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RICHARD KERNER

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Geometrical background for the unified field theories: the Einstein-Cartan theory over a principal fibre bundle

by

Richard KERNER

Département de Mécanique, Université Pierre-et-Marie Curie,
4, Place Jussieu, 75005 Paris, France

ABSTRACT. — The gauge and Lorentz invariant interactions of spinors, vectors, scalar multiplets and gauge fields are constructed using the Einstein-Cartan theory with torsion over a principal fibre bundle. This generalization enables us to get rid of the un-renormalizable terms which are present in the purely Riemannian version of this theory. The decomposition of the generalized spinors over the fibre bundle into the multiplets of Lorentz spinors can provide the theory with the rich mass spectrum.

RÉSUMÉ. — Nous construisons les interactions invariantes sous l'action d'un groupe de jauge et du groupe de Lorentz, comprenant les champs spinoriels, vectoriels, multiplets scalaires et les champs de jauge. La construction se base sur l'utilisation de la théorie d'Einstein-Cartan avec torsion au-dessus d'un fibré principal. Cette généralisation nous permet de supprimer des termes non-rénormalisables qui apparaissaient dans la version Riemannienne de cette théorie.

La décomposition des spineurs généralisés au-dessus du fibré en des multiplets de spineurs lorentziens peut aboutir à l'obtention d'un spectre de masse intéressant.

1. INTRODUCTION

One of the most important problems in the elementary particle physics is to identify the known particles or their hypothetical components (quarks) with the irreducible representations of some Lie group containing both the Lorentz group and what is called the internal symmetry group, or the gauge group. The assignement of well-defined particle properties to the type of the representation of the Lorentz group is well understood: that is how we distinguish the scalar particles, spin 1/2 particles, vector bosons, etc. The situation is less clear when it comes to the symmetries between the different particles of the same family. First of all we are uncertain even as to the choice of the group itself, except that we know that it has to contain at least a U(1) subgroup for the conservation of the electric charge, and a SU(2) subgroup for the isospin conservation; the conservation of the baryonic or leptonic number may come from SU(2) or some larger group.

Even when the group is chosen, the ambiguity persists in the way it mixes with the Lorentz group. The situation is similar to what happens with the spectral lines of the hydrogen atom. We know that this spectrum is invariant under the group $O(4) = O(3) \times O(3)$, but this does not mean at all that the physical system evolves in a 4-dimensional Euclidean space. It just happens that the kinematical and dynamical symmetries of the system mix up in such a curious way. The same may be true with respect to the elementary particles or their constituents.

Now, to describe the vector multiplets of bosons as well as the electromagnetic field, the theory of gauge fields, i. e. the connections on a fibre bundle, has provided an ideal mathematical tool. However the spinor fields are not incorporated into this theory on equal footing; the multiplets of spinors interact with the gauge field via the minimal interaction which is constructed by analogy with the electromagnetic interaction. Moreover, the representation of the gauge group to which such a multiplet belongs is arbitrary. The only leading principle is to obtain the system of equations invariant both under the Lorentz group and the gauge group.

To obtain such an invariant system of equations two complementary approaches are currently in use:

a) the variational one, which consists in finding out a Lagrangian density invariant under the symmetry group we have chosen, and then deriving the equations of motion, which will be invariant under the aforementioned symmetry group.

b) To define some differential operators invariant under the chosen symmetry group, and then apply them to the local sections of the appropriate fibre bundles. In what follows, we shall consider only two types of invariant operators: the Dirac operator and the Laplace-Beltrami operator.

Both approaches have some inconvenience. E. g. in the first case the Lagrangian density is a 4-form over the Minkowski space-time, but we have to know what are the building blocks it was supposed to be constructed with: spinor fields, vector fields, scalar multiplets, tensors, and all other possible cases. There are many ways (« interactions ») in which such a 4-form can be constructed.

In the second case the invariance is built in into the differential operator, but as a rule, the local section of the fibre bundle belongs to a representation of some large group, containing both the Lorentz group and the internal symmetry group, and we have to decompose it into the multiplets of the irreducible representations of the Lorentz group in order to give it an intrinsic physical meaning.

What we propose here is to take the best out of these two approaches and try to construct the invariant models of interactions in a synthetic way. The most important feature of our approach is that the choice of group determines also the choice of its representation, and therefore the exact form of spinor and vector multiplets permitted by the model.

2. GAUGE FIELDS ON A PRINCIPAL FIBRE BUNDLE

Let us remind the fibre bundle formulation of the gauge field theory. Let M_4 be the Minkowskian space-time with the metric form g_M. Let G be a compact, semi-simple Lie group of dimension N. The Cartan-Killing metric form on G will be designed by g_G.

We form a principal fibre bundle $P(M_4, G)$ with the base space M_4 and the structural group G.

The group G acts on the right on $P(M_4, G)$, generating the left-invariant vector fields. The N-dimensional subspace of the tangent space $TP(M_4, G)$ spanned by these fields is called vertical subspace.

The connection on $P(M_4, G)$ is a distribution of 4-dimensional subspaces which are invariant under the group action on $P(M_4, G)$. It is equivalent with defining a left invariant 1-form on $P(M_4, G)$ with values in the Lie algebra of G.

This form is called A; a vector field at the point $p \in P(M_4, G)$ is called horizontal if $A(X) = 0$. The 1-form A is left-invariant, which means that

$$A_{ph} = ad(h^{-1})A_p \tag{2-1}$$

where *ad* means the adjoint representation of G in its Lie algebra and *ph* is the image of $p \in P(M_4, G)$ under the right action of $h \in G$.

Any vector field on $P(M_4, G)$ can be now decomposed into its horizontal and vertical parts relative to the connection A:

$$X = \operatorname{hor} X + \operatorname{ver} X . \tag{2-2}$$

This enables us to define the metric on $P(M_4, G)$ as follows: for any two vectors X, Y we define

$$g_P(X, Y) = g_M(d\pi(X), d\pi(Y)) + g_G(A(X), A(Y)), \tag{2-3}$$

where $d\pi$ is the differential of the canonical projection $\pi : P(M_4, G) \to M_4$.

The connection form A defines also the covariant differentiation of forms on $P(M_4, G)$. Let ω be such a form; then, if its degree is l, then

$$D\omega(X_1, X_2, \ldots, X_{l+1}) = d\omega(\text{hor } X_1, \text{hor } X_2, \ldots, \text{hor } X_{l+1}) \tag{2-4}$$

We define the curvature form of A:

$$F = DA = dA + \frac{1}{2}[A, A] \tag{2-5}$$

where the bracket means the Lie algebra skew product.

Obviously:

$$DF = 0 \tag{2-6}$$

Finally, let φ be a tensorial 1-form of type adG. Then we can define its covariant differential

$$D\varphi(X, Y) = d\varphi(X, Y) + \frac{1}{2}[\varphi(X), A(Y)] + \frac{1}{2}[A(X), \varphi(Y)] \tag{2-7}$$

The metric forms on G and on M_4 define the scalar product not only for vectors belonging to the horizontal or vertical subspaces of $P(M_4, G)$, but also, in a canonical way, the scalar product of two forms of the same degree. Therefore we can define the scalar product of F with itself, symbolically $F \cdot F$. When multiplied by $\sqrt{g_p}$ it gives an invariant Lagrangian density. When integrated over $P(M_4, G)$ with a measure which is the product of the invariant Haar measure on G and the usual volume element on M_4, it will give

$$\int_{P(M_4,G)} \sqrt{|g_P|}\, F \cdot F d^4x dG = V_G \int_{M_4} F \cdot F d^4x \tag{2-8}$$

where V_G is a constant (volume of G).

The principal bundle $P(M_4, G)$ endowed with the canonical metric g_P is a Riemannian space. In a usual way we can introduce the Christoffel connection, which defines the parallel transport and the covariant differentiation of forms and vector fields over $P(M_4, G)$. Let us denote this Christoffel connection by Γ_C. It is a 1-form over $P(M_4, G)$ with values in the Lie algebra of the orthogonal group in $4 + N$ dimensions preserving the metric g_P. This connection is torsionless.

It can be calculated that the Riemann scalar of this new connection, which we note R, is equal to the Lagrangian density of the gauge field $F \cdot F$

plus the scalar curvature K of the base space M_4 (in our case $K = 0$) plus a constant:

$$R = F \cdot F + K + g_{ab}g^{ab} \tag{2-9}$$

In the case when $K = 0$, the Einsteinian variational principle applied to the Riemannian space $P(M_4, G)$ gives once more

$$\int_{P(M_4,G)} R\sqrt{|g_P|}\, dGd^4x = V_G \int_{M_4} \sqrt{|g_{M_4}|}\, F \cdot Fd^4x + \text{Const} \tag{2-10}$$

The equations resulting from this Lagrangian density are the well-known Yang-Mills field equations:

$$DF^* = 0. \tag{2-11}$$

Here F^* means the dual 2-form of F. The duality of forms is defined by the usual Hodge procedure, i. e. by the invariant totally antisymmetric 4-form on M_4, numerically equal to the $\sqrt{|g_{M_4}|}$.

If $K \neq 0$, then the supplementary set of equations gives the definition of the energy-momentum tensor of the Yang-Mills field via Einsteins equations

$$K_{ij} - \frac{1}{2}g_{ij}K = -\varkappa T_{ij}(F) \tag{2-12}$$

where $\varkappa$ is the gravitational constant, $i, j = 0, 1, 2, 3$, and $T_{ij}(F)$ means the energy-momentum tensor of the field F.

3. SPINORS OVER THE BUNDLE

Our next task is to describe the interaction of the gauge field with other fields, namely spinor, scalar or vector fields. Following the suggestions formulated in (1), (2), (3) we shall define spinor fields directly on the fibre bundle $P(M_4, G)$ endowed with the connection form A. Let us remind briefly how the spinor fields are constructed over the Minkowskian space-time M_4 (4), (5), (6). The isometry group of M_4 is $SO(3, 1)$. Construct now the principal fiber bundle $P(M_4, SO(3, 1))$; for simplicity we will assume that it is globally trivial: $P(M_4, SO(3, 1)) \cong M_4 \times SO(3, 1)$. Let $\mathbb{C}^4$ denote the 4-dimensional linear complex space. The *associated spinor bundle* is defined as follows:

$$\frac{P(M_4, SO(3, 1)) \times \mathbb{C}^4}{SO(3, 1)} \tag{3-1}$$

In order to give an explicit meaning to this formula, we have to define the action of the group $SO(3, 1)$ on the linear complex space $\mathbb{C}^4$, i. e. to define the representation. The spinor representation, which is in fact a representation not only of SO(3, 1) but of its covering group Spin (3, 1), is constructed as follows. For a given metric tensor g_{ij} we define the generators of the associated Clifford algebra as follows:

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2g_{ij}\mathrm{I}d \tag{3-2}$$

where Id means the identity operator in the representation space. The lowest faithful representation of the matrices γ_i has the dimension $m = 2^{\left[\frac{N+1}{2}\right]}$, where $i = 1, 2, \ldots, n$, and $[k]$ means the entire part of k. In the case of M_4 $n = 4$ and $m = 2^2 = 4$; that is why we take $\mathbb{C}^4$. It is easy to verify now that the matrices

$$\sigma_{ij} = \frac{1}{8}(\gamma_i\gamma_j - \gamma_j\gamma_i) \tag{3-3}$$

generate the Lie algebra of SO(3, 1). Let $\alpha^{ij} = -\alpha^{ji}$ be the generators of an infinitesimal Lorentz rotation. Then a spinor ψ, which has its values in C^4, transforms under an infinitesimal Lorentz rotation as follows:

$$\delta\psi = \alpha^{ij}\sigma_{ij}\psi \tag{3-4}$$

Define a matrix β by requiring for any $i = 0, 1, 2, 3$

$$\gamma_i^+ = \beta^{-1}\gamma_i\beta \tag{3-5}$$

The Dirac-conjugate spinor $\bar{\psi}$ will be defined then as:

$$\bar{\psi} = \psi^+\beta\,, \tag{3-6}$$

and it transforms under the conjugate transformation law:

$$\delta\bar{\psi} = -\bar{\psi}\sigma_{ij}\alpha^{ij} \tag{3-7}$$

Therefore, the scalar product $\bar{\psi}\psi$ is manifestly invariant under Lorentz transformations.

Let us define the collection of 1-forms spanning a basis at any point of $M_4 : \{\theta^i\}$, $i = 0, 1, 2, 3$. Usually, in holonomic coordinates, one can choose simply $\theta^i = dx^i$. Let the matrices E^l_k form the basis of the Lie algebra of orthogonal transformations of frames. The connection in the bundle of orthogonal frames is a left-invariant 1-form with values in the Lie algebra of orthogonal transformations of frames:

$$\omega = \omega^k_l E^l_k = \Gamma^k_{ml} E^l_k \theta^m \tag{3-8}$$

Defining $\omega_{kl} = g_{km}\omega^m_l$, it is easy to see that for the *orthogonal* frames one must have

$$\omega_{kl} = -\omega_{lk} \tag{3-9}$$

With the connection on the bundle of frames thus defined, we can introduce the covariant derivative in the associated spinor bundle:

$$D\psi = d\psi + \omega^{kl}\sigma_{kl}\psi = (\nabla_k\psi)\theta^k, \tag{3-10}$$

Therefore

$$\nabla_k\psi = \partial_k\psi + \Gamma^m_{kl}g^{jl}\sigma_{mj}\psi \tag{3-11}$$

if $\theta^k = dx^k$.

The covariant derivative of a conjugate spinor is given by

$$D\bar{\psi} = d\bar{\psi} - \bar{\psi}\sigma_{kl}\omega^{kl} \tag{3-12}$$

In a general case of a Riemannian space V_4 replacing M_4, let us define the density η_{ijkl}, a totally antisymmetric tensor such that $\eta_{0123} = |\det g_{ij}|^{1/2}$. Next, we define the associate forms

$$\eta_{ijk} = \theta^l\eta_{ijkl}, \qquad \eta_{ij} = \frac{1}{2}\theta^k \wedge \eta_{ijk} \tag{3-13}$$

$$\eta_i = \frac{1}{3}\theta^j \wedge \eta_{ij}, \qquad \eta = \frac{1}{4}\theta^i \wedge \eta_i \tag{3-14}$$

We can also introduce their dual forms:

$$\eta^k = g^{kl}\eta_l, \ \eta^{ij} = g^{ik}g^{jl}\eta_{kl}, \text{ etc.} \tag{3-15}$$

Finally, we define the Clifford algebra-valued forms: the 1-form $\gamma = \gamma_k\theta^k$, and its dual 3-form $\mu = \gamma_k\eta^k$. Then the Dirac Lagrangian density for the spinor field can be written in the following concise form:

$$\mathscr{L}_\psi = \frac{i}{2}(\bar{\psi}\mu \wedge D\psi + D\bar{\psi} \wedge \mu\psi) + m\eta\bar{\psi}\psi \tag{3-16}$$

which is manifestly hermitian and does not depend on the choice of coordinates.

The corresponding equations can be written in the form

$$\mu D\psi + D(\mu\psi) = 2im\eta\psi, \tag{3-17}$$

which in the case of $D\mu = 0$, (what we assume), explicitly takes on the familiar form

$$i\gamma^k\nabla_k\psi + m\psi = 0 \tag{3-18}$$

and coincides with the Dirac equation in curved space-time.

The conserved current is given by $j = \bar{\psi}\mu\psi$, with $dj = 0$ by virtue of the Dirac equation.

The generalization of the Dirac equation that we propose now means that we should repeat the same construction over the Riemannian manifold $P(V_4, G; g)$. In this notation g means the Riemannian metric (2-3). Thus instead of working with the direct product of the Lorentz group and the internal symmetry group we imbed the two in the isometry group of the metric g_P, and then try to decompose the corresponding spinorial representation.

The spinors will be constructed now as the sections of the associated fiber bundle

$$\frac{P(P(V_4, G; g_P), SO(3+N,1)) \times C^r}{SO(3+N,1)}$$

where $r = 2^{\left[\frac{N+5}{2}\right]}$.

The generalization of the construction of Lorentz spinors to the spinors defined over $P(P(M_4 \times G), SO(3+N,1))$ is obvious. Any Clifford algebra can be decomposed into a tensor product of the four elementary Clifford algebras:

$C(0,1) = \mathbb{C}^1$ = the complex manifold
$C(1,0) = R \oplus R$ = sum of the two real lines
$C(0,2) = \mathbb{H}$ = quaternions
$C(1,1) = \mathrm{Mat}_2 R = 2 \times 2$ real matrices

Here $C(p,q)$ means the Clifford algebra corresponding to the diagonal metric with the signature $(p+, q-)$.

We have also the reduction formulae:

$$\begin{aligned} C(p+1, q+1) &= C(1,1) \otimes C(p,q) \\ C(p+2, q) &= C(2,0) \otimes C(q,p) \\ C(p, q+2) &= C(0,2) \otimes C(q,p) \end{aligned}$$

To give an example, if $G = SU(2)$, $N = 3$ and we have to construct: $C(3+N, 1) = C(6,1)$

$$C(6,1) = [C(3,1) \otimes \mathbb{H}] \oplus [C(3,1) \otimes \mathbb{H}]. \tag{3-19}$$

The lowest-dimensional faithful representation of $C(6,1)$ is given by 16×16 matrices; it is easy to construct it explicitly. If we want to define

$$\Gamma^\alpha = \{ \gamma^i, \gamma^a \} \qquad \text{where } i = 0,1,2,3;\ a = 1,2,3$$

then (up to an equivalence) one of the possible realizations is:

$$\begin{aligned} \Gamma^i &= \gamma^i \otimes \tau^1 \otimes \mathbb{1}_2 \\ \Gamma^a &= \mathbb{1}_4 \otimes \tau^a \otimes \tau^3 \end{aligned} \tag{3-20}$$

Where γ^i are the usual 4×4 Dirac matrices, τ^a are the 2×2 Pauli matrices, and $\mathbb{1}_2$ or $\mathbb{1}_4$ means 2×2 or 4×4 identity matrix; if we omit the $\mathbb{1}$ matrix in the first definition, we would have a representation still, but not a faithful one-it will be realized in 8 dimensions:

$$\Gamma^i = \gamma^i \otimes \tau^1 , \ \Gamma^a = \tau^a \otimes \tau^3 \otimes \mathbb{1}_2 \tag{3-21}$$

In what follows we shall always suppose that the faithful representation is taken. The only thing left now is to generalize the density tensor η for our bundle. For the base space M_4 we take η_{0123}, for the group G we shall take the tensor $\eta_{a_1a_2\ldots a_N}$, for the direct product $M_4 \times G$ we shall take the product tensor $\eta_{0123}\eta_{a_1a_2\ldots a_N}$, and for the fibre bundle $P(M_4, G)$ we will define

$$\eta_{123\ldots N+4} = \eta_{0123}\eta_{a_1a_2\ldots a_N}, \tag{3-22}$$

and other components by antisymmetrization.

The new Lagrangian is analogous to the Dirac Lagrangian on M_4, i. e.

$$\mathscr{L} = \frac{i}{2}(\bar{\psi}\mu \wedge D\psi + D\bar{\psi} \wedge \mu\psi) + m\eta\bar{\psi}\psi \tag{3-23}$$

and the corresponding generalized Dirac equation can be written as:

$$\gamma^\alpha\nabla_\alpha\psi - im\psi = 0 \quad \alpha = (i, a) = 1, 2, \ldots, N + 4 \tag{3-24}$$

Now we shall proceed to explicit calculus of this equation, expressed in local coordinates. The easiest way to do it is by introducing the anholonomic coordinate systems.

4. THE GENERALIZED DIRAC EQUATION

Let us remind that in our notation the indices i, j take on the values 0, 1, 2, 3; the indices a, b take on the values $1, 2, \ldots, N = \dim G$, and the Greek indices α, β take on the values $1, 2, \ldots, N + 4$, and we write symbolically $\alpha = \{ i, a \}$, meaning that the index α means both i and a. Let us take an open subset of $P(M_4, G)$, which is isomorphic to a product of the open subsets of M_4 and G:

$$U \simeq V \times W, \quad \text{where } U \subset P(M_4, G),\ V \subset M_4,\ W \subset G.$$

Without loss in generality we can assume that W is a neighborhood of unity of G. We denote the local coordinates in V by x^i, the local coordinates in W by ξ^a and the local coordinates in U by $p^\alpha = \{ p^i, p^a \}$.

Let us choose a basis in the Lie algebra $\mathscr{A}_G$ such that for any vector $Y \in \mathscr{A}_G$, $Y = Y^a L_a$, with

$$[L_a, L_b] = C^c_{ab} L_c ; \tag{4-1}$$

The structure constants C^c_{ab} satisfy $C^c_{ab} = - C^c_{ba}$ and the Jacobi identity $C^c_{ab}C^e_{cd} + C^c_{bd}C^e_{ca} + C^c_{da}C^e_{cb} = 0$. The bracket above means the skew product in $\mathscr{A}_G$. The action of the group on $P(M_4, G)$ defines N left-invariant wector fields on the bundle, which span a Lie algebra isomorphic to $\mathscr{A}_G$, with the operation being the Lie bracket of vector fields. Let us denote these N vector fields isomorphic to L_a by S_a, $a = 1, 2, \ldots, N$. In local coordinates we can write:

$$S_a = S^\beta_a(p^\alpha)\frac{\partial}{\partial p^\beta} \tag{4-2}$$

The S_a should satisfy $[S_a, S_b] = C^c_{ab}S_c$, therefore

$$S^\beta_a \partial_\beta S^\gamma_b - S^\beta_b \partial_\beta S^\gamma_a = C^c_{ab} S^\gamma_c \tag{4-3}$$

We call S_a the vertical vector fields.

Let π be the canonical projection from $P(M_4, G)$ onto M_4, let $d\pi$ be its differential. Then in local coordinates we can write $d\pi(Y) = X$ when $Y \in TP(M_4, G)$ and $X \in TM_4$. If in local coordinates $X = X^i\partial_i$, $Y = Y^\alpha\partial_\alpha$, then

$$d\pi(Y^\alpha) = d\pi^i_\alpha Y^\alpha = X^i \tag{4-4}$$

We get also the condition of verticality in local coordinates:

$$d\pi^i_\alpha S^\alpha_b = 0 \qquad \text{for all } i, b\,. \tag{4-5}$$

Because any fibre is transformed into a point in M_4 by π, any vertical vector will be transformed into 0 by $d\pi$. With no less in generality we can choose such local coordinates in U, in which $d\pi^i_j = \delta^i_j$, $d\pi^i_a = 0$. This implies that $S^i_a = 0$, and

$$S^b_a \partial_b S^d_c - S^b_c \partial_b S^d_a = C^b_{ac} S^d_b \tag{4-14}$$

The horizontal subspaces are defined by the connection form

$$A = A_\alpha dp^\alpha = A^b_\alpha L_b dp^\alpha\,.$$

The image of a vertical vector field should be the corresponding element in $\mathscr{A}_G$; therefore $A(S_a) = L_a$; this means that in local coordinates

$$A^b_\alpha L_b dp^\alpha(S^\beta_a \partial_\beta) = A^b_\alpha \delta^\alpha_\beta S^\beta_a L_b = A^b_\alpha S^\alpha_a L_b = L_a \tag{4-15}$$

Therefore $A^b_\alpha S^\alpha_a = \delta^b_a$. If $S^i_a = 0$, we have

$$A^b_d S^d_a = \delta^b_a\,, \qquad \text{therefore also } S^b_d A^d_a = \delta^b_a \tag{4-16}$$

(the matrix A^b_d is at every point inverse of S^d_a).

The components $A^a_i(p^\alpha)$ are still arbitrary. It is easy to see that the matrices A^a_b satisfy

$$\partial_a A^b_c - \partial_c A^b_a + C^b_{gh} A^g_a A^h_c = 0 \tag{4-17}$$

A vector $Y \in TP(M_4, G)$ is horizontal if $A(Y) = 0$. In local coordinates it means that $A^a_\alpha Y^\alpha = 0$ for any $a = 1, 2, \ldots, N$.

Let $Y \in TP(M_4, G)$, $Y = Y^\alpha \dfrac{\partial}{\partial p^\alpha}$.

The vertical and horizontal parts of Y are given by:

$$\text{ver}\, Y = S \circ A(Y), \qquad \text{hor} = Y - \text{ver}\, Y\,. \tag{4-6}$$

In local coordinates we have

$$(\text{ver}\, Y)^\alpha = S^\alpha_b A^b_\beta Y^\beta \qquad (\text{hor}\, Y)^\alpha = (\delta^\alpha_\beta - S^\alpha_b A^b_\beta) Y^\beta \tag{4-7}$$

In our coordinates, where $d\pi^i_j = \delta^i_j$, $d\pi^i_a = 0$, we have

$$(\text{ver } Y)^i = 0\,, \qquad (\text{ver } Y)^a = Y^a + S^a_b A^b_i Y^i \tag{4-8}$$

$$(\text{hor } Y)^i = Y^i\,, \qquad (\text{hor } Y)^a = -\, S^a_b A^b_i Y^i \tag{4-9}$$

Finally, let $X = X^i \dfrac{\partial}{\partial x^i}$ be a vector from TM_4. We define the horizontal lift of X as the vector $\tau(X)$ defined at any point of $P(M_4, G)$ by

$$(\tau(X))^i = X^i\,, \qquad (\tau(X))^a = -\, S^a_b A^b_i X^i \tag{4-10}$$

Therefore,

$$(\tau(X))^\alpha = \tau^\alpha_i X^i\,, \tag{4-11}$$

where

$$\tau^j_i = \delta^j_i\,, \qquad \tau^a_i = -\, S^a_b A^b_i \tag{4-12}$$

We have the following relations satisfied:

$$\tau^\alpha_i A^b_\alpha = 0 \qquad \text{for any } b,\, i\,;\ \tau^\alpha_i d\pi^j_\alpha = \delta^j_i \tag{4-13}$$

The form A has a remarkable property of being left-invariant of type *ad* (2-1).

If we take a 1-parameter subgroup of G, $h(t)$, which generates a left-invariant vector field X, the same property can be written infinitesimally:

$$\underset{X}{£} A = Ad(-\, X)A \tag{4-18}$$

where Ad is the adjoint representation of $\mathscr{A}_G$, $\underset{X}{£}$ is the Lie derivative with respect to the field X. The structure constants C^c_{ab} can be regarded upon as N matrices $[C_a]^c_b$ which provide us with the Ad representation of the N left-invariant vector fields: $[C_a, C_b] = C^c_{ab}C_c$, which is nothing else than Jacobi identity. Therefore:

$$\underset{S_a}{£} A = -\, [C_a]A \tag{4-19}$$

or, in local coordinates

$$\underset{S_a}{£} A^b = -\, C^b_{ac} A^c\,, \tag{4-20}$$

and finally:

$$\underset{S_a}{£} A^b_\alpha = -\, C^b_{ac} A^c_\alpha \tag{4-21}$$

$$\underset{S_a}{£} A^b_\alpha = S^\beta_a \partial_\beta A^b_\alpha + A^b_\beta \partial_\alpha S^\beta_a = -\, C^b_{ac} A^c_\alpha \tag{4-22}$$

Because of $S^i_a = 0$, and $A^b_c S^c_d = \delta^b_d$, we can write

$$S^c_a \partial_c A^b_\alpha - S^c_a \partial_\alpha A^b_c = -\, C^b_{ac} A^c_\alpha \tag{4-23}$$

Multiplying both sides by A^a_d we get

$$\partial_d A^b_\alpha - \partial_\alpha A^b_d = -\, C^b_{ac} A^a_d A^c_\alpha \tag{4-24}$$

Now, for $\alpha = g$ we obtain the relation which we have got already

$$\partial_d A^b_g - \partial_g A^b_d + C^b_{ac} A^a_d A^c_g = 0 \tag{4-25}$$

whereas for $\alpha = i$ we get

$$\partial_d A^b_i = - C^b_{ac} A^a_d A^c_i = C^b_{ac} A^a_i A^c_d \tag{4-26}$$

because we assume the coordinate system in which $\partial_i A^a_b = 0$.

The last two formulae, i. e. (4-25) and (4-26) can be interpreted in another way by telling that the curvature form of the connection A is horizontal:

$$F = DA, \qquad \text{and} \qquad F(X, Y) = O \quad \text{whenever}$$

one of the two vectors is vertical.

In local coordinates we have:

$$F = F^a L_a = F^a_{\alpha\beta} L_a dp^\alpha \wedge dp^\beta \tag{4-27}$$

and

$$F^a_{\alpha\beta} = \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha + C^a_{bc} A^b_\alpha A^c_\beta \tag{4-28}$$

If F has to vanish on vertical vectors, its only non-vanishing components will be

$$F^a_{ij} = \partial_i A^a_j - \partial_j A^a_i + C^a_{bc} A^b_i A^c_j \tag{4-29}$$

The formulae (4-25) and (4-26) define the derivatives of A^a_α with respect to the group parameters.

The canonical metric on the principal bundle, introduced in § .2 by (2-3) will give in local coordinates:

$$g(X, Y) = g_{\alpha\beta} X^\alpha Y^\beta = \overset{\circ}{g}_{ij} X^i X^j + \overset{\circ}{g}_{ab}(X^c A^a_c + A^a_i X^i)(Y^d A^b_d + A^b_j Y^j) \tag{4-30}$$

which can be written in a matrix form:

$$g_{\alpha\beta} = \left(\begin{array}{c|c} \overset{\circ}{g}_{ij} + \overset{\circ}{g}_{ab} A^a_i A^b_j & \overset{\circ}{g}_{ab} A^a_d A^b_j \\ \hline g_{ab} A^a_c A^b_i & g_{ab} A^a_c A^b_d \end{array}\right) \tag{4-31}$$

The inverse matrix to (4-31) is given by:

$$g^{\alpha\beta} = \left(\begin{array}{c|c} \overset{\circ}{g}{}^{ij} & - \overset{\circ}{g}{}^{ij} S^b_d A^d_j \\ \hline - \overset{\circ}{g}{}^{ij} S^a_c A^c_i & \overset{\circ}{g}{}^{ab} + \overset{\circ}{g}{}^{ij} S^a_c S^b_d A^c_i A^d_j \end{array}\right) \tag{4-32}$$

With the help of the formulae of derivation (4-25) and (4-26) we can proceed to compute the Christoffel symbols corresponding to our metric:

$$\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} = \frac{1}{2} g^{\alpha\delta}(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}) \tag{4-33}$$

The non-vanishing components of $\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}$ are then:

$$
\begin{aligned}
\left\{ \begin{matrix} a \\ bc \end{matrix} \right\} &= \frac{1}{2} S^a_d(\partial_b A^d_c + \partial_c A^d_b) \\
\left\{ \begin{matrix} i \\ cj \end{matrix} \right\} = \left\{ \begin{matrix} i \\ jc \end{matrix} \right\} &= \frac{1}{2} \overset{\circ}{g}{}^{ik} \overset{\circ}{g}_{ef} A^e_c F^f_{jk} \\
\left\{ \begin{matrix} a \\ bj \end{matrix} \right\} = \left\{ \begin{matrix} a \\ jb \end{matrix} \right\} &= \frac{1}{2} \overset{\circ}{g}{}^{ik} A^c_i S^a_c A^f_b \overset{\circ}{g}_{ef} F^e_{kj} + \frac{1}{2} C^c_{gh} S^a_c A^g_j A^h_b \qquad (4\text{-}34) \\
\left\{ \begin{matrix} a \\ ij \end{matrix} \right\} = \left\{ \begin{matrix} a \\ ji \end{matrix} \right\} &= \frac{1}{2} S^a_d(\partial_i A^d_j + \partial_j A^d_i) + \frac{1}{2} \overset{\circ}{g}{}^{mk} A^c_m S^a_c \overset{\circ}{g}_{df}(A^d_i F^f_{kj} + A^d_j F^f_{ki}) \\
\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \left\{ \begin{matrix} i \\ kj \end{matrix} \right\} &= \frac{1}{2} \overset{\circ}{g}{}^{il} \overset{\circ}{g}_{df}(A^d_j F^f_{kl} + A^d_k F^f_{jl})
\end{aligned}
$$

The calculus of the Riemann tensor is even more complicated. It is much easier to use the anholonomic coordinate system. As a matter of fact, we can interpret the metric tensor $g_{\alpha\beta}$ as an image of the diagonal matrix

$$
\tilde{g}_{\alpha\beta} = \left(\begin{array}{c|c} \overset{\circ}{g}_{ij} & 0 \\ \hline 0 & \overset{\circ}{g}_{ab} \end{array} \right) \qquad (4\text{-}35)
$$

under a local change of basis in the tangent space:

$$
g_{\alpha\beta} = U^\delta_\alpha U^\gamma_\beta \tilde{g}_{\delta\gamma}
$$

and

$$
g^{\alpha\beta} = \overset{-1}{U}{}^\alpha_\delta \overset{-1}{U}{}^\beta_\gamma \tilde{g}^{\delta\gamma} \qquad (4\text{-}36)
$$

where the matrices U^α_β and $\overset{-1}{U}{}^\alpha_\beta$ are given by the following formulae:

$$
U^i_j = \delta^i_j, \quad U^i_a = 0, \quad U^a_i = A^a_i, \quad U^a_b = A^a_b \qquad (4\text{-}37)
$$

and the inverse matrix:

$$
\overset{-1}{U}{}^i_j = \delta^i_j, \quad \overset{-1}{U}{}^i_a = 0, \quad \overset{-1}{U}{}^a_i = - S^a_c A^c_i, \quad \overset{-1}{U}{}^a_b = S^a_b \qquad (4\text{-}38)
$$

It is useful to introduce a « non-holonomic » derivative by:

$$
\mathscr{D}_\alpha = \overset{-1}{U}{}^\beta_\alpha \partial_\beta ; \qquad (4\text{-}39)
$$

of course,

$$
\mathscr{D}_\alpha \mathscr{D}_\beta \neq \mathscr{D}_\beta \mathscr{D}_\alpha .
$$

The connection transforms as usually and in new basis we get

$$\begin{aligned} &\left\{ \begin{matrix} \tilde{a} \\ bc \end{matrix} \right\} = \frac{1}{2} C^a_{bc} \qquad && \left\{ \begin{matrix} \tilde{i} \\ aj \end{matrix} \right\} = \frac{1}{2} g_{ab} g^{im} F^b_{jm} \\ &\left\{ \begin{matrix} \tilde{a} \\ ij \end{matrix} \right\} = -\frac{1}{2} F^a_{ij} && \left\{ \begin{matrix} \tilde{i} \\ ja \end{matrix} \right\} = \frac{1}{2} g_{ab} g^{im} F^b_{jm} \end{aligned} \tag{4-40}$$

and all other components vanish.

In this local basis, the components of the Riemann tensor are the following:

$$\begin{aligned} \tilde{R}_{cd}{}^a{}_b &= \frac{1}{4} C^e_{cd} C^a_{be} \\ \tilde{R}_{ij}{}^a{}_b &= \frac{1}{2} \left[C^a_{cb} F^c_{ij} - \frac{1}{2} F^a_{ki} F^c_{lj} \overset{\circ}{g}{}^{kl} \overset{\circ}{g}_{cb} + \frac{1}{2} F^a_{kj} F^c_{li} \overset{\circ}{g}_{cb} \overset{\circ}{g}{}^{kl} \right] \\ \tilde{R}_{bj}{}^a{}_i &= -\tilde{R}_{jb}{}^a{}_i = \frac{1}{4} \left[C^a_{cb} F^c_{ij} - F^a_{kj} F^d_{il} \overset{\circ}{g}{}^{kl} \overset{\circ}{g}_{db} \right] \\ \tilde{R}_{jk}{}^a{}_i &= -\frac{1}{2} D_i F^a_{kj} = \frac{1}{2} D_i F^a_{jk} \\ \tilde{R}_{bj}{}^i{}_a &= -\tilde{R}_{jb}{}^i{}_a = -\frac{1}{4} \left(C^d_{ba} F^c_{kj} \overset{\circ}{g}_{dc} \overset{\circ}{g}{}^{ik} - F^d_{kj} F^f_{lm} \overset{\circ}{g}_{ad} \overset{\circ}{g}{}^{li} \overset{\circ}{g}{}^{km} \overset{\circ}{g}_{fb} \right) \\ \tilde{R}_{jk}{}^i{}_a &= \frac{1}{2} \overset{\circ}{g}_{ab} D_l F^b_{kj} \overset{\circ}{g}{}^{il} \\ \tilde{R}_{kl}{}^i{}_j &= \frac{1}{4} \left\{ \overset{\circ}{g}_{ab} \overset{\circ}{g}{}^{im} \left[F^a_{jm} F^b_{kl} + F^a_{mk} F^b_{lj} - F^a_{ml} F^b_{kj} \right] \right\} \\ \tilde{R}_{ab}{}^i{}_j &= \frac{1}{2} C^c_{ab} F^d_{jk} \overset{\circ}{g}{}^{ki} \overset{\circ}{g}_{cd} + \frac{1}{4} \overset{\circ}{g}_{ac} \overset{\circ}{g}_{bd} \overset{\circ}{g}{}^{im} \overset{\circ}{g}{}^{kl} \left[F^c_{km} F^d_{jl} - F^c_{jl} F^d_{km} \right] \\ \tilde{R}_{ka}{}^i{}_j &= -\tilde{R}_{ak}{}^i{}_j = \frac{1}{2} D_k F^b_{jl} \overset{\circ}{g}_{ab} \overset{\circ}{g}{}^{li} \end{aligned} \tag{4-41}$$

Here

$$D_i F^a_{jl} = \partial_i F^a_{jl} + C^a_{bc} A^b_i F^c_{jl} \tag{4-42}$$

The invariant scalar curvature is equal to

$$R = -\frac{1}{4} \overset{\circ}{g}{}^{ab} \overset{\circ}{g}_{ab} - \frac{1}{4} \overset{\circ}{g}_{ab} \overset{\circ}{g}{}^{ik} \overset{\circ}{g}{}^{jl} F^a_{ij} F^b_{kl} \tag{4-43}$$

It is easy to write down explicitly the generalized Dirac equation: in our basis it will be:

$$\tilde{\gamma}^\alpha \left(\mathscr{D}_\alpha + \left\{ \begin{matrix} \tilde{\beta} \\ \alpha\gamma \end{matrix} \right\} \tilde{g}_{\beta\delta} \tilde{\sigma}^{\gamma\delta} \right) \psi + im\psi = 0 \tag{4-44}$$

where

$$\mathscr{D}_\alpha = \overset{-1}{\mathrm{U}}{}^\beta_\alpha \partial_\beta$$

The superscript means that the Clifford algebra generators correspond to the diagonal metric $\tilde{g}_{\alpha\beta}$, in which

$$\tilde{\gamma}^i = \overset{\circ}{\gamma}{}^i, \qquad \tilde{\gamma}^a = \overset{\circ}{\gamma}{}^a \tag{4-45}$$

Therefore (4-45) becomes

$$\left[\overset{\circ}{\gamma}{}^i\partial_i - \overset{\circ}{\gamma}{}^i \mathrm{A}^a_i \mathrm{S}^b_a \partial_b + \overset{\circ}{\gamma}{}^a \mathrm{S}^b_a \partial_b + \frac{1}{2}\mathrm{C}^c_{ab} g_{cd} \overset{\circ}{\gamma}{}^a \overset{\circ}{\sigma}{}^{bd} - \frac{1}{2}\overset{\circ}{\gamma}{}^a g_{ab} \mathrm{F}^b_{kl} \overset{\circ}{\sigma}{}^{kl} + im\right]\psi = 0 \tag{4-46}$$

The equation (4-46) can not be used unless we know what means $\partial_b\psi$. In other words, we have to fix the geometrical properties of ψ with respect to the group transformations on $\mathrm{P}(\mathrm{M}_4, \mathrm{G})$. The other problem we face now is the presence of a Pauli-type term $\frac{1}{2}\gamma_a \mathrm{F}^a_{kl}\overset{\circ}{\sigma}{}^{kl}$ which makes our theory unrenormalizable after quantization.

5. METRIC THEORY WITH TORSION OVER THE BUNDLE

Let us return to physics for a while and remind that we are describing an invariant interaction between the gauge field A and the generalized spinor ψ. We know how does the Lie-algebra valued 1-form A transform under the action of the structural group G on the bundle: if X is a left-invariant vector field on $\mathrm{P}(\mathrm{M}_4, \mathrm{G})$ generated by the corresponding element in $\mathscr{A}_{\mathrm{G}}$ (which we denote by the same letter X) then

$$\underset{\mathrm{X}}{\pounds}\mathrm{A} = \mathrm{A}d(-\mathrm{X})\mathrm{A} \tag{4-18}$$

or, if we choose a basis of N left-invariant fields S_a, we have in local coordinates

$$\underset{\mathrm{S}_a}{\pounds}\mathrm{A}^b = -\mathrm{C}^b_{ac}\mathrm{A}^c \tag{4-20}$$

It seems natural, in order to make everything invariant and in order to give an intrinsic meaning to the equation (4-46), to postulate that the spinor ψ transforms in the same way as the form A. This means that ψ satisfies

$$\underset{\mathrm{S}_a}{\pounds}\psi = \mathrm{A}d(+\mathrm{S}_a)\psi\,; \qquad \text{and} \qquad \underset{\mathrm{S}_a}{\pounds}\bar{\psi} = \bar{\psi}\mathrm{A}d(-\mathrm{S}_a) \tag{5-1}$$

Spinors satisfying the relation (5-1) will be called left-invariant spinors of type Ad. We can also introduce another type of spinors on $P(M_4, G)$, namely the spinors satisfying the relation

$$\underset{S_a}{\pounds}\psi = 0 \tag{5-2}$$

The first assumption (5-1), makes our theory entirely covariant, whereas the condition (5-2) breaks the symmetry.

The most general definition of the Lie derivative of a spinor field has been elaborated by A. Lichnerowicz [*5*, *6*] and Y. Kosmann [*4*]. If X is a vector field over a manifold, and if φ_t is a 1-parameter group of transformations generating X, then

$$(\underset{X}{\pounds}\psi)(x) = \lim_{t\to 0}\frac{\psi(\varphi_t(x)) - \psi(x)}{t}. \tag{5-3}$$

On a manifold, with a metric connection (non necessarily riemannian), when expressed in local coordinates, this gives

$$\underset{X}{\pounds}\psi = X^\alpha\nabla_\alpha\psi - \frac{1}{2}(\nabla_\alpha X_\beta - \nabla_\beta X_\alpha)\sigma^{\alpha\beta}\psi \tag{5-4}$$

Finally, if the linear connection form can be written in a coordinate system as $\Gamma_\alpha{}^\beta{}_\gamma$, and if we introduce the notation $\Gamma_{\alpha\gamma\delta} = \Gamma_{\alpha\gamma}{}^\beta g_{\beta\delta}$, then we can write explicitly

$$\underset{X}{\pounds}\psi = X^\alpha\partial_\alpha\psi - \frac{1}{2}(\partial_\alpha X_\beta - \partial_\beta X_\alpha)\sigma^{\alpha\beta}\psi \\ - \frac{1}{2}X^\alpha(\Gamma_{\alpha\beta\gamma} - \Gamma_{\alpha\gamma\beta} + \Gamma_{\beta\gamma\alpha} - \Gamma_{\gamma\beta\alpha})\sigma^{\beta\gamma}\psi \tag{5-5}$$

Here we denote $g_{\beta\delta}\partial_\alpha X^\delta$ by $\partial_\alpha X_\beta$.

Let us remind that the Ad representation of $\mathscr{A}_G$ acts on spinors as

$$Ad(S_a)\psi = \frac{1}{2}C_{abc}\sigma^{bc}\psi \tag{5-6}$$

where $C_{abc} = g_{cd}C^d_{ab}$ and is totally anti-symmetric for any compact and semi-simple Lie group.

What we are interested in is, in fact, the behaviour of ψ under an infinitesimal Lie group transformation, i. e. when X is a left-invariant vector field on $P(M_4, G)$.

Then we can put $X^\alpha = (X^a, 0)(X^i \equiv 0)$, a vertical, left-invariant field, and take any of the fields S_a. In this case the only thing left in (5-5) will be

$$\underset{S_a}{\pounds}\psi = S^b_a\partial_b\psi \tag{5-7}$$

(in non-holonomic coordinates, using the properties of S^a_b and A^b_c established in §.4). The formula (5-7) is so simple because the fields S^b_a are left-invariant and transform into each other under the *ad* action of the group. It is easy to check $[\underset{S_a}{£}, \underset{S_b}{£}] = C^c_{ab}\underset{S_c}{£}$.

Finally, if $\underset{S_a}{£}\psi = Ad(+ S_a)\psi$, it will mean

$$S^b_a\partial_b\psi = \frac{1}{2}C_{abc}\sigma^{bc}\psi \tag{5-8}$$

and

$$\partial_d\psi = \frac{1}{2}A^a_dC_{abc}\sigma^{bc}\psi \tag{5-9}$$

whereas $\underset{S_a}{£}\psi = 0$ means just $\partial_a\psi = 0$ in our coordinate system. The equation (4-47) becomes now:

$$\left[\overset{\circ}{\gamma}{}^i\left(\partial_i - \frac{1}{2}A^a_iC_{abc}\overset{\circ}{\sigma}{}^{bc}\right) - \frac{1}{2}\overset{\circ}{\gamma}{}^a\overset{\circ}{g}_{ab}F^b_{kl}\overset{\circ}{\sigma}{}^{kl} + im\right]\psi = 0 \tag{5-10}$$

in the case when $\underset{S_a}{£}\psi = Ad(+ S_a)\psi$, and

$$\left(\overset{\circ}{\gamma}{}^i\partial_i + \frac{1}{2}C_{abc}\overset{\circ}{\gamma}{}^a\overset{\circ}{\sigma}{}^{bc} - \frac{1}{2}\overset{\circ}{\gamma}{}^a\overset{\circ}{g}_{ab}F^b_{kl}\overset{\circ}{\sigma}{}^{kl} + im\right)\psi = 0 \tag{5-11}$$

in the case when $\underset{S_a}{£}\psi = 0$, i. e. when the symmetry is broken.

Let us remark that the Pauli term $\frac{1}{2}F_{kl}\overset{\circ}{\sigma}{}^{kl}$ will be present also in the abelian case when $G = S^1$ and the gauge field is identified with the electromagnetic field. Now the equations (5-10) and (5-11) are operational, but still describe an unrenormalizable theory because of the Pauli-type term

$$\frac{1}{2}\overset{\circ}{\gamma}{}^a\overset{\circ}{g}_{ab}F^b_{kl}\overset{\circ}{\sigma}{}^{kl}.$$

It is obvious that in this theory only the change of the geometry of the fibre bundle $P(M_4, G)$ can modify the interaction and suppress the Pauli term. We propose to enlarge our theory and to include torsion. This means that now, in general, for any vector fields on the bundle, say X, Y, the quantity

$$\nabla_XY - \nabla_YX - [X, Y] = 2\Theta(X, Y) \neq 0 \tag{5-12}$$

is not null.

The (vector-valued) 2-form Θ is called the torsion form; in local coordinates we use the notation:

$$2\Theta^\alpha(X, Y) = S^\alpha_{\beta\gamma}X^\beta Y^\gamma \tag{5-13}$$

The new connection will be now $\left(\left\{ {\alpha \atop \beta\gamma} \right\} \right.$ being the riemannian connection):

$$\Gamma^\alpha_{\beta\gamma} = \left\{ {\alpha \atop \beta\gamma} \right\} + S^\alpha_{\beta\gamma} \tag{5-14}$$

In general, this connection is not metric; but there exists a unique connection which has the same torsion (the antisymmetric part) and which is metric. That is precisely the connection we want to have. The torsion will be completely defined by the following requirements:

$$\gamma^\alpha S^\beta_{\alpha\gamma} g_{\beta\delta} \sigma^{\gamma\delta} = \frac{1}{2} \gamma^a g_{ab} F^b_{kl} \mathring{\sigma}^{kl} \tag{5-15}$$

because we want to get rid of the Pauli term in the generalized Dirac equation; moreover we want

$$S^\alpha_{\beta\gamma} g_{\alpha\delta} + S^\alpha_{\beta\delta} g_{\alpha\gamma} = 0 \tag{5-16}$$

which will make the new connection metric. This is important if we want to be able to define the spinorial connection and a covariant derivative of a spinor.

It is easy to see that the only independent equations in (5-15) are

$$(S^i_{ad} \mathring{g}_{ij} + 2 S^b_{ja} \mathring{g}_{bd}) = 0 \tag{5-17}$$

and

$$(S^b_{kj} \mathring{g}_{ab} + 2 S^i_{ak} \mathring{g}_{ij}) = \frac{1}{2} \mathring{g}_{ab} F^b_{kj} \tag{5-18}$$

$$S^a_{bc} = 0\,, \qquad S^i_{kj} \text{ arbitrary} \tag{5-19}$$

Combining (5-15) and (5-16) we find easily

$$\begin{gathered} S^d_{ja} = 0\,, \quad S^k_{ab} = 0\,, \quad S^l_{ak} = -\, S^l_{ka} = \frac{1}{6} \mathring{g}_{ab} \mathring{g}^{jl} F^b_{kj} \\ \text{and} \\ S^b_{kj} = \frac{1}{6} \mathring{g}_{ab} F^b_{kj} = -\, S^b_{jk} \end{gathered} \tag{5-20}$$

The new connection, which has torsion but is metric an gives Dirac's equation without Pauli-type term is, in the non-holonomic coordinate system introduced in § 4, the following

$$\begin{aligned} \Gamma^a_{bc} &= \frac{1}{2} C^a_{bc} & \Gamma^l_{ak} &= \frac{2}{3} \mathring{g}_{ab} \mathring{g}^{lj} F^b_{kj} \\ \Gamma^l_{ka} &= \frac{1}{3} \mathring{g}_{ab} \mathring{g}^{lj} F^b_{kj} & \Gamma^a_{ij} &= -\frac{1}{3} F^a_{ij} \end{aligned} \tag{5-21}$$

The new curvature scalar is changed very slightly:

$$R = -\frac{1}{4}\overset{\circ}{g}{}^{ab}\overset{\circ}{g}_{ab} - \frac{1}{3}\overset{\circ}{g}_{ab}\overset{\circ}{g}{}^{ik}\overset{\circ}{g}{}^{jl}F^a_{ij}F^b_{kl} \tag{5-22}$$

which amounts to a change of the scale in the Lagrangian. Thus we are naturally led to the Einstein-Cartan theory, which can be found in a modern version in the works of Trautman, Hehl and Kopczynski [*8*] to [*12*]. In this theory, the energy-momentum tensor is the source of curvature, whereas the spin-density tensor is the source of torsion: in proper units,

$$S^i_{kl} = \left(s^i_{kl} - \delta^i_k s^m_{ml} + \delta^i_l s^m_{mk}\right) \tag{5-23}$$

where by definition

$$s^j_{kl} = i\bar{\psi}(\overset{\circ}{\gamma}{}^j\overset{\circ}{\sigma}_{kl} + \overset{\circ}{\sigma}_{kl}\overset{\circ}{\gamma}{}^j)\psi \tag{5-24}$$

In our version only the space-time components S^i_{kl} are to be determined; it is easy to see that $s^i_{ik} \equiv 0$, therefore

$$S^j_{kl} = i\bar{\psi}(\overset{\circ}{\gamma}{}^j\overset{\circ}{\sigma}_{kl} + \overset{\circ}{\sigma}_{kl}\overset{\circ}{\gamma}{}^j)\psi \tag{5-25}$$

other components of $S^\alpha_{\beta\gamma}$ being determined by (5-20).

The Dirac equation for the left-invariant spinors of A*d* type reduces to a usual one

$$\overset{\circ}{\gamma}{}^k\left(\partial_k - \frac{1}{2}A^a_i C_{abc}\overset{\circ}{\sigma}{}^{bc}\right)\psi + im\psi = 0 \tag{5-26}$$

whereas the spinors constant under group action do not interact at all with the gauge field:

$$\overset{\circ}{\gamma}{}^k\partial_k\psi + \frac{1}{2}C_{abc}\overset{\circ}{\gamma}{}^a\overset{\circ}{\sigma}{}^{bc}\psi + im\psi = 0 \tag{5-27}$$

6. INTERACTION WITH A VECTOR FIELD

To complete our theory, which includes already the generalized spinors and the gauge fields, we want to introduce the equivalent of the Klein-Gordon equation for massive vector fields. The existence of the metric on the fibre bundle $P(M_4, G)$ gives us immediately the natural operator, which is the Laplace-Beltrami operator, and which is introduced as follows:

Define first the Hodge duality operator, which for any p-form ω over an n-dimensional manifold gives an $n-p$-form $*\omega$ which has the same volume as the former one. It is easy to see that $*$ is an isomorphism of the corresponding vector spaces. Then we define the derivation $\delta = *^{-1}d*$; it is easy to see that δ : p-forms into $p-1$-forms. Finally, as $\delta\delta = 0$ (because $dd = 0$) we can define a self-adjoint operator of second order:

$$\Delta = (\delta + d)(\delta + d) = \delta d + d\delta \tag{6-1}$$

acting on any p-form and preserving its type.

The generalized Klein-Gordon equation will be written as

$$(\Delta + M^2)\phi = 0 \tag{6-2}$$

We are interested in the explicit expression for a 1-form over $P(M_4, G)$; let us remind that the metric structure over our fibre bundle gives a one-to-one correspondance between the 1-forms and the vectors.

A 1-form ϕ is decomposed in a local frame $\overset{*}{e}{}^{\alpha}$ as $\phi = \phi_\alpha \overset{*}{e}{}^{\alpha}$. The operator Δ, when expressed in local coordinates, will give

$$(\Delta\phi)_\gamma = g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\phi_\gamma - g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\gamma\phi_\beta + g^{\alpha\beta}\tilde{\nabla}_\gamma\tilde{\nabla}_\alpha\phi_\beta \tag{6-3}$$

where $\tilde{\nabla}_\alpha$ means the covariant derivative with respect to the *Christoffel connection* $\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}$ defined in (4-33).

The same formula can be written as

$$(\Delta\phi)_\gamma = g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\phi_\gamma + g^{\alpha\beta}R_{\alpha\gamma}{}^{\delta}{}_{\beta}\phi_\delta \tag{6-4}$$

where $R_{\alpha\gamma}{}^{\delta}{}_{\beta}$ is the Riemann tensor of the Christoffel connection. Remark that the existence of torsion *does not modify* the definition of the Laplace-Beltrami operator, which is introduced in a unique way by the metric structure only. In order to find out the explicit expression for (6-2) we shall work once more in our non-holonomic system of frames. Our 1-form can be written symbolically as

$$\phi_\alpha = (\phi_a, W_j) \tag{6-5}$$

where ϕ_a is the vertical part of ϕ_α, W_j its horizontal part. Correspondingly, we shall obtain two independent systems of equations.

In order to make them operational, we have to define the behaviour of our form under the action of the structural group G. We are interested in the form of *ad* type, such that

$$\mathcal{D}_b\phi_b = S^a_b\partial_a\phi_c = C^d_{bc}\phi_d \tag{6-6}$$

therefore

$$\mathcal{D}_j\phi_c = \partial_j\phi_c - A^b_jC^d_{bc}\phi_d \tag{6-7}$$

As for the horizontal part W_j, we assume that

$$\mathcal{D}_iW_j = \partial_iW_j, \qquad \text{and} \qquad \mathcal{D}_aW_j \equiv 0 \tag{6-8}$$

After some calculus, we get the following two sets of equations:

$$(\Delta + M^2)\phi_c = \overset{\circ}{g}{}^{ij}\mathcal{D}_i\mathcal{D}_j\phi_c + M^2\phi_c - \overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{kl}\overset{\circ}{g}{}^{cb}F^b_{ik}\mathcal{D}_jW_l \tag{6-9}$$

$$(\Delta + M^2)W_k = \overset{\circ}{g}{}^{ij}\mathcal{D}_i\mathcal{D}_jW_k + M^2W_k + \overset{\circ}{g}{}^{ij}\mathcal{D}_j(F^b_{ik}\phi_b) \tag{6-10}$$

The first two terms in each of these equations describe an invariant interaction of the scalar multiplet ϕ_c with the gauge field, and the Klein-

Gordon equation for the massive vector field. Moreover, next terms describe a very ugly non-minimal couplings which are unrenormalizable.

Let us write down the Lagrangian giving rise to these equations:

$$\begin{aligned}\mathscr{L}_\phi = &-\frac{1}{2}\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{ab}(\mathscr{D}_i\phi_a)(\mathscr{D}_j\phi_b) - \frac{1}{2}\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{kl}(\mathscr{D}_i\mathrm{W}_k)(\mathscr{D}_j\mathrm{W}_l)\\ &-\frac{\mathrm{M}^2}{2}\overset{\circ}{g}{}^{ab}\phi_a\phi_b + \overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{kl}\phi_b\mathrm{F}^b_{ik}\mathscr{D}_j\mathrm{W}_l - \frac{1}{2}\mathrm{M}^2\overset{\circ}{g}{}^{ij}\mathrm{W}_i\mathrm{W}_j\end{aligned} \tag{6-11}$$

Remark that even if ϕ_α had to be vertical, i. e. if $\mathrm{W}_l \equiv 0$, the equation (6-10) would give a strange constraint. We shall see that the symmetry between ϕ_b and W_k can be restored and the non-minimal couplings can be removed once more if we introduce an appropriate terms with torsion $\mathrm{S}^\alpha_{\beta\gamma}$.

The Pauli-term representing the non-minimal coupling in the Dirac equation was removed by introducing the torsion term into the Lagrangian:

$$\bar{\psi}\gamma^\alpha \mathrm{S}^\gamma_{\alpha\beta}g_{\gamma\delta}\sigma^{\beta\delta}\psi \tag{6-12}$$

Let us try to add to the Lagrangian of the vector field on the bundle a similar term including the torsion. If we look closer at our Lagrangian and the corresponding Laplace-Beltrami equation, we see that the Lagrangian can be written as

$$\mathscr{L}_\phi = \frac{1}{2}g^{\alpha\beta}g^{\gamma\delta}(\tilde{\nabla}_\alpha\phi_\alpha)(\tilde{\nabla}_\beta\phi_\delta) + \frac{1}{2}g^{\alpha\beta}\tilde{\mathrm{R}}_{\alpha\gamma}{}^{\rho}{}_{\delta}g^{\gamma\sigma}\phi_\rho\phi_\sigma \tag{6-13}$$

(The tilda means that everything is taken with respect to the Christoffel connection).

Let us prove that we can obtain a new Lagrangian with minimal coupling if we suppress the second term in (6-13) and add instead the term of the following form:

$$\frac{3}{4}\mathrm{S} = \frac{3}{4}g^{\alpha\beta}g^{\gamma\delta}\phi_\delta \mathrm{S}^\rho_{\gamma\alpha}(\tilde{\nabla}_\beta\phi_\rho - \tilde{\nabla}_\rho\phi_\beta) \tag{6-14}$$

As a matter of fact, we have:

$$\begin{aligned}\frac{1}{2}g^{\alpha\beta}g^{\gamma\delta}(\tilde{\nabla}_\alpha\phi_\gamma)(\tilde{\nabla}_\beta\phi_\delta) = &\frac{1}{2}\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{ab}(\mathscr{D}_i\phi_a)(\mathscr{D}_j\phi_b)\\ &+\frac{1}{2}\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{k}(\mathscr{D}_i\mathrm{W}_k)(\mathscr{D}_j\mathrm{W}_l) + \frac{1}{2}\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{kl}\phi_b\mathrm{F}^b_{ik}(\mathscr{D}_j\mathrm{W}_l)\\ &+\frac{1}{4}\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{kl}\phi_b(\mathscr{D}_j\mathrm{F}^b_{ik})\mathrm{W}_l + \frac{1}{8}\mathrm{C}^d_{ac}\mathrm{C}^f_{bd}g^{ab}\phi^c\phi_f\\ &+\frac{1}{8}\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{kl}\mathrm{F}^d_{ik}\mathrm{F}^b_{lj}\phi_b\phi_d\end{aligned} \tag{6-15}$$

whereas a simple calculus gives us

$$\frac{3}{4}S = \frac{3}{4}g^{\alpha\beta}g^{\gamma\delta}\phi_\delta S^\rho_{\gamma\alpha}(\tilde{\nabla}_\beta\phi_\rho - \tilde{\nabla}_\rho\phi_\beta)$$
$$= \frac{1}{4}\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{kl}\phi_b F^b_{ki}\mathscr{D}_j W_l + \frac{1}{4}\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{kl}W_l F^b_{ik}\mathscr{D}_j\phi_b$$
$$+ \frac{1}{8}\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{kl}\phi_b F^b_{ki}F^d_{jl}\phi_d \tag{6-16}$$

The sum of these two expressions gives a Lagrangian with the minimal coupling:

$$\frac{1}{2}g^{\alpha\beta}g^{\gamma\delta}(\tilde{\nabla}_\alpha\phi_\gamma)(\tilde{\nabla}_\beta\phi_\delta) + \frac{3}{4}g^{\alpha\beta}g^{\gamma\delta}\phi_\delta S^\rho_{\gamma\alpha}(\tilde{\nabla}_\beta\phi_\rho - \tilde{\nabla}_\rho\phi_\beta)$$
$$= \frac{1}{2}\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{ab}(\mathscr{D}_i\phi_a)(\mathscr{D}_j\phi_b) + \frac{1}{2}\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{kl}(\mathscr{D}_i W_k)(\mathscr{D}_j W_l)$$
$$+ \frac{1}{8}C^d_{ac}C^f_{bd}g^{ab}\phi^c\phi_f + \frac{1}{4}\mathscr{D}_j(\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{kl}\phi_b F^b_{ik}W_l) \tag{6-17}$$

The last term, being a pure divergence, does not modify the equations of motion. We can write therefore, together with the mass term:

$$\mathscr{L}_\phi = \frac{1}{2}g^{\alpha\beta}g^{\gamma\delta}(\tilde{\nabla}_\alpha\phi_\gamma)(\tilde{\nabla}_\beta\phi_\delta) - \frac{M^2}{2}g^{\alpha\beta}\phi_\alpha\phi_\beta$$
$$+ \frac{3}{4}g^{\alpha\beta}g^{\gamma\delta}\phi_\delta S^\rho_{\gamma\alpha}(\tilde{\nabla}_\beta\phi_\rho - \tilde{\nabla}_\rho\phi_\beta)$$
$$= \frac{1}{2}\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{ab}(\mathscr{D}_i\phi_a)(\mathscr{D}_j\phi_b) + \frac{1}{2}\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{kl}(\mathscr{D}_i W_k)(\mathscr{D}_j W_l)$$
$$- \left(\frac{M^2}{2}\delta^f_c + \frac{1}{8}C^d_{ca}C^f_{bd}g^{ab}\right)\phi^c\phi_f - \frac{M^2}{2}\overset{\circ}{g}{}^{ij}W_iW_j \tag{6-18}$$

In physical applications the structure constants are rescaled and multiplied by some coupling parameter, characterizing the interaction force with the gauge field: $C^c_{ab} \to \lambda C^c_{ab}$. We remind that $C^d_{ca}C^f_{bd}g^{ab} = \delta^f_c$ for any compact and semi-simple Lie group. We see then that the masses of the vector field W_k and the scalar multiplet ϕ_c are different: if the vector field W_k has the mass M, the scalar multiplet will have the mass $\sqrt{M^2 + \frac{\lambda^2}{4}}$ where λ is the coupling constant. The asymmetry in masses is due to the fact that our operator is no longer conformally invariant with respect to the global metric on the fibre bundle. Remark that even if $M = 0$, the scalar field will not be massless anymore and will have the mass of order $\frac{\lambda}{2}$. The only massless scalar particles could exist if there was an abelian subgroup in the structural group G.

It seems that this scheme is an argument to take seriously the Einstein-Cartan theory with torsion as a good tool for the unified theories of interactions. In that case there should also exist an interaction of the vector field W_k with the space-time torsion:

$$\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{kl}W_l S^m_{ki}(\tilde{\nabla}_j W_m - \tilde{\nabla}_m W_j) \tag{6-19}$$

(cf. also [*15*], [*16*], [*17*]).

7. THE REDUCTION OF THE GENERALIZED SPINORS

If we consider the left-invariant generalized spinors on the bundle $P(M_4, G)$, we can calculate in principle the value of the spinor ψ at any point of a fibre if we know this value at another point of the same fibre. Remind that, in our coordinates.

$$\psi(p^\alpha) = \psi(x^i, \xi^a) \tag{7-1}$$

where x^i are the coordinates of a point in M_4, and ξ^a are the coordinates of a point in G (or in a fibre over $x \in M_4$). For the left-invariant spinor

$$\frac{\partial\psi}{\partial\xi^a} = \frac{1}{2} C_{abc} A^d_a \overset{\circ}{\sigma}{}^{bc}\psi \tag{5-9}$$

If we know the value of ψ at $e \in G$, we can have it at $g \in G$. The same is true along the fibres; we can get the value of ψ at (x, g) if we know it at (x, e). The Lie algebra $\mathscr{A}_G$ acts on the fibre bundle spinors through its adjoint representation and via its embedding into the Lie algebra of $SO(N + 3, 1)$ represented in the K-dimensional space ($K = 2^{\left[\frac{N+5}{2}\right]}$):

$$\mathscr{A}_G \to ad\mathscr{A}_G \to \mathscr{A}_{SO(N+3,1)} \to GL(K)$$

The action of a finite translation along a fibre can be obtained by exponentiating the infinitesimal action given by (5-9), because

$$\exp\,[Ad(g)]X = g \exp(X) g^{-1} \tag{7-2}$$

for $g \in \dot{G}$, $X \in \mathscr{A}_G$. We can write, for $p \in P(M_4, G)$,

$$\psi(p) = \psi(x, g) = D(g)\psi(x, e) = D(g)\psi(x) \tag{7-3}$$

where we put for the sake of simplicity $\psi(x, e) = \psi(x)$. The explicit calculus of the representation $D(g)$ may be quite difficult in general.

The same procedure can be applied for the action of the Lorentz group on ψ. If α_{kl} are the infinitesimal parameters of a Lorentz transformation, then the formula

$$\delta\psi = \alpha_{kl}\overset{\circ}{\sigma}{}^{kl}\psi \tag{7-4}$$

defines a *reducible* representation of the Lie algebra of the Lorentz group in a K-dimensional space. This representation can be decomposed into a sum of irreducible representations. In other words, the generalized spinor over the fibre bundle can be represented as a linear combination of the Lorentz spinors. The spinor terms in the Lagrangian density shall represent a sum of Dirac terms with different masses and different charges. This comes out after the integration over the group space (a fibre) with respect to the invariant Haar measure: for example, according to (7-3) we can write the mass term as:

$$\begin{aligned} m\bar{\psi}\psi &= m\overline{\mathrm{D}(g)\psi(x)}\mathrm{D}(g)\psi(x) \\ &= m\bar{\psi}(x)\bar{\mathrm{D}}(g)\mathrm{D}(g)\psi(x) \\ &= \sum_{\mathrm{A,B}} m\begin{pmatrix}\mathrm{A}\\ \mathrm{B}\end{pmatrix} u_{\mathrm{A}}\bar{\mathrm{D}}(g)\mathrm{D}(g)u_{\mathrm{B}} \end{aligned} \tag{7-5}$$

Here u_{A} are the multiplets of Dirac spinirs; the nature of the indices A, B depend on the representation; the decomposition coefficients $\begin{pmatrix}\mathrm{A}\\ \mathrm{B}\end{pmatrix}$ are, in simplest case the $3 - j$ Racah symbols (when $\mathrm{G} = \mathrm{SO}(3)$).

Remark that$\bar{\mathrm{D}}(g)$ does not mean the Hermitian conjugate in the sense of matrices, but the conjugate in the spinor sense:

$$\bar{\mathrm{D}}(g) = \beta^{-1}\mathrm{D}^{+}(g)\beta \tag{7-6}$$

The same decomposition has to be performed with the interaction term

$$-\frac{1}{2}\bar{\psi}\mathrm{A}^a_j\mathrm{C}^c_{ab}g_{cd}\overset{\circ}{\sigma}{}^{bd}\psi = -\frac{1}{2}\sum_{\mathrm{B,C}}\begin{pmatrix}\mathrm{B}\\ \mathrm{C}\end{pmatrix}\bar{u}_{\mathrm{B}}\bar{\mathrm{D}}(g)\mathrm{A}^a_j\mathrm{C}^c_{ab}g_{cd}\overset{\circ}{\sigma}{}^{bd}\mathrm{D}(g)u_{\mathrm{C}} \tag{7-7}$$

Here the integration over the group space will yield the generalized charges corresponding to each multiplet of Dirac spinors.

Such a program has been performed by G. Domokos and S. Kövesi-Domokos [*2*] for the case when $\mathrm{G} = \mathrm{SU}(2)$. For simplicity, they have used an unfaithful representation of C(6, 1) in eight-dimensional space, i. e. realized as a product $\mathrm{H} \otimes \mathrm{C}(3, 1)$.

In this representation the generators of C(6, 1) are chosen to be:

$$\gamma^\alpha = :\{\tau^3 \otimes \gamma^i,\ \tau^1 \otimes \mathbb{1},\ \tau^2 \otimes \mathbb{1},\ \tau^3 \otimes \gamma^5\}$$

here $\alpha = 0, 1, \ldots, 6$, and $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, τ^1, τ^2 and τ^3 are the usual 2×2 Pauli matrices, γ^i are the usual Dirac matrices, and $\mathbb{1}$ means 4×4 identity matrix. Then, if we introduce the projection operators $\frac{1}{2}(\mathbb{1} + \tau^3 \otimes \mathbb{1})$ and $\frac{1}{2}(\mathbb{1} - \tau^3 \otimes \mathbb{1})$, and denote

$$\psi_1 = \frac{1}{2}(\mathbb{1} + \tau^3 \otimes \mathbb{1})\psi$$

$$\psi_2 = \frac{1}{2}(\mathbb{1} - \tau^3 \otimes \mathbb{1})\psi$$

In this case ψ_1 and ψ_2 are the Lorentz spinors embedded in 8-dimensional space. If we denote ψ_1, ψ_2 by ψ_A (A, B) = 1, 2), then $\psi_A(x, g)$ can be expanded into irreducible representations of the group SO(3) as follows:

$$\begin{aligned}\psi_A(x, g) &= \sum_{B,l,m} D^{1/2}_{AB}(g) D^l_{0m}(g) \psi_B^{lm}(x) \\ &= \sum_{l,m,B,j,M} (2j+1)\begin{pmatrix} 1/2 & l & j \\ A & 0 & -A \end{pmatrix}\begin{pmatrix} 1/2 & l & j \\ B & m & -M \end{pmatrix}(-1)^{M-A} D^j_{AM}(g)\psi_B^{lm}(x) \\ &= \sum_{j,M,l} (2j+1)^{1/2}\begin{pmatrix} 1/2 & j-\varepsilon/2 & j \\ A & 0 & -A \end{pmatrix}(-1)^{M-A} D^j_{AM}(g)\chi_{j\varepsilon M}(x)\end{aligned}$$

Where $\varepsilon = \pm 1, |l|, |m| < 2j + 1$, and

$$\chi_{j\varepsilon M}(x) = (2j+1)^{1/2} \sum_{B,l,m} \begin{pmatrix} 1/2 & l & j \\ B & m & -M \end{pmatrix} \delta_{l,j-\varepsilon/2}\psi_B^{lm}(x)$$

We shall not write down the similar expansion for other terms in the Dirac Lagrangian. Instead we give the final result of averaging

$$\bar{\psi}\gamma^\alpha\nabla_\alpha\psi + \nabla_\alpha\bar{\psi}\gamma^\alpha\psi$$

over the group G, in absence of mass m, and of the field A^a_i. We get the following expression:

$$\begin{aligned}&\frac{1}{V_G}\int(\bar{\psi}\gamma^\alpha\nabla_\alpha\psi + \nabla_\alpha\bar{\psi}\gamma^\alpha\psi)dG \\ &= \sum_{j,\varepsilon} [\bar{\Sigma}_{j\varepsilon}(\gamma^i\partial_i)\chi_{j\varepsilon}] + \sum_{j,\varepsilon}\left[\bar{\chi}_{j\varepsilon}\sigma^3\left(\frac{3}{4i}\gamma^5 + \sigma^a C_{abc}\sigma^{bc} + \gamma^5\sigma^3 C_{3bc}\sigma^{bc}\right)\chi_{j\varepsilon}\right]\end{aligned}$$

Here $\chi_{j\varepsilon}$ is a $4(2j + 1)$-column belonging to the irreducible subspace (j) of the gauge group SU(2), i. e. a column of $2j + 1$ Dirac spinors. Let us define new multiplets of Dirac spinors $u_{j\varepsilon}$ by putting

$$\chi_{j\varepsilon} = \frac{1}{2}\left[(1+\gamma^5) + e^{\frac{i\pi\sigma^3}{2}}(1-\gamma^5)\right] e^{\frac{i\pi\sigma^3}{2}(\varepsilon-1)\gamma^5} u_{j\varepsilon}$$

This ansatz eliminates the negative-mass (ghost) terms. As a matter of fact, we get

$$\frac{1}{V_G}\int(\bar{\psi}\gamma^\alpha\nabla_\alpha\psi + \nabla_\alpha\bar{\psi}\gamma^\alpha\psi)dG = \sum_{j,\varepsilon}\left[\frac{1}{2}\bar{u}_{j\varepsilon}\gamma^i\partial_i u_{j\varepsilon} + \left(j+\frac{1}{2}-\frac{\varepsilon}{4}\right)\bar{u}_{j\varepsilon}u_{j\varepsilon}\right]$$

The most important result is therefore the fact that the mass term is given by linear family $\left(j+\frac{1}{2}-\frac{\varepsilon}{4}\right)$, depending on the representation index j. This is similar to the well-known Regge trajectories, whose origin is of course different (here we have the representations of the internal symmetry group instead of the kinematical group in Regge's formalism). Averaging the interaction term over the group

$$\frac{1}{V_G}\int\bar{\psi}\gamma^i A_i^a C_{abc}\overset{\circ}{\sigma}{}^{bc}\psi dG$$

will also give the decomposition into irreducible representations, and the coefficients can be regarded upon as the generalized charges.

The comparison with the experimental data seems to be precarious; anyway, the true symmetry of the theory should be at least SU(2) × SU(1) or SU(3), in order to describe weak and electromagnetic, or the strong interactions.

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