ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 35, nº 4 (1981), p. 259-285
<http://www.numdam.org/item?id=AIHPA_1981__35_4_259_0>

Ann. Inst. Henri Poincaré, Section A :
Vol. XXXV, n° 4, 1981, p. 259-285. *Physique théorique.*

A spectral theory for order unit spaces

by

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ABSTRACT. — We develope a spectral theory for order unit spaces, in terms of F-projections, which generalizes the spectral theories of Alfsen and Shultz. We discuss the relationship between quantum logical or operational descriptions of physical systems and the spectral theory for order unit spaces.

INTRODUCTION

The existence of a spectral theory for order unit spaces has been proved crucial not only from the point of view of the functional analysis, but also in the axiomatic approach to statistical physical theories. In fact, an important problem for axiomatic approaches to statistical theories is to characterize vector spaces which are isomorphic to the Hermitian part of a W*-algebra. In recent papers [*5*] Alfsen, Shultz and Størmer characterized the JBW-algebras in the class of order unit spaces admitting a spectral theory in terms of projective units and gave a Gelfand-type representation theory of JBW-algebras in terms of JC-algebras [*7*] [*28*]. Moreover Alfsen, Hanche-Olsen and Shultz showed how a C*-algebraic structure can be obtained from a given JC-algebra [*5*].

Therefore it is interesting to characterize order unit spaces admitting a spectral theory. In a previous paper the authors developed a spectral theory for a particular class of μ-complete order unit spaces in terms of decision effects [*1*]. These order unit spaces are associated to sum logics admitting a μ-complete set of expectation value functions. The concrete representation of sum logics remains an open question. By adding some conditions on the « spectral » order unit spaces arising from sum logics one could obtain JB-algebras. Then Alfsen's representation theory could be used.

Therefore a spectral theory for order unit spaces may be of interest also in the representation theory for sum logics. However, the μ-completeness requirement on sum logics is not generally satisfied and the duality for quantum logics does not completely correspond to duality for order unit spaces [*18*] [*20*]. Therefore we develope here a spectral theory for not necessarily complete order unit spaces. In this theory we do not assume any duality as in the spectral theories in terms of decision effects or projective units.

1. THE SPECTRAL THEORY

In this section we develope a spectral theory for not necessarily complete order unit spaces. In this theory we do not use any particular duality. The spectral theories of Alfsen and Shultz [*4*], Bonnet [*8*] [*9*] and the authors [*1*] are generalized. We follow the notations adopted in [*1*]. Definitions and generalities on ordered linear spaces can be easily found in the classical literature, f. i. in [*3*] [*19*] and [*22*].

From now on we denote by (E, E_+), or simply by E, an order unit space with positive cone E_+ and order unit e. Let Q be any subset of the order interval $[o, e]$. For every subset D of Q, $\bigwedge^{Q} D$ denotes the infimum of D —if there exists— in the partially ordered set Q. Analogously, $\bigvee^{Q} D$ is defined. In the case $Q = [o, e]$, $\bigwedge^{Q} D$ and $\bigvee^{Q} D$ are simply denoted by $\wedge D$ and $\vee D$, respectively. The map $' : [o, e] \to [o, e]$ is defined, for $r \in [o, e]$, by $r' = e - r$. We denote by Q' the image of the set Q under the map $'$. If the map $'$ is restricted to $Q \cup Q'$ and $o \in Q$, then $(Q \cup Q', \leqslant, ')$ becomes a poset with involution (for the definition, see § 3).

Let Q be a subset of $[o, e]$, with $\{ o, e \} \subseteq Q$. Q is said to be *normalized* provided $q \in Q$, $\| q \| \neq 1$ implies $q = o$.

DEFINITION 1.1. — Let Q be any subset of $[o, e]$ containing o and e. A map $s : \mathbb{R} \to Q$ is a Q-valued resolution of the unit if

i) for real λ, $s(\lambda) = \bigvee^{Q} \{ s(\eta), \eta < \lambda \}$;

ii) there exist η, λ in $\mathbb{R}$ such that $s(\eta) = o$ and $s(\lambda) = e$.

For every Q-valued resolution s of the unit

$$\sigma(s) = \{ \lambda \in \mathbb{R}, s(\lambda - \varepsilon) \neq s(\lambda + \varepsilon), \text{ for every } \varepsilon > 0 \}$$

is a bounded not empty set and there exists $|s| = \sup\{|\lambda|, \lambda \in \sigma(s)\}$. To introduce the natural definition of a Riemann-Stieltijes integral, it is convenient to use the term partition of the real interval $[\alpha, \beta]$ to denote a finite sequence $\Lambda = \{\lambda_n, n = o, n(\Lambda)\}$ such that $\lambda_o = \alpha, \lambda_n < \lambda_{n+1}, \lambda_{n(\Lambda)} = \beta$. The norm $|\Lambda|$ of Λ is defined as $|\Lambda| = \max\{\lambda_n - \lambda_{n-1}, n = 1, n(\Lambda)\}$.

DEFINITION 1.2. — Let s be a Q-valued resolution of the unit. A real function f is norm integrable with respect to s on some real interval $[\alpha, \beta)$, provided there exists $r(f, s)$ in E with the following property: for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left\| r(f, s) - \sum_{\Lambda} f(\alpha_n)(s(\lambda_{n+1}) - s(\lambda_n)) \right\| < \varepsilon \tag{1.1}$$

whenever Λ is a partition of $[\alpha, \beta]$ with norm $|\Lambda| < \delta$ and the α_n's belong to $[\lambda_n, \lambda_{n+1}]$ for $0 \leqslant n < n(\Lambda)$.

If f is norm integrable with respect to s on $[\alpha, \beta)$, the element $r(f, s)$ in formula (1.1) is unique. Then $r(f, s)$ is called the norm integral of f with respect to s on $[\alpha, \beta)$ and is denoted by $\int_\alpha^\beta f(\lambda) ds(\lambda)$. In particular, if f is the identity map $i : \lambda \mapsto \lambda$, the corresponding net in (1.1) is a Cauchy net. Thus, if E is norm complete the identity map is norm integrable with respect to every Q-valued resolution of the unit on every interval $[\alpha, \beta)$.

The norm integral of the identity map i with respect to s on the interval $[-|s|, |s| + \varepsilon)$ does not depend on ε, for $\varepsilon > 0$. It is called the norm mean of s and denoted by $\int \lambda ds(\lambda)$. If E is not complete, we consider E as canonically embedded in his Cauchy completion $\bar{E}$ and the Cauchy nets in (1.1) correspond to norm integrals in $\bar{E}$. These definitions can be obviously extended to the case $\beta < \alpha$.

The following lemma concerning Q-valued resolutions of the unit with Q and Q′ normalized sets, will be very useful.

LEMMA 1.1. — Let Q be a normalized subset of $[o, e]$ and a be the norm mean of some Q-valued resolution s of the unit. Then

i) $s(0) = o$ if and only if $a \geqslant o$;
ii) $\|a\| \leqslant |s|$.
If also Q′ is normalized, then
iii) $s(\lambda) = e$ for all $\lambda > 0$ if and only if $a \leqslant o$;
iv) $\|a\| = |s|$.

Proof. — *ii)* We remark that for $|\eta| < |s|$ and $\varepsilon > 0$

$$a = \int_{-|s|}^{|s|+\varepsilon} \lambda ds(\lambda) = \int_{-|s|}^{\eta} \lambda ds(\lambda) + \int_{\eta}^{|s|+\varepsilon} \lambda ds(\lambda)$$

where

$$- | s | s(\eta) \leqslant \int_{-|s|}^{\eta} \lambda ds(\lambda) \leqslant \eta s(\eta) \tag{1.2}$$

and

$$\eta s'(\eta) \leqslant \int_{\eta}^{|s|+\varepsilon} \lambda ds(\lambda) \leqslant (| s | + \varepsilon) s'(\eta) \tag{1.3}$$

so that statement (*ii*) follows from

$$- | s | e \leqslant - | s | s(0) \leqslant \int_{-|s|}^{0} \lambda ds(\lambda) \leqslant \int_{-|s|}^{|s|+\varepsilon} \lambda ds(\lambda) \leqslant \int_{0}^{|s|+\varepsilon} \lambda ds(\lambda)$$
$$\leqslant (| s | + \varepsilon) s'(0) \leqslant (| s | + \varepsilon) e .$$

i) First, assume that $s(0) = 0$. By appling formulae (1.2) and (1.3) in the case $\eta = 0$, we obtain $\int_{-|s|}^{0} \lambda ds(\lambda) = o$ and $o \leqslant \int_{0}^{|s|+\varepsilon} \lambda ds(\lambda) = a$.

Conversely, let a be positive and let $s(0) \neq o$. Then there exists $\eta < 0$ with $s(\eta) \neq o$. Since Q is normalized, there exists x in the dual cone E^*_+, $\| x \| = 1$ such that $x(s(\eta)) = 1$, i. e. $x(s'(\eta)) = 0$. By applying (1.3) and (1.2) we obtain

$$x(a) = x\left(\int_{-|s|}^{\eta} \lambda ds(\lambda)\right) \leqslant \eta x(s(\eta)) = \eta < 0$$

contradicting the hypothesis $a \geqslant o$. We conclude that $a \geqslant o$ implies $s(0) = o$ and statement *i*) is proved.

iii) The proof is analogous to *i*) and is omitted.

iv) We remark that $a + \| a \| e$ is the norm mean of the resolution of the unit $\lambda \mapsto s(\lambda - \| a \|)$.

Analogously, $a - \| a \| e$ is the norm mean of the resolution of the unit $\lambda \mapsto s(\lambda + \| a \|)$. Hence $| s | \leqslant a$ follows from

$$a - \| a \| e \leqslant o \leqslant a + \| a \| e$$

and the statements *i*) and *iii*). ///

We recall that a *projection* of E is any extreme point p of $[o, e]$ or, equivalentely, every $p \in [o, e]$ such that $V(p) \cap V(p') = \{ o \}$, where $V(p)$ denotes the order ideal generated by p. The set P(E) of all projections of E is a normalized subset of $[o, e]$ and the map $'$ is an orthocomplementation on P(E). For every projection p, the linear manifold $C(p) := V(p) + V(p')$ is the direct sum of $V(p)$ and $V(p')$. Elements of $C(p)$ are said to be *compatible* with p. We denote by π_p the corresponding canonical projection of $C(p)$ into its direct summand $V(p)$. Let p and q be mutually compatible projections. We say that p *commutes* with q if $\pi_q p \in V_+(p)$, $\pi_q p' \in V_+(p')$, $\pi_{q'} p \in V_+(p)$ and $\pi_{q'} p' \in V_+(p')$. In this case q commutes with p and p' and $o \leqslant \pi_q p = \pi_p q$. Whenever $p \leqslant q$, then p commutes with q, moreover $\pi_q p = p$, $\pi_q p' = q - p$, $\pi_{q'} p = o$ and $\pi_{q'} p' = q'$ [*1*].

The following proposition characterizes Q-valued resolutions of the unit, whose norm means belong to P(E).

PROPOSITION 1.2. — Let $Q \cup Q'$ be a normalized subset of $[o, e]$. Let $p \in P(E)$ be the norm mean of some Q-valued resolution s of the unit. Then s is the unique Q-valued resolution of the unit representing p and satisfies

i) $s(\lambda) = o$ for $\lambda \leqslant 0$;
ii) $s(\lambda) = p'$ for $0 < \lambda \leqslant 1$;
iii) $s(\lambda) = e$ for $1 < \lambda$.

Moreover p belongs to Q'.

Proof. — Since $p \in [o, e]$, by Lemma 1.1 we obtain *i)* and *iii)*. To prove *ii)*, first we prove that s is constant on $(0, 1)$ and hence on $(0, 1]$. Let $0<\eta<\tau<1$. Then

$$0 \leqslant \eta(s(\tau) - s(\eta)) \leqslant \int_\eta^\tau \lambda ds(\lambda) \leqslant p$$

implies that $s(\tau) - s(\eta)$ belongs to $V(p)$. From

$$p' = \int_0^{1+\varepsilon} (1 - \lambda)ds(\lambda) \geqslant \int_0^\tau (1 - \lambda)ds(\lambda) \geqslant (1 - \tau)s(\tau) \geqslant o \qquad \text{for } \varepsilon > 0,$$

we obtain that $s(\tau) \in V(p')$ and that $s(\tau) - s(\eta) \in V(p) \cap V(p') = \{ o \}$ so that $s(\tau) = s(\eta)$. It follows easily that $\int \lambda ds(\lambda) = s'(1)$. Hence *ii)* holds and p belongs to Q'. ///

The following lemma will be useful for our spectral theory.

LEMMA 1.3. — Let E be an order unit space and P be a subset of P(E). Assume moreover that $(P, \leqslant, ')$ is an orthocomplemented lattice such that for p, q in P $p \wedge q$ exists and $p \wedge q = p \overset{P}{\bigwedge} q$.

i) $q \in V(p)$ for $p, q \in P$ implies $q \leqslant p$.

ii) Two elements p, q in the lattice P are compatible if and only if p commutes with q. In this case, $p \wedge q = \pi_p q$ and $p \vee q = (\pi_{p'} q')'$. In particular, if p and q are orthogonal $p \vee q = p + q$.

iii) P is orthomodular.

Proof. — *i)* We first prove that $q \in V(p)$ for $p, q \in P$ implies $q \in V(p \wedge q)$. Actually, $q \in V(p)$ if and only if there exists $\alpha \in (0, 1)$ such that $o \leqslant \alpha q \leqslant p$. Due to the assumptions on P, we get $\alpha q \leqslant p \wedge q$, so that $q \in V(p \wedge q)$. Since $o \leqslant q - p \wedge q \leqslant q \in V(p \wedge q)$, there exists $\alpha \in (0, 1)$ such that $o \leqslant \alpha(q - p \wedge q) \leqslant (p \wedge q) \wedge (e - p \wedge q)$.

Now, P is an orthocomplemented lattice. Then $p \wedge p' = o$ for every $p \in P$. We conclude that $q = p \wedge q \leqslant p$ and *i)* holds.

ii) Let p commute with q. Then $p = \pi_q p + \pi_{q'} p$ with $o \leqslant \pi_q p \leqslant p \wedge q$ and $o \leqslant \pi_{q'} p \leqslant p \wedge q'$. Therefore, $p \leqslant p \wedge q + p \wedge q'$. Moreover, $p \wedge q + p \wedge q' \leqslant q + q' = e$ and $p \wedge q + p \wedge q' \in V(p)$ and, since p is a maximal element of $V(p) \cap [o, e]$, $p = p \wedge q + p \wedge q'$. Since $p \wedge q \in V(q)$ and $p \wedge q' \in V(q')$, $p \wedge q$ equals $\pi_q p$. Therefore, $p \vee q = (p' \wedge q')'$ equals $(\pi_{q'} p')' = p + q - \pi_q p$.

Moreover, $\pi_q p$, $\pi_{q'} p$ and $\pi_{p'} q$ are orthogonal elements in P such that $p = \pi_{q'} p + \pi_q p$ and $q = \pi_{p'} q + \pi_q p$. Therefore there exist in P mutually orthogonal elements p_o, q_o and r such that $p_o + r = p$ and $q_o + r = q$, i. e. p and q are compatible elements in the lattice P.

Conversely, suppose that p and q are compatible in P. Standard arguments show that $p_o = p \wedge q' \in V_+(p) \cap V_+(q')$, $q_o = q \wedge p' \in V_+(q) \cap V_+(p')$ and $r = p \wedge q \in V_+(p) \cap V_+(q)$.

We conclude that p commutes with q. In particular, orthogonal p and q commute, moreover $\pi_q p = o$ holds. Hence $p \vee q = p + q$.

iii) For $p \leqslant q$ we get $q = p + (q - p) = p + \pi_{p'} q = p \vee (p' \wedge q)$, i. e. P is an orthomodular lattice. ///

Let P satisfy the assumptions of Lemma 1.3. For every P-valued resolution s of the unit and $p \in P$ commuting with $s(\lambda)$ for all real λ, the map s_p is defined by

$$s_p(\lambda) = \begin{cases} s(\lambda) \wedge p \text{ for } \lambda \leqslant 0 \\ s(\lambda) \wedge p + p' \text{ for } \lambda > 0 . \end{cases}$$

The meaning of s_p is clarified by the following proposition.

PROPOSITION 1.4. — *i*) The map $\lambda \mapsto s(\lambda)$ above defined is a P-valued resolution of the unit. Moreover, $|s| = \text{Max}(|s_p|, |s_{p'}|)$.

ii) If s and s_p have means in E and $V(p)$ and $V(p')$ are closed, then the norm mean a of s belongs to $C(p)$ and $\pi_p a = \int \lambda ds_p(\lambda)$; if moreover $a \geqslant o$, then $\pi_p a \geqslant o$.

iii) If π_p is a continuous map and the norm mean of s belongs to $C(p)$ then s_p has norm mean in E.

Proof. — *i*) By Lemma 1.3 the map $\lambda \mapsto s_p(\lambda)$ has range in $P \subseteq P(E)$. Therefore, to prove the first statement in *i*) it is enough to prove that for every real λ

$$s(\lambda) \wedge p = \bigvee^{P} \{ s(\eta) \wedge p, \eta < \lambda \} .$$

For $\eta < \lambda$, $s(\eta) \leqslant s(\lambda)$ implies $s(\eta) \wedge p \leqslant s(\lambda) \wedge p$. Let $s(\eta) \wedge p \leqslant r$ for some $r \in P$ and every $\eta < \lambda$. Then $s(\eta) \wedge p \leqslant r \wedge p$ and

$$s(\eta) = s(\eta) \wedge p + s(\eta) \wedge p' \leqslant (r \wedge p) + p' = (r \wedge p) \vee p'$$

which implies $s(\lambda) \leqslant (r \wedge p) \vee p'$ so that $s(\lambda) \wedge p \leqslant ((r \wedge p) \vee p') \wedge p = r \wedge p \leqslant r$, as required. Using the formula

$$s(\lambda) = \begin{cases} s_p(\lambda) + s_{p'}(\lambda) \text{ for } \lambda \leqslant 0 \\ s_p(\lambda) + s_{p'}(\lambda) - e \text{ for } \lambda > 0 \end{cases} \tag{1.4}$$

one easily obtains that $\sigma(s) \cup \{ 0 \} = \sigma(s_p) \cup \sigma(s_{p'}) \cup \{ 0 \}$, so that statement *i*) follows.

ii) Using Formula (1.4) it is easily checked that $\int \lambda ds_{p'}(\lambda)$ also exists and $a = \int \lambda ds_{p'}(\lambda) + \int \lambda ds_{p'}(\lambda)$. Since V($p$) is closed, $\int \lambda ds_p(\lambda) \in \mathrm{V}(p)$ and, similarly, $\int \lambda ds_{p'}(\lambda) \in \mathrm{V}(p')$.

Let a be positive. Then by *i*), Lemma 1.1 $s(0) = o$, so that $s_p(0) = o$ and hence $\pi_p a \geqslant o$. This proves *ii*).

The proof of *iii*) is trivial. ///

Remark that $s'(0)$ commutes with $s(\lambda)$ for all real λ. Hence the resolution $s_+ := s_{s'(0)}$ of the unit is defined and has the simple form:

$$s_+(\lambda) = \begin{cases} o \text{ for } \lambda \leqslant 0 \\ s(\lambda) \text{ for } \lambda > 0 . \end{cases}$$

In the sequel we are particularly interested in some special projections, introduced by Bonnet [8] [9]. A $p \in \mathrm{E}_+$ such that $\mathrm{V}(p) \cap [-e, e] = [-p, p]$ is said to be an *F-projection*. We denote by F(E) the set of the F-projections of E. For $p \in \mathrm{F(E)}$ the order interval $[-p, p]$ equals the relativized unit ball and therefore the ideal V(p) is a norm closed subspace of E.

In this section we shall characterize those order unit spaces, every element of which is the norm mean of some P-valued resolution of the unit, where P is assumed to be a set of F-projections such that $o \in \mathrm{P} = \mathrm{P}'$.

Remark. — Let $a \in \mathrm{E}_+$ and let $-a$ be the norm mean of some P-valued resolution s of the unit, with P as above. As a consequence of *iii*), Lemma 1.1, it is easily obtained that $-a = \int_{-|s|}^{0} \lambda ds(\lambda)$ and for every integer n

$$-a = \int_{-|s|}^{-\frac{1}{n}} \lambda ds(\lambda) + \int_{-\frac{1}{n}}^{0} \lambda ds(\lambda) \leqslant \int_{-|s|}^{-\frac{1}{n}} \lambda ds(\lambda) \leqslant -\frac{1}{n} s\left(-\frac{1}{n}\right)$$

so that $o \leqslant s\left(-\frac{1}{n}\right) \leqslant na$. Hence $\left\{ s\left(-\frac{1}{n}\right), n \in \mathbb{N} \right\}$ is contained in V(a). Moreover, we obtain

$$s(0) = \bigwedge^{\mathrm{P}} \{ q \in \mathrm{P}, a \in \mathrm{V}(q) \} .$$

Assume indeed, $q \in \mathrm{P}$ and $a \in \mathrm{V}(q)$. Then $s\left(-\frac{1}{n}\right) \in \mathrm{V}(a) \subseteq \mathrm{V}(q)$. Since $q \in \mathrm{F(E)}$, we get $s\left(-\frac{1}{n}\right) \leqslant q$. By $s(0) = \bigvee^{\mathrm{P}} \left\{ s\left(-\frac{1}{n}\right), n \in \mathbb{N} \right\}$, $s(0) \leqslant q$ follows. We deduce that there exists $\bigvee^{\mathrm{P}} (\mathrm{P} \cap \mathrm{V}(a)) = s(0)$ and $a \in \mathrm{V}(s(0))$.

From now on we assume that E satifies the following axiom.

Axiom I. — There exists a subset P of F(E) such that $\mathrm{P} \subseteq \mathrm{P}'$ and for every $a \in \mathrm{E}_+$

i) there exists $\bigvee^{\mathrm{P}} (\mathrm{P} \cap \mathrm{V}(a))$;

ii) a belongs to the order ideal generated by $\bigvee^{\mathrm{P}} (\mathrm{P} \cap \mathrm{V}(a))$.

If Axiom I is assumed, the element $\bigvee^{\mathrm{P}} (\mathrm{P} \cap \mathrm{V}(a))$ is denoted by $e(a)$.

Then there exists $\bigwedge^{\mathrm{P}} \{ q \in \mathrm{P}, a \in \mathrm{V}(q) \}$ and equals $e(a)$.

Proposition 1.5. — Let E be an order unit space satisfying Axiom I.

i) P is an orthocomplemented lattice and consists exactly of the projections p such that $e(p)$ belongs to $\mathrm{V}(p)$. For every subset D of P

a) there exists $\wedge$ D if and only if $\bigwedge^{\mathrm{P}}$ D exists; in this case,

$$\wedge \mathrm{D} = \bigwedge^{\mathrm{P}} \mathrm{D}.$$

b) there exists $\vee$ D if and only if $\bigvee^{\mathrm{P}}$ D exists; in this case,

$$\vee \mathrm{D} = \bigvee^{\mathrm{P}} \mathrm{D}.$$

ii) P equals F(E) and is orthomodular.
iii) If E is a Banach space, then P is a σ-lattice.

Proof. — *i*) In the case $a = e$, condition *i*) of Axiom I guarantees that there exists $p \in P$ such that $V(p) = E$. Therefore e belongs to P. Since $P \subseteq P'$ implies $P' \subseteq P'' = P$, the map $'$ is an involution on $(P, \leqslant)$, where $\leqslant$ denotes the ordering induced on P by E_+. From $e = p \overset{P}{\bigvee} p'$ for every $p \in P$ we conclude that $(P, \leqslant, ')$ is an orthocomplemented poset. Let $D \subseteq P$ admit the infimum in $[o, e]$, denoted by p. Then $p \leqslant e$ and $e(p) \in F(E)$ imply $p \leqslant e(p)$. From $p \leqslant q$, for $q \in D$ we obtain $p \in V(q)$ and hence $e(p) = \overset{P}{\bigwedge} \{ r \in P, p \in V(r) \} \leqslant q$ for all $q \in D$ so that $e(p) \leqslant p$. We get $p = e(p) \in P$ so that $p = \wedge D = \overset{P}{\bigwedge} D$. Conversely, suppose that there exists $\overset{P}{\bigwedge} D$. For every r in $[o, e]$, with $r \leqslant q$ for all $q \in D$, we get $e(r) \in V(q)$ so that $r \leqslant e(r) \leqslant q$. We deduce that there exists $\wedge D$ and equals $\overset{P}{\bigwedge} D$. The dual statements in *b*) hold, since the map $'$ is an involution on $[o, e]$ and on the subset P of $[o, e]$.

To prove that P is a lattice, one needs just to show that, for p, q in P, there exists $p \vee q$, since in this case $p \wedge q$ exists and equals $(p' \vee q')'$. From $\{ p, q \} \subseteq V(p + q) \subseteq V(e(p + q))$ one gets $p, q \leqslant e(p + q)$. For every $r \in P$ such that $p, q \leqslant r$, $p + q$ belongs to $V(r)$ and hence $e(p + q) \leqslant r$. We conclude that there exists $p \vee q$ and equals $e(p + q)$.

Finally, let $e(q) \in V(q)$ for some projection q. From $q \leqslant e(q) \in V(q)$ and *iv*), Proposition 2.1 of [*1*], one obtains $q = e(q) \in P$.

Conversely, for $p \in P \subseteq F(E)$, condition *ii*) of Axiom I implies that $e(p) = p \in V(p)$.

ii) Let $p \in F(E)$. For every $q \in P \cap V(p)$, $q \leqslant p$. Hence

$$e(p) = \vee (P \cap V(p)) \leqslant p .$$

Therefore, $e(p) \in V(p)$. By *i*) of this proposition we obtain that $p \in P$ so that P equals F(E). By Lemma 1.3 F(E) is an orthomodular lattice.

iii) Finally, let E be complete. For every sequence $\{ p_n, n \in \mathbb{N} \} \subseteq P$ there exists in E_+ the element $a = \sum_n 2^{-n} p_n$. For every integer n, $p_n \leqslant e(a)$. Suppose conversely that for some $r \in P$ and all integer n, $p_n \leqslant r$. Then $a \in V(r)$ implies $e(a) \leqslant r$. We conclude that $e(a) = \vee \{ p_n, n \in \mathbb{N} \}$ and that P is a σ-lattice. ///

Let s be an F(E)-valued resolution of the unit. Suppose that the norm means a and a_+ of s and s_+, respectively, belong to E. Then one easily shows that

$$a = a_+ - (a - a_+) \quad \text{with} \quad a_+ \in V_+(s'(0)) \quad \text{and} \quad a - a_+ \in V_+(s(0)).$$

If E is consistent with a suitable spectral theory (f. i. if E is a Banach space), then a_+ belongs to E for every $a \in E$. Therefore, the following definition is of interest for every spectral theory in terms of F-projections.

DEFINITION 1.3. — We say that $a \in E$ admits an orthogonal decomposition if an F-projection p exists such that $a \in C(p)$ and $\pi_{p'}a \leqslant o \leqslant \pi_p a$. Then p is said to induce an orthogonal decomposition of a.

Let p induce an orthogonal decomposition of $a \in E$. Then $e(\pi_p a) \leqslant p$ and $e(-\pi_{p'}a) \leqslant p' \leqslant e'(\pi_p a)$ so that $e(\pi_p a)$ induces the same orthogonal decomposition as p. Moreover, one easily shows that

$$a \geqslant o \text{ if and only if } \pi_{p'}a = o \quad \text{and} \quad a \leqslant o \text{ if and only if } \pi_p a = o. \tag{1.5}$$

The next lemma will be crucial for our spectral theory.

LEMMA 1.6. — Let p, q be F-projections such that $a \in C(p) \cap C(q)$ with $\pi_p a \leqslant o \leqslant \pi_q a$. Then $p \wedge e(\pi_q a) = o$.

Proof. — Since $\pi_{q'}a \leqslant \pi_q a + \pi_{q'}a = a = \pi_p a + \pi_{p'}a \leqslant \pi_{p'}a$, both a and $\pi_q a = a - \pi_{q'}a$ belong to $V(p' \vee q')$. Therefore $e(\pi_q a) \leqslant p' \vee q'$, i. e. $p \wedge q \leqslant e'(\pi_q a)$. But $e(\pi_q a) \leqslant q$ and hence

$$p \wedge e(\pi_q a) \leqslant p \wedge q \leqslant e'(\pi_q a)$$

which implies $p \wedge e(\pi_q a) = o$. ///

COROLLARY 1.7. — Let $a = a_1 - a_2 = a_3 - a_4$ be orthogonal decompositions of some a in E such that $e(a_1)$ commutes with $e(a_3)$. Then $a_1 = a_3$ and $a_2 = a_4$.

Proof. — By Lemma 1.6, $e(a_1) \wedge e'(a_3) = e(a_3) \wedge e'(a_1) = o$. Hence, by *ii*) Lemma 1.3

$$e(a_1) = e(a_1) \wedge e(a_3) + e(a_1) \wedge e'(a_3) = e(a_1) \wedge e(a_3) = e(a_3)$$

and the statements follows. ///

PROPOSITION 1.8. — Let $a = a_+ - a_-$ be an orthogonal decomposition of a. Then

$$e(a_+) = \wedge \{p \in F(E), p \text{ commuting with } e(a_+), a \in C(p) \text{ and } \pi_{p'}a \leqslant o\}.$$

Proof. — By Lemma 1.6, $e(a_+) \wedge p' = o$ for every p commuting with $e(a_+)$, such that $a \in C(p)$ and $\pi_{p'}a \leqslant o$. By *ii*) Lemma 1.3, $e(a_+) \wedge p \leqslant p$. ///

PROPOSITION 1.9. — Assume that every $a \in E$ admits a unique orthogonal decomposition and that

$$a \in V(p), \quad p \in F(E) \quad \text{implies} \quad a_+, a_- \in V(p). \tag{1.6}$$

Then, for every F-projection q

i) $a \in C(q)$ if and only if a_+ and a_- belong to $V_+(q) + V_+(q')$. In this case

$$a_+ = (\pi_q a)_+ + (\pi_{q'} a)_+ \quad \text{and} \quad a_- = (\pi_q a)_- + (\pi_{q'} a)_- ;$$

ii) the map π_q is a closed continuous positive linear map of $C(q)$ onto $V(q)$;

iii) a projection p belongs to $C(q)$ if and only if p commutes with q.

Proof. — For every $a \in C(q)$, let p_+ and p_- denote the F-projections $e((\pi_q a)_+) + e((\pi_{q'} a)_+$ and $e((\pi_q a)_- + e((\pi_{q'} a)_-)$, respectively. We remark that $p_+ + p_- = e((\pi_q a)_+) + e((\pi_{q'} a)_+) + e((\pi_q a)_-) + e((\pi_{q'} a)_-)$. By (1.6), $p_+ + p_- \leqslant q + q' = e$, so that p_+ and p_- are orthogonal. Moreover, $a = \pi_q a + \pi_{q'} a = (\pi_q a)_+ + (\pi_{q'} a)_+ - (\pi_q a)_- - (\pi_{q'} a)$. Since

$$(\pi_q a)_+ + (\pi_{q'} a)_+ \in V_+(p_+) \quad \text{and} \quad (\pi_q a)_- + (\pi_{q'} a)_- \in V_+(p_-) \subseteq V_+(p'_+),$$

their difference a belongs to $C(p_+)$ with

$$o \leqslant \pi_{p_+} a = (\pi_q a)_+ + (\pi_{q'} a)_+ \quad \text{and} \quad \pi_{p_-} a = -((\pi_q a) + (\pi_{q'} a)_- \leqslant o .$$

By the uniqueness of the orthogonal decomposition of a we obtain *i)*.

ii) Let $a \in [-e, e] \cap C(q)$, for some $q \in F(E)$. By statement *i)*, π_q is a positive linear map of $C(q)$ onto $V(q)$ and $-e \leqslant a \leqslant e$ implies $-q \leqslant \pi_q a \leqslant q$. Therefore π_q is a continuous map. Let $\{ a_n, n \in \mathbb{N} \} \subseteq C(q)$ be a sequence converging to $a \in E$ and such that $\pi_q(a_n)$ converges in E. Since $V(q)$ is closed, $\lim_n \pi_q(a_n)$ belongs to $V(q)$. Also $\lim_n \pi_{q'} a_n$ exists and belongs to $V(q')$. By $a = \lim_n \pi_q a_n + \lim_n \pi_{q'} a_n$ we obtain that $\pi_q a = \lim_n \pi_q a_n$. Hence, π_q is a closed map.

iii) By statement *i)* of this proposition, $p = p_+ = (\pi_q p)_+ + (\pi_{q'} p)_+$ for all $p \in P(E) \cap C(q)$. By statement *ii)*

$$\pi_q p = (\pi_q p)_+ \quad \text{and} \quad \pi_{q'} p = (\pi_{q'} p)_+ .$$

It follows that $o \leqslant \pi_q p \leqslant p$ and $o \leqslant \pi_{q'} p$. Hence $\pi_q p$ and $\pi_{q'} p$ belong to $V_+(p)$. Similarly, $\pi_q p'$ and $\pi_{q'} p'$ belong to $V_+(p')$. We conclude that p commutes with q, as required. ///

We are interested in orthogonal decompositions $a = a_+ - a_-$ where $e(a_+)$ « bicommutes » with a according to the following definition.

DEFINITION 1.4. — Let $p \in F(E)$ and $a \in E$. We say that p bicommutes with a if p commutes with every $q \in F(E)$ such that $a \in C(q)$.

If $a \in E$ admits an orthogonal decomposition $a = a_+ - a_-$ with $e(a_+)$

bicommuting with a, then by Corollary 1.7, a admits a unique orthogonal decomposition and

$$e(a_+) = \wedge \{ p \in \mathrm{F(E)} : a \in \mathrm{C}(p) \quad \text{and} \quad \pi_{p'} a \leqslant o \} .$$

Moreover, one easily proofs that

$$a \in \mathrm{V}(p), \ p \in \mathrm{F(E)} \quad \text{implies} \quad a_+, a_- \in \mathrm{V}(p) ; \tag{1.6}$$
$$e(a) = \wedge \{ p \in \mathrm{F(E)} : a \in \mathrm{V}(p) \} , \tag{1.7}$$

where $e(a)$ denotes the F-projection $e(a_+) + e(a_-)$.

From now on, E is supposed to satisfy the following axiom.

AXIOM II. — E is an order unit space satisfying Axiom I and every element a of E admits an orthogonal decomposition $a = a_+ - a_-$, where $e(a_+)$ bicommutes with a.

For every a in E and real λ, $s^a(\lambda) = e((\lambda e - a)_+)$ is defined. When a is understood, s^a is simply denoted by s and the corresponding maps $\pi_{(s^a(\lambda))}$ and $\pi_{(s^a(\lambda))'}$ are denoted by π_λ and $\pi_{\lambda'}$, respectively. The maps $s^a : \mathbb{R} \to \mathrm{F(E)}$, $s^a(\lambda) = e((\lambda e - a)_+)$ are the basic tools for the spectral theory as the following propositions show.

LEMMA 1.10. — For every a in E and real η, λ, with $\eta < \lambda$

$$o \leqslant (\lambda - \eta)s^a(\eta) \leqslant (\lambda e - a)_+ - (\eta e - a)_+ \leqslant (\lambda - \eta)s^a(\lambda) .$$

Proof. — Since $\lambda e - a = (\lambda - \eta)e + (\eta e - a)$ then, for each F-projection p, $\lambda e - a \in \mathrm{C}(p)$ if and only if $\eta e - a \in \mathrm{C}(p)$ and in this case,

$$\pi_p(\lambda e - a) = (\lambda - \eta)p + \pi_p(\eta e - a) \geqslant \pi_p(\eta e - a) .$$

In particular, $\lambda e - a \in \mathrm{C}(s(\eta))$ and by i), Proposition 1.9

$$(\lambda e - a)_+ = (\pi_\eta(\lambda e - a))_+ + (\pi'_\eta(\lambda e - a))_+ \geqslant (\pi_\eta(\lambda e - a))_+$$

and

$$(\eta e - a)_- = (\pi_\lambda(\eta e - a))_- + (\pi'_\lambda(\eta e - a))_- \geqslant (\pi'_\lambda(\eta e - a))_- \geqslant -\pi'_\lambda(\eta e - a)$$

and therefore

$$o \leqslant (\lambda - \eta)s(\eta) = \pi_\eta(\lambda e - a) - \pi_\eta(\eta e - a) \leqslant (\pi_\eta(\lambda e - a))_+ - (\eta e - a)_+ \leqslant (\lambda e - a)_+ - (\eta e - a)_+$$

and

$$o \leqslant (\lambda - \eta)s'(\lambda) = \pi'_\lambda(\lambda e - a) - \pi'_\lambda(\eta e - a) = -(\lambda e - a)_- - \pi'_\lambda(\eta e - a) \leqslant (\eta e - a)_- - (\lambda e - a)_- .$$

We conclude that

$$o \leqslant (\lambda - \eta)s(\eta) \leqslant (\lambda e - a)_+ - (\eta e - a)_+ = \lambda e - a - (\eta e - a) + (\lambda e - a)_- - (\eta e - a)_-$$
$$= (\lambda - \eta)e + (\lambda e - a)_- - (\eta e - a)_- \leqslant (\lambda - \eta)s(\lambda),$$

as required. ///

Thus, we obtain the main theorem of our spectral theory.

PROPOSITION 1.11. — For every element a of E

i) the map s^a is an F(E)-valued resolution of the unit, with

$$s^a(\lambda) = o \quad \text{for} \quad \lambda \leqslant -\|a\| \quad \text{and} \quad s^a(\lambda) = e \quad \text{for} \quad \lambda > \|a\|;$$

ii) for every real τ there exists in E

$$\int_{-\|a\|}^{\tau} \lambda ds^a(\lambda)$$

and equals $\pi_\tau a$;

iii) a is the norm mean of s^a, i. e.

$$a = \int \lambda ds^a(\lambda) = \int_{-\|a\|}^{\|a\|+\varepsilon} \lambda ds^a(\lambda), \qquad \text{for every } \varepsilon > 0.$$

Proof. — *i)* We notice that $(\lambda - \|a\|)e \leqslant \lambda e - a \leqslant (\lambda + \|a\|)e$. Hence, for $\lambda \leqslant \|a\|$, $\lambda e - a$ is negative and therefore $(\lambda e - a)_+ = o$ and $s(\lambda) = o$. For $\lambda > \|a\|$, $\lambda e - a \geqslant (\lambda - \|a\|)e$ implies $(\lambda e - a)_+ \geqslant (\lambda - \|a\|)e$. Hence e belongs to $V((\lambda e - a)_+)$ and $s(\lambda)$ equals e. For $\eta < \lambda$, Lemma 1.10 implies that $s(\eta) \leqslant s(\lambda)$ and that

$$o \leqslant (\lambda e - a)_+ - (\eta e - a)_+ \leqslant (\lambda - \eta)s(\lambda) \leqslant (\lambda - \eta)e.$$

Therefore $(\lambda e - a)_+ = \lim_{\eta \to \lambda_-} (\eta e - a)_+$ and hence, for any $p \in F(E)$ such that $s(\eta) \leqslant p$ for all $\eta < \lambda$, we obtain that $(\lambda e - a)_+ \in V(p)$ and that $s(\lambda) \leqslant p$. We conclude that $s(\lambda) = \vee \{ s(\eta), \eta < \lambda \}$ so that s is an F(E)-valued resolution of the unit.

ii) For all partition $\wedge = \{ \lambda_n, 0 \leqslant n \leqslant n(\wedge) \}$ of the interval $[-\|a\|, \tau]$, for $\tau > -\|a\|$, we set

$$s_\wedge = \sum_\wedge \lambda_n(s(\lambda_{n+1}) - s(\lambda_n)) \quad \text{and} \quad s^\wedge = \sum_\wedge \lambda_{n+1}(s(\lambda_{n+1}) - s(\lambda_n)).$$

By statement *i)* of this proposition, we must only show that

$$s_\wedge \leqslant \pi_\tau(a) \leqslant s^\wedge \quad \text{with} \quad \|s^\wedge - s_\wedge\| < |\wedge|.$$

By Lemma 1.10 and the formula

$$\pi_\lambda a = \pi_\lambda(\lambda e - (\lambda e - a)) = \lambda s(\lambda) - (\lambda e - a)_+,$$

we get

$$o \leqslant \eta(s(\lambda) - s(\eta)) = \lambda s(\lambda) - \eta s(\eta) - (\lambda - \eta)s(\lambda) \leqslant \lambda s(\lambda) - \eta s(\eta) - (\lambda e - a)_+ + (\eta e - a)_+$$
$$= \pi_\lambda a - \pi_\eta a \leqslant \lambda s(\lambda) - \eta s(\eta) - (\lambda - \eta)s(\eta) = \lambda(s(\lambda) - s(\eta))$$

and therefore,

$$\eta(s(\lambda) - s(\eta)) \leqslant \pi_\lambda a - \pi_\eta a \leqslant \lambda(s(\lambda) - s(\eta)) . \tag{1.9}$$

Since $\pi_{\lambda_o} a = o$, we get $\pi_\tau a = \sum_\wedge (\pi_{\lambda_{n+1}}(a) - \pi_{\lambda_n}(a))$. We obtain that, for every partition $\wedge$ of the interval $[- \| a \|, \tau]$

$$s_\wedge \leqslant \pi_\tau a \leqslant s^\wedge$$

with

$$o \leqslant s^\wedge - s_\wedge = \sum_\wedge (\lambda_{n+1} - \lambda_n)(s(\lambda_{n+1}) - s(\lambda_n)) \leqslant | \wedge | e ,$$

i. e. $\| s^\wedge - s_\wedge \| < | \wedge |$, as required. For $\tau \leqslant - \| a \|$ the statement is obvious.

iii) follows by choosing $\tau > \| a \|$ in *ii*). ///

COROLLARY 1.12. — P(E) = F(E).

Proof. — Immediate from Proposition 1.11 and Proposition 1.2. ///

PROPOSITION 1.13. — Let s be any P(E)-valued resolution of the unit with norm mean a and such that $a \in C(s(\lambda))$, for every real λ. Then $s = s^a$.

Proof. — By Corollary 1.12, s is an F(E)-valued resolution of the unit. For real η, λ $s(\eta) \in C(s(\lambda))$ so that, for every partition $\wedge$ of the interval $[- | s |, | s | + \varepsilon]$, $\varepsilon > 0$ $s^\wedge$ belongs to $C(s(\lambda))$. Moreover,

$$\pi_{s(\lambda)}(s^\wedge) = \sum_{\lambda_{n+1} \leqslant \lambda} \lambda_{n+1}(s(\lambda_{n+1}) - s(\lambda_n)) \leqslant \lambda s(\lambda)$$

and

$$\pi_{s'(\lambda)}(s^\wedge) \geqslant \sum_{\lambda \leqslant \lambda_n} \lambda_{n+1}(s(\lambda_{n+1}) - s(\lambda_n)) \geqslant \lambda s'(\lambda) .$$

By *ii*), Proposition 1.9, the map $\pi_{s(\lambda)}$ is continuous. From $a \in C(s(\lambda))$ and $a = \lim_{|\wedge| \to 0} s^\wedge$ we obtain

$$\pi_{s'(\lambda)}(\lambda e - a) \leqslant o \leqslant \pi_{s(\lambda)}(\lambda e - a) .$$

Hence $s(\lambda)$ induces an orthogonal decomposition of $\lambda e - a$ and therefore $s^a(\lambda) \leqslant s(\lambda)$. We remark that $\{ s^\wedge \}$ is a monotone decreasing net converging to a, so that

$$(\lambda e - a)_+ = \pi_{s(\lambda)}(\lambda e - a) = \lambda s(\lambda) - \pi_{s(\lambda)} a$$

is the limit of the monotone increasing net

$$\{ \Sigma(\lambda - \lambda_{n+1})(s(\lambda_{n+1}) - s(\lambda_n)), \lambda_i < \lambda \text{ and } \lambda_i \in \wedge \} .$$

From $(\lambda - \lambda_1)(s(\lambda_1) - s(\lambda_0)) = (\lambda - \lambda_1)s(\lambda_1) \leqslant (\lambda e - a)_+ \in V(s^a(\lambda))$ we obtain

$s(\lambda_1) \leqslant s^a(\lambda)$ for every partition $\wedge$ and every $\lambda_1 < \lambda$. We conclude that $s(\lambda) \leqslant s^a(\lambda)$, as required. ///

Remark. — If for every $p \in P(E)$, $C(p)$ is closed (f. i. if E is a Banach space), then the requirement $a \in C(s(\lambda))$ in Proposition 1.13 can be removed. In this case, s^a is the unique P(E)-valued resolution of the unit with norm mean a. Actually, for every real λ, the vectors $s^\wedge$ —and hence $a = \lim_\wedge s^\wedge$ — belong to $C(s(\lambda))$.

We conclude this section with a simple example of a not complete order unit space E satisfying Axiom II, with F(E) a complete lattice and F(E)* order determining on E (a definition of F(E)* is given in § 2). Consider any abelian W*-algebra A. The Hermitian part A_h of A is an order unit space whose F-projections are exactly the projections of the algebra A and the states in the predual of A are an order determining set of completely additive measures on $P(A_h)$. Consider now the linear manifold E in A_h generated by $P(A_h)$. When ordered by the cone E_+ consisting of linear combinations with positive coefficients of elements of $P(A_h)$, E results an order unit space satisfying Axiom II. Actually, F-projections of E are exactly the projections of A and the predual cone induces on E an order determining set of F(E)-states.

2. *Q*-VALUED MEASURES

Let E be an order unit space with order unit e and let Q be a subset of $[o, e]$ such that $o \in Q = Q'$. In this section we investigate the relationship between Q-valued resolutions of the unit and Q-valued measures on the natural σ-algebra $B(\mathbb{R})$ of Borel sets of the real line.

DEFINITION 2.1. — A map $m : B(\mathbb{R}) \to Q$ is said to be a Q-valued measure if

i) $m(\mathbb{R}) = e$;

ii) if Δ_1 and Δ_2 are disjoint Borel sets,

$$m(\Delta_1 \cup \Delta_2) = m(\Delta_1) + m(\Delta_2);$$

iii) for every increasing sequence $\{ \Delta_n \}$ of Borel sets,

$$\bigvee^{Q} \{ m(\Delta_n), n \in \mathbb{N} \} \quad \text{exists and equals} \quad m\Big(\bigcup_n \Delta_n\Big).$$

The *support* $\sigma(m)$ of m is defined as the intersection of all closed subsets Δ of $\mathbb{R}$ such that $m(\Delta) = e$. If $\sigma(m)$ is a compact set, the measure m is said to be *bounded* and $\sup \{ | \lambda |, \lambda \in \sigma(m) \}$ is denoted by $| m |$.

PROPOSITION 2.1. — For every bounded Q-valued measure m, the mapping $\mathcal{S}(m)$ defined for real λ by

$$\mathcal{S}(m)(\lambda) = m((-\infty, \lambda))$$

is a Q-valued resolution of the unit. Moreover,

i) $\sigma(m) = \sigma(\mathcal{S}(m))$;
ii) $|m| = |\mathcal{S}(m)|$;
iii) $\sigma(m) \subseteq [0, +\infty)$ if and only if $\mathcal{S}(m)(0) = o$.

Proof. — The proof is standard. ///

PROPOSITION 2.2. — For every Q, $\mathcal{S}$ is an injective map.

Proof. — Let m, n be Q-valued measures such that $\mathcal{S}(m) = \mathcal{S}(n)$. We denote by $B(n, m)$ the family of all Borel subsets Δ of the real line, such that $m(\Delta) = n(\Delta)$. For $\Delta \in B(n, m)$,

$$m(\Delta^c) = e - m(\Delta) = e - n(\Delta) = n(\Delta^c),$$

so that also Δ^c belongs to $B(n, m)$. By hypothesis $(-\infty, \lambda)$ belongs to $B(n, m)$ for all real λ. Since

$$m([\eta, \lambda)) = m((-\infty, \lambda)) - m((-\infty, \eta)),\ \eta < \lambda,\ [\eta, \lambda) \in B(n, m)$$

and, by additivity, $B(n, m)$ contains the ring of all disjoint unions of intervals of the form $(-\infty, \lambda)$, $[\eta, \lambda)$, $[\lambda, +\infty)$. Clearly, $B(n, m)$ is a monotone class. Hence by the lemma on monotone classes [*17*, § 6, Theorem B], $B(n, m) = B(\mathbb{R})$ as required. ///

There are difficulties to extend any Q-valued resolution of the unit to a Q-valued measure. Under the assumption that Q equals the interval $[o, e]$ of a μ-complete order unit space, admitting an order determining set of σ-normal functionals, the map $\mathcal{S}$ was proved to be a bijection of the set of all bounded Q-valued measures onto the set of all Q-valued resolutions of the unit [*2*]. Before to prove a similar statement for Q-valued measures, we introduce some definitions.

We call *Q-state* every positive linear functional x on E such that $x(e) = 1$ and that, if $q = \bigvee^{Q} \{q_n\}$, with $\{q_n\} \subseteq Q$ a monotone sequence, then

$$x(q) = \sup_n x(q_n).$$

We denote by Q^* the convex set of all Q-states.

The map $\tilde{}: E \to A^b(Q^*)$, defined for $x \in Q^*$ and $a \in E$ by

$$\tilde{a}(x) = x(a)$$

is a linear positive map onto a subspace $\tilde{E}$ of the μ-complete order unit space $A^b(Q^*)$ of all bounded real valued affine functions on Q^*; it is an order isomorphism if Q* is order determining on E. In this case, for every increasing sequence $\{ q_n \} \subseteq Q$, such that there exists $\bigvee^{Q} \{ q_n \} = q$, the supremum of $\{ q_n \}$ exists in E and equals q; thus, every σ-normal linear functional on E is a Q-state.

PROPOSITION 2.3. — Let Q be a σ-poset (for the definition, see § 3) with Q* order determining on Q. Then $\mathscr{S}$ is a bijection onto the set of Q-valued resolutions of the unit if and only if :

$$p, q \in Q \quad \text{and} \quad p + q \leqslant e \quad \text{imply} \quad p + q \in Q. \qquad (2.1)$$

Proof. — Assume Formula (2.1). By Proposition 2.2 we must only prove that $\mathscr{S}$ is onto. We denote by $\tilde{Q}$ the image of Q under the above defined map

$$\tilde{} : E \to A^b(Q^*).$$

Note that if $p, q \in Q$ and $p \leqslant q$, then $p + q' \leqslant e$ so $p + q' \in Q$ and $(q - p)' = p + q' \in Q$ and therefore $\tilde{q} - \tilde{p} \in \tilde{Q}' = \tilde{Q}$.

For every Q-valued resolution s of the unit and every $x \in Q^*$, we denote by μ_x the unique real valued measure such that, for real λ

$$\mu_x((-\infty, \lambda)) = x(s(\lambda)).$$

Let $\tilde{m} : B(\mathbb{R}) \to A^b(Q^*)$:

$$\tilde{m}(\Delta)(x) = \mu_x(\Delta), \qquad \text{for every } x \in Q^*.$$

Actually, $o \leqslant \tilde{m}(\Delta) \leqslant 1$.

Let $B(s) = \{ \Delta \in B(\mathbb{R}) : \tilde{m}(\Delta) \in \tilde{Q} \}$. Arguing as in the proof of Proposition 2.2 one easily proves that B(s) contains the ring generated by the intervals of the form $(-\infty, \lambda)$, $[\eta, \lambda)$ and $[\lambda, +\infty)$ for all real η, λ with $\eta < \lambda$. As Q is a σ-poset, B(s) is a monotone class. By the lemma on monotone classes [*17*], $B(s) = B(\mathbb{R})$. By assumption, Q* is order determining on Q. So the map $\tilde{}$ is injective on Q and it makes sense to define $m : B(\mathbb{R}) \to Q$ by $m(\Delta)^{\sim} = \tilde{m}(\Delta)$. Then, by definition :

$$x(m(\Delta)) = \mu_x(\Delta), \qquad \text{for all } \Delta \in B(\mathbb{R}) \qquad \text{and } x \in Q^*,$$

so the map $\Delta \to m(\Delta)$ is the required bounded Q-valued measure extending s.

Conversely, if $\mathscr{S}$ is onto and $p + q \leqslant e$, $p, q \in Q$, define

$$s(\lambda) = \begin{cases} o & \text{if} \quad \lambda \leqslant 0 \\ p & \text{if} \quad 0 < \lambda \leqslant 1 \\ q' & \text{if} \quad 1 < \lambda \leqslant 2 \\ e & \text{if} \quad 2 < \lambda. \end{cases}$$

If $s = \mathscr{S}(m)$, then $m(\{ 1 \}) = q' - p = (p + q)'$, so $p + q \in Q' = Q$. ///

COROLLARY 2.4. — Let $P \subseteq P(E)$ be a σ-lattice satisfying the conditions of Lemma 1.3 and such that P^* is order determining on P. Then

i) P is an orthomodular σ-lattice and the restrictions to P of P-states are an order determining set of probability measures on $(P, \leqslant, ')$ (for the definition, see § 3) :

ii) the map $\mathcal{S}$ is a bijection of the set of all bounded P-valued measures onto the set of all Q-valued resolutions of the unit.

Proof. — It is an obvious consequence of Lemma 1.3 and Proposition 2.3. ///

Let E be a Banach space satisfying Axiom I. By Proposition 1.4 F(E) is a σ-lattice satisfying the conditions of Lemma 1.3. If $F(E)^*$ is order determining on F(E) we can assume $P = F(E)$ in the above corollary.

For every bounded Q-valued measure m and $x \in Q^*$, a real valued probability measure m_x on $\mathbb{R}$ is defined, for $\Delta \in B(\mathbb{R})$, by

$$m_x(\Delta) : = x(m(\Delta)) .$$

Assume that Q^* is order determining on E. Then, we can introduce a weak integration with respect to bounded Q-valued measures, according to the following definition.

DÉFINITION 2.2. — Let m be a bounded Q-valued measure. A measurable real function f is said to be weakly integrable with respect to m on the Borel set Δ if, for all $x \in Q^*$, f is m_x-integrable on Δ and there exists in E a vector m_f such that

$$x(m_f) = \int_\Delta f dm_x . \tag{2.2}$$

The vector m_f in (2.2) is unique and is called the weak integral of f with respect to m on Δ, denoted by

$$m_f = \int_\Delta f dm .$$

The weak integral on $\mathbb{R}$ of the identity map i is called the weak mean of m.

Remark. — For a bounded Q-valued measure m on $\mathbb{R}$ the norm mean of $\mathcal{S}(m)$ exists in the completion $\bar{E}$ of E. By standard arguments one obtains that the weak mean of m exists in $\bar{E}$ and

$$\int_{\mathbb{R}} idm = \int \lambda d\mathcal{S}(m)(\lambda) .$$

In particular, the norm mean of $\mathcal{S}(m)$ belongs to E if and only if the weak mean of m belongs to E.

We denote by Exp the map which associates to every bounded Q-valued measure m, the weak mean of m, i. e.

$$\text{Exp}\, m = \int_{\mathbb{R}} i\,dm = \int \lambda\, d\mathcal{S}(m)(\lambda) \in \bar{\text{E}} \subseteq \text{A}^b(\text{Q}^*)\,.$$

The following corollary of Proposition 2.3 concerns order unit spaces E satisfying Axiom II. Every a in E is the norm mean of the F(E)-valued resolution s^a of the unit.

COROLLARY 2.5. — Let E be an order unit space satisfying Axiom II, with F(E)* order determining.

i) Let m be a bounded P(E)-valued measure such that the weak mean a of m exists in E and belongs to $\text{C}(m(\Delta))$, for all Borel set Δ. Then $\mathcal{S}(m) = s^a$. If C(p) is closed for all $p \in \text{F(E)}$, then every element a of E admits only one bounded P(E)-valued measure with weak mean a.

ii) If E is complete, the map $m \to \int_{\mathbb{R}} i\,dm$ is a bijection of the set of all bounded P(E)-valued measures onto E.

Proof. — The norm mean of the P(E)-valued resolution $\mathcal{S}(m)$ of the unit equals a and P(E) equals F(E) by Proposition 1.12. Hence, by point *a)*, statement *i)* of Proposition 1.5, by Proposition 1.2 and Proposition 2.2 we conclude that $\mathcal{S}(m)$ equals s^a.

For every bounded P(E)-valued measure m, the norm mean a of $\mathcal{S}(m)$ equals the weak mean of m. The second part of *i)* follows by Remark under Proposition 1.13 and by the injectivity of the map $\mathcal{S}$.

ii) The weak mean of every bounded P(E)-valued measure exists in the Banach space E. By *ii)*, Proposition 1.9, the map π_p is a continuous closed positive linear map of C(p) onto V(p), for every $p \in \text{F(E)} = \text{P(E)}$. Hence, C($p$) is closed and the statement follows by Corollary 2.4 and by the statement *i)* of this corollary. ///

An interesting question is to characterize order unit spaces with F(E)* order determining. Any answer to this question will require some generalization of Lodkin Theorem [*20*].

3. QUANTUM LOGICAL AND OPERATIONAL DESCRIPTIONS FOR PHYSICAL SYSTEMS

In this section we discuss the relationship between the spectral theory of order unit spaces and the quantum logical or operational descriptions for physical systems.

In § 1 we proved that the posets of F-projections of order unit spaces satisfying Axiom I are orthomodular lattices. In the sequel we show how operational descriptions naturally arise from the quantum logical descriptions. The relevant mathematical object in this procedure is the order unit space of the expectation value functions of observables in a quantum logical description.

Here we recall some definitions. Let $(L, \leqslant)$ be a bounded poset. A map $' : L \to L$ is said to be an *involution* on L if

i) $p \leqslant q$ implies $q' \leqslant p'$ for $p, q \in L$;
ii) $p'' = p$ for $p \in L$.

The triple $(L, \leqslant, ')$ is said to be a poset with involution. Let $(L, \leqslant, ')$ be a poset with involution. Two elements p, q of L are said to be *orthogonal* if $p \leqslant q'$; they are said to be *compatible*, provided there exists in L pairwise orthogonal elements p_o, q_o and r such that

$$p = p_o \vee r \qquad \text{and} \qquad q = q_o \vee r\,.$$

A *state* on $(L, \leqslant, ')$ is a map $x : L \to \mathbb{R}_+$ such that

i) $x(I) = 1$;
ii) $x(r) = x(p) + x(q)$ whenever $r = p \vee q$, with $p \leqslant q'$.

A *probability measure* on L is a state x such that, for every sequence $\{ q_n \}$ of pairwise orthogonal elements of L admitting supremum,

$$x(\vee \{ q_n \}) = \sum_n x(q_n)\,.$$

If there exists in L the supremum of any increasing sequence, L is said to be a σ-poset. If, for every $p \in L$ there exists $p \vee p'$ and equals I, then the map $'$ is called an *orthocomplementation.* If, moreover, there exists in L the supremum of any orthogonal pair, then L is called an *orthoposet.* If a σ-poset is an orthoposet, it is called a *σ-orthoposet.*

An orthoposet L is said to be orthomodular, provided $p \leqslant q$ implies that there exists $r \in L$ such that $p \leqslant r'$ and $p \vee r = q$. One can show that $r = p' \wedge q$. It is well known that any orthoposet admitting a separating set of states is orthomodular [*14*].

Let L be an orthomodular orthoposet. We denote by $B(\mathbb{R})$ the natural σ-algebra of Borel subsets of $\mathbb{R}$. A map $m : B(\mathbb{R}) \to L$ is an *observable* provided :

i) $m(\mathbb{R}) = I$;
ii) for disjoint Borel sets Δ_1 and Δ_2, $m(\Delta_1) \leqslant (m(\Delta_2))'$;
iii) for every sequence $\{ \Delta_n \}$ of pairwise disjoint Borel sets, there exists

$$\vee \{ m(\Delta_n) \} \qquad \text{and equals} \qquad m\Big(\bigcup_n \Delta_n\Big).$$

The support $\sigma(m)$ of an observable m is defined as the intersection of all closed subsets Δ of $\mathbb{R}$ such that $m(\Delta) = \mathrm{I}$. If $\sigma(m)$ is a compact set, then m is said to be a bounded observable.

The observable m is said to be compatible with $p \in \mathrm{L}$ if for all $\Delta \in \mathrm{B}(\mathbb{R})$, $m(\Delta)$ is compatible with p. Then the map m_p is defined, for $\Delta \in \mathrm{B}(\mathbb{R})$ by

$$m_p(\Delta) = \begin{cases} m(\Delta) \wedge p & \text{if} \quad 0 \notin \Delta ; \\ (m(\Delta) \wedge p) \vee p' & \text{if} \quad 0 \in \Delta . \end{cases}$$

For $p = m([0, +\infty))$, m_p is denoted by m_+. Thus, $\sigma(m_+) \subseteq [0, +\infty)$. By statement *i*), Proposition 3.8 of [*29*], m_p is an observable whenever L is an orthoposet. In this case, m_p is interpreted as the observable « m conditioned by p ».

We denote by u the unique observable such that $u(\{1\}) = \mathrm{I}$. Then u is compatible with every $p \in \mathrm{L}$ and u_p, called the « simple observable » corresponding to p, is characterized by the property that $u_p(\{0\}) = p'$ and $u_p(\{1\}) = p$.

In every standard quantum logical description for a physical system a triple $(\mathscr{L}, \mathscr{S}, \mathscr{O})$ is defined, where $\mathscr{L}$, the « logic », is an orthoposet representing the set of the questions or propositions, which correspond to the set of all physically admissible simple observables; in the description of statistical systems it is natural to assume that $\mathscr{L}$ is a σ-orthoposet. $\mathscr{S}$ is assumed to be an order determining convex set of probability measures on $\mathscr{L}$ and represents the set of all physical states of the system. $\mathscr{O}$ is a set of bounded observables, representing the set of all physically meaningful bounded observables. Then, it is natural to assume that $\mathscr{O}$ contains the observable u and, for any proposition p commuting with an observable m in $\mathscr{O}$, also m_p belongs to $\mathscr{O}$.

The space $\mathrm{A}^b(\mathscr{S})$ of all real valued affine functions on $\mathscr{S}$, with pointwise ordering and supremum norm is a pointwise μ-complete order unit space, with order unit the constant 1 function on $\mathscr{S}$. There is a natural embedding of $\mathscr{L}$ into $\mathrm{A}^b(\mathscr{S})$. Actually, the map $\hat{}: \mathscr{L} \to \mathrm{A}^b(\mathscr{S})$, defined for $x \in \mathscr{S}$ and $p \in \mathscr{L}$ by

$$\hat{p}(x) = x(p),$$

is a involution preserving monotone injective map with range $\hat{\mathscr{L}}$ contained in the interval $[o, 1]$. The poset $\hat{\mathscr{L}}$ is order isomorphic to $\mathscr{L}$. Thus, probability measures on $\mathscr{L}$ correspond exactly to $\hat{\mathscr{L}}$-states, so that the set of $\hat{\mathscr{L}}$-states is order-determining on $\mathrm{A}^b(\mathscr{S})$. To every $m \in \mathscr{O}$, the $\hat{\mathscr{L}}$-valued measure $\hat{m}$ is associated, where for $x \in \mathscr{S}$ and $\Delta \in \mathrm{B}(\mathbb{R})$

$$(\hat{m}(\Delta))(x) = x(m(\Delta)).$$

A map $\mathrm{Exp} : \mathscr{O} \mapsto \mathrm{A}^b(\mathscr{S})$ is defined, which associates to every $m \in \mathscr{O}$ the weak mean $\mathrm{Exp}\ \hat{m}$ of $\hat{m}$, i. e.

$$\mathrm{Exp}\, m = \mathrm{Exp}\, \hat{m} = \int_{\mathbb{R}} id\hat{m}.$$

Exp m is called the expectation value functional of m. For any simple observable u_p, Exp u_p equals $\hat{p}$.

Rüttimann [26] called $\mathscr{L}$ a « spectral logic », provided Exp is a surjective map of the set of *all* bounded observables onto $A^b(\mathscr{S})$, where $\mathscr{S}$ is the set of *all* probability measures on $\mathscr{L}$. We are interested in more general descriptions, where $\mathscr{L}$ and $\mathscr{S}$ are not necessarily supposed to be total and the range E of the map Exp may be strictly contained in $A^b(\mathscr{S})$. However, some other natural assumptions on E will be made.

The following proposition shows how « spectral » order unit spaces of the type introduced in § 1 arise from quantum logical descriptions.

PROPOSITION 3.1. — Let $\mathscr{L}$ be a σ-orthoposet, $\mathscr{S}$ be an order determining convex set of probability measures on $\mathscr{L}$ and $\mathscr{O}$ be a set of bounded observables containing the observable u and with the properties that

1) the range E of the map Exp is a linear manifold in $A^b(\mathscr{S})$;

2) for $m \in \mathscr{O}$ and $p \in \mathscr{L}$ compatible with m, m_p belongs to $\mathscr{O}$. (3.1)

Then

i) E is an order unit space with the constant 1 function on $\mathscr{S}$ as order unit and the map $\hat{}$ is an involution preserving order isomorphism of $\mathscr{L}$ into $[o, 1]$;

ii) E satisfies Axiom I with $P = \hat{\mathscr{L}}$ if and only if :

$$\text{for all } p, q \text{ in } \mathscr{L} \text{ and real } \alpha \in (0, 1), \quad \alpha x(p) \leqslant x(q), \quad \forall\, x \in \mathscr{S} \text{ implies } p \leqslant q; \tag{3.2}$$

iii) if (3.2) holds, then E satisfies Axiom II if and only if :

$$m \text{ compatible with } p \in \mathscr{L} \quad \text{is equivalent to} \quad \text{Exp } m \in C(\hat{p}). \tag{3.3}$$

Proof. — *i*) is obvious.

ii) Condition (3.2) implies that the image $\hat{\mathscr{L}}$ of the set of all simple observables is a normalized set. Actually, $\hat{p} \leqslant \alpha \hat{I}$, $\alpha \in (0, 1)$ implies $(1 - \alpha)\hat{I} \leqslant \hat{p}'$ and hence, by (3.2), $\hat{p}' = \hat{I}$, i. e. $\hat{p} = o$.

By Remark below Definition 2.2, every element in E is the norm mean with to some $\hat{\mathscr{L}}$-valued resolution of the unit. Assume now Condition (3.2) and consider an element a in $[o, 1] \cap V(\hat{q})$, for some $q \in \mathscr{L}$. Every $\hat{p} \in V(a)$, $p \in \mathscr{L}$, belongs to $V(\hat{q})$. Hence there exists $\alpha \in (0, 1)$ such that $\alpha\hat{p} \leqslant \hat{q}$ so that (3.2) implies $\hat{p} \leqslant \hat{q}$. Arguing as in the Remark below Proposition 1.4 we obtain that there exists $\bigvee^{\hat{\mathscr{L}}} (\hat{\mathscr{L}} \cap V(a)) \leqslant \hat{q}$. From

$$a \leqslant \|a\| \bigvee^{\hat{\mathscr{L}}} (\hat{\mathscr{L}} \cap V(a)) \leqslant \hat{q}$$

we obtain $\hat{q} \in F(E)$ and Axiom I is satisfied for $P = \hat{\mathscr{L}}$.

Conversely, assume that E satisfies Axiom I, with $P = \hat{\mathscr{L}}$. Thus, $\alpha\hat{p} \leqslant \hat{q}$ for $p, q \in \mathscr{L}$ and $\alpha > 0$ implies $o \leqslant \hat{p} \leqslant \alpha^{-1}q$ and $\hat{p} \in V(\hat{q})$. From $\hat{\mathscr{L}} = F(E)$ we obtain $\hat{p} \leqslant \hat{q}$ and (3.2) holds.

iii) If conditions (3.1) and (3.2) hold, then $m \in \mathcal{O}$, m compatible with $p \in \mathscr{L}$ implies Exp $m \in C(\hat{p})$. Actually, for every $m \in \mathcal{O}$ and every $p \in \mathscr{L}$ compatible with m, m_p belongs to $\mathcal{O}$ and $\mathscr{S}(\hat{m}_p) = (\mathscr{S}(\hat{m}))_p$ has norm mean in E. By *ii)*, Proposition 1.4, Exp $m \in C(\hat{p})$. Therefore, it is enough to prove that Axiom II is equivalent to the condition that

$$\text{Exp } m \in C(\hat{p}) \quad \text{implies} \quad m \text{ compatible with } p\,.$$

We first assume that Exp $m \in C(\hat{p})$ implies m compatible with p. Let $m \in \mathcal{O}$. Then Exp m = Exp m_+ − (Exp m − Exp m_+) is an orthogonal decomposition of Exp m with

$$e(\text{Exp } m_+) = \hat{m}((0, +\infty))\,.$$

We must only prove that $e(\text{Exp } m_+)$ bicommutes with Exp m. By *ii)* of this proposition and *ii)*, Proposition 1.5, any F-projection of E can be represented as $\hat{p}$ for some $p \in \mathscr{L}$. Then $\hat{p}$ commutes with $\hat{m}((0, +\infty))$ if and only if p and $m((0, +\infty))$ are compatible in the lattice $\mathscr{L}$. Hence Exp $m \in C(\hat{p})$ implies that $m((0, +\infty))$ is compatible with p, i. e. $e(\text{Exp } m_+)$ bicommutes with $\hat{p}$.

Suppose now that E satisfies Axiom II and that Exp $m \in C(\hat{p})$ for some $m \in \mathcal{O}$ and $p \in \mathscr{L}$. Then, for every real λ, λe − Exp $m \in C(\hat{p})$ so that Axiom II implies that $e((\lambda e - \text{Exp } m)_+)$ commutes with $\hat{p}$. For every Borel subset Δ of the real line, the corresponding observable $m_{m(\Delta)}$ has mean in E. By *ii)*, Proposition 1.4, Exp m belongs to $C(\hat{m}(\Delta))$ and by Proposition 1.13 we deduce that

$$\mathscr{S}(\hat{m})(\lambda) = \hat{m}((-\infty, \lambda)) = e((\lambda e - \text{Exp } m)_+)$$

so that $\mathscr{S}(\hat{m})(\lambda)$ commutes with $\hat{p}$ for every real λ. By Proposition 3.11 and Proposition 3.12 [*29*], there exists a Boolean sub σ-algebra B of $\hat{\mathscr{L}}$ containing $\{ \hat{p} \}$ and $\{ \mathscr{S}(\hat{m})(\lambda), \lambda \in \mathbb{R} \}$ so that, for every Borel set Δ, $\hat{m}(\Delta) \in B$. Hence, m is compatible with p and Condition (3.3) holds. ///

Conditions (3.1), (3.2) and (3.3) characterize linear manifolds in $A^b(\mathscr{S})$ satisfying our spectral axioms. Condition (3.1) is a natural physical requirement for quantum theories. Condition (3.2) generalizes the classical « projection postulate » of von Neumann and enable us to regard $\hat{\mathscr{L}}$ as a set of F-projections. It is evident that, if this condition is satisfied, $\mathscr{S}$ is order determining on $\mathscr{L}$. Conversely, if $\mathscr{S}$ is strongly order determining on $\mathscr{L}$ [*14*], Condition (3.2) is satisfied. Condition (3.3) assures that for $p \in \mathscr{L}$ and $m \in \mathcal{O}$, compatibility of m and p corresponds to compatibility of Exp m and $\hat{p}$.

We propose to call a triple $(\mathscr{L}, \mathscr{S}, \mathcal{O}_{\mathscr{L}})$ a *quantum logical description*, provided :

i) $\mathscr{L}$ is a σ-orthoposet;

ii) $\mathscr{S}$ is a convex set of probability measures on $\mathscr{L}$;

iii) $\mathcal{O}_{\mathscr{L}}$ is a set of observables containing u and such that Exp $\mathcal{O}_{\mathscr{L}}$ is a linear manifold;

iv) Conditions (3.1), (3.2) and (3.3) are satisfied.

Examples of such descriptions are the ones associated to the projections of W*-algebras or JW-algebras [*4*].

A quantum logical description is, however, not the unique way to describe a statistical system. At first Jordan [*15*], Segal [*27*], Haag and Kastler [*16*] and Ludwig [*21*] emphasized the importance of order theoretical methods in the algebraic approach to statistical theories. More recently the « operational » approach to the description of classical and quantum mechanical statistical systems has been proposed by Davies and Lewis [*10*], Edwards [*11*] [*13*], Mielnik [*23*] [*24*] [*25*] and others. This approach was successful in characterizing in terms of order properties state spaces of JB-algebras or C*-algebras [*5*] [*6*] [*21*]. Thus, physically meaningful conditions can be formulated in the operational setting in order to obtain the usual algebraic descriptions of physical systems.

The question of the more profitable operational setting remains still open. In all operational descriptions of statistical systems the starting point is, however, the set of all physical states, which is supposed to be a convex subset S of some real vector space. The space $A^b(S)$ of all real valued affine functions on S, with pointwise ordering and supremum norm, is a complete order unit space with the constant 1 function on S as unit. The set of all simple observables is supposed to correspond to some separating convex set Q in the order interval $[o, 1]$ of the space $A^b(S)$. The set Q contains the zero element and $1 - q$ for every $q \in Q$. The assumption that Q equals $[o, 1]$ is strong, because it would exclude the classical statistical mechanics [*13*]. Elements of Q are interpreted as « effects » or « counters ». Given an effect q and a state x, the number $q(x)$ means the statistically averaged answer of q to the individuals of the « statistical ensemble » x. Then $q_1 \leqslant q_2$ means that the counter corresponding to q_1 is « more sensitive » than the one corresponding to q_2. The counters q and αq, $\alpha \in (0, 1)$ differ only by the scale factor α : the counter αq is supposed to detect, for every ensemble x, only the fraction αx of x. As for observables, they are represented by Q-valued measures.

A fundamental problem in any approch to statistical systems is to give a suitable notion of « propositions ». In the quantum logical approach they are given *a priori* and correspond exactly to the « properties » of the physical system. In the W*-algebra or in the JW-algebra model they are the extreme points in $[o, e]$. In the operational approach one needs some additional requirements to characterize counters testing whether the statistical ensamble satisfies or not some physical property. For instance, Alfsen and Shultz introduced the « projective units » [*4*], Wittstock defined the analogue notion of *nh*-projections [*31*] and the authors [*1*] defined « decision effects » generalizing the classical notion introduced by Ludwig [*21*]. Here we use the weaker notion of F-projections, more natural in spectral theories which not require duality. The condition characterizing F-projections appears as a generalization of Ludwig's « Sensitivity Increasing Axiom » and means that, when compairing the sensitivity of counters associated to « propositions », their sensitivity does not depend on scale factors. Thus, we assume that the propositions set L consists of F-projections of the space $E = \text{lin } Q$. In the description of statistical systems it is natural to assume that L is a σ-poset and that L^* contains S [*4*] [*11*] [*12*] [*13*].

We are therefore interested in those operational descriptions (S, Q, L, O) where

i) S is a convex set;

ii) Q is a separating convex set in $A^b(S)$ such that $o \in Q = Q' \subseteq [o, 1]$;

iii) $L = F(E)$ is a σ-poset such that L^* contains S, where E is the linear span of Q;

iv) O is a set of bounded Q-valued measures and the subset O_L of all L-valued measures in O contains the one extending $\mathscr{S}(m)_p$, for every $m \in O$ and $p \in L$ commuting with $\mathscr{S}(m)$. Moreover, $E = \text{Exp } O_L$.

We call (S, Q, L, O) a « spectral operational description ». These descriptions are characterized by the quite natural requirement that every $q \in Q$ appears as the integration of elementary measurament's acts.

We conclude our paper with a trivial statement concerning the equivalence between quantum logical and spectral operational descriptions.

PROPOSITION 3.2. — *i*) Let $(\mathscr{L}, \mathscr{S}, \mathcal{O}_{\mathscr{L}})$ be a quantum logical description. Then the map $\hat{}$ is an involution preserving order isomorphism of $\mathscr{L}$ onto F(E), with $E = \text{Exp } \mathcal{O}_{\mathscr{L}}$. The quadruple $(\mathscr{S}, \mathscr{Q}, F(E), \mathcal{O})$ is a spectral operational description, where

$\mathscr{Q}$ is any convex subset of $[o, 1]$ with linear span E, such that $o \in \mathscr{Q} = \mathscr{Q}'$,
$\mathcal{O}$ is any set of $\mathscr{Q}$-valued measures containing $\hat{\omega}$ for every $\omega \in \mathcal{O}_{\mathscr{L}}$.

Moreover, E satisfies Axiom II.

ii) Let, conversely, (S, Q, L, O) be a spectral operational description. Then the pair (L, S) is a quantum logic satisfying Condition (3.2).

If, moreover, the linear span E of Q satisfies Axiom II, then the triple (L, S, O_L) is a quantum logical description, with $E = \text{Exp } O_L$.

This proposition is an improvement of an analogue statement of the authors [2], where it was assumed that L consists of « decision effects » or, equivalently, that L is a logic and S a strongly order determining set of states of L.

ACKNOWLEDGEMENTS

We acknowledge the referee for improvements and useful suggestions.

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(Reçu le 28 janvier 1981. Manuscrit révisé, reçu le 21 septembre 1981)