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Section A :
Physique théorique.

The n-field-irreducible part of a n-point functional

by

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RÉSUMÉ. — La notion de partie irréductible d'ordre n pour une fonction à n points est introduite à l'aide d'une décomposition en secteurs de champs des fonctions à n points. Cette décomposition peut être considérée comme une généralisation du processus de troncation. Afin de motiver et de justifier la définition de la partie irréductible d'ordre n d'une fonction à n points on isole les éléments « essentiellement indépendants » de la décomposition en secteur d'une suite de fonctions à n points et on étudie certains de leurs propriétés. Une application possible à la théorie quantique des champs est discutée.

ABSTRACT. — The notion of the « n-field-irreducible part of a n-point functional » is introduced in terms of the « field-sector decomposition » of the n-point functionals. This field-sector decomposition can in one respect be viewed as a generalization of the truncation procedure. In order to motivate and to justify the definition of the « n-field-irreducible part of a n-point functional » the « essentially independent » elements of the « field-sector decomposition » of a sequence of n-point functionals are isolated and some properties of them are investigated. A possible application to relativistic quantum field theory is discussed.

0. INTRODUCTION

If $A \in L^h_{S.C.C.}(E, L(D, D))$ is a « field » over a suitable space E of test functions on $\mathbb{R}^4$ on a dense subspace $D = D_A$ of some separable Hilbert space

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$\mathcal{H} = \mathcal{H}_A$ with cyclic unit vector ϕ_0 its n-point functionals are defined according to ($n \in \mathbb{N}$)

$$T_n^A(x_1 \otimes \ldots \otimes x_n) = \langle \phi_0, A(x_1) \ldots A(x_n)\phi_0 \rangle \; x_j \in E\,, \quad j = 1, \ldots, n$$

Canonically associated with every field there is a hermitian representation $\underline{A}$ of the tensor algebra $\mathscr{I}(E) = \bigoplus_{n=0}^{\infty} E^{\otimes n}$ (locally convex direct sum) over E such that

$$\underline{A}(\underline{1}) = \mathrm{id}_{\mathcal{H}}\,, \qquad \underline{A}(x_1 \otimes x_2) = A(x_1)A(x_2)$$

hold. The sequence

$$T^A = \{ 1, T_1^A, T_2^A, T_3^A, \ldots \}$$

of n-point functionals of the field A can then be expressed in terms of this representation according to

$$T^A(\underline{x}) = \langle \phi_0, \underline{A}(\underline{x})\phi_0 \rangle \qquad \text{for all} \qquad \underline{x} \in \mathscr{I}(E)$$

Therefore, T^A is a normalized monotone (e. g. $T(\underline{x}^* \cdot \underline{x}) \geq 0$ for all $\underline{x} \in \mathscr{I}(E)$) linear functional on $\mathscr{I}(E)$. The usual assumptions on E are

i) E is nuclear,

ii) E_n ($\equiv$ completion of $E^{\otimes n}$) is barrelled for all $n \in \mathbb{N}$ (1).

Then it follows that T^A has a unique continuous linear extension to the completion $\underline{E}$ of $\mathscr{I}(E)$, e. g. $T \in \underline{E}'_{+,1}$. Now it is well-known and very easy to prove that the « essential » part of the 2-point functional $T_2 = T_2^A$ is

$$T_{1,1}^1 = T_2 - T_1 \otimes T_1$$

which is again monotone in the sense that

$$T_{1,1}^1(x^* \otimes x) \geq 0 \qquad \text{for all} \qquad x \in E$$

holds. Thus we have a decomposition of the 2-point functional into two monotone continuous linear functionals $T_{1,1}^0 = T_1 \otimes T_1$ and $T_{1,1}^1$, $T_2 = T_{1,1}^0 + T_{1,1}^1$, where the first summand $T_{1,1}^0$ is fixed by the 1-point functional while the second summand $T_{1,1}^1$ is not (and is indeed completely independent of T_1). The analogue is true for all T_{2n}, $n = 2, 3, \ldots$.

A well-known and very important application of a symmetrized version of this observation has been made by R. Haag [*1*] in relativistic quantum field theory (r. q. f. t.) when he introduced the notion of « truncated » n-point functionals in order to eliminate symmetrically the contribution of the « vacuum-state » in these functionals. These truncated functionals in turn play a very important role in the analysis of general r. q. f. t., in particular in scattering theory [*2*].

The mathematical basis of the possibility of this subtraction is the following simple consequence

$$| T_{n+m}(x_n^* \otimes x_m) |^2 \leqslant T_{2n}(x_n^* \otimes x_n) T_{2m}(x_m^* \otimes x_m)\,, \qquad n, m = 0, 1, 2, \ldots$$

of the monotonicity condition $T \in \underline{E}'_{+,1}$ for the sequence of n-point functionals. But this monotonicity condition is by no means exhausted by the above sequence of inequalities. For instance there is a similar sequence of inequalities

$$| T_{n,m}(x_n^* \otimes x_m) |^2 \leqslant T_{n,n}(x_n^* \otimes x_n) T_{m,m}(x_m^* \otimes x_m)$$
$$n, m = 0, 1, 2, \ldots, T_{n,m} = T_{n+m} - T_n \otimes T_m$$

which thus allows a corresponding subtraction.

Here we intend to give a systematic analysis of this aspect of a sequence of n-point functionals of a field. This analysis will generalize the concept of truncation as introduced by R. Haag in the sense that we will give a general « field-sector decomposition » of the m-point functionals (notice that the one dimensional subspace spanned by the vacuum vector in r. q. f. t. can be interpreted as the 0-field-sector). But there is an important difference. As the vacuum state is in the domain of all field operators the elimination of the 0-field-sector contribution can be done symmetrically; but this is in general not the case for the n-field-sector contribution for $n \geq 1$. A class of fields for which this however is possible is known, this is the class of Jacobi-fields as introduced in [3].

The way the « field-sector decomposition » is introduced is very simply and at first this decomposition seems to be of no use. But nevertheless the field-sector decomposition has several interesting properties:

1. First of all it is as good as the sequence of n-point functionals to characterize a field.

2. The elements $T^j_{n,m} \in E'_{n+m}$ of the field-sector decomposition of a sequence $T = \{ 1, T_1, T_2, \ldots \}$ of n-point functionals satisfy convolution-type equations. This then allows to prove in terms of which n-point functionals the element $T^j_{n,m}$ of this decomposition is fixed and to give a precise meaning to this notion.

3. The main advantage of this decomposition is that it allows to distinguish which parts of a sequence of n-point functionals are independent and which are not. Specifically this decomposition allows to distinguish that part T^0_n of a n-point functional T_n which is « fixed » in terms of the lower m-point functionals $T_1, \ldots, T_{n-1}$ and a rest $T_n - T^0_n = T^{\text{irr}}_n$ which is not.

4. An interesting application of this decomposition is the solution of the following problem. How to calculate $T_n \in E'_n$, $n \geq 2N + 1$, such that

$$T = \{ 1, T_1, T_2, \ldots \} \in \underline{E}'_{+,1}$$

if $T_1, \ldots, T_{2N}$ are given such that $T_{(2N)} = \{ 1, T_1, \ldots, T_{2N} \}$ is monotone and $T^{\text{irr}}_{2N} = 0$. The solution of this problem says that the whole sequence of n-point functionals is fixed in terms of $T_1, \ldots, T_{2N-1}$ if the irreducible part T^{irr}_{2N} of the 2N-point functional vanishes.

5. To characterize a field to be a Jacobi-field demands continuity and hermiticity properties of the n-field-sector decomposition of the sequence $T = \{ 1, T_1, T_2, \dots \}$ of the n-point functionals, not of the n-point functionals directly [3]. If this n-field-sector decomposition is applied to r. q. f. t. it is possible to give a more precise description of the energy-momentum spectrum of the theory. This in turn is important to analyze the « particle content » of the theory in question. In particular one can look for the connection of the n-field-irreducible part of a n-point functional and the n-particle-irreducible Green's functions as analyzed by J. Bros and Lassalle [4].

I. THE FIELD-SECTOR DECOMPOSITION OF THE n-POINTS-FUNCTIONALS OF A FIELD

The following definition will be motivated by proposition 2 and the well-known relation between normalized continuous linear functionals on $\mathscr{I}(E)$ and fields over E (GNS-construction).

DÉFINITION 1. — A system $\{ T^j_{n,m} \mid 0 \leq n, m \}$ of functionals $T^j_{n,m} \in E'_{n+m}$ is called a (normalized) *field-sector decomposition* over E iff

a) $T^0_{0,0} = 1$ and $T^j_{n,m} = 0$ if $j > n$ or $j > m$

b) $$\sum_{j=0}^{\infty} T^j_{n,m} = \sum_{j=0}^{\infty} T^j_{n',m'}, \quad \text{if} \quad n + m = n' + m'$$

c) $$\sum_{n,m=j}^{N} T^j_{n,m}(x_n^* \otimes x_m) \geq 0 \quad \text{for all} \quad x_n \in E_n \quad \text{and all} \quad j, N \in \mathbb{N} .$$

As usual the positivity conditions *c*) of a field-sector decomposition imply the following hermiticity- and continuity-properties of its elements:

$$\begin{aligned} &a) \qquad (T^j_{n,n})^* = T^j_{m,n} \\ &b) \qquad | T_{n,m}(x_n^* \otimes x_m) |^2 \leq T^j_{n,n}(x_n^* \otimes x_n) T^j_{m,m}(x_m^* \otimes x_m) \end{aligned} \tag{2}$$

for all $x_n \in E_n$, $x_n \in E_m$, and all $0 \leq j \leq n, m$.

Here we used the definition $S^*(x) = \overline{S(x^*)}$ for $S \in E'_n$. The relations (2) will become apparent in the course of the proof of

PROPOSITION 2. — There is a one-to-one correspondance between normalized field-sector decompositions over E and normalized monotone continuous linear functionals on the tensoralgebra $\mathscr{I}(E)$ over E.

Proof. — Suppose $\{ T^j_{n,m} \mid 0 \leq j \leq n, m \}$ to be a normalized field-sector decomposition over E. According to constraints *a*) and *b*) above

$$T_N = \sum_{j=0}^{\infty} T^j_{n,m}, \qquad n + m = N \tag{3}$$

is a well-defined continuous linear functional on E_N for all $N = 0, 1, 2, \ldots$. In particular we have $T_0 = 1$. Thus $T = \{ 1, T_1, T_2, \ldots \} \in E'$ satisfies $T(\underline{1}) = 1$. The positivity constraint *c*) implies monotonicity of this functional T on $\underline{E}$. Whenever $\underline{x} \in \underline{E}$ the following chain of equations holds:

$$T(\underline{x}^*\underline{x}) = \sum_{n,m=0}^{\infty} T_{n+m}(x_n^* \otimes x_m) = \sum_{n,m=0}^{\infty} \sum_{j=0}^{n \wedge m} T^j_{n,m}(x_n^* \otimes x_m)$$
$$= \sum_{j=0}^{\infty} \sum_{n,m \geq j} T^j_{n,m}(x_n^* \otimes x_m)$$

and thus $T \in \underline{E}'_{+,1}$.

Conversely suppose $T \in \underline{E}'_{+,1}$ is given. Then according to the GNS-construction there is a strongly continuous hermitian representation $\underline{A}$ of $\underline{E}$ with a cyclic unit vector Φ_0 in some separable Hilbert space $\mathcal{H}$ such that

$$T(\underline{x}) = \langle \Phi_0, \underline{A}(\underline{x})\Phi_0 \rangle$$

holds for all $\underline{x} \in \underline{E}$. More precisely there is a whole class of unitarily equivalent such representations. The restriction of $\underline{A}$ to E defines a field $A \in \mathcal{S}'^h_{scc}(E, L(D, D))$ (on $D = \underline{A}(\mathcal{S}(E))\Phi_0$ for instance) over E. The various n-field sectors $\mathcal{H}_n$ of this field A are defined according to

$$\mathcal{H}_n = \mathcal{H}_{(n)} \ominus \mathcal{H}_{(n-1)}, \qquad n = 1, 2, 3, \ldots$$

$$\mathcal{H}_{(n)} = \text{closure of} \quad \underline{A}\left(\bigoplus_{\nu=0}^{n} E_\nu\right)\Phi_0 \quad \text{in } \mathcal{H}.$$

These sectors define an orthogonal decomposition of the state-space $\mathcal{H}$ of the field A:

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \qquad \text{if the field A is of infinite order}$$

or (5)

$$\mathcal{H} = \bigoplus_{n=0}^{N} \mathcal{H}_n, \qquad \text{if the field is of finite order N}$$

in both cases $\mathcal{H}_0 = \mathbb{C}\Phi_0$ is used. Now let

$$\Phi^j_n(x_n) = Q_j \Phi_n(x_n)$$

denote the component of $\Phi_n(x_n) = \underline{A}(x_n)\Phi_0$, $x_n \in E_n$, in the j-field sector $\mathcal{H}_j$ (Q_j denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_j$) and define

$$T^j_{n,m}(x^*_n \otimes x_m) = \langle \Phi^j_n(x_n), \Phi^j_m(x_m) \rangle_j \tag{6}$$

for all $x_n \in E_m$ where $\langle \cdot, \cdot \rangle_j$ denotes the scalar product in $\mathcal{H}_j$. As the representation $\underline{A}$ is strongly continuous the functionals $T^j_{n,m}$ are separably continuous on $E_n \times E_m$ and therefore according to the assumptions on E and the nuclear theorem belong indeed to E'_{n+m}. By construction $\phi^j_n(\cdot) = 0$ holds for $j > n$. Therefore *a*) of definition 1 follows. Condition *b*) is implied by $\phi_n = \bigoplus_{j=0}^{n} \phi^j_n$ and $T_{n+m}(x^*_n \otimes x_m) = \langle \phi_n(x_n), \phi_m(x_m) \rangle$. The representation (6) of the functionals $T^j_{n,m}$ in terms of scalar-products yields the positivity condition *c*). Thus the system $\{ T^j_{n,m} \mid 0 \leq j \leq n, m \}$ of functionals $T^j_{n,m}$ defined according to (6) is a field-sector decomposition over E such that relation (3) holds.

Furthermore it is easy to check that any other representation of $\underline{E}$ which is unitarily equivalent to that one we used above yields the same system of functionals $\{ T^j_{n,m} \}$. This then proves proposition 2.

The remainder of this section is used to discuss some simple but important properties of field sector decompositions. The first point is to show convolution type equations for the elements of a field-sector decomposition. In the case $E = \mathcal{S}(\mathbb{R}^4)$ (the Schwartz-space of rapidly decreasing $\mathcal{C}^\infty$-functions on $\mathbb{R}^4$) such a convolution equation reads in a formal manner

$$T^j_{n,m}(f_n \otimes f_m) = \int \dots \int d^{4j}\xi_j d^{4j}\eta_j T^j_{n,j}(f_n, \xi_j) K_j(\xi_j, \eta_j) T^j_{j,m}(\xi_j, f_m)$$
$$\equiv T^j_{n,j} \textcircled{j} T^j_{j,m}(f_n \otimes f_m) \tag{7}$$

with a suitable kernel K_j of j-variables.

In order to prove such a relation we will use nuclearity of E in an essential way. Note first that the n-field-sector $\mathcal{H}_n$ is spanned by

$$\{ Q_n \phi_n(x_n) \mid x_n \in E_n \} \qquad n = 1, 2, \dots .$$

Furthermore nuclearity of E implies nuclearity of $\underline{E}$ and thus of

$$\phi : \underline{E} \to \underline{A}(\underline{E})\phi_0 , \qquad \phi(\underline{x}) = \underline{A}(\underline{x})\phi_0$$

because $\underline{A}(\underline{E})\phi_0$ is dense in [5]. This in turn implies that for all $n = 1, 2, \dots$ there is a countable set of vectors of the form $\{ Q_n \Phi_n(x^n_j) \mid x^n_j \in E_n, j \in \mathbb{N} \}$ which spans $\mathcal{H}_n$. The Gram-Schmidt orthonormalization procedure together with the linearity of $\Phi^n_n = Q_n \Phi_n$ yield the existence of an orthonormal basis of the n-field-sector $\mathcal{H}_n$ of the form

$$\{ \varphi^n_j = \Phi^n_n(h^n_j) \mid h^n_j \in E_n, j \in \mathbb{N} \} .$$

The relation

$$\langle \Phi(\underline{x}), \Phi(\underline{y}) \rangle = T(\underline{x}^* \cdot \underline{y}) = T_{(2n)}(\underline{x}^* \cdot \underline{y}) \quad \text{for all} \quad \underline{x}, \underline{y} \in \underline{E}_{(n)} = \bigoplus_{j=0}^{n} E_j$$

shows that the restriction of the scalarproduct of $\mathscr{H}$ to $\mathscr{H}_{(n)}$ is determined by $T_{(2n)} = \{ 1, T_1, \ldots, T_{2n} \}$. This implies that every orthogonal basis of $\mathscr{H}_n$ is determined by $T_{(2n)}$ but is independent of possible choices of T_j for $j > 2n$. We thus get that the orthogonal projections $Q_n \upharpoonright \mathscr{H}_{(n)}$ are determined by $T_{(2n)}$ and that these projections can be expressed in terms of the basis $\varphi_j^n = \Phi_n^n(h_j^n)$, $j = 1, 2, \ldots$ according to

$$Q_n \psi = \sum_{j=1}^{\infty} \langle \varphi_j^n, \psi \rangle \varphi_j^n \qquad \text{for all} \qquad \psi \in \mathscr{H}_{(n)} .$$

Now introducing this into the definition of $T_{n,m}^j$ yields

$$T_{n,m}^j(x_n^* \otimes y_m) = \sum_{k=1}^{\infty} T_{n,j}^j(x_n^* \otimes h_k^j) T_{j,m}^j(h_k^{j*} \otimes y_m)$$

for all $x_n \in E_n$ and all $y_m \in E_m$. Our considerations above show that the right hand side of this equation does not depend on the special choice of the orthonormal basis in $\mathscr{H}_j$, but only on the projection onto this space as the left hand side does. Therefore a ⓙ-*convolution* of $T_{n,j}^j$ and $T_{j,m}^j$ is well defined according to ($x_n \in E_n$, $y_m \in E_m$):

$$T_{n,j}^j \, ⓙ \, T_{j,m}^j(x_n \otimes y_m) = \sum_{k=1}^{\infty} T_{n,j}^j(x_n \otimes h_k^j) T_{j,m}^j(h_k^{j*} \otimes y_m) \tag{8}$$

where $\{ \Phi_j^j(h_k^j) \mid k \in \mathbb{N} \}$ is any orthonormal basis of $\mathscr{H}_j$. Thus the following convolution-type equations result:

$$T_{n,m}^j = T_{n,j}^j \, ⓙ \, T_{j,m}^j \qquad 0 \leq j \leq n, \ m \tag{9}$$

In the case $E = \mathscr{S}(\mathbb{R}^4)$ one can define formally

$$K_j(\xi_j, \eta_j) = \sum_{k=1}^{\infty} h_k^j(\xi_j) h_k^{j*}(\eta_j) \tag{10}$$

to get equation (7) out of (9).

Note that by construction the orthonormal basis $\{ \Phi_j^j(h_k^j) \mid k \in \mathbb{N} \}$ of $\mathscr{H}_j$ only depends on $T_{(2j)} = \{ 1, T_1, T_2, \ldots, T_{2j} \}$, but is independent of T_n for $n > 2j$. Therefore the ⓙ-convolution itself is determined by $T_{(2j)}$. Having this in mind we discuss some applications of these ⓙ-convolutions.

Thus we have shown the first part of

PROPOSITION 3. — *a*) The elements $T^j_{n,m}$ of a field-sector decomposition $\{ T^j_{n,m} \mid 0 \leq j \leq n, m \}$ satisfy the following convolution type equations

$$T^j_{n,m} = T^j_{n,j} \textcircled{j} T^j_{j,m} \qquad 0 \leq j \leq n,\ m .$$

b) If $\{ T^j_{n,m} \mid 0 \leq j \leq n, m \}$ is the field-sector decomposition of $T = \{ 1, T_1, T_2 \ldots \} \in \underline{E}'_{+,1}$ then the element $T^j_{n,m}$ is fixed in terms of

$$\{ T_1, \ldots, T_{2j}, T_n, \ldots, T_{n+j}, T_m, \ldots, T_{m+j} \}$$

Proof. — We prove statement *b*) by induction on *j*. If $j = 0$ then $T^0_{n,m} = T_n \otimes T_m$ and thus $T^0_{n,m}$ is fixed in terms of $\{ T_n, T_m \}$. Now suppose the statement to hold for all $j \in \{ 0, \ldots, j_0 - 1 \}$ and all $n, m \geq j$ for some fixed $j_0 \geq 1$. For all $n \geq j_0$ equation (3) implies

$$T_{n+j_0} = \sum_{j=0}^{j_0-1} T^j_{n,j_0} + T^{j_0}_{n,j_0}$$

By induction hypothesis the first summand is fixed in terms of

$$\{ T_1, \ldots, T_{2j_0-1}, T_n, \ldots, T_{n+j_0-1} \} .$$

Therefore $T^{j_0}_{n,j_0}$ is fixed in terms of $\{ T_1, \ldots, T_{2j_0-1}, T_n, \ldots, T_{n+j_0} \}$. Similarly $T^{j_0}_{j_0,m}$ is fixed in terms of $\{ T_1, \ldots, T_{2j_0-1}, T_m, \ldots, T_{m+j_0} \}$. Therefore, according to equation (9), $T^{j_0}_{n,m}$ is fixed in terms of

$$\{ T_1, \ldots, T_{2j_0-1}, T_n, \ldots, T_{n+j_0}, T_m, \ldots, T_{m+j_0} \}$$

and the j_0-convolution, that is in terms of

$$\{ T_1, \ldots, T_{2j_0}, T_n, \ldots, T_{n+j_0}, T_m, \ldots, T_{m+j_0} \}$$

according to our remark above.

Remark. — The *j*-convolutions, $j = 1, 2, \ldots$, are well-defined by equations (8). But these definitions are not very convenient. Therefore one would prefer to give the more elegant but formal relations (7) and (10) a precise meaning. This then demands further investigations.

II. THE *n*-FIELD IRREDUCIBLE PART OF A *n*-POINT FUNCTIONAL

The field-sector decomposition of a sequence $T = \{ 1, T_1, T_2, \ldots \} \in E'_{+,1}$ of *n*-point functionals associates with this sequence another countable set of functionals which looks even more complicated accordings to relations (3) and (4) and thus hardly useful. But the number of elements of the field-sector decomposition allows a considerable reduction to « essentially independent » elements. This in turn allows to distinguish that part of a *n*-point functional T_n which is already fixed in terms of $\{ T_1, \ldots, T_{n-1} \}$

and a rest which is not (a part which is « essentially independent » of $T_1, \ldots, T_{n-1}$). To have such a characterization in terms of « essentially independent » functionals is of considerable interest for the construction of a sequence of n-point functionals. Together with another result (theorem 6) this reduction to « essentially independent » functionals will motivate and justify our definition of the n-field irreducible part of a n-point functional.

The following construction shows which elements of a n-field-sector decomposition have to be given and which properties they must have in order to construct all elements of this n-field-sector decomposition and the associated n-point functionals.

For two sequences of functionals $\{ R_j \mid j \in \mathbb{N} \}$ and $\{ S_j \mid j \in \mathbb{N} \}$ one can demand the following properties:

$$R_j \in E'_{2j,+}, \qquad \text{e. g.} \qquad R_j(x^* \otimes x) \geq 0 \qquad \text{for all} \quad x \in E_j \qquad (11\,a)$$

$S_j \in E'_{2j+1}$ such that for all $x \in E_j$ and all $y \in E_{j+1}$

$$| S_j(x^* \otimes y) | \leq q_j(x) p^j_{j+1}(y) \qquad (11\,b)$$

with the continuous semi-norm q_j on E_j

$$q_j(x) = \sqrt{R_j(x^* \otimes x)} \qquad (11\,c)$$

and some continuous semi-norm p^j_{j+1} on E_{j+1}.

Furthermore the recursively defined functionals $T^j_{j,n} \in E'_{n+j}$, $n > j + 1$ (see below) satisfy

$$| T^j_{j,n}(x^* \otimes y) | \leq q_j(x) p^j_n(y) \qquad \forall x \in E_j \quad \forall y \in E_n \qquad (12\,a)$$

with some continuous semi-norm p^j_n on E_n; and the recursively defined functionals $T^i_{j,j+1}$, $0 \leq i \leq j - 1$ (see below), satisfy

$$S^*_j - S_j = \sum_{i=0}^{j-1} \{ T^i_{j,j+1} - (T^i_{j,j+1})^* \} \qquad (12\,b)$$

With this specifications we are going to show

Proposition 4. — Any two sequences $\{ R_j \mid j \in \mathbb{N} \}$ and $\{ S_j \mid j \in \mathbb{N} \}$ of functionals which have the properties (11) and (12) and any $T_1 = T^*_1 \in E'_1$ determine exactly one sequence $T = \{ 1, T_1, T_2, \ldots \} \in \underline{E}'_{+,1}$ of n-point functionals whose field-sector decomposition satisfies

$$T^j_{j,j} = R_j \qquad \text{and} \qquad T^j_{j,j+1} = S_j \qquad j = 1, 2, \ldots \qquad (13)$$

Proof. — *a*) Each $R_j \in E'_{2j,+}$ has a canonical pre-Hilbert space realization $V_j = (\Phi^j_j(E_j), \langle\, ,\, \rangle_j)$ where Φ^j_j denotes the canonical quotient map from E_j onto the factor-space of E_j with respect to the kernel $q_j^{-1}(0)$ of the continuous semi-norm q_j on E_j defined according to

$$q_j(x) = \sqrt{R_j(x^* \otimes x)} \qquad \forall x \in E$$

and where the scalar product $\langle \, , \rangle_j$ is defined according to

$$\langle \Phi_j^j(x), \Phi_j^j(y) \rangle_j = R_j(x^* \otimes y) \qquad \forall x \in E_j \quad \forall y \in E_j . \tag{14}$$

As E_j is nuclear the completion $\mathscr{H}_j$ of the pre-Hilbert space V_j is separable. Assumptions (11 *b*) and (11 *c*) then imply (using a well known theorem of Riesz and Frechet)

$$S_j(x \otimes y) = \langle \Phi_j^j(x^*), \Phi_{j+1}^j(y) \rangle_j \tag{15 a}$$

for all $x \in E_j$ and all $y \in E_{j+1}$ with some linear continuous function

$$\Phi_{j+1}^j : E_{j+1} \to \mathscr{H}_j \tag{15 b}$$

b) Now we are prepared to define recursively all elements of the field-sector decomposition we are looking for. This will be done by induction together with the definition of the *n*-point-functionals of this field sector decomposition. To this end we prove by induction on N.

If $\{ T_1, S_1, R_1, \ldots, R_N, S_N \}$ are given and if these functionals satisfy the constraints (11) and the chain of hypothesis (12) up to the corresponding order $j = N$ there is a unique way to construct

$$(I_N) \qquad \{ T_1, T_2, \ldots, T_{2N}, T_{2N+1}, T_{2N+2}^0 \} , \qquad T_n \in E_n' ,$$

and

$$\{ T_{n,m}^j \mid 0 \leq j \leq N, j \leq n, m \leq 2N + 1 - j \}$$

and that the conditions *a*), *b*), *c*) of a field sector decomposition are fulfilled up to the corresponding order and such that equation (2) holds. Here

$$T_{2N+2}^0 = \sum_{j=0}^{N} T_{N+1,N+1}^j$$

denotes that part of T_{2N+2} which is fixed by $T_1, \ldots, T_{2N+1}$ according to proposition 3 *b*).

We start with $N = 1$. Define first

$$\begin{aligned} T_{0,0}^0 &= 1 , \qquad T_{0,1}^0 = T_{1,0}^0 = T_1 , \qquad T_{1,1}^0 = T_1 \otimes T_1 \\ T_{1,1}^1 &= R_1 \\ T_2 &= T_{1,1} + T_{1,1}^1 \end{aligned} \tag{16}$$

$R_1 \in E_{2,+}'$ and $T_1 = T_1 \in E_1'$ imply that T_2 belongs to $E_{2,+}'$. If we define

$$T_{1,2}^0 = T_1 \otimes T_2 \qquad T_{2,1}^0 = (T_{1,2}^0)^* = T_2 \otimes T_1 \tag{17 a}$$

$$T_{1,2}^1 = S_1 \qquad T_{2,1}^1 = S_1^* \tag{17 b}$$

$$T_3 = T_{1,2}^0 + T_{1,2}^1 \tag{17 c}$$

it follows $T_3 \in E'_3$ and assumption (12 *b*) for $j = 1$ implies

$$T_3^* = T_{2,1}^0 + (T_{1,2}^1)^* = T_3 . \tag{17 d}$$

Furthermore we define

$$T_{2,2}^0 = T_2 \otimes T_2 \in E'_{4,+}, \qquad T_{3,3}^0 = T_3 \otimes T_3 \in E'_{6,+} \tag{18 a}$$

$$T_{2,2}^1 = T_{2,1}^1 \, (\!\!\!\!\bigcirc\!\!\! j\,) \, T_{1,2}^1 \in E'_{4,+} \tag{18 b}$$

$$T_4^0 = T_{2,2}^0 + T_{2,2}^1 \tag{18 c}$$

where we used assumption (11 *b*) for $j = 1$ in order to be sure that the 1-convolution is well-defined. This way we succeeded in constructing

$$\{ T_1, T_2, T_3, T_4^0 \} \quad \text{and} \quad \{ T_{n,m}^j \,|\, 0 \le j \le 1, j \le n, m \le 3 - j \} .$$

such that equation (2) holds. The only non-trivial point which has to be checked in the list of defining properties of a field sector decomposition is the positivity condition *c*) for $j = 0$ and $j = 1$ and $j \le n$, $m \le 2$. But this follows from the representation of the functionals involved in terms of scalar products according to eq. (14)-(15) for $j = 1$. Thus (I_1) holds.

Suppose now I_N to hold for some $N \ge 1$. Then we may assume that

$$\{ T_1, \ldots, T_{2N}, T_{2N+1}, T_{2N+2}^0 \} \quad \text{and} \quad \{ T_{n,m}^j \,|\, 0 \le j \le N, j \le n, m \le 2N+1-j \}$$

are constructed according I_N. In order to prove I_{N+1} the functionals T_{2N+2}, T_{2N+3}, T_{2N+4}^0 and

$$T_{n,2n+2-j}^j, \qquad T_{n,2N+3-j}^j, \qquad j \le n \le 2N + 3 - j; \quad j = 1, \ldots, N, N + 1$$

have to be constructed. This is done in the following way. We construct T_{2N+2}, T_{2N+3} and

$$T_{j,2N+2-j}^j, \qquad T_{j,2N+3-j}^j, \qquad j = 2, \ldots, N, N + 1$$

in such a way that these functionals admit a *j*-convolution and then define

$$T_{n,2N+i-j}^j = T_{n,j}^j \, (\!\!\!\!\bigcirc\!\!\! j\,) \, T_{j,2N+i-j}^j , \qquad i = 2, 3 .$$

To begin with we use T_{2N+2}^0 to define

$$T_{N+1,N+1}^{N+1} := R_{N+1} \tag{19 a}$$

$$T_{2N+2} = T_{2N+2}^0 + R_{N+1} \tag{19 b}$$

to get functionals $T_{N+1,N+1}^{N+1}$ and T_{2N+2} in $E'_{2N+2,+}$.

The consistency relations *b*) of definition 1 are used to define the missing functionals, e. g. we define for $j = 1, \ldots, N$

$$T_{j,2N+2-j}^j := T_{2N+2} - \sum_{i=0}^{j-1} T_{j,2N+2-j}^i \tag{20 a}$$

$$T_{2N+2-j,j}^j := (T_{j,2N+2-j}^j)^* = T_{2N+2} - \sum_{i=0}^{j=1} T_{2N+2-j,j}^i \tag{20 b}$$

Checking the range of the indices involved shows that all functionals on the righthand side of equation (20) are given by I_N. Assumption (12 *a*) in this order implies that there is exactly one continuous linear function

$$\Phi^j_{2N+2-j} : E_{2N+2-j} \to \mathcal{H}_j$$

such that

$$T^j_{j,2N+2-j}(x^* \otimes y) = \langle \Phi^j_j(x), \Phi_{2N+2-j}(y) \rangle_j \qquad \forall x \in E_j \quad \forall \in E_{2N+2-j} \quad (21)$$

holds and similarly for $T^j_{2N+2-j,j}$. Then these functionals admit the j-convolution for $j = N$. Thus

$$T^j_{n,2N+2-j} = T^j_{n,j} \circledS T^j_{j,2N+2-j}, \qquad j=1, \ldots, N; \quad j \le n \le 2N+2-j \quad (22\,a)$$

and

$$T^0_{n,m} = T_n \otimes T_m, \qquad 0 \le n, m \le 2N + 2 \quad (22\,b)$$

are well defined in $E'_{n+2N+2-j}$ respectively in E'_{n+m}. In defining (22 *a*) the functionals $T^j_{n,j}$, $j \le n \le 2N + 1 - j$ are given according to I_N. The others are constructed in (20) respectively in (21).

Now by I_N the functionals $T^j_{N+1,j}$, $j = 0, 1, \ldots, N$ and T^j_{j+N+2}, $j = 0, 1, \ldots, N - 1$ are given and admit the j-convolution. Thus the functionals

$$T^j_{N+1,N+2} = T^j_{N+1,j} \circledS T^j_{j,N+2}, \qquad j = 0, 1, \ldots, N - 1 \quad (23\,a)$$

result. By equation (20 *a*) for $j = N$ $T^N_{N,N+2}$ is constructed and admits a N-convolution according to equation (21). This allows to construct

$$T^N_{N+1,N+2} = T^N_{N+2,N} \circledN T^N_{N,N+2} \quad (23\,b)$$

and therefore to define

$$T^0_{2N+3} = \sum_{j=0}^{N} T^j_{N+1,N+2} \quad (23\,c)$$

in E'_{2N+3}. Now it is possible to construct the next n-point functional $T_{2N+3} \in E'_{2N+3}$:

$$T^{N+1}_{N+1,N+2} = S_{N+1}, \qquad T^{N+1}_{N+2,N+1} = S^*_{N+1} \quad (24\,a)$$

$$T_{2N+3} = T^0_{2N+3} + S_{N+1} \quad (24\,b)$$

Assumption (12 *b*) for $j = N + 1$ implies $T^*_{2N+3} = T_{2N+3}$.

Again the missing functionals of the field sector decomposition in this step are defined according to the corresponding consistency relation *b*), e. g.

$$T^j_{j,2N+3-j} = T_{2N+3} - \sum_{i=0}^{j-1} T^i_{j,2N+3-j} \quad (25\,a)$$

$$T^j_{2N+J-j,j} = T_{2N+3} - \sum_{i=0}^{j-1} T^i_{2N+3-j,j} \quad (25\,b)$$

with

$$T^i_{2N+3-j,j} = (T^i_{j,2N+3-j})^* = T^i_{2N+3-j,i} ⓘ T^i_{i,j} \quad \text{for} \quad j=1, \ldots, N+1 . \quad (25\,c)$$

The functionals which are used to define $T^i_{2N+3-j,j}$, $0 \leq i \leq j-1$ in (25 *c*) are given according to I_N respectively according to the construction (20) and these functionals admit the corresponding *i*-convolutions. Again by assumption (12 *a*) in this order there exist continuous functions

$$\Phi^j_{2N+3-j} : E_{2N+3-j} \rightarrow \mathscr{H}_j$$

such that

$$T^j_{j,2N+2-j}(x^* \otimes y) = \langle \Phi^j_j(x), \Phi^j_{2N+3-j}(y) \rangle_j \qquad \forall x \in E_j \quad \forall y \in E_{2N+3-j} \quad (26)$$

holds. Therefore these functionals again admit the *j*-convolution for $j = 1, \ldots, N+1$ and thus the functionals $T^j_{n,2N+3-j}$ can be constructed

$$T^j_{N,2+3-j} = T^j_{n,j} ⓙ T^j_{j,2N+3-j} \qquad j \leq n \leq 2N+3-j, \quad j=1, \ldots, N+1 . \quad (27)$$

Collecting the results of the various steps we conclude that all functionals

$$\{ T^j_{n,m} \mid 0 \leq j \leq N+1 ; j \leq n, m \leq 2N+3-j \} \qquad (28\,a)$$

have been constructed such that conditions *a*) and *b*) of definition 1 hold for these functionals up to the corresponding order. Again the positivity condition *c*) for the functionals in (28 *a*) results from the representation of these functionals in terms of scalarproducts.

In particular the functionals $T^j_{N+2,N+2}$, $j = 0, 1, \ldots, N+1$ have been constructed so that

$$T^0_{2N+4} = \sum_{j=0}^{N+1} T^j_{N+2,N+2} \qquad (28\,b)$$

is also well-defined and belongs to $E'_{2(N+2),+}$. Thus I_{N+1} is shown and the proof is complete.

As the converse of proposition 4 is trivial one gets

THEOREM 5. — Any sequence of *n*-point-functionals

$$T = \{ 1, T_1, T_2, \ldots \} \in \underline{E}'_{+,1}$$

is uniquely characterized in terms of $T_1 = T_1^* \in E'_1$ and a sequence $\{ (R_j, S_j) \}_{j \in N}$ of « essentially independent » functions (R_j, S_j) which satisfy the constraints (11) and (12).

Remarks. — *a*) One might think that the complicated constraints (12) imply that the functionals $\{ R_j \}$ and $\{ S_j \}$ depend on each other in a very strong way. We want to argue that this is not the case.

b) A simple way to see this is to consider the class of Jacobi-fields. Within this class of fields one can show

i) to prescribe $R_j = T^j_{j,j}$ means to prescribe $A_{j,j-1}$ and $A_{j-1,j}$
ii) to prescribe $S_j = T^j_{j,j+1}$ means to prescribe $A_{j,j}$

if $A_{ik} \in L^h_{sc}(E, L(D^k, D^i))$ are given for $i, k \leq j - 1$ and $| i - k | \leq 1$.

As the various A_{ij} of a Jacobi-field are independent (up to natural domain restrictions) we conclude a corresponding independence for the functionals $\{ R_j = T^j_{j,j} \}$ and $\{ S_j = T^j_{j,j+1} \}$ and therefore call them « essentially independent » functionals.

c) As the constraints (11) and (12) on the sequence of functionals (R_j, S_j) are rather complicated it is not obvious that in concrete cases a construction of a sequence $T \in \underline{E}'_{+,1}$ of n-point-functionals is practicable at all along the lines of proposition 4. Such a construction will be presented together with some other applications in a forth coming paper.

d) Clearly according to the proof of proposition 4 the chain of indirect assumptions (12) can be translated into a chain of direct assumptions for the sequence $\{ R_j, S_j \}_{j \in \mathbb{N}}$ itself, but these then look horrable.

The following theorem clarifies another aspect of the importance of the functionals $\{ T^j_{j,j} \}$ for a sequence of n-point functionals. It generalizes in particular the well-known fact that $T = \{ 1, T_1, T_2, \ldots \} \in \underline{E}'_{+,1}$ and $T_2 - T_1 \otimes T_1 = T^1_{1,1} = 0$ imply $T_n = T_1^{\otimes n}$ for all n.

THEOREM 6. — Suppose that for $T = \{ 1, T_1, T_2, \ldots \} \in \underline{E}'_{+,1}$ there is a minimal $n \in \mathbb{N}$ such that $T^n_{n,n} = 0$ holds. Then all T_m, $m \geq 2n$, are fixed in terms of

$$T_{(2n-1)} = \{ 1, T_1, T_2, \ldots, T_{2n-1} \} .$$

In particular one knows how to calculate all the functionals T_m, $m \geq 2n$, in terms of $T_{(2n-1)}$.

Proof. — *a*) $T^n_{n,n} = 0$ implies by definition (3) $\phi^n_n = 0$ and thus

$$\mathscr{H}_{(n)} = \mathscr{H}_{(n-1)} \tag{29}$$

We want to show that

$$\mathscr{H}_{(n+1)} = \mathscr{H}_{(n-1)} \qquad \forall l \in \mathbb{N} \tag{30}$$

holds. Now according to its definition T is the sequence of n-point functionals of a field A with dense domain D. We establish (30) by proving

$$\Phi_{n+1}(E^{\otimes l} \otimes E_n) \subset D \cap \mathscr{H}_{(n-1)} \qquad \forall l \in \mathbb{N} \tag{31}$$

By definition $\Phi_n(E_n) \subset D \cap \mathscr{H}_{(n)}$ and thus by (29) $\phi_n(E_n) \subset D \cap \mathscr{H}_{(n-1)}$ and therefore for all x in E and all y in E_n

$$\Phi_{n+1}(x \otimes y) = A(x)\Phi_n(y) \in D \cap \mathscr{H}_{(n)} = D \cap \mathscr{H}_{(n-1)} .$$

This implies relation (31) for $l = 1$. Suppose now that for some $l_0 \geq 2$ and $1 \leq l \leq l_0 - 1$

$$\Phi_{n+1}(E^{\otimes l} \otimes E_n) \subset D \cap \mathscr{H}_{(n-1)}$$

holds. As above we get for all x in E and all y in $E^{\otimes(l_0-1)} \otimes E_n$

$$\Phi_{n+l_0}(x \otimes y) = A(x)\Phi_{n+l_0-1}(y) \in A(x)D \cap \mathscr{H}_{(n-1)} \subset D \cap \mathscr{H}_{(n)} = D \cap \mathscr{H}_{(n-1)}$$

and therefore relation (31) follows by induction on l. A density argument then yields the claim (30).

b) Equation (30) implies

$$\Phi^j_{n+l} = 0 \qquad \text{for all} \qquad j \leq n \quad \text{and all} \quad l = 0, 1, 2, \ldots \tag{32}$$

Application of equation (2 *a*) then yields

$$T_{2(n+l)} = \sum_{j=0}^{n+1} T^j_{n+l,n+l} = \sum_{j=0}^{n-1} T^j_{n+l,n+l} \qquad \text{for} \qquad l = 0, 1, 2, \ldots .$$

Proposition 3 thus says that $T_{2(n+l)}$ is fixed in terms of

$$\{ T_1, \ldots, T_{2n-2}, T_{n+l}, \ldots, T_{2n+l-1} \} \qquad \text{for} \qquad l = 0, 1, 2, \ldots .$$

In particular T_{2n} is fixed in terms of $\{ T_1, \ldots, T_{2n-1} \}$. For every $l \geq 1$ we have $2n - 1 + l \leq 2(n + l - 1)$, that is $T_{2(n+l)}$ is fixed in terms of $\{ T_1, \ldots, T_{2(n+l-1)} \}$ for $l = 1, 2, \ldots$. This relation then allows to prove by induction on l that all functionals $T_{2(n+l)}$, $l = 1, 2, \ldots$, are fixed in terms of $\{ T_1, \ldots, T_{2n-1} \}$.

Similarly we proceed for $T_{2(n+l)+1}$, $l = 0, 1, 2, \ldots$. Equations (2) and (32) imply

$$T_{2(n+l)+1} = \sum_{j=0}^{n-1} T^j_{n+l,n+l+1} \qquad l = 0, 1, 2, \ldots .$$

Proposition 3 thus shows that $T_{2(n+l)+1}$ is fixed in terms of

$$\{ T_1, \ldots, T_{2n+l} \}.$$

Again we have $2n + l \leq 2(n + l - 1)$ for $l \geq 1$. Therefore an induction proof on l works if we can show the statement for $l = 0$; now as T_{2n+1} is fixed in terms of $\{ T_1, \ldots, T_{2n} \}$ and as T_{2n} is shown to be fixed in terms of $\{ T_1, \ldots, T_{2n-1} \}$ the result follows by induction.

c) The calculation of the functionals T_m, $m \geq 2n$, uses the various ⓙ-convolutions and proceeds successively in a complicated way. An example is discussed below.

Remark. — To say $T^n_{n,n} = 0$ is equivalent to say $\Delta_n = 0$, where Δ_n is the function on E_n defined according to [*6*]. Therefore this theorem is another version of theorem 2.3 of [*6*].

Examples. — We want to give two examples how to calculate the functionals T_m, $m \geq 2n$, in the case $T^n_{n,n} = 0$. We treat the simplest cases $T^1_{1,1} = 0$ and $T^2_{2,2} = 0$. Suppose that $T = \{ 1, T_1, T_2, \dots \} \in \underline{E}'_{+,1}$ is a given sequence of n-point functionals.

a) $T^1_{1,1} = 0$. Equation (2) implies $T_2 = T_1 \otimes T_1 + T^1_{1,1} = T_1 \otimes T_1$ and $T_{n+1} = T^0_{n,1} + T^1_{n,1} = T_n \otimes T_1 + T^1_{n,1} \, ① \, T^1_{1,1} = T_n \otimes T_1$. Therefore it is trivial to conclude

$$T_n = T_1^{\otimes n} \qquad \text{for all} \qquad n = 1, 2, \dots .$$

b) $T^2_{2,2} = 0$. In this case only the ①-convolution occurs and allows (in principle) to compute all the functionals T_m, $m \geq 4$. To this end we have to use equations (2) and (9) again and again. It is clear that in this case the basic functionals are

$$T_1, T^1_{1,1} \qquad \text{and} \qquad T^1_{2,1} = (T^1_{1,2})^* .$$

They are fixed in terms of $\{ T_1, T_2, T_3 \}$ according to equations (2) and (9):

$$T_2 = T_1 \otimes T_1 + T^1_{1,1} \qquad \text{and} \qquad T_3 = T_1^{\otimes 3} + T_1 \otimes T^1_{1,1} + T^1_{1,2}$$

As in the proof of proposition 4 all functionals T_m, $m \geq 4$, can be expressed in terms of the basic functionals; but the degree of complexity grows considerably with m. To give an impression we write down the formula for $m = 4$ explicitly:

$$T_4 = T_1^{\otimes 4} + T_1^{\otimes 2} \otimes T^1_{1,1} + T^1_{1,1} \otimes T_1^{\otimes 2} + T^1_{1,1} \otimes T^1_{1,1} + T^1_{2,1} \, ① \, T^1_{1,2}$$

We indicate how to go on. Defining $T^1_{1,3} = T_4 - T_1 \otimes T_3$ we get the next two functionals explicitly:

$$T_5 = T_2 \otimes T_3 + T^1_{2,1} \, ① \, T^1_{1,3} \qquad \text{and} \qquad T_6 = T_3 \otimes T_3 + T^1_{3,1} \, ① \, T^1_{1,3} .$$

Theorems 5 and 6 prepare and motivate the following definition.

DEFINITION 7. — For a given sequence of n-point functionals

$$T = \{ 1, T_1, T_2, \dots \} \in \underline{E}'_{+,1}$$

with the field-sector decomposition

$$\{ T^j_{n,m} \mid 0 \leq j \leq n, m \}$$

the functional

$$T^{[n/2]}_{[n/2],n-[n/2]} \in E'_n$$

is called the *n-field-irreducible part* T_n^{irr} of the n-point functional T_n, $n = 2, 3, \dots$.

Now we can state the following splitting of a n-point functional T_n. Each n-point functional T_n allows a decomposition into two parts

$$T_n = T_n^0 + T_n^{irr} \tag{33 a}$$

T_n^0 is that part of T_n which is already fixed in terms of $\{ T_1, \ldots, T_{n-1} \}$ while the n-field-irreducible part T_n^{irr} is not. In terms of the n-field-sector decomposition of T the following representations hold:

$$T_n^0 = \sum_{j=0}^{[n/2]-1} T^j_{[n/2],n-[n/2]} \qquad T_n^{irr} = T^{[n/2]}_{[n/2],n-[n/2]} \tag{33 b}$$

III. SOME APPLICATIONS TO QFT

We want to finish with a short outlook of a possible application to relativistic quantum field theory. To be definite we suppose that A is a relativistic quantum field over $E = \mathscr{S}(\mathbb{R}^4)$ and that $T_n = T_n^A$ are its n-point functionals. The Poincaré-covariance of the theory is expressed in terms of a strongly continuous unitary representation U of the Poincaré-group $P_+^\uparrow$ in the state-space $\mathscr{H} = \mathscr{H}_A$ of the field such that

$$U(\alpha)\Phi_0 = \Phi_0 \qquad U(\alpha)A(f)U(\alpha)^{-1} = A(f_\alpha) \tag{34}$$

holds for all $\alpha \in P_+^\uparrow$ and all $f \in E$. Here Φ_0 denotes the cyclic vacuum-state of the theory.

The relations (34) easily imply that all the n-field-sectors $\mathscr{H}_n$ of A are invariant with respect to the action U of $P_+^\uparrow$ in $\mathscr{H}$:

$$U(\alpha)\mathscr{H}_n = \mathscr{H}_n \qquad \forall\alpha \in P_+^\uparrow , \qquad n = 0, 1, 2, \ldots \tag{35}$$

This then implies that each functional of the n-field-sector decomposition of $T = \{ 1, T_1, T_2, \ldots \}$ is $P_+^\uparrow$-invariant and that a strongly continuous unitary representation U_n of $P_+^\uparrow$ on the n-field-sector $\mathscr{H}_n$ is well defined according to

$$U_n(\alpha) = U(\alpha) \upharpoonright \mathscr{H}_n , \qquad \forall\alpha \in P_+^\uparrow , \qquad n = 1, 2, \ldots \tag{36}$$

Let us denote by Σ_n the spectrum of the generator of the space-time-translations in the representation U_n. As usual we want to characterize these spectral properties of the energy-momentum operator on the various n-field-sectors in terms of support-properties of the Fourier transforms of the elements $T^j_{n,m}$ of the field-sector decomposition. To this end let us denote by

$$\tilde{T}^j_{n,m}(p_1, \ldots, p_n ; q_1, \ldots, q_m) \in E'_{n+m}$$

the Fourier transform of $T^j_{n,m}$. A simple modification of the usual analysis of the support-properties of the n-point functionals in momentum-space yields:

$$\operatorname{supp} \tilde{T}^j_{n,m} = \Big\{ (p_1, \ldots, p_n ; q_1, \ldots, q_m) \mid \sum_{i=1}^{n} p_i + \sum_{i=1}^{m} q_i = 0 ,$$

$$- \sum_{i=1}^{n} p_i \in \Sigma_j , \quad \sum_{i=1}^{m} q_i \in \Sigma_j , \quad - \sum_{i=1}^{r} p_i \in \bigcup_{i=0}^{r} \Sigma_i , \quad r = 1, \ldots, n-1 ,$$

$$\sum_{i=m-1}^{m} q_i \in \bigcup_{i=0}^{l+1} \Sigma_i , \quad l = 0, 1, \ldots, m-2 \Big\} \tag{37}$$

Thus, evidently, the Fourier transforms of the elements of the field-sector decomposition of the n-point functionals contain a much more detailed information about the energy-momentum spectrum of the associated theory.

This is of particular interest for those cases where one has assured a particle-field association such that the n-field-sectors correspond the n-particle-sectors. In this context the problem arises to clarify the connection of the n-field-irreducible part T_n^{irr} of a n-point functional T_n and the n-particle-irreducible Greens function of the associated Greens function as investigated by Bros and Lassalle [*4*].

The last linear constraint which characterizes a relativistic quantum field theory is the locality condition. It is the only constraint which couples the various elements $T^j_{n,m}$ of the field-sector decomposition of T. Because of

$$T_{n+m}(x_1, \ldots, x_n ; y_1, \ldots, y_m) = \sum_{j=0}^{n \wedge m} T^j_{n,m}(x_1, \ldots, x_n ; y_1, \ldots, y_m)$$

the locality condition on the n-point functionals is easily translated into constraints for the functionals $\{ T^j_{n,m} \}$. The result is:

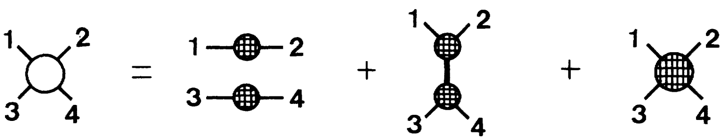
a) $T^j_{n,m}(x_1, \ldots, x_n ; y_1, \ldots, y_m)$ is local in the usual sense with respect to the variables $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ separately.

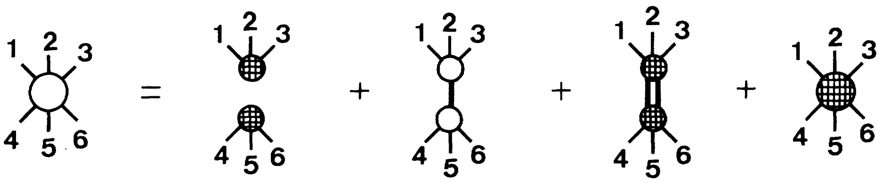
b) $\displaystyle\sum_{j=0}^{n \wedge m} \{ T^j_{n,m}(x_1, \ldots, x_n ; y_1, \ldots, y_m) - T^j_{n,m}(x_1, \ldots, y_1 ; x_n, \ldots, y_m) \} = 0$

whenever $(x_n - y_1)^2 < 0$.

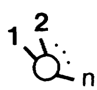
In particular the relations *b*) express a strong dependence of the n-field-irreducible parts T_n^{irr} of the various n-point functionals on each other and thus a considerable reduction in the possible choices of these functionals has to be expected. A detailed analysis of these points would be helpful to understand the structure of a non-trivial relativistic quantum field theory.

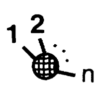
As a last point we comment on a special interpretation of the field-sector decomposition of the sequence of VEV's of a relativistic quantum field. The interpretation we have in mind is to represent the various functionals in terms of certain graphs. In order to be short we only give examples:





Here it is assumed that the 1-point functional vanishes.

 denotes the VEV of order n.

 denotes the n-field irreducible part of the VEV of order n

|| . . . | (j-times) represents the j-convolution.

This decomposition and the convolutions involved have the following advantages:

i) It is well-defined in a natural way (with respect to positivity) without any additional assumptions.

ii) It respects the covariance properties.

iii) It isolates more details on the energy-momentum spectrum.

iv) It respects positivity.

But as mentioned above the elements of this decomposition are not local with respect to all variables. This is a disadvantage compared to the decompositions and convolutions as proposed by Epstein-Glaser-Stora [*7*] and Bros-Lasalle [*4*]. On the other side these suggestions do not satisfy *i*) and *iv*).

The fact that the decomposition suggested above does not respect locality but is irreducible with respect to positivity offers the chance of considering the following type of important questions: Which 2-point-functionals admit local 3-point-functionals ? and then Which 2- and 3-point-functionals admit local 4-point-functionals ? and so on. Which part of the

4-field-irreducible part of the 4-point-functionals is fixed by the 2-point-functional and which part by the 3-point-functional through locality? Clearly in order to be able to attack this kind of problems successfully more details of the kernels K_j which define the j-convolutions have to be known.

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