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Compatibility and partial compatibility in quantum logics

by

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ABSTRACT. — Compatibility relation, commensurability of observables and existence of joint distributions in quantum logics are considered. A weakened form of compatibility, so-called partial compatibility of propositions is introduced and its connections with a relativized commensurability of observables and with the existence of joint probability distributions of Gudder's type are studied.

1. INTRODUCTION

In the quantum logic approach to quantum theory, the structure of the set of all yes-no measurements (called also propositions, questions, events), which is called the logic of a physical system, is of a primary importance.

The logic of a classical system is found to be the Boolean lattice of all Borel subsets of the phase space of the system, while the logic of a standard quantum mechanical system is the complete ortholattice of all closed sub-spaces of a (complex, separable) Hilbert space corresponding to the system.

For a general physical system its logic L is assumed to be an orthomodular σ-orthoposet, i. e. L is a partially ordered set with 0 and 1 and with the orthocomplementation $\perp : \mathrm{L} \to \mathrm{L}$ such that *i)* $\vee\, a_i \in \mathrm{L}$ for any sequence of pairwise orthogonal elements of L (we say that $a, b \in \mathrm{L}$ are orthogonal and write $a \perp b$ if $a \leq b^{\perp}$) and *ii)* $a \leq b$ ($a, b \in$ L) implies $b = a \vee c$, where $c \in$ L, $c \leq a^{\perp}$.

If the logic L is given, we can identify the states of the physical system with probability measures on L and the observables with σ-homomorphisms from Borel subsets of the real line R^1 into L. (See e. g. Mackey [*1*], Varadarajan [*2*] and [*3*]).

2. COMPATIBILITY RELATION

Let L be an orthomodular σ-orthoposet. In the following we shall call L briefly a logic. A subset K of L is a sublogic of L if *i)* $a \in$ K implies $a^{\perp} \in$ K and *ii)* $\vee\, a_i \in$ K for any sequence $\{ a_i \}$ of mutually orthogonal elements of K. A subset K of L is a Boolean subalgebra of L if *i)* $a \in$ K implies $a^{\perp} \in$ K, *ii)* for any $a, b \in$ K, $a \vee b \in$ K ($a \wedge b \in$ K) and *iii)* for any $a, b, c \in$ K,

$$a \wedge (b \vee c) = a \wedge b \vee a \wedge c \qquad (a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)).$$

K is a Boolean sub-σ-algebra of L if it is a Boolean sub-algebra and $\vee\, a_i \in$ K ($\wedge\, a_i \in$ K) for any sequence $\{ a_i \}$ of elements of K.

Two elements $a, b \in$ L are said to be compatible ($a \leftrightarrow b$ in symbols) if there exist three pairwise orthogonal elements a_1, b_1, c in L such that $a = a_1 \vee c$ and $b = b_1 \vee c$. Varadarajan [*2*] proved the following:

$a \leftrightarrow b$ iff there exist an observable $x : B(R^1) \to$ L and Borel subsets E, F of R^1 such that $a = x(E)$ and $b = x(F)$.

As a direct consequence we obtain:

$a \leftrightarrow b$ implies $a \leftrightarrow b^{\perp}$.

The following statement may help to clarify the significance of the relation $\leftrightarrow$ [*2*]:

$a \leftrightarrow b$ iff there exists a Boolean subalgebra of L containing both a and b.

Thus, if $a, b \in$ L are compatible, they can be treated as classical propositions. As the most important feature of quantum mechanical physical system is considered the existence of propositions that are not compatible.

The following statements were proved by Varadarajan [*2*] and Mackey [*1*].

(1) If $a \leftrightarrow b$, that is $a = a_1 \vee c$, $b = b_1 \vee c$, $a_1, b_1, c \in$ L are mutually orthogonal, then there exist $a \vee b$ and $a \wedge b$ and $c = a \wedge b$.

(2) Let $a_1, a_2, \ldots$ are elements of L. If $a \leftrightarrow a_i$ for all $i = 1, 2, \ldots$, and if $\vee\, a_i$ and $\vee\, (a \wedge a_i)$ both exist, then $a \leftrightarrow \vee\, a_i$ and $a \wedge (\vee\, a_i) = \vee\, a \wedge a_i$.

(3) The logic L is a Boolean σ-algebra iff $a \leftrightarrow b$ for any $a, b \in$ L.

Guz [*4*] showed that $\leftrightarrow$ is the strongest one in the family of all relations $C \subset L \times L$ such that

i) C is symmetric and reflexive,

ii) aCb implies $aCb^{\perp}$,
iii) $a \leq b$ implies aCb,
iv) aCb, aCc, $b \perp c$ imply $aC(b \vee c)$.

If the relation $\leftrightarrow$ has the following property *c)* for any triple a, b, c of mutually compatible elements of L one has $a \leftrightarrow b \vee c$, we say that $\leftrightarrow$ is regular. The logic L is said to be regular if the relation $\leftrightarrow$ in it is regular. Examples, which have been found by Pool [*6*] and independently by Ramsay [7] show that not every logic is regular. If L is a lattice, then property (2) implies that it is regular.

Let A be a subset of L, we say that A is compatible if $a \overset{L_0}{\leftrightarrow} b$ for any $a, b \in A$. The following statement is true: the logic L is regular iff for any compatible subset A of L there is a Boolean sub-σ-algebra of L containing A (see e. g. Guz [*6*]).

If the logic L is not regular, then a stronger definition of compatibility is needed for the existence of a Boolean σ-algebra containing a compatible set. Such a condition was found by Guz [*5*] and, independently, by Neubrunn [*8*]. We shall call it strong compatibility (s-compatibility). Given a set $A \subset L$, the smallest sublogic L_0 of L containing it always exists. The set A is said to be strongly compatible if any two elements $a, b \in A$ are compatible in L_0. (The compatibility of a, b in L_0, denoted by $a \overset{L_0}{\leftrightarrow} b$ means that there are mutually orthogonal elements a_1, b_1, c in L_0 such that $a = a_1 \vee c$ and $b = b_1 \vee c$). In [*5*] and [*8*] the following theorem is proved.

THEOREM 2.1. — If a subset A of L is strongly compatible, then there is a Boolean sub-σ-algebra B such that $A \subset B \subset L$.

Moreover, Neubrunn [*8*] proved that the sublogic generated by an s-compatible set A coincides with the generated Boolean sub-σ-algebra.

Another strenghthening of compatibility has been introduced by Brabec [*9*]. To distinguish this notion we shall call it full compatibility (f-compatibility). A finite set $\{ a_1, a_2, \ldots, a_n \}$ of elements of L is said to be fully compatible in L if there exists a finite collection of pairwise orthogonal elements $\{ e_i : 1 \leq i \leq k \}$ of L such that for any element $a_i (1 \leq i \leq n)$ there exists a finite subcollection $\{ e_{ij} \}_j$ of $\{ e_i \}_i$ such that $a_i = \vee_j e_{ij}$. The collection $\{ e_i \}_i$ is called an orthogonal covering of the set $\{ a_1, \ldots, a_n \}$. A set $A \subset L$ is said to be f-compatible in L if any finite subset of A is f-compatible in L.

Using f-compatibility, the following result was proved in [*9*].

THEOREM 2.2. — If $A \subset L$ is f-compatible, then there exists a Boolean sub-σ-algebra B such that $A \subset B \subset L$.

Relations among s-compatibility, f-compatibility and pairwise compatibility are discussed in [*10*]. It can be easily seen that s-compatibility implies f-compatibility and f-compatibility implies the pairwise compatibility. The logic L is regular iff f-compatibility is equivalent to the pairwise compatibility.

If $\{ e_i \}_{i=1}^k$ and $\{ f_j \}_{j=1}^l$ are two orthogonal coverings of a finite set $M \subset L$, we say that $\{ e_i \}_{i=1}^k$ is less than $\{ f_j \}_{j=1}^l$ if for any e_i, $1 \leq i \leq k$, there is a subcollection $\{ f_{is} \}_s$ of $\{ f_j \}_{j=1}^l$ such that $e_i = \bigvee_s f_{js}$.

If $M \subset L$, we write $M^\perp = \{ a^\perp : a \in M \}$. The following statement is a consequence of the fact that to any f-compatible subset of L there is a Boolean sub-σ-algebra containing it.

LEMMA 2.3. — Let $M = \{ a_1, a_2, \ldots, a_n \}$ be f-compatible in L. Then the collection $F = \{ a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} : d \in D^n \}$, where $D = \{ 0, 1 \}$, $d = (d_1, d_2, \ldots, d_n) \in D^n$, $a^{d_j} = \begin{cases} a & \text{if } d_j = 1 \\ a^\perp & \text{if } d_j = 0, \end{cases}$ $(a \in L)$, is the minimal covering of the set $M \cup M^\perp$.

LEMMA 2.4. — The set $F = \{ a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} : d \in D^n \}$ is an orthogonal covering of the set $M = \{ a_1, \ldots, a_n \}$ iff

$$\bigvee_{d \in D_n} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} = 1 .$$

Proof. — Necessity follows by Lemma 2.3. To prove sufficiency, let

$$\bigvee_{d \in D_n} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} = 1 .$$

Let $a_j \in M$ be fixed. Clearly, $a_j \leftrightarrow b$ for any $b \in F$. As the elements

$$\{ b \wedge a_j : b \in F \}$$

are mutually orthogonal, we get by (2) that

$$a_j = a_j \wedge \left(\bigvee_{b \in F} b \right) = \bigvee_{b \in F} b \wedge a_j .$$

But

$$b \wedge a_j = \begin{cases} b & \text{if } d_j = 1 \\ 0 & \text{if } d_j = 0 \end{cases},$$

$b = a_1^{d_1} \wedge \ldots \wedge a_j^{d_j} \wedge \ldots \wedge a_n^{d_n}$. From this we have that F is the orthogonal covering of M.

COROLLARY 2.5. — The set $A \subset L$ is f-compatible in L iff for any finite subset $\{ a_1, a_2, \ldots, a_n \}$ of A the elements $a_1^{d_1} \wedge \ldots \wedge a_n^{d_n}$, $d \in \mathrm{D}^n$, all exist and $\bigvee_{d \in \mathrm{D}^n} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} = 1$.

It can be easily seen that a subset A of L is contained in a Boolean sub-σ-algebra of L iff it is f-compatible. The minimal Boolean sub-σ-algebra containing A can be found in the following way. For any finite subset $\mathrm{M} = \{ a_1, \ldots, a_n \}$ of A the set $\mathrm{F} = \{ a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} : d \in \mathrm{D}^n \}$ is orthogonal and its lattice sum equals to one. From this it follows that the set of all lattice sums over all subsets of F is a Boolean subalgebra of L (see e. g. [7]). Let us denote it by B(M). Now let $\mathrm{B}' = \cup \{ \mathrm{B(M)} : \mathrm{M}$ is a finite subset of A $\}$, then B′ is a Boolean subalgebra of L. Indeed, if a, $b \in \mathrm{B}'$ then there are M_1 and M_2 such that $a \in \mathrm{B}(\mathrm{M}_1)$, $b \in \mathrm{B}(\mathrm{M}_2)$. But $\mathrm{M}_1 \cup \mathrm{M}_2$ is a finite subset of A and a, $b \in \mathrm{B}(\mathrm{M}_1 \cup \mathrm{M}_2)$. From this it follows that $a \vee b$, $a \wedge b \in \mathrm{B}(\mathrm{M}_1 \cup \mathrm{M}_2) \subset \mathrm{B}'$. Similarly we show the distributivity. Evidently, B′ is s-compatible, so that by [8] the least sublogic B containing B′ is a Boolean sub-σ-algebra of L. Clearly, B is the minimal Boolean sub-σ-algebra of L containing A.

A set of observables $\{ x_\alpha \}_\alpha$ is said to be commensurable if there is an observable x and Borel functions $f_\alpha : \mathrm{R}^1 \to \mathrm{R}^1$ such that $x_\alpha = f_\alpha \circ x$. (By $f \circ x$, where f is a Borel function we mean the observable

$$f \circ x : \mathrm{E} \to x(f^{-1}(\mathrm{E})),\ \mathrm{E} \in \mathrm{B}(\mathrm{R}^1)).$$

THEOREM 2.6. — A set $\{ x_n \}_{n=1}^{\infty}$ of observables on a logic L is commensurable iff the set $\bigcup_{n=1}^{\infty} \mathrm{R}(x_n)$, where $\mathrm{R}(x_n) = \{ x_n(\mathrm{E}) : \mathrm{E} \in \mathrm{B}(\mathrm{R}^1) \}$ is the range of the observable x_n, is f-compatible in L.

Proof. — The statement follows from the fact that $\{ x_n \}_{n=1}^{\infty}$ are commensurable iff $\bigcup_{n=1}^{\infty} \mathrm{R}(x_n)$ is contained in a Boolean sub-σ-algebra of L (see [3]) iff $\bigcup_{n=1}^{\infty} \mathrm{R}(x_n)$ is f-compatible.

The commensurability of observables enables us to construct joint probability distributions for observables [3].

COROLLARY 2.7. — Let $\{ x_\alpha \}_\alpha$ be a set of observables on L. The joint probability distribution for $\{ x_\alpha \}_\alpha$ exists iff the set $\bigcup_\alpha \mathrm{R}(x_\alpha)$ is f-compatible iff

$$\bigvee_{d \in \mathrm{D}^n} x_1(\mathrm{E}_1)^{d_1} \wedge \ldots \wedge x_n(\mathrm{E}_n)^{d_n} = 1$$

for any $n \in \mathrm{N}$, any $x_1, x_2, \ldots, x_n \in \{ x_\alpha \}_\alpha$ and any $\mathrm{E}_1, \mathrm{E}_2, \ldots, \mathrm{E}_n \in \mathrm{B}(\mathrm{R}^1)$.

Proof. — Let the joint distribution exist. Then for any finite subset $\{x_1, \ldots, x_n\} \subset \{x_\alpha\}_\alpha$ there is a σ-homomorphism $h : \mathrm{B}(\mathrm{R}^n) \to \mathrm{L}$ such that $h(\mathrm{E}_1 x \ldots x \mathrm{E}_n) = x_1(\mathrm{E}_1) \wedge \ldots \wedge x_n(\mathrm{E}_n)$ [3]. From this it follows that

$$\bigvee_{d \in \mathrm{D}^n} x_1(\mathrm{E}_1)^{d_1} \wedge \ldots \wedge x_n(\mathrm{E}_n)^{d_n} = 1 \tag{2.1}$$

is satisfied. It can be easily seen that (2.1) is equivalent to the f-compatibility. Indeed, if $\{a_1, \ldots, a_n\}$ is any subset of $\bigcup_\alpha \mathrm{R}(x_\alpha)$, then any element $a_1^{d_1} \wedge \ldots \wedge a_n^{d_n}$ can be written in the form $x_1(\mathrm{E}_1)^{d_1} \wedge \ldots \wedge x_k(\mathrm{E}_k)^{d_k}$ for some $x_1, \ldots, x_k \in \{x_\alpha\}_\alpha$, $\mathrm{E}_1, \ldots, \mathrm{E}_k \in \mathrm{B}(\mathrm{R}^1)$ and some $d \in \mathrm{D}^k$. The equivalence then follows by Corollary 2.5. Now, if $\bigcup_\alpha \mathrm{R}(x_\alpha)$ is f-compatible, then $\bigcup_{i=1}^n \mathrm{R}(x_i)$ is f-compatible for any $x_1, \ldots, x_n \in \{x_\alpha\}_\alpha$. Hence, $x_1, \ldots, x_n$ are commensurable. From this it follows that the joint distribution exists for them.

3. PARTIAL COMPATIBILITY

DEFINITION 3.1. — Let L be an orthomodular σ-orthoposet. A subset A of L is said to be partially compatible (p. c.) with respect to some $a_0 \in \mathrm{L}$ $(a_0 \neq 0)$ if

i) $a_0 \leftrightarrow a$ for any $a \in \mathrm{A}$,
ii) the set $\{a_0 \wedge a : a \in \mathrm{A}\}$ is f-compatible in L.

PROPOSITION 3.2. — A set $\mathrm{A} \subset \mathrm{L}$ is partially compatible with respect to a_0 iff $a_0 \leftrightarrow a$ for all $a \in \mathrm{A}$ and the elements $a_0 \wedge a, a \in \mathrm{A}$ are f-compatible in the logic $\mathrm{L}_{[0,a_0]} = \{b \in \mathrm{L} : b \leq a_0\}$.

Proof. — Let A be p. c. with respect to a_0 and let $\{a_1, \ldots, a_n\}$ be any finite subset of A. Let $b_i = a_i \wedge a$, $1 \leq i \leq n$. The set $\{b_i, b_i^\perp, 1 \leq i \leq n\}$ is f-compatible in L and $\{b_1^{d_1} \wedge \ldots \wedge b_n^{d_n} : d \in \mathrm{D}^n\}$ is the minimal orthogonal covering of it. If $d_j = 1$ for some j, then

$$b_1^{d_1} \wedge \ldots \wedge b_j^{d_j} \wedge \ldots \wedge b_n^{d_n} \leq b_j \leq a_0,$$

and

$$b_1^\perp \wedge \ldots \wedge b_n^\perp = (b_1 \vee b_2 \vee \ldots \vee b_n)^\perp \geq a_0^\perp,$$

so that all the elements $b_1^{d_1} \wedge \ldots \wedge b_n^{d_n}$, $d \in \mathrm{D}^n$, are compatible with a_0. From this it follows that

$$\bigvee_{d \in \mathrm{D}^n} b_1^{d_1} \wedge \ldots \wedge b_n^{d_n} \wedge a_0 = \bigvee_{d \in \mathrm{D}^n} b_1^{d_1} \wedge a_0 \wedge \ldots \wedge b_n^{d_n} \wedge a_0$$
$$= \left(\bigvee_{d \in \mathrm{D}^n} b_1^{d_1} \wedge \ldots \wedge b_n^{d_n} \right) \wedge a_0 = a_0,$$

where

$$b_i^{d_i} \wedge a_0 = \begin{cases} b_i = a_i \wedge a_0 & \text{if } d_i = 1 \\ b_i^\perp \wedge a_0 = a_i^\perp \wedge a_0 & \text{if } d_i = 0 . \end{cases}$$

Since $b_i^\perp \wedge a_0$ is the orthocomplement of b_i in $L_{[0,a_0]}$, we get by Lemma 2.4 that $\{ b_1, \ldots, b_n \}$ are f-compatible in $L_{[0,a_0]}$.

On the other hand, if $a_0 \leftrightarrow a_i$ for any $a \in A$ and $a_0 \wedge a_i$ $1 \leq i \leq n$ are f-compatible in $L_{[0,a_0]}$, then there is an orthogonal covering of the set $\{ a_0 \wedge a_i : 1 \leq i \leq n \}$ in $L_{[0,a_0]}$, which is also an orthogonal covering in L.

COROLLARY 3.4. — A set $A \subset L$ is partially compatible with respect to $a_0 \in L$ iff for any $\{ a_1, \ldots, a_n \} \subset A$ all the elements $a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge a_0$, $d \in D^n$, exist and

$$\bigvee_{d \in D^n} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge a_0 = a_0 .$$

THEOREM 3.5. — Let $a_1, a_2, \ldots, a_n \in L$ be such that all the elements $a_1^{d_1} \wedge \ldots \wedge a_n^{d_n}$, $d \in D^n$ exist and

$$a_0 = \bigvee_{d \in D^n} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \neq 0 .$$

Then $a_1, \ldots, a_n$ are p. c. with respect to a_0.

Proof. — The elements $a_1^{d_1} \wedge \ldots \wedge a_n^{d_n}$, $d \in D^n$, are mutually orthogonal and $a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \leq a_j$ or $a_j^\perp$ for any $1 \leq j \leq n$, so that

$$a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \leftrightarrow a_j$$

for any $d \in D^n$ and $1 \leq j \leq n$. From this it follows that $a_j \leftrightarrow a_0$, $1 \leq j \leq n$. Moreover,

$$a_j \wedge a_0 = \bigvee_{d \in D^n} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge a_j ,$$

so that $\{ a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge a_j : d \in D^n, 1 \leq j \leq n \}$ is the orthogonal covering of $\{ a_1 \wedge a_0, \ldots, a_n \wedge a_0 \}$.

PROPOSITION 3.6. — If the logic L is regular, then the elements $a_1, \ldots, a_n$ of L are p. c. with respect to a_0 iff a_i, a_j are p. c. with respect to a_0 for any $1 \leq i, j \leq n$.

Proof. — Necessity is clear. To prove sufficiency, let a_i, a_j be p. c. with respect to a_0 for any $1 \leq i, j \leq n$. This implies that $a_i \wedge a_0 \leftrightarrow a_j \wedge a_0$ for all i, j in the logic $L_{[0,a_0]}$. By regularity of the logic $L_{[0,a_0]}$ then

$$a_1 \wedge a_0, \ldots, a_n \wedge a_0$$

are f-compatible in $L_{[0,a_0]}$, hence $a_1, \ldots, a_n$ are p. c. with respect to a_0.

Let A be a subset of L. We set $A^c = \{ b \in L : b \leftrightarrow A \}$, where $b \leftrightarrow A$ means that $b \leftrightarrow a$ for all $a \in A$.

THEOREM 3.7. — If a finite subset M of L is p. c. with respect to a_0, then M^{cc} is also p. c. with respect to a_0.

Proof. — Let $M = \{ a_1, a_2, \ldots, a_n \}$ and let $b \in M^{cc}$. As $a_0 \leftrightarrow a_i$, $1 \leq i \leq n$, $a_0 \in M^c$, so that $a_0 \leftrightarrow b$. Let $\{ e_1, e_2, \ldots, e_k \}$ be the minimal orthogonal covering of the set $\{ a_1 \wedge a_0, \ldots, a_n \wedge a_0 \}$. Clearly, $e_i \leftrightarrow a_j \wedge a_0$, $1 \leq j \leq n$, and $e_i \leq a_0 \leq (a_j \wedge a_0^\perp)^\perp$ implies $e_i \leftrightarrow a_j \wedge a_0^\perp$, $1 \leq j \leq n$ for any $1 \leq i \leq k$. Hence $e_i \leftrightarrow a_j \wedge a_0 \vee a_j \wedge a_0^\perp$, $1 \leq j \leq n$, so that $b \leftrightarrow e_i$, $1 \leq i \leq k$. As $\bigvee_{i=1}^k e_i \leq a_0$, we have $a_0 = \bigvee_{i=1}^k e_i \vee \left(\bigvee_{i=1}^k e_i\right)^\perp \wedge a_0$. But $b \leftrightarrow a_0$ implies $b \leftrightarrow a_0^\perp$, so that $b \leftrightarrow \bigvee_{i=1}^k e_i \vee a_0^\perp$ $\left(\text{because } b \leftrightarrow \bigvee_{i=1}^k e_i,\ b \leftrightarrow a_0^\perp \text{ and } \bigvee_{i=1}^k e_i \perp a_0^\perp\right)$. Then

$$a_0 \wedge b = \left(\bigvee_{i=1}^k e_i\right) \wedge b \vee \left(\bigvee_{i=1}^k e_i\right)^\perp \wedge a_0 \wedge b$$
$$= \bigvee_{i=1}^k (e_i \wedge b) \vee \left(\bigvee_{i=1}^k e_i\right)^\perp \wedge b \wedge a_0 .$$

Since $b \leftrightarrow e_i$, we get $e_i = b \wedge e_i \vee b^\perp \wedge e_i$, $1 \leq i \leq k$, so that

$$\left\{ \{ b \wedge e_i, b^\perp \wedge e_i \}_{i=1}^k,\ b \wedge a_0 \wedge \left(\bigvee_{i=1}^k e_i\right)^\perp \right\}$$

is an orthogonal covering of the set $\{ a_1 \wedge a_0, \ldots, a_n \wedge a_0, b \wedge a_0 \}$. We have shown that $M \cup \{ b \}$ is p. c. with respect to a_0. We proceede further by induction: let $M \cup \{ b_1, \ldots, b_n \}$, $b_1, \ldots, b_n \in M^{cc}$ be p. c. with respect to a_0, and let $b_{n+1} \in M^{cc}$. As $M \subset M \cup \{ b_1, \ldots, b_n \} \subset M^{cc}$ implies $(M \cup \{ b_1, \ldots, b_n \})^{cc} = M^{cc}$, we get by the above part of proof that $M \cup \{ b_1, \ldots, b_n, b_{n+1} \}$ is p. c. with respect to a_0. From this it follows that any finite subset $\{ b_1, \ldots, b_n \}$ of M^{cc} is p. c. with respect to a_0, i. e. M^{cc} is p. c. with respect to a_0.

Let $A \subset L$ be p. c. with respect to a_0 ($a_0 \neq 0$). By Zorn's lemma, there is a maximal set Q p. c. with respect to a_0 and such that $A \subset Q \subset L$.

THEOREM 3.8. — Let $A \subset L$ be p. c. with respect to a_0. Then the maximal set Q p. c. with respect to a_0 and containing A is a sublogic of L.

Proof. — Let $a_i \in Q$, $i = 1, 2, \ldots$ be mutually orthogonal. We show that $\bigvee_{i=1}^{\infty} a_i \in Q$. From $a_i \leftrightarrow a_0$, $i = 1, 2, \ldots$, we have $\bigvee_{i=1}^{\infty} a_i \leftrightarrow a_0$ and $\left(\bigvee_{i=1}^{\infty} a_i\right) \wedge a_0 = \bigvee_{i=1}^{\infty} (a_i \wedge a_0)$. By Proposition 3.2, the set $Q \wedge a_0 = \{ a \wedge a_0 : a \in Q \}$ is f-compatible in $L_{[0,a_0]}$. By Brabec [9] the set $Q \wedge a_0 \vee \left\{ \bigvee_{i=1}^{\infty} (a_i \wedge a_0) \right\}$ is f-compatible in $L_{[0,a_0]}$. From

$$\bigvee_{i=1}^{\infty} (a_i \wedge a_0) = \left(\bigvee_{i=1}^{\infty} a_i\right) \wedge a_0$$

we then get that $Q \vee \left\{ \bigvee_{i=1}^{\infty} a_i \right\}$ is p. c. with respect to a_0, i. e. $\bigvee_{i=1}^{\infty} a_i \in Q$ by the maximality of Q. Clearly, $a \in Q$ implies $a^{\perp} \in Q$, hence Q is a sublogic of L.

REMARK 3.9. — If L is a lattice, then Q is a lattice, too. Indeed, if $a_i \in Q$, $i = 1, 2, \ldots, n$, the elements $\bigvee_{i=1}^{n} a_i$ and $\bigvee_{i=1}^{n} (a_i \wedge a_0)$ exist and $a_0 \leftrightarrow a_i$ implies $a_0 \leftrightarrow \bigvee_{i=1}^{n} a_i$ and $\left(\bigvee_{i=1}^{n} a_i\right) \wedge a_0 = \bigvee_{i=1}^{n} (a_i \wedge a_0)$. By [9], then $\left\{ \bigvee_{i=1}^{n} a_i \wedge a_0 \right\} \cup Q \wedge a_0$ are f-compatible in $L_{[0,a_0]}$, hence $Q \cup \left\{ \bigvee_{i=1}^{n} a_i \right\}$ are p. c. with respect to a_0, i. e. $\bigvee_{i=1}^{n} a_i \in Q$. In this case the set

$$Q \wedge a_0 = \{ a \wedge a_0 : a \in Q \}$$

is a Boolean sub-σ-algebra of $L_{[0,a_0]}$.

In what follows we shall suppose that the logic L is a lattice. We recall that the logic L is separable if any subset of mutually orthogonal elements is at most countable.

THEOREM 3.10. — Let M be a subset of a separable lattice logic L. For any finite set $N \subset M$ let us set

$$a(N) = \bigvee_{d \in D^n} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n},$$

where

$$N = \{ a_1, \ldots, a_n \}, \; D = \{ 0, 1 \}, \; d = (d_1, \ldots, d_n) \in D^n,$$

$$a^{d_j} = \begin{cases} a & \text{if } d_j = 1 \\ a^{\perp} & \text{if } d_j = 0 \end{cases}.$$

Then the element $a_0 = \bigwedge_{N \subset M} a(N)$ exists in L. If $a_0 \neq 0$, M is p. c. with respect to a_0.

Proof. — We show that $N_1 \subset N_2$ implies $a(N_2) \leq a(N_1)$. Indeed, let $N_1 = \{ a_1, \ldots, a_n \}$, $N_2 = \{ a_1, \ldots, a_n, a_{n+1}, \ldots, a_m \}$. Then for any fixed $(d_1, \ldots, d_n) \in D^n$ we have

$$\bigvee_{d_{n+1}, \ldots, d_m} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge a_{n+1}^{d_{n+1}} \wedge \ldots \wedge a_m^{d_m} \leq a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge \bigvee_{d \in D^{m-n}} a_{n+1}^{d_{n+1}} \wedge \ldots \wedge a_m^{d_m}$$

and

$$a(N_2) = \bigvee_{d \in D^m} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge a_{n+1}^{d_{n+1}} \wedge \ldots \wedge a_m^{d_m}$$
$$\leq \bigvee_{d \in D^n} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge \bigvee_{d \in D^{n-m}} a_{n+1}^{d_{n+1}} \wedge \ldots \wedge a_m^{d_m}$$
$$\leq \bigvee_{d \in D^n} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} = a(N_1).$$

Now for any $b \in M'$, $a(N \cup \{ b \}) \leq a(N)$ for any $N \subset M$. By Zierler [*11*] there is a sequence $N_1, N_2, \ldots$ such that $a_0 = \bigwedge_{i=1}^{\infty} a(N_i)$. Then

$$\bigwedge_{i=1}^{\infty} a(N_i \cup \{ b \}) \leq \bigwedge_{i=1}^{\infty} a(N_i) = a_0 .$$

On the other hand, $a_0 = \bigwedge_{N \subset M} a(N)$, so that $a_0 \leq a(N \cup \{ b \})$ for any finite $N \subset M$, hence $a_0 = \bigwedge_{i=1}^{\infty} a(N_i \cup \{ b \})$. As $a(N_i \cup \{ b \}) \leftrightarrow b$ (Theorem 3.5) for any $i = 1, 2, \ldots,$ we get $a_0 \leftrightarrow b$. Now let $\mathrm{N} = \{ a_1, \ldots, a_n \} \subset \mathrm{M}$. Then $a_0 \leftrightarrow a_i$ implies $a_0 \leftrightarrow a_1^{d_1} \wedge \ldots \wedge a_n^{d_n}$ for all $d \in \mathrm{D}^n$. Further, $a_i \wedge a(\mathrm{N}) = \bigvee_{d \in \mathrm{D}^n} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge a_i$. From this it follows that

$$a_i \wedge a_0 = a_i \wedge a(\mathrm{N}) \wedge a_0 = \bigvee_{d \in \mathrm{D}^n} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge a_i \wedge a_0 ,$$

hence $\{ a_1^{d_1} \wedge a_0, \ldots, a_n^{d_n} \wedge a_0 : d \in \mathrm{D}^n \}$ is the orthogonal covering of the set $\{ a_1 \wedge a_0, \ldots, a_n \wedge a_0 \}$. That is, M is p. c. with respect to a_0.

DEFINITION 3.11. — We say that a set $\mathrm{A} \subset \mathrm{L}$ is relatively compatible with respect to a state m if for any finite subset N of A there holds $m(a(\mathrm{N})) = 1$.

From $m(a(\mathrm{N})) = 1$ it follows that $a(\mathrm{N}) \neq 0$, so that by Theorem 3.5 N is p. c. with respect to $a(\mathrm{N})$.

The notion of relative compatibility in Hilbert space logics was introduced by Hardegree [*12*]. In [*13*] it is shown that the relative compatibility is closely connected to the existence of joint distributions of type-1, introduced by Gudder [*14*]: We say that observables $x_1, \ldots, x_n$ on a logic L have a type-1 joint distribution in a state m if there is a measure μ on $\mathrm{B}(\mathrm{R}^n)$ (i. e. the Borel subsets of R^n) such that for any rectangle set

$$\mathrm{E}_1 \times \mathrm{E}_2 \times \ldots \times \mathrm{E}_n \in \mathrm{B}(\mathrm{R}^n)$$

there holds

$$\mu(\mathrm{E}_1 \times \ldots \times \mathrm{E}_n) = m(x_1(\mathrm{E}_1) \wedge \ldots \wedge x_n(\mathrm{E}_n)) .$$

It was proved in [*15*] that $\mathrm{M} \subset \mathrm{L}$ is relatively compatible with respect to a state m iff the two-valued observables corresponding to the elements of M have a type 1 joint distribution in the state m.

It can be shown that if L is a separable lattice logic then a subset M of L is relatively compatible with respect to a state m iff $m(a_0) = 1$, where a_0 is the element defined in Theorem 3.8. Indeed, let M be relatively compatible and let $\{ \mathrm{N}_i \}_i$ be such that $a_0 = \bigwedge_{i=1}^{\infty} a(\mathrm{N}_i)$. Let us set $\mathrm{Q}_1 = \mathrm{N}_1, \mathrm{Q}_i = \bigcup_{i \leq j} \mathrm{N}_j$, then $\mathrm{Q}_1 \subset \mathrm{Q}_2 \subset \ldots, a_0 = \bigwedge_{i=1}^{\infty} a(\mathrm{Q}_i), a(\mathrm{Q}_1) \geq a(\mathrm{Q}_2) \geq \ldots,$ and the continuity from above of m yelds that $m(a_0) = 1$. The converse is straightforward.

Let $\{ x_\alpha : \alpha \in \Lambda \}$ be a set of observables on L. Let us set $\mathrm{M} = \bigcup_\alpha \mathrm{R}(x_\alpha)$, where $\mathrm{R}(x_\alpha) = \{ x_\alpha(\mathrm{E}) : \mathrm{E} \in \mathrm{B}(\mathrm{R}^1) \}$ is the range of x_α. Then for any

$$\{ a_1, \ldots, a_n \} \subset \mathrm{M}$$

the elements $a_1^{d_1} \wedge \ldots \wedge a_n^{d_n}$, $d \in D^n$, can be expressed in the form $x(E_1)^{d_1} \wedge \ldots \wedge x_k(E_k)^{d_k}$ for some $E_1, \ldots, E_k \in B(R^1)$, some $x_1, \ldots, x_k$ and some $(d_1, \ldots, d_k) \in D^k$. By [*15*] we then obtain the following theorem.

THEOREM 3.12. — Observables $\{ x_\alpha \}_\alpha$ on a lattice logic L have a type 1 joint distribution in a state m iff the set $M = \bigcup_\alpha R(x_\alpha)$ is relatively compatible withe respect to m iff

$$m\left(\bigvee_{d \in D^n} x_1(E_1)^{d_1} \wedge \ldots \wedge x_n(E_n)^{d_n}\right) = 1$$

for any $n \in N$, any $x_1, \ldots, x_n \in \{ x_\alpha \}_\alpha$ and any $E_1, \ldots, E_n \in B(R^1)$.

Next theorem shows a connection between relative compatibility and « relative commensurability » of observables.

THEOREM 3.13. — Let the observables $\{ x_\alpha \}_\alpha$ on a separable lattice logic L have a type 1 joint distribution in a state m. Then there are an observable z and Borel functions $f_\alpha : R^1 \to R^1$ such that $m(x_\alpha(E)) = m((f_\alpha \circ z)(E))$ for any $E \in B(R^1)$ and any α.

Proof. — The existence of joint distribution implies that the set

$$M = \bigcup_\alpha R(x_\alpha)$$

is relatively compatible with respect to m. From this it follows that the set M is p. c. with respect to a_0, where a_0 is defined as in Theorem 3.10. Hence, the set $M \wedge a_0 = \{ b \wedge a_0 : b \in M \}$ is f-compatible in $L_{[0,a_0]}$. Let us set $\tilde{x}_\alpha(E) = x_\alpha(E) \wedge a_0$, $E \in B(R^1)$, then $\tilde{x}_\alpha : B(R^1) \to L_{[0,a_0]}$ are compatible observables on the logic $L_{[0,a_0]}$. By [*3*], there exist an observable $\tilde{z}$ on $L_{[0,a_0]}$ and Borel functions $f_\alpha : R^1 \to R^1$ such that $\tilde{x}_\alpha = f_\alpha \circ \tilde{z}$ for any α. Let us set $z(E) = \tilde{z}(E) \vee w(E) \wedge a_0^\perp$, where w is an observable on L defined by

$$w(E) = \begin{cases} 1 & \text{if } c \in E \\ 0 & \text{if } c \notin E \end{cases}$$

for some $c \in R^1$. It can be easily checked that z is an observable on L. Then for any $E \in B(R^1)$,

$$\begin{aligned}(f_\alpha \circ z)(E) &= z(f_\alpha^{-1}(E)) = \tilde{z}(f_\alpha^{-1}(E)) \vee w(f_\alpha^{-1}(E)) \wedge a_0^\perp \\ &= (f_\alpha \circ \tilde{z})(E) \vee (f_\alpha \circ w)(E) \wedge a_0^\perp = \tilde{x}_\alpha(E) \vee (f_\alpha \circ w)(E) \wedge a_0^\perp .\end{aligned}$$

As $m(a_0) = 1$, we obtain $m((f_\alpha \circ z)(E)) = m(x_\alpha(E))$.

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