3)$$

By the same method of proof we have

$$U(t, s)\psi = U_0(t, s)\psi - i\int_s^t U(t, r)V_1(r)U_2(r, s)\psi dr \qquad (2.4)$$

and

$$U(t, s)\psi = U_0(t, s)\psi - i\int_s^t U(t, r)V_2(r)U_1(r, s)\psi dr . \qquad (2.5)$$

If we add (2.4) to (2.5) and substract (2.3) we obtain

$$\begin{aligned} U(t, s)\psi = U_0(t, s)\psi - i\int_s^t U(t, r)V_1(r)[U_2(r, s) - U_0(r, s)]\psi dr \\ - i\int_s^t U(t, r)V_2(r)[U_1(r, s) - U_0(r, s)]\psi dr \\ = U_0(t, s)\psi - \int_s^t dr_1 \int_s^{r_1} dr_2 U(t, r_1)V_1(r_1)U_2(r_1, r_2)V_2(r_2)U_0(r_2, s)\psi \\ - \int_s^t dr_1 \int_s^{r_1} dr_2 U(t, r_1)V_2(r_1)U_1(r_1, r_2)V_1(r_2)U_0(r_2, s)\psi \end{aligned} \quad (2.6)$$

Estimates from Section 3 together with some density arguments show that (2.6) is valid for any $\psi \in \mathcal{S}$.

The estimates from Section 3 also show that the iterates of equation (2.6) converge for all $\psi \in \mathcal{S}$ if s is large and $t > s$, or if $-s$ is large and $t < s$. However, the iterates of (2.6) give the same result as equation (2.2) by some trivial algebra. So, we conclude from Section 3 that the right hand side of (2.2) converges to the left hand side whenever $\psi \in \mathcal{S}$, $|s|$ is large, and either $0 < s < t$ or $t < s < 0$.

Remarks 1. — Equation (2.2) is physically interpreted as a « multiple collision expansion ». The propagator is written as a sum of terms in which particle 3 alternately interacts with particles 1 and 2. This heuristic picture makes it clear that (2.2) should converge for $|s|$ large and $0 < s < t$ or $t < s < 0$ since it should be very unlikely for particle 3 to oscillate very many times between particles 1 and 2 as time evolves between s and t.

2. In Section 5 we prove that this multiple collision expansion converges for $s = 0$ and all t if $|v_1 - v_2|$ is large. Intuitively this is also very reasonable.

3. CONVERGENCE OF THE FADDEEV SERIES

In this section we prove the estimates necessary for establishing convergence of the series (2.2). These estimates bear considerable resemblance to those used to prove asymptotic completeness for the usual three body problem [*3*] [*5*] [*7*] [*16*].

The crucial result we will prove is Lemma 3.6, which shows that the operator valued function

$$\int_s^t A_j(x - a_j - v_j t)U_k(t, r)V_k(x - a_k - v_k r)U_j(r, s)B_j(x - a_j - v_j s)dr \qquad (j \neq k)$$

belongs to $L^1((s, \infty), dt)$ for large s, and that its L^1 norm tends to zero as s tends to infinity.

We being with some standard results:

LEMMA 3.1. — Assume hypothesis (H1), and suppose $\psi \in L^1 \cap L^2$. Let $A_j(x) = (1 + x^2)^{-\delta/2}V_j(x)$ and $B_j(x) = (1 + x^2)^{-\delta/2}$, where δ is as in hypothesis (H1). Then $\int_s^\infty || A_j(x - a_j - v_jt)U_0(t, s)\psi || \, dt < \infty$ and

$$\int_s^\infty || A_j(x-a_j-v_jt)U_0(t, s)B_k(x-a_k-v_ks) || \, dt<\infty \qquad \text{for } j=1,2, \ k=1,2,$$

and any value of s (The norm in the first integral is the L^2 norm; the norm in the second is the operator norm from L^2 to L^2).

Proof. — The operator $U_0(t, s)$ has a well known integral kernel [*10*] with which one can show by an interpolation argument that

$$U_0(t, s) : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$$

is bounded with bound dominated by $[c\,|\,t - s\,|\,]^{-n(p^{-1}-2^{-1})}$ for $s \neq t$, $1 \leq p \leq 2$, and $p^{-1} + q^{-1} = 1$. Since $\psi \in L^p$ for all $p \in [1, 2]$, we therefore have $|| U_0(t, s)\psi ||_q \leq [c\,|\,t - s\,|\,]^{-n(p^{-1}-2^{-1})}$. Since hypothesis (H1) implies $A_j \in L^r(\mathbb{R}^n)$ for $n - \varepsilon < r < n + \varepsilon$, A_j maps L^q into L^2 for all q in a neighborhood of $(2^{-1}-n^{-1})^{-1}$ $(n \geq 3)$. So, for all p in a neighborhood of $(2^{-1}+n^{-1})^{-1}$ we have $|| A_j(x - a_j - v_jt)U_0(t, s)\psi || \leq [c\,|\,t - s\,|\,]^{-n(p^{-1}-2^{-1})}$. By choosing p so that $p^{-1} - 2^{-1} > n^{-1}$ for $|\,t - s\,|$ large and $p^{-1} - 2^{-1} < n$ for $|\,t - s\,|$ small we see that the first integral of the lemma converges.

The second integral is dealt with in a similar manner after remarking that $B_k(x - a_k - v_ks)$ maps L^2 into $L^p(\mathbb{R}^n)$ by Hölder's inequality for all p in a neighborhood of $(2^{-1} + n^{-1})^{-1}$. ■

Henceforth we will write $A_j(t)$, $B_j(t)$, and $V_j(t)$ for multiplication by $A_j(x - a_j - v_jt)$, $B_j(x - a_j - v_jt)$, and $V_j(x - a_j - v_jt)$, respectively. We also will occasionally work in different frames of reference in which $v_j = 0$ for $j = 1$ or 2. By abusing notation somewhat we will use $U(t, s)$, $U_1(t, s)$, etc. for all of the obvious operators in the different frames of reference, rather than subject the reader to more notation.

LEMMA 3.2. — Assume hypotheses (H1) and (H2). Then for $j \neq k$,

$$\int_s^t || A_j(r)U_k(r, s)P_k(s)B_k(s) || \, dr \leq F(t, s),$$

where $F(t, s)$ is uniformly bounded and monotone decreasing in s for $t > s$. Furthermore $\lim_{s\to\infty} (\sup_{t>s} F(t, s)) = 0$.

Proof. — Since B_k is bounded, it suffices to prove the lemma with B_k replaced by 1. The norm in the integrand is then dominated by

$$\sum_{l=1}^{N} || A_j(x - a_j - (v_j - v_k)r)\psi_{l,k}(x) ||_{L^2},$$

where $\{ \psi_{l,k} \}_{l=1}^{N}$ is an orthonormal basis for Ran P_k composed of eigenfunctions of H_k (we have chosen a frame of reference in which $P_k(s)$ and $H_k(s)$ are independent of s). By Hölder's inequality this quantity is dominated by

$$\sum_{l=1}^{N} \| A_j(x - a_j - v_j r)(1 + [x - a_j - (v_j - v_k)r]^2)^{\delta/2} \|_n$$
$$\| (1 + [x - a_j - (v_j - v_k)r]^2)^{-\delta/2}(1 + x^2)^{-\delta/2} \|_\infty \cdot \| (1 + x^2)^{\delta/2}\psi_{l,k}(x) \|_p ,$$

where $p^{-1} = 2^{-1} - n^{-1}$. Simon [15] has proved that $e^{a|x|}\psi_{l,k}(x) \in D(H_0)$ for some $a > 0$, so ([10], Theorem IX, 28) the third factor in each of these terms is bounded. As remarked in Section 1, the first factors are bounded, so we need only show that the second factors are in $L^1(dr)$. However, that follows by elementary computation since $\delta > 1$. ■

LEMMA 3.3. — Assume hypotheses (H1) and (H2). Then for any s, $\int_s^\infty \| A_j(t)U_j(t, s)(1 - P_j(s))B_j(s) \| dt \leq C$ for some C.

Proof. — In the frame of reference in which A_j, P_j and B_j are time independent, the integral in question has no s dependence, so the uniformity in s is obvious. So, we need only show $A_jU_j(t, 0)(1 - P_j)B_j \in L^1((0, \infty), dt)$.

Let $\chi(t)$ denote the characteristic function of the set $[0, \infty)$, and define

$$K_1(t) = \chi(t)A_jU_0(t, 0)B_j ,$$
$$K_2(t) = \chi(t)A_jU_0(t, 0)(1 - P_j)B_j ,$$
$$G_2(t) = \chi(t)A_j(1 - P_j)U_0(t, 0)(1 - P_j)B_j .$$

All three of these operator valued functions belong to $L^1((0, \infty), dt)$ by trivial modifications of Lemma 3.1. All three have operator valued Fourier transforms which are analytic compact operator valued functions in the open upper half plane and which are continuous in the closed upper half plane [3]. Furthermore all three Fourier transforms tend to zero in norm as $|z| \to \infty$ in the closed upper half plane.

The function $F(t) = \chi(t)A_jU_j(t, 0)(1 - P_j)B_j$ satisfies

$$F(t) = K_2(t) + (K_1 * F)(t)$$

and

$$F(t) = G_2(t) + (F * K_2)(t) ,$$

where $(a * b) = \int_{-\infty}^{\infty} a(t - s)b(s)ds$. Thus the Fourier transform of F satisfies

$$\hat{F}(z) = \hat{K}_2(z) + \hat{K}_1(z)\hat{F}(z) = (1 - \hat{K}_1(z))^{-1}\hat{K}_2(z)$$

and

$$\hat{F}(z) = \hat{G}_2(z) + \hat{F}(z)\hat{K}_2(z) = \hat{G}_2(z)(1 - \hat{K}_2(z))^{-1} .$$

By hypothesis (H2) and standard calculations [14], $(1 - \hat{K}_1(z))^{-1}$ exists except when z is an eigenvalue of H. All such eigenvalues are negative and there are finitely many of them.

In contrast we will now show that $(1 - \hat{K}_2(z))^{-1}$ exists for all negative z. The analytic Fredholm theorem shows that $(1 - \hat{K}_2(z_0))^{-1}$ fails to exist only if there is a non-zero $\phi \in L^2(\mathbb{R}^n)$ such that $\phi = \hat{K}_2(z_0)\phi$. Using the explicit formula for $\hat{K}_2(z_0)$ we have $\phi = A_j(z_0 - H_0)^{-1}(1 - P_j)B_j\phi$. If $z_0 < 0$, then we have $\phi_1 = (z_0 - H_0)^{-1}(1 - P_j)B_j\phi$ in L^2. Clearly $\phi = A_j\phi_1$ and $\phi_1 = (z_0 - H_0)^{-1}(1 - P_j)V_j\phi_1$. Applying $z_0 - H_0$ to both sides of this equation we have $H_0\phi_1 + (1 - P_j)V_j\phi_1 = z_0\phi_1$.

Thus, $(1 - P_j)z_0\phi_1 = (1 - P_j)[H_0 + V_j]\phi_1$ and $\phi_1 = P_j\phi_1$ since

$$(1 - P_j)[H_0 + V_j] = [H_0 + V_j](1 - P_j)$$

has no eigenvalues. Now we have

$$z_0 \langle \phi_1, \phi_1 \rangle = \langle \phi_1, [H_0 + (1 - P_j)V_j]\phi_1 \rangle = \langle \phi_1, H_0\phi_1 \rangle .$$

Since ϕ_1 is non-zero, this contradicts $z_0 < 0$.

To prove $F(t)$ belongs to $L^1((0, \infty), dt)$ we mimic the proof of Theorem XVIII of [9], with certain alterations to accommodate the poles of $(1 - \hat{K}_1(z))^{-1}$ and $(1 - \hat{K}_2(z))^{-1}$.

Choose $\varepsilon > 0$ so that $-\varepsilon$ is greater than all eigenvalues of H_j. Define

$$w_1(x) = \begin{cases} 0 & \text{if} \quad x \le -\varepsilon \quad \text{or} \quad x > 2b \\ (x + \varepsilon)/\varepsilon & \text{if} \quad -\varepsilon < x \le 0 \\ 1 & \text{if} \quad 0 < x \le b \\ 2 - x/b & \text{if} \quad b < x \le 2b \end{cases}$$

$$w_2(x) = \begin{cases} 0 & \text{if} \quad x < -2b \quad \text{or} \quad 0 \le x \\ 2 + x/b & \text{if} \quad -2b \le x < -b \\ 1 & \text{if} \quad -b \le x < -\varepsilon \\ -x/\varepsilon & \text{if} \quad -\varepsilon \le x < 0 \end{cases}$$

Let $w_3(x) = 1 - w_1(x) - w_2(x)$. Here b is a large number to be chosen later.

By repeating the arguments on page 62 of [9], $(1 - \hat{K}_1(x))^{-1}\hat{K}_1(x)w_3(x)$ is the Fourier transform of some function $Q_3(t)$ which belongs to $L^1((-\infty, \infty), dt)$ if b is sufficiently large. Fix such a number b so that $Q_3(t)$ is in $L^1((-\infty, \infty))$. Note that the poles of $(1 - \hat{K}_1(z))^{-1}$ will cause no trouble in these arguments since $w_3(x)$ is zero on an open set of real numbers containing the poles.

Next we repeat the arguments of the proof of Theorem 9 D of Chapter 2 of [4] to see that $(1 - \hat{K}_1(x))^{-1}w_1(x) = \hat{Q}_1(x)$ for some $Q_1(t)$ which also belongs to $L^1((-\infty, \infty))$. Similarly, there exists $Q_2 \in L^1((-\infty, \infty))$ such that $\hat{Q}_2(x) = (1 - \hat{K}_2(x))^{-1}w_2(x)$. Here we have used the information obtained above concerning the location of poles of $(1 - \hat{K}_1(z))^{-1}$ and $(1 - \hat{K}_2(z))^{-1}$.

Due to the exponential fall off of eigenfunctions, of H_j,

$$(1 - P_j) \cdot B_j = B_j(1 - \tilde{P}_j),$$

where $\tilde{P}_j = B_j^{-1}P_jB_j$ is a bounded operator on $L^2(\mathbb{R}^n)$. Consequently,

$$\begin{aligned}\hat{F}(z) &= (1 - \hat{K}_1(z))^{-1}\hat{K}_2(z) \\ &= (1 - \hat{K}_1(z))^{-1}\hat{K}_1(z)(1 - \tilde{P}_j)\,.\end{aligned}$$

We now define

$$f(t) = (Q_1 * K_2)(t) + (G_2 * Q_2)(t) + Q_3(t)(1 - \tilde{P}_j)\,.$$

This function belongs to $L^1((-\infty, \infty))$, and its Fourier transform equals

$$\begin{aligned}\hat{f}(x) &= \hat{Q}_1(x)\hat{K}_2(x) + \hat{G}_2(x)\hat{Q}_2(x) + \hat{Q}_3(x)(1 - \tilde{P}_j) \\ &= \hat{F}(x)(w_1(x) + w_2(x) + w_3(x)) \\ &= \hat{F}(x)\,.\end{aligned}$$

It follows ([*9*], p. 62-63) that $f(t) = 0$ a. e. for $t < 0$, and that $F(t) = f(t)$.

Remark. — Theorem 9 D of Chapter 2 of [*4*] says that on a compact set, an analytic function of the Fourier transform of an L^1 function is equal to the Fourier transform of an L^1 function. Theorem XVIII of [*9*] is the same result for the whole real line (not just a compact set) when the analytic function is $z/(1 - z)$. Extensions of these theorems to operator valued functions which are Bochner measurable is a routine exercise.

COROLLARY 3.4. — Assume hypotheses (H1) and (H2). Then for any s and $j \neq k$, we have

a) $$\int_s^\infty \| A_j(t)U_k(t, s)(1-P_k(s))B_k(s) \| \, dt \leq C_1\,.$$

b) $$\int_s^\infty \| A_j(t)U_k(t, s)B_k(s) \| \, dt \leq C_2\,.$$

c) $$\int_s^\infty dt \int_s^t dr \, \| A_j(t)U_k(t, r)(1-P_k(r))V_k(r)U_j(r, s)(1-P_j(s))B_j(s) \| \leq C_3\,.$$

d) $$\int_s^\infty dt \int_s^t dr \, \| A_j(t)U_k(t, r)P_k(r)V_k(r)U_j(r, s)(1-P_j(s))B_j(s) \| \equiv F_4(s) \leq C_4\,.$$

e) $$\int_s^\infty dt \int_s^t dr \, \| A_j(t)U_k(t, r)(1-P_k(r))V_k(r)U_j(r, s)P_j(s)B_j(s) \| \equiv F_5(s) \leq C_5\,.$$

f) $$\int_s^\infty dt \int_s^t dr \, \| A_j(t)U_k(t, r)P_k(r)V_k(r)U_j(r, s)P_j(s)B_j(s) \| \equiv F_6(s) \leq C_6\,.$$

Furthermore, in *d*), *e*) and *f*), $\lim_{s \to \infty} F_i(s) = 0$.

Proof. — To prove *a*) we replace $U_k(t, s)$ by

$$U_0(t, s) + i\int_s^t U_0(t, r)V_k(r)U_k(r, s)dr$$

and dominate the integral of the norm of the sum by the integral of the sum of the norms. Furthermore, we can put the norms inside the r integral. The term involving a single integral is uniformly bounded by a trivial modification of Lemma 3.1. Lemmas 3.1 and 3.3 show that the remaining term is uniformly bounded.

Parts *b*)-*f*) follow from Lemma 3.2 and part *a*). ■

Most of the rest of this section will be devoted to proving a result similar to *c*), which is stronger in some sense. We will prove

$$\lim_{s\to\infty}\int_s^\infty \left\| \int_s^t A_j(t)U_k(t, r)(1-P_k(r))V_k(r)U_j(r, s)(1-P_j(s))B_j(s)dr \right\| dt = 0$$

This result, combined with parts *d*)-*f*), will be enough to show that (2.2) converges and that multiple scatterings are unlikely after some large time s.

Lemma 3.5. — Assume hypotheses (H1) and (H2). Then for $j \neq k$ we have

a) $$\lim_{s\to\infty}\int_s^\infty dt\int_s^t dr \,\| A_j(t)U_0(t, r)V_j(r)U_0(r, s)B_k(s) \| = 0$$

b) $$\lim_{s\to\infty}\int_s^\infty \left\| \int_s^t A_j(t)U_0(t, r)V_k(r)U_0(r, s)B_j(s)dr \right\| dt = 0 .$$

Proof. — Consider first part *a*). Choose a frame of reference in which V_j has no time dependence, and replace the operator inside the norm by its adjoint. Next, change variables by replacing t by $q = t - s$ and r by $p = r - s$. The integral in *a*) is then equal to

$$\int_0^\infty dq\int_0^q dp \,\| B_k(s)U_0(0, p)V_jU_0(p, q)A_j \| .$$

By the proof of Lemma 3.1, the integrand is dominated by a fixed L^1 function, and so the dominated convergence theorem shows that we need only show for fixed p, q that $\lim_{s\to\infty} \| B_k(s)U_0(0, p)V_jU_0(p, q)A_j \| = 0$.

From the proof of Lemma 3.1 it is not hard to prove [*3*] that $B_jU_0(p, q)A_j$ is compact for $q \neq p$. Furthermore, it is clear from that proof that $B_k(s)U_0(0, p)A_j$ tends to zero strongly as $s \to \infty$. Thus,

$$B_k(s)U_0(0, p)V_jU_0(p, q)A_j = B_k(s)U_0(0, p)A_jB_jU_0(p, q)A_j$$

tends to zero in norm as $s \to \infty$. Thus, *a*) is proved.

To prove *b*) we first note that the estimates in the proof of Lemma 3.1 show that it is sufficient to prove *b*) when A_j, B_j, and V_k have been replaced by functions in $\mathscr{S}(\mathbb{R}^n)$ (which we again denote by A_j, B_j, and V_k, respectively).

Furthermore, those estimates show that it is sufficient to prove the result with the $\int_s^t dr$ replaced by $\int_{s+\varepsilon}^{t-\varepsilon} dr$ and the $\int_s^\infty dt$ replaced by $\int_{s+2\varepsilon}^\infty dt$ for arbitrarily small $\varepsilon > 0$.

Choose a frame of reference in which A_j and B_j are time independent. Then by changing variables we have

$$\int_{s+2\varepsilon}^{\infty} \left\| \int_{s+\varepsilon}^{t-\varepsilon} A_j U_0(t, r) V_k(r) U_0(r, s) B_j dr \right\| dt$$
$$= \int_{2\varepsilon}^{\infty} \left\| \int_{\varepsilon}^{p-\varepsilon} A_j U_0(p, q) V_k(q + s) U_0(q, 0) B_j dq \right\| dp,$$

so that all the s dependence is in the V_k factor. Since Hilbert-Schmidt norms dominate operator norms, we have

$$\left\| \int_{\varepsilon}^{p-\varepsilon} A_j U_0(p, q) V_k(q+s) U_0(q, 0) B_j dq \right\|^2$$
$$\leq \left\| \int_{\varepsilon}^{p-\varepsilon} A_j U_0(p, q) V_k(q+s) U_0(q, 0) B_j dq \right\|_{H-S}^2 = \int_{\mathbb{R}^{2n}} dxdy \, | A_j(x) |^2 \, | B_j(y) |^2$$
$$\left| \int_{\varepsilon}^{p-\varepsilon} dq \int_{\mathbb{R}^n} dz (4\pi i(p-q))^{-n/2} (4\pi i q)^{-n/2} \right.$$
$$\left. \exp \{ i \, | x - z |^2 / 4(p-q) \} \, V_k(q+s, z) \exp \{ i \, | z - y |^2 / 4q \} \right|^2$$
$$\leq \int_{\mathbb{R}^{2n}} dxdy \, | A_j(x) |^2 \, | B_j(y) |^2 \left(\int_{\mathbb{R}^n} dz \, | V_k(z) | \right)^2 \left(\int_{\varepsilon}^{p-\varepsilon} (16\pi(p-q)q)^{-n/2} dq \right)^2,$$

which is the square of an L^1 function of p, independent of s. So by the dominated convergence theorem, we need only show that

$$\left\| \int_{\varepsilon}^{p-\varepsilon} A_j U_0(p, q) V_k(q + s) U_0(q, 0) B_j dq \right\|$$

tends to 0 as $s \to \infty$ for each fixed positive p. However, by the above calculations and several more applications of dominated convergence, we need only show that for each fixed value of p, q, x, y we have

$$\lim_{s \to \infty} \left| \int_{\mathbb{R}^n} V_k(q + s, z) \exp \{ i \, | x - z |^2 / 4(p-q) + i \, | z - y |^2 / 4q \} \, dz \right| = 0 . \quad (3.1)$$

The quantity on the left hand side is equal to

$$\lim_{s \to \infty} \left| \int_{\mathbb{R}^n} V_k(q, z) \exp \{ i \, | x - z - s(v_k - v_j) |^2 / 4(p - q) \right.$$
$$\left. + \, i \, | z + s(v_k - v_j) - y |^2 / 4q \} \, dz \right|$$
$$= \lim_{s \to \infty} \left| \int_{\mathbb{R}^n} [V_k(q, z) \exp \{ i \, | z |^2 (1/4(p - q) + 1/4q) \}] \right.$$
$$\left. \cdot \exp \{ 2iz \cdot ((s(v_k - v_j) - x)/4(p - q) + (s(v_k - v_j) - y)/4q) \} \, dz \right| .$$

This tends to 0 as $s \to \infty$ by the Riemann-Lebesgue Lemma since $v_k \neq v_j$, $p - q > 0$, and $q > 0$. ■

Remark. — In the last step of the above proof it is essential that $(s/4(p-q) + s/4q)(v_k - v_j)$ is never zero. In some vague sense the non-vanishing of this factor is related to the fact that it is extremely unlikely for the particle to be near V_j at time s, near V_k at time r and near V_j at time t. Geometrically, if this were likely, the particle would have to more or less reverse the direction of its momentum during the collision with V_k.

LEMMA 3.6. — Assume hypotheses (H1) and (H2) and choose $j \neq k$. Then $\lim_{s\to\infty} \int_s^\infty \left\| \int_s^t A_j(t)U_k(t, r)V_k(r)U_j(r, s)B_j(s)dr \right\| dt = 0.$

Proof. — It follows from parts *d*), *e*), *and f*) of Corollary 3.4 that we need only show

$$\lim_{s\to\infty} \int_s^\infty \left\| \int_s^t A_j(t)U_k(t, r)(1-P_k(r))V_k(r)U_j(r, s)(1-P_j(s))B_j(s)dr \right\| = 0 .$$

The integrand in the inner integral can be rewritten as I + II + III + IV, where

$$I(t, r, s) = A_j(t)(1 - P_k(t))U_0(t, r)V_k(r)U_0(r, s)(1 - P_j(s))B_j(s) ,$$

$$\begin{aligned} &II(t, r, s) \\ &= -i\int_r^t A_j(t)(1 - P_k(t))U_k(t, u)V_k(u)U_0(u, r)V_k(r)U_0(r, s)(1 - P_j(s))B_j(s)du , \end{aligned}$$

$$\begin{aligned} &III(t, r, s) \\ &= i\int_s^r A_j(t)(1 - P_k(t))U_0(t, r)V_k(r)U_0(r, w)V_j(w)U_j(w, s)(1 - P_j(s))B_j(s)dw , \end{aligned}$$

$$\begin{aligned} &IV(t, r, s) \\ &= \int_r^t du\int_s^r dw A_j(t)(1 - P_k(t))U_k(t, u)V_k(u)U_0(u, r)V_k(r)U_0(r, w)V_j(w) \\ &\qquad \cdot U_j(w, s)(1 - P_j(s))B_j(s) . \end{aligned}$$

We will show $\lim_{s\to\infty} \int_s^\infty \left\| \int_s^t F(t, r, s)dr \right\| dt = 0$ for F equal to I, II, III and IV. This will imply the lemma. We can rewrite I(t, r, s) as a sum of four terms:

$$\begin{aligned} I(t, r, s) = {} & A_j(t)U_0(t, r)V_k(r)U_0(r, s)B_j(s) \\ & + [A_j(t)P_k(t)B_k(t)^{-1}][B_k(t)U_0(t, r)V_k(r)U_0(r, s)B_j(s)] \\ & + [A_j(t)U_0(t, r)V_k(r)U_0(r, s)B_j(s)][B_j(s)^{-1}P_j(s)B_j(s)] \\ & + [A_j(t)P_k(t)B_k(t)^{-1}][B_k(t)U_0(t, r)V_k(r)U_0(r, s)B_j(s)][B_j(s)^{-1}P_j(s)B_j(s)] \end{aligned}$$

Each term here is a product of uniformly bounded factors times an operator valued function which has all the desired properties by trivial modifications of Lemma 3.5. Thus, $I(t, r, s)$ is controlled.

Using the same ideas (rewriting as above) one may control II, III and IV after using Corollary 3.4 *a*) to see that all factors involving U_j with $j \neq 0$ are integrable with L^1 norms bounded independent of s. ■

PROPOSITION 3.7. — For sufficiently large values of $s > 0$ and all $\psi \in \mathcal{S}$, hypotheses (H1) and (H2) imply the convergence in L^2 of the series (2.2) for $U(t, s)\psi$ uniformly for all $t > s$.

Proof. — As outlined in Section 2, one obtains equation (2.2) by iterating equation (2.6) and collecting terms. If we do this several times we obtain $U_0(t, s)\psi$ plus two terms containing a single integral plus two terms containing a double integral, and so on, until we reach two « error terms » which contain an N-fold integral whose integrands contain factors of $U(t, r_1)$. Lemma 3.1 shows that the inner-most integral (in all of the terms containing integrals) is convergent and yields a result whose norm is bounded uniformly by a constant, C, which may be taken to be the same constant for each term. Corollary 3.4 *b*) then shows that all of the integrals are convergent. However, Lemma 3.6 shows that if s is sufficiently large, then as we evaluate each consecutive pair of integrals after the inner-most one, our result has its norm reduced by a factor of ε, where $\varepsilon > 0$ is arbitrary. Thus, a term containing $2m + 1$ integrals has its norm bounded by $\varepsilon^m C$. A term containing $2m$ integrals can be dealt with in the same way, except that we must use Corollary 3.4 *b*) to estimate the outermost integral. Thus, such a term is bounded by $C_2\varepsilon^{m-1}C$.

From this analysis it is clear that for large s and $t > s$ the « error terms » tend to zero as we iterate (2.6) more and more times. Furthermore, as we take the number of iterations to infinity, the series (2.2) applied to ψ is convergent in norm for $t > s$ and s large. The sum of the norms of the individual terms is bounded by

$$\| \psi \| + 2C + 2C_2C + 2\varepsilon C + 2\varepsilon C_2 C + \dots$$
$$\leq \| \psi \| + (2C + 2C_2C) \sum_{n=0}^{\infty} \varepsilon^n \leq \| \psi \| + 2C(1 + C_2)(1 - \varepsilon)^{-1} .$$

So, for s large, $t > s$, and $\psi \in \mathcal{S}$, the right hand side of equation (2.2) converges to $U(t, s)\psi$. ■

Remarks 1. — Lemma 3.3 is indispensable for the proof of our theorem. The critical idea for this proof comes from Theorem XVIII of the classical book of Paley and Wiener [*9*]. We feel that it is remarkable that it is not widely used in scattering theory, particularly because Percy Deift has informed us that it plays a role in inverse scattering theory.

2. Although we assumed above that ψ belonged to $\mathcal{S}$, a density argument shows that the result remains true if we only require $\psi \in L^1 \cap L^2$. We will use this fact in the next two sections without comment.

4. ASYMPTOTIC COMPLETENESS

In this section we prove the final statement of the theorem in Section 1. We so do by more or less explicitly constructing the adjoints of $\Omega_i^{\pm}(s)$ for large values of s.

We begin the proof by remarking that it suffices to show that any $\psi \in L^1 \cap L^2$ can be written as $\psi = \Omega_0^{\pm}(s)\phi_{0,\pm}(s) + \Omega_1^{\pm}(s)\phi_{1,\pm}(s) + \Omega_2^{\pm}(s)\phi_{2,\pm}(s)$ because $L^1 \cap L^2$ is dense in L^2. Next we notice that it is sufficient to do so for only one value of s because $[\text{Ran } \Omega_i^{\pm}(t)] = U(t, s)\,[\text{Ran } \Omega_i^{\pm}(s)]$ and $U(t, s)$ is unitary. Finally, we will only consider $\Omega_i^{-}(s)$. The proofs for $\Omega_i^{+}(s)$ are similar.

By Proposition 3.7 we may fix s so large that the series (2.2) converges to $U(t, s)\psi$ for all $\psi \in L^1 \cap L^2$ and all $t > s$. For fixed $\psi \in L^1 \cap L^2$ we notice that equation (2.2) has the form

$$U(t, s)\psi = U_0(t, s)\psi + \int_s^t U_0(t, r)F_0(r)dr + \int_s^t U_1(t, r)F_1(r)dr + \int_s^t U_2(t, r)F_2(r)dr\,,$$

where F_0, F_1 and F_2 are $L^1((s, \infty))$ functions with values in $L^2(\mathbb{R}^n)$. In this equation we replace $U_j(t, r)$ by

$$U_j(t, r)P_j(r) + U_0(t, r)(1 - P_j(r)) + i\int_r^t U_0(t, u)V_j(u)U_j(u, r)(1 - P_j(r))du \qquad \text{for} \qquad j = 1, 2\,.$$

Using the explicit form of the F_i's and Lemma 3.3 we can conclude that for $t > s$,

$$U(t, s)\psi = U_0(t, s)\psi + \int_s^t U_0(t, r)G_0(r)dr + \int_s^t U_1(t, r)P_1(r)G_1(r)dr + \int_s^t U_2(t, r)P_2(r)G_2(r)dr\,, \tag{4.1}$$

where G_0, G_1 and G_2 are $L^1((s, \infty))$ functions with values in $L^2(\mathbb{R}^n)$.

We define $\phi_0(s)$, $\phi_1(s)$ and $\phi_2(s)$ by

$$\phi_0(s) = \psi + \int_s^\infty U_0(s, r)G_0(r)dr , \tag{4.2}$$

$$\phi_1(s) = \int_s^\infty U_1(s, r)P_1(r)G_1(r)dr , \tag{4.3}$$

$$\phi_2(s) = \int_s^\infty U_2(s, r)P_2(r)G_2(r)dr , \tag{4.4}$$

Then from equation (4.1) we have

$$\begin{aligned} &\| \psi - U(s, t)U_0(t, s)\phi_0(s) - U(s, t)U_1(t, s)\phi_1(s) - U(s, t)U_2(t, s)\phi_2(s) \| \\ &\quad = \| U(t, s)\psi - U_0(t, s)\phi_0(s) - U_1(t, s)\phi_1(s) - U_2(t, s)\phi_2(s) \| \\ &\quad = \left\| \int_t^\infty U_0(t, r)G_0(r)dr + \int_t^\infty U_1(t, r)P_1(r)G_1(r)dr + \int_t^\infty U_2(t, r)P_2(r)G_2(r)dr \right\| \\ &\quad \leq \int_t^\infty (\| G_0(r) \| + \| G_1(r) \| + \| G_2(r) \|)dr . \end{aligned}$$

As t tends to infinity this quantity tends to zero. Thus,

$$\psi = \Omega_0^-(s)\phi_0(s) + \Omega_1^-(s)\phi_1(s) + \Omega_2^-(s)\phi_2(s) . \qquad \blacksquare$$

Remark. — In the above proof, our definitions of $\phi_0(s)$, $\phi_1(s)$ and $\phi_2(s)$ may seem unmotivated. Our motivation comes from experience with N-body problems. See Section III of [5].

5. CONVERGENCE OF THE FADDEEV SERIES FOR $\Omega_i^\pm(0)$

In Section 4 we used a convergent series for $U(t, s)$ to construct $\langle \psi, \Omega_i^\pm(s)^*\phi \rangle$ more or less explicitly for large values of s. (See equations (4.2)-(4.4).) In this section we show for any s that if $| v_1 - v_2 |$ is large, then $\langle \psi, \Omega_i^\pm(s)\phi \rangle$ can be constructed by using the series for $U(t, s)$. It is our hope that the series we obtain for $\langle \psi, \Omega_i^\pm(s)\phi \rangle$ will be used to study the behavior of cross sections for various processes in the impact parameter approximation.

By formally taking adjoints of the operators which occur in equations (4.2)-(4.4), one obtains formal expressions for $\Omega_i^-(s)$. More explicitly, after some rearranging we have

$$
\begin{aligned}
\Omega_0^-(s) = 1 &+ i\int_s^\infty dr U_0(s, r)V_1(r)(1 - P_1(r))U_0(r, s) \\
&+ i\int_s^\infty dr U_0(s, r)V_2(r)(1 - P_2(r))U_0(r, s) \\
&+ \int_s^\infty dr_1 \int_s^{r_1} dr_2 U_0(s, r_2)V_1(r_2)(1 - P_1(r_2))U_1(r_2, r_1)V_1(r_1)U_0(r_1, s) \\
&+ \int_s^\infty dr_1 \int_s^{r_1} dr_2 U_0(s, r_2)V_2(r_2)(1 - P_2(r_2))U_2(r_2, r_1)V_2(r_1)U_0(r_1, s) \\
&- \int_s^\infty dr_1 \int_s^{r_1} dr_2 U_0(s, r_2)V_2(r_2)U_2(r_2, r_1)V_1(r_1)(1 - P_1(r_1))U_0(r_1, s) \\
&- \int_s^\infty dr_1 \int_s^{r_1} dr_2 U_0(s, r_2)V_1(r_2)U_1(r_2, r_1)V_2(r_1)(1 - P_2(r_1))U_0(r_1, s) \\
&+ i\int_s^\infty dr_1 \int_s^{r_1} dr_2 \int_s^{r_2} dr_3 U_0(s, r_3)V_2(r_3)U_2(r_3, r_2)V_1(r_2) \\
&\qquad\qquad (1 - P_1(r_2))U_1(r_2, r_1)V_1(r_1)U_0(r_1, s) \\
&+ i\int_s^\infty dr_1 \int_s^{r_1} dr_2 \int_s^{r_2} dr_3 U_0(s, r_3)V_1(r_3)U_1(r_3, r_2)V_2(r_1) \\
&\qquad\qquad (1 - P_2(r_2))U_2(r_2, r_1)V_2(r_1)U_0(r_1, s) \\
&+ \dots \qquad (5.1)
\end{aligned}
$$

$$
\begin{aligned}
\Omega_1^-(s) = i&\int_s^\infty dr U_0(s, r)V_1(r)P_1(r)U_1(r, s) \\
&- \int_s^\infty dr_1 \int_s^{r_1} dr_2 U_0(s, r_2)V_2(r_2)U_2(r_2, r_1)V_1(r_1)P_1(r_1)U_1(r_1, s) \\
&+ \dots \qquad (5.2)
\end{aligned}
$$

$$
\begin{aligned}
\Omega_2^-(s) = i&\int_s^\infty dr U_0(s, r)V_2(r)P_2(r)U_2(r, s) \\
&- \int_s^\infty dr_1 \int_s^{r_1} dr_2 U_0(s, r_2)V_1(r_2)U_1(r_2, r_1)V_2(r_1)P_2(r_1)U_2(r_1, s) \\
&+ \dots \qquad (5.3)
\end{aligned}
$$

We argue below that the right hand sides of (5.1)-(5.3) make sense if the integrals are interpreted as improper integrals, $\lim_{t\to\infty}\int_s^t dr_1 \dots$, and the limits are taken in the very weak sense that matrix elements between $L^1 \cap L^2$ functions are supposed to converge. With this interpretation one may use the proof of the orthogonality of the ranges of the wave operators to rewrite (5.2) and (5.3) as

$$\Omega_1^-(s) = P_1(s) - i\int_s^\infty U_2(s, r)V_2(r)U_1(r, s)P_1(s)dr + \dots \qquad (5.2')$$

$$\Omega_2^-(s) = P_2(s) - i\int_s^t U_1(s, r)V_1(r)U_2(r, s)P_2(s)dr + \dots \qquad (5.3')$$

We should note that the terms which we have explicitly written in formulas (5.1)-(5.3′) are precisely those which arise from those terms in (2.2) which we have explicitly written. More terms in (5.1)-(5.3′) may be generated by computing more terms in (2.2). Similar formulas can be obtained for $\Omega_i^+(s)$.

If we are content to study $\langle \psi, \Omega_i^\pm(\mp s)\phi \rangle$ for $\phi, \psi \in L^1 \cap L^2$ and large values of s, then our proof of Theorem 1.1 shows that the Faddeev series (which arises from (5.1)-(5.3) by taking the matrix elements) converge. However, to compute cross sections it is not sufficient to have information about $\Omega_i^\pm(\mp s)$ for large $s > 0$.

For simplicity, take $s = 0$. By mimicking the proof of Theorem 1.1, it is clear that Theorem 1.2 can be proved by constructing

$$\langle \psi, \Omega_i^-(0)\phi \rangle = \langle \Omega_i^{-*}(0)\psi, \phi \rangle$$

if we can show that

$$\lim_{|v_1 - v_2| \to \infty} \int_0^\infty dt \int_0^t dr \, \| A_j(t)U_0(t, r)V_j(r)U_0(r, 0)B_k(0) \| = 0 \qquad (5.4)$$

and

$$\lim_{|v_1 - v_2| \to \infty} \int_0^\infty dt \left\| \int_0^t dr A_j(t)U_0(t, r)V_k(r)U_0(r, 0)B_j(0) \right\| = 0 \qquad (5.5)$$

To prove Theorem 1.2, one simply uses the construction in the proof of Theorem 1.1, with Lemma 3.5 replaced by (5.4) and (5.5). An analog of Lemma 3.2 is required, but its proof is essentially the same as that of Lemma 3.2. The s independent estimates in the other lemmas of Section 3 are also uniform in $|v_1 - v_2|$.

To prove (5.4) we follow the proof of Lemma 3.5 *a*) to see that it is sufficient to prove $\underset{|v_1 - v_2| \to \infty}{s\text{-}\lim} B_k(0)U_0(0, p)A_j(p) = 0$ for each $p \neq 0$ in the frame of reference in which V_j has no p dependence. In that frame $B_k(0)$ is multiplication by a function of $x - a_k + p(v_j - v_k)$, and as $|v_j - v_k| \to \infty$, $B_k(0)U_0(0, p)A_j$ converges to zero strongly.

To prove (5.5) we mimic the proof of Lemma 3.5 *b*). This shows that it is sufficient to prove equation (3.1) remains valid when $\lim_{s \to \infty}$ is replaced by $\lim_{|v_1 - v_2| \to \infty}$ and $s = 0$. That is,

$$\lim_{|v_1 - v_2| \to \infty} \left| \int_{\mathbb{R}^n} V_k(q, z) \exp\{ i|x - z|^2/4(p - q) + i|z - y|^2/4q \} dz \right| = 0 .$$

The quantity on the left hand side may be written as

$$\lim_{|v_1-v_2|\to\infty} \left| \int_{\mathbb{R}^n} V_k(0, z) \exp\{ i|x - z - q(v_k - v_j)|^2/4(p-q) + i|z + q(v_k - v_j) - y|^2/4q \} dz \right|$$
$$= \lim_{|v_1-v_2|\to\infty} \left| \int [V_k(z) \exp\{ i|z|^2(1/4(p-q) + 1/4q) \}] \cdot \exp\{ 2iz\cdot((q(v_k - v_j) - x)/4(p-q) + (q(v_k - v_j) - y)/4q) \} dz \right| .$$

This limit is zero by the Riemann-Lebesgue Lemma for $q \neq 0$, $p - q > 0$, and $q > 0$. So, (5.5) is valid.

With equations (5.4) and (5.5) verified, it is a routine exercise to show that all the lemmas of Section 3 hold when $\lim_{s\to\infty}$ is replaced by $\lim_{|v_1-v_2|\to\infty}$ and s is set equal to 0. The arguments in Section 4 then show how to construct $\Omega_i^-(0)^*\psi$ by the Faddeev series for $\psi \in L^1 \cap L^2$ when $|v_1 - v_2|$ is large. Thus, $\langle \psi, \Omega_i^-(0)\phi \rangle = \langle \Omega_i^-(0)^*\psi, \phi \rangle$ is equal to its convergent Faddeev series for large $|v_1 - v_2|$, $\phi \in L^2$, and $\psi \in L^1 \cap L^2$. The argument for $\Omega_i^+(0)$ is similar.

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