ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 34, nº 2 (1981), p. 173-229
<http://www.numdam.org/item?id=AIHPA_1981__34_2_173_0>

Ann. Inst. Henri Poincaré,
Vol. XXXIV, n° 2, 1981, p. 173-229.

Section A :
Physique théorique.

On the phase diagram of a $P(\phi)_2$ quantum field model

by

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ABSTRACT. — The phase diagram of a two-dimensional Bose quantum field model (with polynomial self-interaction of degree six) is rigorously verified, except in a neighbourhood of the expected critical points, by the construction of distinct states satisfying the Osterwalder-Schrader axioms coexisting along the expected phase transition lines of the diagram. Perturbation theory in the respective states is proven to be asymptotic (without the use of a convergent cluster expansion), yielding asymptotic expansions to arbitrary order for the generalized Schwinger functions throughout the diagram. A strong estimate on the positions (in parameter space) of the double and triple points is given.

RÉSUMÉ. — Le diagramme de phase d'un modèle de Bose dans un champ quantique à deux dimensions (avec une self-interaction de polynôme du sixième degré) est rigoureusement vérifié, sauf dans le voisinage des points critiques attendus, en construisant des états distincts répondant aux axiomes d'Osterwalder-Schrader qui coexistent le long des lignes de transition de phase. On démontre que la théorie perturbative dans les états respectifs est asymptotique (sans utiliser « d'expansion cluster » convergente) et conduit les expansions asymptotiques à un ordre arbitraire dans le cas des fonctions généralisées de Schwinger dans tout le diagramme. On donne une bonne estimation des positions (dans l'espace des paramètres) pour les points double et triple.

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1. INTRODUCTION

This work examines phase transitions in a particular quantum field model, i. e., a quantum field theory with a given polynomial self-interaction $P(\phi)$ in two space-time dimensions. We study a specific polynomial of degree six, but the methods of analysis and proof are not limited to this model.

The existence of phase transitions in a quantum field theory was first proven by Glimm, Jaffe and Spencer [GJS2, 3]. They showed that $P_0(\phi) - \phi^2 \equiv P(\phi)$ models (in two space-time dimensions) have at least two phases, when the interaction density is given by

$$P_0(\phi) = \lambda \sum_{j=2}^{N} c_{2j}\phi^{2j} \quad , \quad c_{2N} > 0,$$

and λ is sufficiently small (the proof was given explicitly for ϕ^4). This phase transition is associated with the breaking of the discrete symmetry of the action, $\phi \leftrightarrow -\phi$. Later, phase transitions were proven to exist in ϕ_3^4 models, including continuous symmetry breaking [FSS], and the methods of [GJS3] were extended to apply to phase transitions without symmetry breaking [Fr2]. In addition, while this work was still in its beginnings, the existence of three phases in the model we shall consider was proven [Ga], and in [CR] the existence of three phases in a two-component ϕ^4 model was established. All of these results and the corresponding methods of proof were motivated by conjectures based on the classical limit, mean field theory.

The model studied in this paper is determined by the following interaction polynomial (see fig. 1):

(1.1)
$$\begin{aligned} P(\phi) &= (\lambda\phi^2 - 1)^2\phi^2 + \sigma\phi^2 - h\phi \\ &= \lambda^2\phi^6 - 2\lambda\phi^4 + (1 + \sigma)\phi^2 - h\phi. \end{aligned}$$

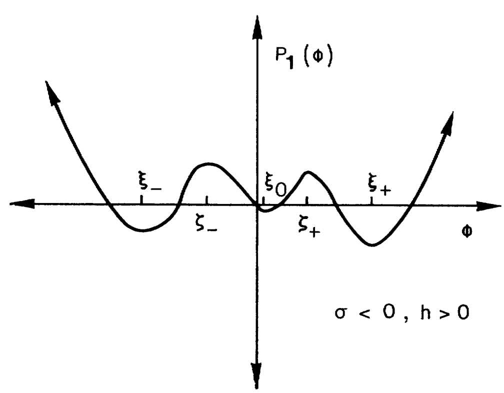

FIG. 1. — The graph of the interaction polynomial.

This polynomial has three minima $\xi_{+,0,-}$, given by

$$(1.2\,a)\quad \xi_0 = \frac{h}{2(1+\sigma)} + 0(h^3\lambda),$$

$$(1.2\,b)\quad \xi_\pm = \pm\,\lambda^{-1/2}\left(2/3 + \frac{1}{3}\sqrt{1-3\sigma}\right)^{1/2} + \frac{h}{2c_2} + 0(h^2\lambda^{1/2}),$$

where $c_2 = 4/3 - 4\sigma + 8(1+3\sigma)^{1/2}/3$, and two maxima, given by

$$(1.2\,c)\qquad \zeta_\pm = \pm\,\lambda^{-1/2}\left(\frac{2}{3} - \frac{1}{3}\sqrt{1-3\sigma}\right)^{1/2} + 0(h).$$

At the maxima the polynomial is $0(\lambda^{-1})$. Thus, as $\lambda \downarrow 0$, the minima are widely separated by a large potential barrier. Moreover, this polynomial has a mean field limit (see [GJS2]). That is to say, expressing the polynomial in terms of variables $\psi_{+,0,-} = \phi - \xi_{+,0,-}$ centered at the local minima,

$$(1.3)\qquad P(\phi) = \sum_{i=3}^{6} c_i^{+,0,-}\psi^i_{+,0,-} + \frac{m^2_{+,0,-}}{2}\psi^2_{+,0,-} + E_{+,0,-},$$

one sees that the interaction coefficients satisfy the following relation:

$$|\,c_i^{+,0,-}\,| \ll m^2_{+,0,-}\quad,\quad 3 \le i \le 6,$$

as $\lambda \downarrow 0$. Thus, quantum corrections to the free field theory with (classical) mass $m_{+,0,-}$ and (classical) mean $\xi_{+,0,-}$ are small for field values near $\xi_{+,0,-}$. And field values away from small neighborhoods of the minima should be suppressed by the large potential peaks.

Assuming that, due to the smallness of the quantum corrections, the classical picture is approximately correct, this model has the phase diagram indicated in figure 2. One observes that for $h \neq 0$, one expects the existence of double points $\sigma_D(\lambda, h)$ at which the $+$ and 0 ($-$ and 0) states coexist,

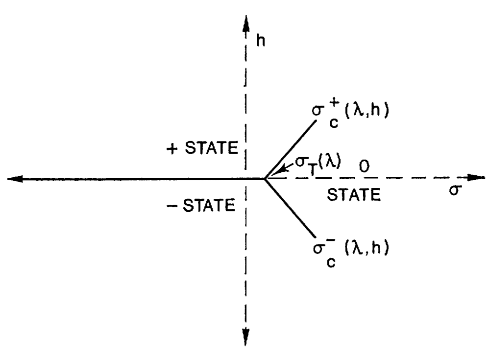

FIG. 2. — The phase diagram in parameter space for fixed, sufficiently small coupling constant.

and that for $h = 0$ and $\sigma \leq \sigma_T(\lambda) = \sigma_D(\lambda, h = 0)$ the $+$ and $-$ states coexist (Heuristically, the $+$, 0, $-$ state is the state given by the small perturbation

$$P(\psi_{+,0,-}) \equiv \sum_{i=3}^{6} c_i^{+,0,-}(\phi - \xi_{+,0,-})^i \tag{1.4}$$

to the free state of mass $m_{+,0,-}$ and mean $\xi_{+,0,-}$). At $\sigma = \sigma_T(\lambda)$, $h = 0$, the $+$, 0 and $-$ states all coexist, and at $\sigma = \sigma_c^{+,-}(\lambda, h)$, one expects a critical point where the classical mass vanishes and $\xi_{+,-} = \xi_0$. Only the phase transition at $h = 0$, $\sigma \leq \sigma_T(\lambda)$ (for the $+$ and $-$ states) is associated with symmetry breaking.

The existence of the triple point $\sigma_T(\lambda)$ for small enough λ has been established in [Ga] by a construction somewhat different than that used here. However, the problems of determining the properties of the states at the triple point and constating the rest of the phase diagram were not addressed. In this paper we verify the phase diagram as indicated, for all σ and $|h| \leq \lambda^{-1/2+\varepsilon}$, $\varepsilon > 0$ and arbitrarily small (see fig. 3), by constructing distinct states satisfying the Osterwalder-Schrader axioms, including clustering, that coexist along the expected phase transition lines. Perturbation theory in the respective states is proven to be asymptotic, yielding expansions asymptotic to arbitrary order in $\lambda^{1/2}$ for the generalized Schwinger functions.

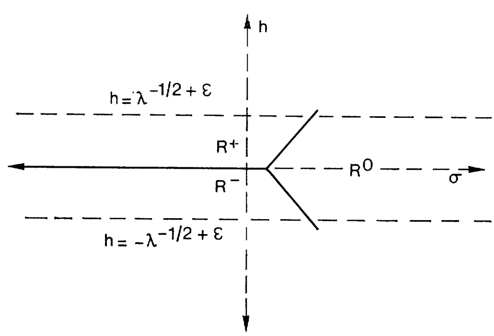

FIG. 3. — The regions $R^{+,0,-}$ in parameter space.

An expansion asymptotic to third order in $\lambda^{1/2}$ for the positions (in the parameter space) of the double points and triple point is given. Furthermore, the phase diagram, that is to say the position of the phase transition lines, is shown to be independent of the boundary conditions originally placed on the system. The critical points are not within the region of parameter space that is studied here.

The paper is constructed as follows. Chapter II gives certain basic definitions and states the major results. Chapter III presents an outline of the proof of the results, providing motivation for the technical proofs to come. Essential vacuum energy estimates are proven in chapter IV and chapter V uses them to prove the existence of the phase transitions indicated in figure 3, In chapter VI the estimates on the positions of the phase transition lines in parameter space are demonstrated. An argument appearing in [Su2] is utilized in chapter VII to prove the asymptotic nature of perturbation theory.

We shall work with the Euclidean formulation of quantum field theory (see [N, Si2] for details), but all results will have a natural translation into results on physical (Minkowski space) objects through the Osterwalder-Schrader reconstruction theorem [OS].

2. THE MAIN RESULTS

We study the quantum field model with interaction polynomial

$$\text{(2.1 a)} \qquad P_1(\phi) = \lambda^2\phi^6 - 2\lambda\phi^4 + (1 + \sigma)\phi^2 - h\phi,$$

and we define, as well,

$$\text{(2.1 b)} \qquad P_2(\phi) = \lambda^2\phi^6 - 2\lambda\phi^4 + (1 + \sigma)\phi^2 - h\phi - E_c,$$

where E_c is chosen such that $\inf_{\phi} P_2(\phi) = 0$. We note that the classical masses defined in (1.3) have the following values:

$$\text{(2.2 a)} \qquad m^2_{+,-} = 8 - 16\sigma + 0(\sigma^2) + 0(\lambda^{1/2}h)$$

and

$$\text{(2.2 b)} \qquad m_0^2 = 2 + 2\sigma + 0(\lambda^{1/2}h).$$

We define the following finite volume interacting measures:

$$\text{(2.3)} \quad d\phi^{+,0,-}_{i,\Lambda} = e^{-:P_i(\phi)_\Lambda: + \frac{m^2_{+,0,-}}{2} :(\phi - \xi_{+,0,-})^2_\Lambda: + c_W^{+,0,-}|\Lambda|} d\mu(\phi - \xi_{+,0,-})_{m^2_{+,0,-}}$$

$i = 1, 2$, where

$$\text{(2.4)} \qquad c_W^{+,0,-} = \lim_{\Lambda \uparrow R^2} \frac{1}{|\Lambda|} \ln \int e^{\left(1 - \frac{m^2_{+,0,-}}{2}\right):\phi^2_\Lambda:} d\mu(\phi)_2$$
$$= \frac{m^2_{+,0,-}}{8\pi} \ln \frac{m^2_{+,0,-}}{2} - \left(\frac{m^2_{+,0,-} - 2}{8\pi}\right).$$

The subscript Λ signifies that the quantity is integrated over the (bounded) space-time region $\Lambda \subset R^2$; : : denotes Wick ordering with respect to a Gaussian measure with covariance $(-\Delta + 2)^{-1}$ and mean zero (Δ is the Laplacian in two dimensions); $d\mu(\phi - \xi_{+,0,-})_a$ signifies a Gaussian measure with covariance $(-\Delta + a)^{-1}$ and mean $\xi_{+,0,-}$. The choice $i = 1, 2$ is made for the sake of convenience in the later estimates; of course, the normalized expectations defined below will not depend on this choice nor on the constants $c_W^{+,0,-}$, which are chosen so that the vacuum energy densities $\alpha_\infty^{+,0,-}$ defined below are equal.

In the measures (2.3), the second term of the exponent cancels the mass and the mean of the Gaussian measure in the region Λ and leaves an external field of strength $\xi_{+,0,-}$ in $R^2\backslash\Lambda$. Formally, the infinite volume limit of (2.3) is

$$e^{-:P_i(\phi)_{R^2}:}d\mu(\phi)_0^{+,0,-},$$

where the +, 0, — signifies the boundary conditions at infinity. The primary motive for the definition of the +, 0, — measure is to assure that, when the polynomial is expressed in terms of $\psi_{+,0,-}$, the interaction exponent will be a small perturbation to the free field theory with classical mass $m_{+,0,-}$ and mean (in terms of ϕ) $\xi_{+,0,-}$, as indicated in the Introduction.

We define the vacuum energy densities corresponding to the measures (2.3) as

$$\text{(2.5)} \qquad \alpha_{\Lambda,i}^{+,0,-} = \frac{1}{|\Lambda|} \ln \int d\phi_{i,\Lambda}^{+,0,-} \quad , \quad i = 1, 2,$$

and

$$\text{(2.6)} \qquad \alpha_{\infty,i}^{+,0,-} = \lim_{\Lambda \uparrow R^2} \alpha_{\Lambda,i}^{+,0,-} \quad , \quad i = 1, 2.$$

It is known that this limit exists [Gu1]. We comment that the classical energy densities $E_{+,0,-}$ defined in (1.3) have the values

$$\text{(2.7 a)} \quad E_{+,-} = \lambda^{-1}\left(\frac{2}{27} + \frac{2\sigma}{3} - \frac{2}{27}\sqrt{1-3\sigma} + \frac{2\sigma\sqrt{1-3\sigma}}{9}\right) - h\xi_{+,-}(0) - \frac{h^2}{4c_2} + 0(h^3\lambda^{1/2}),$$

and

$$\text{(2.7 b)} \qquad E_0 = \frac{-h^2}{4(1+\sigma)} + 0(h^4\lambda)$$

$(\xi_{+,-}(0) \equiv \xi_{+,-}(h = 0))$. Finally, for $F(\phi)$ a function of the field ϕ, we will define the following expectations (if they exist):

$$\langle F(\phi) \rangle^{+,0,-} = \lim_{\Lambda \uparrow R^2} \langle F(\phi) \rangle_\Lambda^{+,0,-} = \lim_{\Lambda \uparrow R^2} \int F(\phi) d\phi_\Lambda^{+,0,-} \Big/ \int d\phi_\Lambda^{+,0,-}.$$

Occasionally, we shall place subscripts on the expectations to emphasize certain interaction parameters.

If we define the following function spaces

$$\begin{aligned} L_{1,p} &= L_1(R^2) \cap L_p(R^2),\ p < \infty, \\ L_{1,\infty,\varepsilon} &= L_1(R^2) \cap \{ f \mid \| f \|_\infty < \varepsilon \}, \end{aligned}$$

we know from [GJ1] that the infinite volume Schwinger functions

$$\left\langle \prod_{i=1}^{n} \phi(x_i) \right\rangle^{+,0,-}$$

exist and are moments of a unique measure $d\phi^{+,0,-}$ on $\mathscr{S}'(R^2)$. Moreover, they satisfy the Osterwalder-Schrader axioms [OS], excluding possibly clustering and Euclidean invariance (however, time-translation invariance holds). In addition, if $1 \leq m_i \leq 6$, the generalized Schwinger functions

$$\int \prod_{i=1}^{n} : \phi^{m_i} : (x_i) d\phi^{+,0,-}$$

are continuous as multilinear forms on $\prod_{i=1}^{n} L_{1,6/6-m_i}$ and are functional derivatives of

$$Z(f_1, \ldots, f_6) = \int e^{\sum_{j=1}^{6} :\phi^j:(f_j)} d\phi^{+,0,-}, \tag{2.8}$$

which is bounded and analytic in $f_j \in L_{1,6/6-j}$. This is the basic existence theorem we shall use to insure that the infinite volume expectations appearing below exist. The indicated restriction on the degree of the Wick monomials in the generalised Schwinger functions (which does not, however, restrict the total degree of the product) will be tacitly assumed in the rest of the paper.

We will state the primary results of this work. Let us define for some $K > 0$, $\varepsilon > 0$ and fixed λ, h,

$$\sigma_D(\lambda, h) = \inf \{ \sigma \mid \lambda \langle : \phi^2 : (\Delta) \rangle_\sigma \leq K\lambda^{1/2} + \lambda \xi_0^2 \}.$$

THEOREM 2.1. — For fixed $K > 0$, $\varepsilon > 0$, there is a $\lambda_0(K, \varepsilon) > 0$ such that for all $0 \leq \lambda \leq \lambda_0$ and $|h| \leq \lambda^{-1/2+\varepsilon}$, $\sigma_D(\lambda, h)$ exists and is independent of the (classical) boundary conditions placed on the expectation $\langle : \phi^2 : (\Delta) \rangle$.

We define the following regions of parameter space:

$$\begin{aligned} R^+ &= \{ \sigma, h \mid 0 \leq h \leq \lambda^{-1/2+\varepsilon} &, \quad \sigma \leq \sigma_D(\lambda, h) \} \\ R^0 &= \{ \sigma, h \mid |h| \leq \lambda^{-1/2+\varepsilon} &, \quad \sigma \geq \sigma_D(\lambda, h) \} \\ R^- &= \{ \sigma, h \mid 0 \geq h \geq -\lambda^{-1/2+\varepsilon} &, \quad \sigma \leq \sigma_D(\lambda, h) \} \end{aligned}$$

(see fig. 3).

THEOREM 2.2. — Given $\varepsilon > 0$, there exists a $\lambda_0 > 0$ such that for all $0 \leq \lambda \leq \lambda_0$,

i) for $h = 0$, $\sigma \leq \sigma_T(\lambda)$, the + and − states,
ii) for $0 \leq h \leq \lambda^{-1/2+\varepsilon}$, $\sigma = \sigma_D(\lambda, h)$, the + and 0 states,
iii) and for $0 \geq h \geq -\lambda^{-1/2+\varepsilon}$, $\sigma = \sigma_D(\lambda, h)$, the − and 0 states

satisfy *all* the Osterwalder-Schrader axioms (in particular, with clustering, except possibly the 0 state at $\sigma = \sigma_T(\lambda)$, $h = 0$).

In other words, the states defined at the phase transition lines are pure states, with the possible exception noted. We comment that the +, 0, − state at the phase transition lines is defined through a limit—see section 5.2—of +, 0, − states as σ, h are suitably manipulated. These limits are formally unnecessary (the boundary conditions produced by the $\xi_{+,0,-}$ external field in the measures (2.3) should suffice to pick out the correct pure state), but we cannot do without them at the present. In [Su1] it was shown, through the convergence of a mean field cluster expansion, that the additional limits are indeed unnecessary (and that there is a nonzero mass gap (exponential clustering)). But the convergence is not known in a small neighbourhood of the triple point $\sigma_T(\lambda)$. In appendix 2 the generating functionals $Z(f_1)$ of these limit states are shown to have the previously stated analyticity properties and the existence and continuity of their generalized Schwinger functions are proven.

The next theorem states that the perturbation series for the generalized Schwinger functions is asymptotic to arbitrary order.

THEOREM 2.3. — For $0 \leq \lambda \leq \lambda_0$, for any n, $\{ m_i \}$ and r positive integers, and $\sigma, h \in \mathbb{R}^{+,0,-}$,

$$\left\langle \prod_{i=1}^{n} :(\phi - \xi_{+,0,-})^{m_i} : (x_i) \right\rangle^{+,0,-} = \sum_{i=1}^{r} \alpha_i^{+,0,-}(\sigma, h)\lambda^{i/2} + 0(\lambda^{r+1/2}).$$

The coefficients $\{ \alpha_i^{+,0,-}(\sigma, h) \}$ are independent of λ and continuous in σ, h. They are, in fact, precisely those given by perturbation theory calculated about the minimum $\xi_{+,0,-}$. $0(\lambda^{r+1/2})$ depends on $N(A) = \Sigma_{i=1}^{r} m_i$ and on r.

Thus, in particular,

THEOREM 2.4. — For all $0 \leq \lambda \leq \lambda_0$, and $\sigma, h \in \mathbb{R}^{+,0,-}$,

i) $$\langle \phi(x) \rangle^{+,0,-} = \xi_{+,0,-} + 0(\lambda),$$

ii) $$\langle \phi(x)\phi(y) \rangle^{+,0,-} - \langle \phi(x) \rangle^{+,0,-} \langle \phi(y) \rangle^{+,0,-} = (-\Delta + m^2_{+,0,-})^{-1}(x, y) + \lambda G_2^{+,0,-}(x, y),$$

iii) $$\langle \phi(x_1) ; \ldots ; \phi(x_n) \rangle_T^{+,0,-} = \lambda^{1/2} G_n^{+,0,-}(x_1, \ldots, x_n), \; n \geqslant 3,$$

iv) $$\left\langle \prod_{i=1}^{n} :(\phi - \xi_{+,0,-})^{m_i}:(x_i) \right\rangle_{\mathrm{T}}^{+,0,-} = \left\langle \prod_{i=1}^{n} :\psi^{m_i}_{+,0,-}:(x_i) \right\rangle_{0,\mathrm{T}}^{+,0,-} + \lambda^{1/2} \mathrm{N}_n^{+,0,-}(x_1, \ldots, x_n).$$

$\langle \,.\, \rangle_{\mathrm{T}}$ denotes the truncated (connected part of the) expectation value. $m_{+,0,-}$ is the bare mass of the $+$, 0, $-$ state (see above). The functions, $\mathrm{G}_n^{+,0,-}$, $\mathrm{N}_n^{+,0,-}$ are bounded and continuous in λ (including $\lambda = 0$), $x_1, \ldots, x_n$, and σ, $h \in \mathrm{R}^{+,0,-}$ (one-sided limits taken at boundaries). $\langle \,.\, \rangle_0^{+,0,-}$ is the expectation in the Gaussian measure $d\mu(\phi - \xi_{+,0,-})_{m^2_{+,0,-}}$.

Remark — A quick glance at (1.2) and (*i*) above shows that the $+$, 0 and $-$ states are indeed distinct, and their coexistence along the phase transition lines indicated confirms the phase diagram shown in figure 3. We comment that because the 0 state at $(\sigma_{\mathrm{T}}(\lambda), 0)$ is not necessarily pure, it could *a priori* be a convex superposition of the $+$ and $-$ states. However, theorem 2.3 entails that at $(\sigma_{\mathrm{T}}(\lambda), 0)$,

$$\langle :\phi^2:(x) \rangle^{+,0,-} = \xi^2_{+,0,-} + 0(\lambda),$$

which, with (*i*) above, excludes that possibility.

Finally, the positions of the phase transition lines can be rather precisely determined:

THEOREM 2.5. — For $0 \leq \lambda \leq \lambda_0$ and $| h | \leq \lambda^{-1/2+\varepsilon}$, the double point is given by

$$\sigma_{\mathrm{D}}(\lambda, h) = \lambda\left(\frac{1}{\pi} \ln 4 - \frac{3}{4\pi}\right) + \lambda \,|\, h \,|\, \xi_+(0) + 0(\lambda h^2) + 0(\lambda^{3/2}).$$

The h-dependence can be calculated more precisely, see chapter VI.

The next chapter sketches the proof of these results.

3. OUTLINE OF PROOF OF RESULTS

Because the proofs are somewhat involved, an outline is given here in order to organize and motivate beforehand the many technical details to follow. For the sake of clarity, we will sketch the approach to attain the indicated results only in the region of parameter space defined by the condition $| h | \leq \lambda^{1/2}$. The arguments to establish the above mentioned claims in the rest of $\mathrm{R}^{+,0,-}$ are not essentially different, but keeping track of the effect of large external fields will serve only to lengthen and obfuscate this outline of the essential points.

In order to establish the validity of the phase diagram, it is shown that one can construct three states $\langle\ .\ \rangle^{+,0,-}$ from the finite volume interacting measures in (2.3) that yield the following theorem.

THEOREM 3.1. — There exists a $\lambda_0 > 0$ such that for all $0 \leq \lambda \leq \lambda_0$, $|h| \leq \lambda^{1/2}$, there is a $\sigma_D(\lambda, h)$ so that if Δ is an arbitrary unit lattice square,

i) for $\sigma \geq \sigma_D(\lambda, h)$, $\lambda \langle :\phi^2:(\Delta)\rangle_\sigma^0 \leq K\lambda^{1/2}$,

ii) for $\sigma \leq \sigma_D(\lambda, h)$, $\lambda \langle :\phi^2:(\Delta)\rangle_\sigma^{+,-} \geq \lambda\xi_+^2 - K\lambda^{1/2}$,

iii) for $\sigma \leq \sigma_D(\lambda, h)$, $h \geq 0$, $\lambda^{1/2}\langle \phi(\Delta)\rangle_\sigma^+ \geq \lambda^{1/2}\xi_+ - K\lambda^{1/2}$,

iv) for $\sigma \leq \sigma_D(\lambda, h)$, $h \leq 0$, $\lambda^{1/2}\langle \phi(\Delta)\rangle_\sigma^- \leq \lambda^{1/2}\xi_- + K\lambda^{1/2}$,

for some $K > 0$, independent of λ, σ, h and Δ.

Note. — In *i)-iv)*, λ and h are viewed as fixed; σ is varied. Thus, at $\sigma = \sigma_D(\lambda, h)$, $h > 0$ ($h < 0$), the *distinct* + and 0 (— and 0) states coexist; at $\sigma = \sigma_T(\lambda)$, $h = 0$, the *distinct* +, 0 and — states coexist, and for $\sigma < \sigma_T(\lambda)$, $h = 0$, the *distinct* + and — states coexist. This theorem thereby expresses the existence of the indicated phase transition lines and provides some relatively crude bounds on the first two moments of the coexisting states.

This result was established in [Ga] for $h = 0$ and with $K\lambda^{1/2}$ replaced by a strictly positive and small constant δ. We have extended and refined the arguments in that work to apply to $|h| \leq \lambda^{-1/2+\varepsilon}$, ε small and positive, and then have utilized this theorem as indispensable input to the proofs of the results described in chapter II.

The proof of this theorem involves showing that for all small enough λ and all σ and h as above,

$$\lambda\langle :\phi^2:(\Delta)\rangle^{+,0,-} \notin [K\lambda^{1/2}, \lambda\xi_+^2 - K\lambda^{1/2}]$$

($\lambda\xi_+^2 \equiv \omega_+^2 = 1 - \sigma/2 + 0(\sigma^2) + 0(\lambda^{1/2}h)$), which is expected because field values lying outside the wells of the polynomial (fig. 1) are strongly suppressed by the potential peaks. This is shown by establishing, with vacuum energy estimates and the chessboard estimate [FS] (see section 5.1), that

$$|\lambda^2\langle :\phi^2:(\Delta_\alpha)(\xi_+^2 - :\phi^2:(\Delta_\beta))\rangle^{+,0,-}| \leq K\lambda^{1/2}, \tag{3.1}$$

uniformly in Δ_α and Δ_β and in the parameters σ and h, and that, for $0 < \sigma_0$,

$$\lim_{\lambda\downarrow 0} \lambda\langle :\phi^2:(\Delta)\rangle_{\sigma_0}^{+,0,-} = 0, \tag{3.2}$$

and

$$\lim_{\lambda\downarrow 0} \lambda\langle :\phi^2:(\Delta)\rangle_{-\sigma_0}^{+,0,-} = \omega_+^2(h = 0) \tag{3.3}$$

the subscript indicates the sign of σ_0 in the interaction polynomial). However, because $\alpha_\infty(\lambda, \sigma, h)$ is convex in σ and h, it is continuously differentiable in σ and h at all but (at most) countably many locations in parameter space. This is significant because, among other reasons, wherever the vacuum energy density is differentiable in h [Gu2],

$$\lim_{|x-y|\to\infty} \langle \phi(x)\phi(y) \rangle - \langle \phi(x) \rangle \langle \phi(y) \rangle = 0,$$

and where it is differentiable in σ [Ga],

$$\lim_{|x-y|\to\infty} \langle :\phi^2:(x):\phi^2:(y) \rangle - \langle :\phi^2:(x) \rangle \langle :\phi^2:(y) \rangle = 0.$$

Thus, wherever $\alpha_\infty(\lambda, \sigma, h)$ is differentiable in σ, (3.1) establishes our claim.

Moreover, if we define, for fixed h and fixed, sufficiently small λ,

$$S^{+,0,-}_{\lambda,h} = \{ \sigma \mid \lambda \langle :\phi^2:(\Delta) \rangle^{+,0,-} \leq K\lambda^{1/2} \},$$

(3.2) entails that $S^{+,0,-}_{\lambda,h}$ is nonempty and (3.3) entails that it is bounded from below. Therefore,

$$\sigma_D^{+,0,-}(\lambda, h) = \inf S^{+,0,-}_{\sigma,h}$$

exists. And because the second Griffiths' inequality entails that

$$\langle :\phi^2:(\Delta) \rangle^{+,0,-}$$

is monotone decreasing in σ, we have for any $\sigma_0 > \sigma_D^{+,0,-}(\lambda, h)$,

$$(3.4)\quad \lambda \langle :\phi^2:(\Delta) \rangle^{+,0,-}_{\sigma_0} \leq \lim_{\sigma_n \downarrow \sigma_D^{+,0,-}(\lambda,h)} \lambda \langle :\phi^2:(\Delta) \rangle^{+,0,-}_{\sigma_n} \equiv \lambda \langle :\phi^2:(\Delta) \rangle^{+,0,-}_{\sigma_D^{+,0,-}(\lambda,h)+0} \leq K\lambda^{1/2},$$

and, for any $\sigma_0 < \sigma_D^{+,0,-}(\lambda, h)$,

$$(3.5)\quad \lambda \langle :\phi^2:(\Delta) \rangle^{+,0,-}_{\sigma_0} \geq \lim_{\sigma_n \uparrow \sigma_D^{+,0,-}(\lambda,h)} \lambda \langle :\phi^2:(\Delta) \rangle^{+,0,-}_{\sigma_n} \equiv \lambda \langle :\phi^2:(\Delta) \rangle^{+,0,-}_{\sigma_D^{+,0,-}(\lambda,h)-0} \geq \omega_+^2 - K\lambda^{1/2},$$

by taking limits through σ_n's at which the vacuum energy density is differentiable. We mention that with (3.4) and (3.5) and convexity atguments, we show that

$$\sigma_D^+(\lambda, h) = \sigma_D^0(\lambda, h) = \sigma_D^-(\lambda, h),$$

and, in fact, $\sigma_D(\lambda, h)$ is independent of the (classical) boundary conditions that are placed on $\langle :\phi^2:(\Delta) \rangle$ (see lemma 5.2.2 and appendix 1).

Arguments related to those establishing (3.1) are used, along with (3.5), to show, for $h \geq 0$,

$$(3.6)\quad \lambda \langle \phi(\Delta_\alpha)\phi(\Delta_\beta) \rangle^+_{\sigma_D(\lambda,h)-0,h+0} \equiv \lim_{h' \downarrow h} \lambda \langle \phi(\Delta_\alpha)\phi(\Delta_\beta) \rangle^+_{\sigma_D(\lambda,h)-0,h'} \geq \omega_+^2 - K\lambda^{1/2},$$

uniformly in Δ_α and Δ_β. Using the convexity in h of the vacuum energy density and the fact that $\langle \phi(\Delta) \rangle$ is monotone increasing in h (monotone decreasing in σ), we can conclude that for every $h_0 > h$,

$$(3.7) \quad \lambda^{1/2} \langle \phi(\Delta) \rangle^{+}_{\sigma_D(\lambda,h)-0,h_0} \geqslant \lambda^{1/2} \langle \phi(\Delta) \rangle^{+}_{\sigma_D(\lambda,h)-0,h_n} \geq \lambda^{1/2} \langle \phi(\Delta) \rangle^{+}_{\sigma_D(\lambda,h)-0,h+0} \geq \omega_+ - K\lambda^{1/2},$$

choosing $\{ h_n \} \downarrow h$ such that $\partial\alpha^+(\lambda, \sigma, h)/\partial h$ exists at each h_n $(h_n \leq h_0)$. By defining the $+$, 0, $-$ state at $(\sigma_D(\lambda, h), h)$ to be the appropriate limit, (3.4)-(3.7) and the $h \leq 0$ counterpart of (3.7) prove the theorem 3.1.

The estimates of theorem 3.1 will be used to prove the following bounds on expectations of the spin characteristic functions $\chi_{+,0,-}(\Delta)$ (see (4.3)), that hold the average value of the field to lie within the corresponding potential well around $\xi_{+,0,-}$ (fig. 1):

$$(3.8\,a) \quad \text{for} \quad \sigma \geq \sigma_D(\lambda, h), \langle \chi_0(\Delta) \rangle^0_\sigma \geq e^{-K\lambda^{1/4}},$$

$$(3.8\,b) \quad \text{for} \quad \sigma \leq \sigma_D(\lambda, h), h \geq 0, \langle \chi_+(\Delta) \rangle^+_\sigma \geq e^{-K\lambda^{1/4}},$$

$$(3.8\,c) \quad \text{for} \quad \sigma \leq \sigma_D(\lambda, h), h \leq 0, \langle \chi_-(\Delta) \rangle^-_\sigma \geq e^{-K\lambda^{1/4}},$$

for some $K > 0$. In addition, an input to the proof of (3.1) and (3.6), proven by a Peierls' argument and the Gaussian domination bound [FSS] is: whenever $\sigma(\Delta_\alpha) \neq \sigma(\Delta_\beta)$ ($\sigma(\Delta) = +$, 0 or $-$),

$$(3.9) \qquad \langle \chi_{\sigma(\Delta_\alpha)}(\Delta_\alpha)\chi_{\sigma(\Delta_\beta)}(\Delta_\beta) \rangle^{+,0,-} \leq e^{-c\lambda^{-1/2}},$$

for some $c > 0$, uniformly in Δ_α, Δ_β, σ and h. Moreover, it will be seen that it is possible to redefine the $+$, 0, $-$ states (actually, only the 0 state at $\sigma_T(\lambda)$ need be redefined, see section 5.2) at the phase transition lines as limits of $+$, 0, $-$ states $\langle \,.\, \rangle_n^{+,0,-}$ in the $+$, 0, $-$ region of parameter space, $R^{+,0,-}$, such that $\alpha_\infty(\lambda, \sigma, h)_n$ (the vacuum energy density corresponding to the state $\langle \,.\, \rangle_n^{+,0,-}$) is differentiable in h for each n. Let us consider such a limit for the 0 state:

$$\langle \,.\, \rangle^0_{\sigma_D(\lambda,h)} \equiv \lim_{n\to\infty} \langle \,.\, \rangle^0_n.$$

Due to a theorem by Simon [Si1], one knows that if

$$\lim_{|x-y|\to\infty} \langle \phi(x)\phi(y) \rangle - \langle \phi(x) \rangle \langle \phi(y) \rangle = 0,$$

then the state clusters, i. e., is pure. Thus, by (3.9) for each n, we have

$$\langle \chi_+(\Delta) \rangle^0_n \langle \chi_0(\Delta) \rangle^0_n \leq e^{-c\lambda^{-1/2}},$$

and (3.8 *a*) implies, for each n,

$$\langle \chi_+(\Delta) \rangle^0_n \leq e^{-c\lambda^{-1/2}}[\langle \chi_0(\Delta) \rangle^0_n]^{-1} \leq e^{-c\lambda^{-1/2}+K\lambda^{1/4}} \leq e^{-c_1\lambda^{-1/2}}.$$

But, as

$$\langle \chi_+(\Delta) \rangle^0_{\sigma_D(\lambda,h)} = \lim_{n\to\infty} \langle \chi_+(\Delta) \rangle^0_n,$$

we conclude that

$$\langle \chi_+(\Delta) \rangle^0_{\sigma_D(\lambda,h)} \leq e^{-c\lambda^{-1/2}}.$$

Similar arguments for the other states and correlation inequalities (see section 5.2) entail:

THEOREM 3.2. — For all $0 \leq \lambda \leq \lambda_0$, σ real, $|h| \leq \lambda^{1/2}$, and Δ an arbitrary unit lattice square, there exists a $c > 0$, independent of λ, σ, h and Δ, such that

i) for $\sigma \geq \sigma_D(\lambda, h)$, $\langle \chi_+(\Delta) + \chi_-(\Delta) \rangle^0_\sigma \leq e^{-c\lambda^{-1/2}}$,

ii) for $\sigma \leq \sigma_D(\lambda, h)$, $h \geq 0$, $\langle \chi_0(\Delta) + \chi_-(\Delta) \rangle^+_\sigma \leq e^{-c\lambda^{-1/2}}$,

iii) for $\sigma \leq \sigma_D(\lambda, h)$, $h \leq 0$, $\langle \chi_+(\Delta) + \chi_0(\Delta) \rangle^-_\sigma \leq e^{-c\lambda^{-1/2}}$.

These will be seen to be very useful bounds. They express the fact that the probability that the average value of the field lies near the « wrong » minimum of the polynomial is extremely small.

A beautiful result of Fröhlich and Simon [FS] informs us that whenever the vacuum energy density is differentiable in the external field, its corresponding state satisfies all the Osterwalder-Schrader axioms, including clustering, and is independent of the (classical) boundary conditions. These properties, except the clustering and the linear growth condition of Osterwalder and Schrader [OS], can be seen, at once, to carry over to the limit states, i. e., the $+$, 0, $-$ states at the phase transition lines. In fact, it will be possible to choose the sequences $\langle \, . \, \rangle_n$ such that their limits, except possibly for the 0 state at $h = 0$, $\sigma = \sigma_T(\lambda)$, are continuous from the right (or left) in h, and, thus, by employing an argument of Fröhlich and Simon, it will be possible to show that these limits also cluster. The linear growth condition will be proven separately.

Although the limit states are independent of the boundary conditions, they do depend (in principle) on the choice of defining sequence $\langle \, . \, \rangle_n$. Because the Schwinger functions are monotonically decreasing in σ and increasing in h ($h \geq 0$), the $+$, 0, $-$ state (except the 0 state at $h = 0$, $\sigma = \sigma_T(\lambda)$) is independent of any appropriate choice of defining sequence (see section 5.2). For example, with $h \geq 0$, the $+$ state at $\sigma = \sigma_D(\lambda, h)$ is independent of the choice of any sequence $(\sigma_n, h_n) \to (\sigma_D(\lambda, h), h)$ such that $\{ (\sigma_n, h_n) \}$ eventually lies in the second quadrant of parameter space, where $(\sigma_D(\lambda, h), h)$ is regarded as the origin.

We will now quickly suggest how we obtain a strong bound on the position of $\sigma_D(\lambda, h)$; indeed, we determine its leading coefficient in λ. Defining the approximate vacuum energies as

$$\alpha^{+,0,-}_{\infty,1}(\chi_{+,0,-}) = \lim_{\Lambda \uparrow \mathbb{R}^2} \frac{1}{|\Lambda|} \ln \int \prod_{\Delta \subset \Lambda} \chi_{+,0,-}(\Delta) d\phi^{+,0,-}_{1,\Lambda},$$

and noting that theorem 3.2 implies

(3.10 *a*) for $\sigma \geq \sigma_D(\lambda, h), \langle \chi_0(\Delta) \rangle_\sigma^0 \geq e^{-Ke^{-c\lambda^{-1/2}}}$,

(3.10 *b*) for $\sigma \leq \sigma_D(\lambda, h), h \geq 0, \langle \chi_+(\Delta) \rangle_\sigma^+ \geq e^{-Ke^{-c\lambda^{-1/2}}}$,

(3.10 *c*) for $\sigma \leq \sigma_D(\lambda, h), h \leq 0, \langle \chi_-(\Delta) \rangle_\sigma^- \geq e^{-Ke^{-c\lambda^{-1/2}}}$,

we can prove with the chessboard estimate (5.1.1) that for $\sigma, h \in \mathrm{R}^{+,0,-}$,

$$(3.11) \qquad 0 \geq \alpha_\infty^{+,0,-}(\chi_{+,0,-}) - \alpha_\infty \geq - Ke^{-c\lambda^{-1/2}}.$$

Moreover, one has the lower bounds:

$$(3.12) \qquad \alpha_{\infty,1} \geq \begin{cases} - E_0 \\ - E_{+,-} + \frac{1}{\pi}\ln 4 - \frac{3}{4\pi} - K\lambda^{1/2} \end{cases}$$

and the upper bounds:

$$(3.13\,a) \qquad \alpha_{\infty,1}^{+;-}(\chi_{+,-}) \leq - E_{+,-} + \frac{1}{\pi}\ln 4 - \frac{3}{4\pi} + K\lambda^{1/2}$$

and

$$(3.13\,b) \qquad \alpha_{\infty,1}^0(\chi_0) \leq K\lambda^{1/2}.$$

Thus (3.11) and (3.13 *a*) entail, when $h \geq 0$, $\sigma = \sigma_D(\lambda, h)$,

$$- E_+(\sigma_D(\lambda, h)) + \frac{1}{\pi}\ln 4 - \frac{3}{4\pi} + K_1\lambda^{1/2} \geq \alpha_{\infty,1}^+(\chi_+)$$
$$\geq \alpha_{\infty,1} - K_2e^{-c\lambda^{-1/2}} \geq - E_0 - K_2e^{-c\lambda^{-1/2}}$$

(by (3.12)). This implies, since $E_0 = 0(\lambda)$ whenever $|h| \leq \lambda^{1/2}$ and $E_+(\sigma_D(\lambda, h)) = (\sigma_D(\lambda, h) + 0(\sigma_D(\lambda, h)^2))/\lambda - h\xi_+(h = 0) + 0(h^2)$, that

$$(3.14) \quad \lambda h\xi_+(0) + 0(\sigma_D(\lambda, h)^2) + K_3\lambda^{3/2} - \lambda\left(\frac{1}{\pi}\ln 4 - \frac{3}{4\pi}\right) \geq \sigma_D(\lambda, h).$$

Similarly,

$$K_4\lambda^{1/2} \geq \alpha_{\infty,1}^0(\chi_0) \geq \alpha_{\infty,1} - K_5e^{-c\lambda^{-1/2}}$$
$$\geq - E_+ + \frac{1}{\pi}\ln 4 - \frac{3}{4\pi} - K_6\lambda^{1/2}$$

implies that

$$(3.15) \quad \sigma_D(\lambda, h) \geq \lambda h\xi_+(0) + 0(\sigma_D(\lambda, h)^2) - K_7\lambda^{3/2} + \lambda\left(\frac{1}{\pi}\ln 4 - \frac{3}{4\pi}\right).$$

Thus, (3.14) and (3.15), with a similar calculation for $h \leq 0$, imply that

$$\sigma_D(\lambda, h) = \lambda\left(\frac{1}{\pi}\ln 4 - \frac{3}{4\pi}\right) + \lambda\,|\,h\,|\,\xi_+(0) + 0(\lambda^{3/2})$$

(This can be calculated somewhat more precisely, see chapter VI). This rather exact knowledge of the position of the double (and triple) points is of interest not only of itself, but is essential in the proof in [Su1] of the convergence of the cluster expansion in most of parameter space and is believed by the author to be essential in the proof of convergence in the small neighborhood of the triple point that the results of [Su1] do not include.

Referring back to (1.1), it is easy to see that the classical triple point occurs at $\sigma = 0$, $h = 0$. $\lambda \mid h \mid \xi_+(0)$ above is the leading classical contribution of the external field to the position of the double points. The term $\ln 4/\pi - 3/4\pi$, which is the difference in Wick ground state energies (at $h = 0$) between the $+$, $-$ state and the 0 state (determined solely by the classical masses of the three states), is the leading quantum term. The higher order quantum effects (and the next highest order classical contribution, which is $0(\lambda h^2)$) are subsumed in $0(\lambda^{3/2})$.

To close this chapter, we discuss the proof of the asymptotic nature of perturbation theory. Although we have elsewhere [Su2] presented the essential points of this argument, in application to the ϕ_2^4 model deep in the two-phase region, and have there considered its application to all $P(\phi)_2$ models with mean field limits, we will review the approach briefly for the benefit of the reader not familiar with [Su2].

Because integration by parts is permitted in the infinite volume limit for $P(\phi)_2$ theories [GJ1], i. e., for a theory with interaction polynomial P and bare mass m_0,

$$\langle :\phi^j:(x)R(\phi)\rangle = j\left\langle \int :\phi^{j-1}:(x)(-\Delta + m_0^2)^{-1}(x, y)\left[\frac{\delta R}{\delta\phi(y)} - R(\phi):P'(y):\right]dy \right\rangle,$$

we may apply it to the generalized Schwinger functions of the $+, 0, -$ state. Repeated integration by parts, as described in chapter VII, yields, e. g.,

$$(3.16) \quad \left\langle \prod_{i=1}^{n} :(\phi - \xi_+)^{m_i}:(x_i) \right\rangle^+ = \sum_{i=1}^{M-1} \alpha^+ \lambda^{i/2} + \sum_i \langle R_i(\psi_+)\rangle^+,$$

where the second term on the right-hand side is a finite sum over expectations of quantities of the form of derivatives of the original product of Wick monomials, contracted through the free covariance to products of derivatives of the polynomial $\Sigma_{i=3}^{6} c_i^+ : \psi_+^i :$ (themselves possibly contracted to each other). The integration by parts is carried out until at least M polynomials have been brought down into each term $R_i(\psi_+)$. The constants α_i^+ are those generated by the perturbation theory about the minimum ξ_+. The crux of the matter is, of course, to show that the remainder term is $0(\lambda^{M/2})$. Here we shall consider a simple example and shall ignore a few technical fine points.

We examine the following term:

$$(3.17)\quad \int (-\Delta + m_+^2)^{-1}(x, y) \left\langle :\psi_+^2:(x) \left(\sum_{i=3}^{6} i c_i^+ :\psi_+^{i-1}:(y) \right) \right\rangle^+ dy.$$

Let $\Delta_x(\Delta_y)$ denote the unit lattice square centered at $x(y)$. Using the identity (see chapter IV)

$$1 = \chi_+(\Delta) + \chi_0(\Delta) + \chi_-(\Delta)$$

at each square Δ_x, Δ_y, we see

$$i c_i^+ \langle :\psi_+^2:(x) :\psi_+^{i-1}:(y) \rangle^+ = \sum_{\sigma(\Delta)} i c_i^+ \langle :\psi_+^2:(x) \chi_{\sigma(\Delta_x)}(\Delta_x) :\psi_+^{i-1}:(y) \chi_{\sigma(\Delta_y)}(\Delta_y) \rangle^+,$$

where $\sigma(\Delta)$ takes the values $+$, 0, $-$; the sum is over all possible choices of $\sigma(\Delta_x)$, $\sigma(\Delta_y)$. We apply Hölder's inequality to each term in this sum that contains $\chi_0(\Delta)$ or $\chi_-(\Delta)$, estimating its absolute value by

$$(3.18)\quad i \,|\, c_i^+ \,|\, \langle \chi_{-\text{ or } 0}(\Delta) \rangle^{+1/2} \times \langle (:\psi_+^2:(x))^2 \chi_{\sigma(\Delta_x)}(\Delta_x) (:\psi_+^{i-1}:(y))^2 \chi_{\sigma(\Delta_y)}(\Delta_y) \rangle^{+1/2}.$$

The one term with $\sigma(\Delta_x) = \sigma(\Delta_y) = +$ is spared this dissection.

Thus, we are interested in estimating expectations of the form

$$\left\langle \prod_{i=1}^{n} F_i(\Delta_i) \chi_{\sigma_i}(\Delta_i) \right\rangle.$$

where $F_i(\Delta_i)$ is a Wick monomial of the field ψ_+, localized in a unit lattice square Δ_i. However, because $\chi_{\sigma_i}(\Delta_i)$ restrains $\phi(\Delta_i)$ to lie close to ξ_{σ_i}, we will be able to bound such expectations by the following:

$$(3.19)\quad \prod_{i=1}^{n} \{ K(N(F_i))(1 + |\, \xi_+ - \xi_{\sigma_i} \,|^{N(F_i)}) \},$$

where $N(F_i)$ is the total degree of $F_i(\Delta_i)$ and K is a constant uniform as $\lambda \downarrow 0$. When $\sigma_i \neq +$, $|\, \xi_{\sigma_i} - \xi_+ \,|^{N(F_i)} = 0(\lambda^{-N(F_i)/2})$. But, in that case, we have from (3.18) and theorem 3.2 the strong suppression factor $\exp\{ - c\lambda^{-1/2} \}$. When $\sigma_i = +$, the i-th contribution in (3.19) is $0(1)$. Remarking that the interaction coefficients $c_i^{+,0,-}$ satisfy the following bounds:

$$|\, c_i^{+,0,-} \,| = 0\left(\lambda^{\frac{i-2}{2}}\right) \quad , \quad i \geq 3,$$

we obtain an estimate for (3.17):

$$\left| \int (-\Delta + m_+^2)^{-1}(x, y) \left\langle :\psi_+^2:(x)\left(\sum_{i=3}^{6} c_i^+ i :\psi_+^{i-1}:(y)\right)\right\rangle^+ dy \right|$$

$$\leq \left[0(\lambda^{1/2}) + (3^2 - 1)0(\lambda^{-1})\left(\sum_{i=3}^{6} i \mid c_i^+ \mid \lambda^{-(i-1)/2}\right)e^{-c\lambda^{-1/2}} \right] \times \int (-\Delta + m_+^2)^{-1}(x, y)dy$$

$$\leq 0(\lambda^{1/2}) + 0(\lambda^{-3/2})e^{-c\lambda^{-1/2}},$$

where $3^2 - 1$ comes from the sum over choices of $\sigma(.)$, and we have recalled that x is fixed, so that the integral over y is 0(1).

In the general case (3.16), if $R_i(\psi_+)$ has M polynomials in it, the one term in the spin configuration sum that has $\sigma_i(\Delta_i) = +$, for all i, gives a contribution $0(\lambda^{M/2})$, and all the rest give contributions of the order

$$0(\lambda^{-M/2-N(A)/2}) \times \exp\{-c\lambda^{1/2}\} \; (N(A) = \Sigma_{i=1}^n m_i),$$

proving that perturbation theory is indeed asymptotic in the coupling constant.

4. VACUUM ENERGY ESTIMATES

In this chapter essential vacuum energy estimates that are uniform as $\lambda \downarrow 0$ are proven. In order to prove such bounds we must restrict our attention to subsets of path space ($\mathcal{S}'(R^2)$). To understand this, we point out the fact that, if $\phi_\kappa(x)$ denotes the ultraviolet cutoff field, the ultraviolet cutoff interaction density (e. g., for the + state),

$$:P_2(\phi_\kappa):(x) - \frac{m_+^2}{2}:(\phi_\kappa - \xi_+)^2:(x), \tag{4.1}$$

is not uniformly bounded from below as $\lambda \downarrow 0$. In fact, when $\phi_\kappa(x) \approx \xi_-$, (4.1) is $-0(\lambda^{-1})$, since $P_2(\xi_-) = E_- - E_c$, which does not provide the necessary control. However, when $\phi_\kappa \geq \zeta_+$, (4.1) *is* uniformly bounded from below as $\lambda \downarrow 0$. Thus, we define the following « spin » characteristic functions. Let

$$\chi_{[a,b]}(x) = \begin{cases} 1, & x \in [a, b] \\ 0, & x \notin [a, b] \end{cases}, \tag{4.2}$$

and let

$$\chi_+(x) = \chi_{[\zeta_+,\infty)}(x),$$
$$\chi_0(x) = \chi_{(\zeta_-,\zeta_+)}(x),$$
$$\chi_-(x) = \chi_{(-\infty,\zeta_-]}(x).$$

We define the average field in a unit lattice square Δ (Δ is a unit square from a lattice with bonds of unit length which covers R^2) to be

$$\phi(\Delta) = \int_\Delta \phi(x)dx.$$

Then, we write

$$\chi_{+,0,-}(\Delta) = \chi_{+,0,-}(\phi(\Delta)). \tag{4.3}$$

Note that

$$1 = \chi_+(\Delta) + \chi_0(\Delta) + \chi_-(\Delta). \tag{4.4}$$

We also wish to define a « spin configuration » function $\sigma(.)$, which is constant on unit lattice squares and takes values in $\{ +, 0, - \}$. The $\chi_{\sigma(\Delta)}(\Delta)$ holds the average value of the field in Δ to lie in a neighborhood of $\xi_{\sigma(\Delta)}$ that excludes the other minima of the polynomial. Of course, only the average value of the field is restricted, and

$$\phi(x) = \phi(\Delta) + \delta\phi(x) \quad , \quad x \in \Delta,$$

so it will be necessary to control an error term due to the high momentum part $\delta\phi$.

In fact, occasion will arise to consider path space in yet smaller pieces. We define the « shrunken » spin characteristic functions that restrain the average value of the field to lie very close to the minima of the polynomial:

$$\chi_{+,s}(\Delta) = \chi_{[\xi_+ - \lambda^{1/4}\xi_+ ,\xi_+ + \lambda^{1/4}\xi_+]}(\phi(\Delta)), \tag{4.5 a}$$

$$\chi_{0,s}(\Delta) = \chi_{[\xi_0 - \lambda^{1/4}\xi_+ ,\xi_0 + \lambda^{1/4}\xi_+]}(\phi(\Delta)), \tag{4.5 b}$$

$$\chi_{-,s}(\Delta) = \chi_{[\xi_- - \lambda^{1/4}\xi_+ ,\xi_- + \lambda^{1/4}\xi_+]}(\phi(\Delta)). \tag{4.5 c}$$

The « peak » characteristic functions are:

$$\chi_{\substack{+\\ \underline{0}},p}(\Delta) = \chi_{\substack{+\\ \underline{0}}}(\Delta) - \chi_{\substack{+\\ \underline{0}},s}(\Delta). \tag{4.6}$$

We shall prove the necessary vacuum energy estimates in the subset of parameter space defined by

$$T = \{ \sigma, h \mid | h | \leq \lambda^{-1/2+\varepsilon} \quad , \quad | \sigma | \leq 10^{-1} \},$$

where $\varepsilon > 0$ is arbitrarily small and fixed. We shall not show the corresponding estimates for interaction parameters lying in $(R^+ \cup R^0 \cup R^-)\backslash T$, since T will be seen to contain the most interesting portions of the phase diagram. In any case, all bounds in $(R^+ \cup R^0 \cup R^-)\backslash T$ are proven in full detail in [Su1]. Whenever we speak of « sufficiently small λ » below, we shall mean all $0 \leq \lambda \leq \lambda_0(\varepsilon)$, where $\lambda_0(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$.

LEMMA 4.1. — Let $0 < \eta \leq 10^{-2}$, $\zeta \geq 10^2$. Then there are constants $a = a(\zeta) > 0$, $b = b(\zeta) > 0$, such that for any large κ, $x \in \Delta$, and $\sigma(\Delta)$, the following inequalities hold for all sufficiently small λ and σ, $h \in T$.

$$i)\ :P_2(\phi_\kappa):(x) - \frac{m^2_{+,0,-}}{2}\eta :(\phi_\kappa - \xi_{\sigma(\Delta)})^2:(x) + \zeta :\delta\phi_\kappa^2:(x)$$
$$- \ln \chi_{\sigma(\Delta)}(\Delta) + E_c - E_{\sigma(\Delta)} \geq - a \ln^4 \kappa.$$
$$ii) :P_2(\phi_\kappa):(x) - \frac{m^2_{+,0,-}}{2}\eta :(\phi_\kappa - \xi_{\sigma(\Delta)})^2:(x) + \zeta :\delta\phi_\kappa^2:(x)$$
$$- \ln \chi_{\sigma(\Delta),p}(\Delta) \geq b\lambda^{-1/2} - a \ln^4 \kappa.$$

Remarks. — 1. Since $\chi_{\sigma(\Delta),p}(\Delta)$ restrains the field to take values where the polynomial (2.1) is large, *ii*) is reasonable.

2. We subtract the factor $E_{\sigma(\Delta)} - E_c$ because of the inequality of the classical vacuum energy densities, $E_{+,-}$ and E_0, and in anticipation of arguments to be made in the next chapter.

3. We use the ultraviolet cutoff of [GJS4], which has the convenient property that $\phi(\Delta) = \phi_\kappa(\Delta)$.

Proof. — Denote by c_κ the ultraviolet cutoff Wick constant and define

$$\tau(x) = :P_2(\phi_\kappa):(x) - \frac{m_+^2}{2}\eta :(\phi_\kappa - \xi_{\sigma(\Delta)})^2:(x) + \zeta :\delta\phi_\kappa^2:(x) + E_c - E_{\sigma(\Delta)}$$
$$= P_2(\phi_\kappa) - \frac{m_+^2}{2}\eta(\phi_\kappa - \xi_{\sigma(\Delta)})^2 - 15\lambda^2 c_\kappa \phi_\kappa^4 + [45\lambda^2 c_\kappa^2 + 12\lambda c_\kappa]\phi_\kappa^2$$
$$- 15\lambda^2 c_\kappa^3 - 6\lambda c_\kappa^2 - (1+\sigma)c_\kappa + \frac{m_+^2}{2}\eta c_\kappa + \zeta\delta\phi_\kappa^2 - \zeta\delta c_\kappa + E_c - E_{\sigma(\Delta)}$$

($\delta c_\kappa = 0(\ln \kappa)$ is the Wick constant for $\delta\phi_\kappa^2$). We therefore wish to show, in order to prove *i*), that

$$\tau(x) - \ln \chi_{\sigma(\Delta)}(\Delta) \geq - 0(1) \ln^4 \kappa. \tag{4.7}$$

But for any $\delta > 0$,

$$- (15\lambda^2 c_\kappa + 3\delta\lambda^2)\phi_\kappa^4 \geq - \delta\lambda^3\phi_\kappa^6 - (4\delta)^{-1}\lambda[15c_\kappa + 3\delta]^2\phi_\kappa^2$$
$$\geq - \delta\lambda^3\phi_\kappa^6 - \delta\lambda^2\phi_\kappa^4 - \left[\frac{(15c_\kappa + 3\delta)^2}{8\delta^{3/2}}\right]^2$$
$$\geq - \delta\lambda^3\phi_\kappa^6 - \delta\lambda^2\phi_\kappa^4 - 0(\delta^{-3}) \ln^4 \kappa,$$

since $c_\kappa = 0$ (ln κ). Therefore,

$$- 15\lambda^2 c_\kappa\phi_\kappa^4 \geq - \delta\lambda^3\phi_\kappa^6 + 2\delta\lambda^2\phi_\kappa^4 - 0(\delta^{-3}) \ln^4 \kappa$$
$$\geq - \delta\lambda^3\phi_\kappa^6 + 2\delta\lambda^2\phi_\kappa^4 - \delta\lambda(1+\sigma)\phi_\kappa^2 + \delta\lambda h\phi_\kappa + \delta\lambda E_c - 0(\delta^{-3}) \ln^4 \kappa,$$

since $\delta\lambda E_c = 0(1)$. Thus,

$$\tau(x) \geq (1 - \delta\lambda)P_2(\phi_\kappa) - \frac{m_+^2}{2}\eta(\phi_\kappa - \xi_{\sigma(\Delta)})^2 + \zeta\delta\phi_\kappa^2$$
$$+ E_c - E_{\sigma(\Delta)} - 0(\delta^{-3}) \ln^4 \kappa.$$

I. Pick Δ, fix $x \in \Delta$, specify $\sigma(\Delta) = -$. The proof will proceed by examining the four cases in which ϕ_κ is restricted to have a value that lies (1) within the right-hand external well of the polynomial, (2) within the middle well of the polynomial, (3) within the left-hand external well of the polynomial, (4) outside of the wells altogether.

CASE 1. — $| \phi_\kappa - \xi_+ | \leq 2\xi_+/3$.

Since $\phi_\kappa = \phi(\Delta) + \delta\phi_\kappa$, one must have either (*a*) $\delta\phi_\kappa \geq 2\xi_+/3$ or (*b*) $\phi(\Delta) \geq -\xi_+/3$. However, for $\sigma, h \in \mathrm{T}$ and for the indicated range of ϕ_κ, we have

$$\mathrm{P}_2(\phi_\kappa) + \mathrm{E}_c - \mathrm{E}_- \geq \mathrm{E}_+ - \mathrm{E}_- \geq -1.1(-h\xi_+(0) + h\xi_-(0))$$

(see (2.7))

$$\geq -3\lambda^{-1+\varepsilon}.$$

Therefore, as $(\phi_\kappa - \xi_-)^2 \leq 9\xi_+^2$,

$$\tau(x) \geq -3\lambda^{-1+\varepsilon} - 4.5m_+^2\eta\xi_+^2 + \zeta\delta\phi_\kappa^2 - 0(\delta^{-3}) \ln^4 \kappa.$$

In subcase (*a*), $\delta\phi_\kappa^2 \geq 4\xi_+^2/9$, so that

$$\begin{aligned}\tau(x) &\geq -3\lambda^{-1+\varepsilon} - 4.5m_+^2\eta\xi_+^2 + \zeta 4\xi_+^2/9 - 0(\delta^{-3}) \ln^4 \kappa \\ &\geq b_1(\zeta)\lambda^{-1} - 0(\delta^{-3}) \ln^4 \kappa,\end{aligned}$$

for all sufficiently small λ and $b_1(\zeta) > 0$ (We set $\delta = 10^{-3}$). Because $-\ln \chi_-(\Delta) \geq 0$, (4.7) is confirmed. In subcase (*b*), $\phi(\Delta) \geq -\xi_+/3$, so that $\chi_-(\Delta) = 0$ ($\zeta_- < -\xi_+/3$; see (1.2)). Thus, (4.7) again follows.

CASE 2. — $| \phi_\kappa | \leq \xi_+/3$.

One must have either (*a*) $\delta\phi_\kappa \geq \xi_+/6$ or (*b*) $\phi(\Delta) \geq -\xi_+/2$. In subcase (*b*) $\chi_-(\Delta) = 0$, and in subcase (*a*) $\delta\phi_\kappa^2 \geq \xi_+^2/36$. But, for ϕ_κ restricted as shown,

$$\begin{aligned}\mathrm{P}_2(\phi_\kappa) + \mathrm{E}_c - \mathrm{E}_- \geq \mathrm{E}_0 - \mathrm{E}_- \geq -1.1\left(\frac{\sigma}{\lambda} - h\xi_-(0) + h^2/4(1+\sigma)\right) \\ \geq -1.5(|\sigma|\lambda^{-1} + \lambda^{-1+\varepsilon}),\end{aligned}$$

and $(\phi_\kappa - \xi_-)^2 \leq 4\xi_+^2$. Thus,

$$\tau(x) \geq -1.5(|\sigma|\lambda^{-1} + \lambda^{-1+\varepsilon}) - 2m_+^2\eta\xi_+^2 + \zeta\delta\phi_\kappa^2 - 0(\delta^{-3}) \ln^4 \kappa.$$

In subcase (*b*), (4.7) follows at once and in subcase (*a*),

$$\tau(x) \geq b_2(\zeta)\lambda^{-1} - 0(\delta^{-3}) \ln^4 \kappa,$$

for $b_2(\zeta) > 0$.

CASE 3. — $| \phi_\kappa + \xi_+ | \leq 2\xi_+/3$.

By (1.3), in this case,

$$\tau(x) \geq \frac{1}{4}(1 - \delta\lambda)m_+^2(\phi_\kappa - \xi_-)^2 - \frac{m_+^2}{2}\eta(\phi_\kappa - \xi_-)^2 - 0(\delta^{-3}) \ln^4 \kappa$$
$$\geq - 0(\delta^{-3}) \ln^4 \kappa,$$

by hypothesis, verifying (4.7).

CASE 4. — $|\phi_\kappa| \geq 5\xi_+/3$.

For these values of the field $P_2(\phi_\kappa)$ is very large; in fact, one can readily see that for σ, $h \in T$ and η as hypothesized,

$$\tau(x) \geq 4\lambda^{-1} - 0(\delta^{-3}) \ln^4 \kappa.$$

Thus, (4.7) obtains in this case.

II. For $\sigma(\Delta) = +$, the arguments are similar.

III. Consider $\sigma(\Delta) = 0$. Then,

$$\tau(x) \geq (1 - \delta\lambda)P_2(\phi_\kappa) - \frac{m_+^2}{2}\eta(\phi_\kappa - \xi_0)^2 + \zeta\delta\phi_\kappa^2 + E_c - E_0 - 0(\delta^{-3}) \ln^4 \kappa.$$

CASE 1. — $|\phi_\kappa - \xi_+| \leq \xi_+/10$.

One must have either (*a*) $\delta\phi_\kappa \geq \xi_+/10$ or (*b*) $\phi(\Delta) \geq 4\xi_+/5$. For the indicated values of ϕ_κ,

$$\tau(x) \geq E_+ - E_0 - \frac{m_+^2}{2}\eta\left(\frac{6}{5}\xi_+\right)^2 + \zeta\delta\phi_\kappa^2 - 0(\delta^{-3}) \ln^4 \kappa$$
$$\geq - 1.5(|\sigma|\lambda^{-1} + \lambda^{-1+\varepsilon}) - \frac{m_+^2}{2}\eta\left(\frac{6}{5}\xi_+\right)^2 + \zeta\delta\phi_\kappa^2 - 0(\delta^{-3}) \ln^4 \kappa.$$

In subcase (*a*),

$$\tau(x) \geq b_3(\zeta)\lambda^{-1} - 0(\delta^{-3}) \ln^4 \kappa,$$

with $b_3(\zeta) > 0$, and in subcase (*b*), $\chi_0(\Delta) = 0$. Thus, (4.7) holds in this case.

CASE 2. — $|\phi_\kappa| \leq 9\xi_+/10$.

Here, using (1.3),

$$\tau(x) \geq \frac{1}{2}(1 - \delta\lambda)(\phi_\kappa - \xi_0)^2 - \frac{m_+^2}{2}\eta(\phi_\kappa - \xi_0)^2 - 0(\delta^{-3}) \ln^4 \kappa$$
$$\geq - 0(\delta^{-3} \ln^4 \kappa,$$

by hypothesis.

CASE 3. — $| \phi_\kappa + \xi_+ | \leq \xi_+/10$.

One must have either (a) $\delta\phi_\kappa \leq -\xi_+/10$ or (b) $\phi(\Delta) \leq -4\xi_+/5$. The argument is thus similar to case 1.

CASE 4. — $| \phi_\kappa | \geq 11\xi_+/10$.

In this case (e. g., for $h \leq 0$)

$$\begin{aligned}\tau(x) &\geq -\frac{m_+^2}{2}\eta\left(\frac{11}{10}\right)^2\xi_+^2 + (1-\delta\lambda)\mathrm{P}_2\left(\frac{11}{10}\xi_+\right) + \mathrm{E}_c - \mathrm{E}_0 - 0(\delta^{-3})\ln^4\kappa \\ &\geq \lambda^{-1}/2 - 0(\delta^{-3})\ln^4\kappa.\end{aligned}$$

This completes the proof of i) with the choice m_+^2. It is clear that the argument is the same for the choice $m_{0,-}^2$.

The proof of ii) similar. Consider I, i. e., the choice $\sigma(\Delta) = -$. Because $\chi_{-,p}(\Delta) \leq \chi_-(\Delta)$, the proof of cases 1, 2 and 4 above shows, in fact (note $\mathrm{E}_{\sigma(\Delta)} - \mathrm{E}_c \geq 0$),

$$\tau(x) - \ln\chi_{-,p}(\Delta) \geq b\lambda^{-1} - 0(\delta^{-3})\ln^4\kappa.$$

We will consider case 3 in several subcases:

i) $-5\xi_+/3 \leq \phi_\kappa \leq \xi_- - \lambda^{1/4}\xi_+/2$,
ii) $| \phi_\kappa - \xi_- | \leq \lambda^{1/4}\xi_+/2$, and
iii) $\xi_- + \lambda^{1/4}\xi_+/2 \leq \phi_\kappa \leq -\xi_+/3$.

Subcases i) and iii) are similar, so we consider only subcase i):

$$\begin{aligned}\tau(x) - \mathrm{E}_c + \mathrm{E}_{\sigma(\Delta)} &\geq (1-\delta\lambda)\mathrm{P}_2(\phi_\kappa) - \frac{m_+^2}{2}\eta(\phi_\sigma - \xi_-)^2 + \zeta\delta\phi_\kappa^2 - 0(\delta^{-3})\ln^4\kappa \\ &\geq (1-\delta\lambda)\mathrm{P}_2(\xi_- - \lambda^{1/4}\xi_+/2) - \frac{m_+^2}{2}\eta(\lambda^{1/4}\xi_+/2)^2 - 0(\delta^{-3})\ln^4\kappa\end{aligned}$$

(since $\partial\tau(x)/\partial\phi_\kappa < 0$, for $\phi_\kappa \leq \xi_- - \lambda^{1/4}\xi_+/2$ and the parameters as assumed in the hypothesis)

$$\geq \frac{1}{2}\lambda^{1/2}\xi_+^2 - 0(\delta^{-3})\ln^4\kappa \geq b\lambda^{-1/2} - 0(\delta^{-3})\ln^4\kappa.$$

For subcase ii), when $\phi(\Delta) \in [\xi_- - \lambda^{1/4}\xi_+, \xi_- + \lambda^{1/4}\xi_+]$, $\chi_{-,p}(\Delta) = 0$, so that this range of average field values is excluded. We thus have either (a) $\delta\phi_\kappa \geq \lambda^{1/4}\xi_+/2$ or (b) $\delta\phi_\kappa \leq -\lambda^{1/4}\xi_+/2$ (i. e. when $\phi(\Delta) < \xi_- - \lambda^{1/4}\xi_+$ or $\phi(\Delta) > \xi_- + \lambda^{1/4}\xi_+$). The conclusion thus follows, since

$$\tau(x) - \mathrm{E}_c + \mathrm{E}_{\sigma(\Delta)} \geq -\frac{m_+^2}{2}\eta(\lambda^{1/6}\xi_+/2)^2 + \zeta\delta\phi_\kappa^2 - 0(\delta^{-3})\ln^4\kappa$$

(note : $\mathrm{P}_2(\phi_\kappa) \geq 0$). II and III are treated similarly. This completes the proof of lemma 4.1.

We comment that the coefficients $c_i^{+,0,-}$ of $:P(\psi_{+,0,-}):$ (see (1.4)) and $d_i^{+,0,-}$ of

$$W^{+,0,-}(x) = :P_2(\phi):(x) - \frac{m^2_{+,0,-}}{2}\eta :(\phi - \xi_{+,0,-})^2:(x) + E_c - E_{+,0,-}$$
$$= \sum_{i=0}^{6} d_i^{+,0,-} : \psi^i_{+,0,-} : (x)$$

satisfy the following bounds.

LEMMA 4.2. — For all small λ and σ, $h \in T$,

$$| d_i^{+,0,-} | = | c_i^{+,0,-} | = 0\left(\lambda^{\frac{i-2}{2}}\right) \quad , \quad i \geq 3,$$
$$| d_2^{+,0,-} | = 0(1)$$
$$d_1^{+,0,-} = d_0^{+,0,-} = 0.$$

Proof. — By direct calculation, using the fact that $| \xi_{+,0,-} | \leq 0(\lambda^{-1/2})$. If we define $W_\kappa^{+,0,-}(x)$ by the substitution $\phi \to \phi_\kappa$ in $W^{+,0,-}(x)$ and $\delta W_\kappa(x) = W(x) - W_\kappa(x)$, we have

LEMMA 4.3. — There are positive constants K and δ such that if $\{ m(\Delta) \mid \Delta \subset R^2 \}$ is a set of nonnegative integers and $\{ \kappa(\Delta) \mid \Delta \subset R^2 \}$ is a set of positive numbers, then for any $Y \subset R^2$,

$$i) \quad \left| \int \prod_{\Delta \subset Y} \delta W_{\kappa(\Delta)}^{+,0,-}(\Delta)^{m(\Delta)} d\mu(\psi_{+,0,-})_{m^2_{+,0,-}} \right| \leq \prod_{\Delta \subset Y} [(6m(\Delta))\,!\,(K\kappa(\Delta)^{\delta-})^{m(\Delta)}].$$

K and δ are uniform in λ as $\lambda \downarrow 0$. Moreover, for q even,

$$ii) \quad \left| \int \prod_{\Delta \subset Y} :P(\psi_{+,0,-}):(\Delta)^q d\mu(\psi_{+,0,-})_{m^2_{+,0,-}} \right| \leq ((6q)\,!\,(K\lambda)^{q/2})^{|Y|}.$$

Proof. — As in [DG], using lemma 4.2.

It is now possible to prove the desired vacuum energy bounds.

PROPOSITION 4.4. — The are strictly positive constants $a(\eta)$, $b(\eta)$, such that for all $\eta > 0$ sufficiently small, all λ sufficiently small, all σ, $h \in T$ and $1 \leq p \leq 1 + \eta/30$, the following estimates obtain.

$$i) \quad \int e^{-p\left(:P_2(\phi)_\Lambda: - \frac{m^2_{\sigma_0}}{2}:(\phi-\xi_{\sigma_0})^2_\Lambda: + (E_c - E_{\sigma_0})|\Lambda|\right)} \prod_{\Delta \subset \Lambda} \chi^p_{+,0,-}(\Delta) d\mu(\phi - \xi_{\sigma_0}) \leq e^{a\lambda^{1/2}|\Lambda|}$$

and

$$ii) \quad \int e^{-p\left(:P_2(\phi)_\Lambda: - \frac{m^2_{\sigma_0}}{2}:(\phi-\xi_{\sigma_0})^2_\Lambda:\right)} \prod_{\Delta \subset \Lambda} \chi^p_{\sigma_0,0}(\Delta) d\mu(\phi - \xi_{\sigma_0})_{m^2_{\sigma_0}} \leq e^{-b\lambda^{-1/2}|\Lambda|},$$

for $\sigma_0 = +$, 0 or $-$.

Proof. — By Hölder's inequality, e. g., for $\sigma_0 = +$,

$$(4.8) \quad \int e^{-p\left(:P_2(\phi)_\Lambda: - \frac{m_+^2}{2} :(\phi-\xi_+)_\Lambda^2: + [E_c - E_+]|\Lambda|\right)} \prod_{\Delta \subset \Lambda} \chi_+(\Delta)^p d\mu(\phi - \xi_+)_{m_+^2}$$

$$\leq \left(\int e^{-pq'\left(:P_2(\phi)_\Lambda: + \zeta:\delta\phi_\Lambda^2: - \eta \frac{m_+^2}{2} :(\phi-\xi_+)_\Lambda^2: + [E_c - E_+]|\Lambda|\right)} \prod_{\Delta \subset \Lambda} \chi_+^{pq'}(\Delta) d\mu(\phi - \xi_+) \right)^{1/q'}$$

$$\times \left(\int e^{pq\left(\zeta:\delta\phi_\Lambda^2: + (1-\eta)\frac{m_+^2}{2} :(\phi-\xi_+)_\Lambda^2:\right)} d\mu(\phi - \xi_+)_{m_+^2} \right)^{1/q}.$$

By conditioning with respect to Neumann boundary conditions [GJS4, GRS1] on the bonds of some square lattice over R^2, the second factor is estimated by

$$\left(\int e^{pq\left[\zeta:\delta\phi_\Delta^2: + (1-\eta)\frac{m_+^2}{2} :(\phi-\xi_+)_\Delta^2:\right]} d\mu(\phi - \xi_+)_{\partial\Delta}^{N} \right)^{\frac{|\Lambda|}{q|\Delta|}}$$

(Δ is not necessarily a *unit* lattice square, here) (1). A standard calculation yields for this factor (note $\delta\phi = \delta\psi_{+,0,-}$):

$$(4.9) \quad (\det{}_2 [1 - pq(2\zeta(1 - P_\Delta) + (1 - \eta)m_+^2)(-\Delta_\Delta^N + m_+^2)^{-1}])^{-\frac{|\Lambda|}{2q|\Delta|}},$$

where P_Δ is the projection in $L_2(\Delta)$ onto χ_Δ (the characteristic function of the square Δ); Δ_Δ^N is the Laplacian with Neumann boundary conditions on $\partial\Delta$, the boundary of Δ. But since $(1 - P_\Delta)\chi_\Delta = 0$ and because

$$-\Delta_N^\Delta \upharpoonright \{\chi_\Delta\}^\perp \geq \pi^2/|\Delta|,$$

we have

$$1 - pq(2\zeta(1 - P_\Delta) + (1 - \eta)m_+^2)(-\Delta_\Delta^N + m_+^2)^{-1}$$
$$\geq 1 - qp(1 - \eta) - 2\zeta qp/(\pi^2/|\Delta| + m_+^2) > 0,$$

if we choose $\eta = 10^{-3}$, $q = 1 + \eta/60$, $\zeta = 10^2$, $|\Delta| = 10^{-6}$ (these constants have been chosen so that the hypothesis of lemma 4.1 is also satisfied). Thus (4.9) is finite and is bounded by

$$(4.10) \quad e^{K_1|\Lambda|}$$

for some constant K_1.

Lemmas 4.1 and 4.3 and standard arguments [DG] permit us to bound the first factor of (4.8) by

$$(4.11) \quad e^{K_2|\Lambda|}$$

for a constant K_2 uniform in λ. Use of lemma 4.1 *ii*) entails that if $\prod_{\Delta \subset \Lambda} \chi_+(\Delta)$ is replaced by $\prod_{\Delta \subset \Lambda} \chi_{+,p}(\Delta)$, the above bound is replaced by

$$\exp\{-K_2\lambda^{-1/2}|\Lambda|\}.$$

(1) See note added in proof.

To complete the proof of *i*), one employs a perturbation argument due to [GJS4] using the identity

$$e^{-p\mathrm{V}(\Delta)} = 1 - p\mathrm{V}(\Delta)\int_0^1 e^{-pt\mathrm{V}(\Delta)}dt.$$

In fact, with $t \equiv \{ t(\Delta) \mid \Delta \subset \mathrm{Y} \}$, we have (e. g., for $\sigma_0 = +$):

$$\int e^{-p\left(:\mathrm{P}_2(\phi)_\Lambda: - \frac{m_+^2}{2}:(\phi-\xi_+)^2_\Lambda: + (\mathrm{E}_c - \mathrm{E}_+)|\Lambda|\right)} \prod_{\Delta\subset\Lambda} \chi_+(\Delta)^p d\mu(\phi-\xi_+)_{m_+^2}$$

$$= \int e^{-p:\mathrm{P}(\psi_+)_\Lambda:} \prod_{\Delta\subset\Lambda} \chi_+(\Delta)^p d\mu(\phi-\xi_+)_{m_+^2}$$

$$\leq \sum_{\mathrm{Y}\subset\Lambda} \int_0^1 \int \left| \prod_{\Delta\subset\mathrm{Y}} p : \mathrm{P}(\psi_+) : (\Delta) \right| e^{-pt:\mathrm{P}(\psi_+)_\mathrm{Y}:} \prod_{\Delta\subset\Lambda} \chi_+(\Delta)^p d\mu(\psi_+)_{m_+^2} dt$$

$$\leq \sum_{\mathrm{Y}\subset\Lambda} \left\{ \left[\int \left| \prod_{\Delta\subset\mathrm{Y}} p : \mathrm{P}(\psi_+) : (\Delta) \right|^{q'} d\mu(\psi_+)_{m_+^2} \right]^{1/q'} \right.$$

$$\left. \times \sup_t \left[\int e^{-pqt:\mathrm{P}(\psi_+)_\mathrm{Y}:} \prod_{\Delta\subset\Lambda} \chi_+(\Delta)^{pq} d\mu(\psi_+)_{m_+^2} \right]^{1/q} \right\}$$

(we choose $1 < q$ small enough that $pq \leq 1 + \eta/30$)

$$\leq \sum_{\mathrm{Y}\subset\Lambda} (\mathrm{K}_3\lambda^{1/2})^{|\mathrm{Y}|} e^{(\mathrm{K}_1+\mathrm{K}_2)|\mathrm{Y}|}$$

(we have used (4.8)-(4.11) and lemma 4.3)

$$= \sum_{r=0}^{|\Lambda|} \binom{|\Lambda|}{r} (\mathrm{K}_3\lambda^{1/2})^r e^{\mathrm{K}_4(\eta)r}$$

$$= \prod_{\Delta\subset\Lambda} (1 + \mathrm{K}_5\lambda^{1/2}) \leq e^{a\lambda^{1/2}|\Lambda|}.$$

Thus, the proof of the proposition is completed.

We have seen in proposition 4.4 that the vacuum energy densities corresponding to the measures (2.3) are bounded uniformly as $\lambda \downarrow 0$, if the integral is restrained to be taken over only those fields whose average values lie « close » to the appropriate classical mean. In order to patch together these estimates to obtain a bound on an integral over all of $\mathscr{S}'(\mathrm{R}^2)$, we will use the fact that the Gaussian measure is $\mathscr{S}$-quasi-invariant (e. g. [Fr1]) and its Radon-Nikodym derivative is given by

$$\frac{d\mu(\phi-f)_a}{d\mu(\phi)_a} = e^{-\phi((-\Delta+a)f) - \frac{1}{2}\langle f,(-\Delta+a)f\rangle}, \qquad (4.12)$$

where $\langle .,. \rangle$ signifies the real L_2 inner product. In order to define an admissible shift that will also accomplish the desired translation of the mean of the Gaussian measure from, e. g., ξ_+ to ξ_- in Λ, we define:

$$g_{h_1}(x) = \eta_v \int (-\Delta + a)^{-1}(x-y) v\left(\frac{x-y}{L}\right) h_1(y) dy, \tag{4.13}$$

where

$$v(x) = \begin{cases} 0 \ , & \text{if} \ |x| > 1/2 \\ 1 \ , & \text{if} \ |x| \leq 1/4 \end{cases} \ , \quad 0 \leq v(x) \leq 1 \ , \quad v(x) \in C_0^\infty,$$

$$\eta_v^{-1} = \int (-\Delta + a)^{-1}(y) v(y/L) dy,$$

$$h_1(y) = \begin{cases} \xi_+ \ , & y \in R^2 \backslash (\Lambda \cup N(\partial\Lambda)) \\ \xi_- \ , & y \in \Lambda \cup N(\partial\Lambda) \end{cases} ,$$

where $N(\partial\Lambda) = \{ \Delta \subset R^2 \mid \text{dist}(\Delta, \partial\Lambda) \leq L \}$, $1 \leq L < \infty$ and fixed. It is important to notice that $g \restriction \Lambda = \xi_-$ and $g(x) = \xi_+$ for $x \in R^2 \backslash \Lambda$ such that dist $(x, \partial\Lambda) \geq 2L$.

We also introduce a space-dependent mass for a Gaussian measure, as in [GJS4], which will permit us to shift masses between the interaction exponent and the Gaussian measure. Here we note only that for $\omega(x)$ satisfying

$$0 < \omega(x) \leq a \quad , \quad \inf_x \omega(x) \equiv \underline{\omega} > 0,$$

such that $a - \omega(x)$ has compact support, then for

$$Z_\omega = \int e^{\frac{1}{2}\int (a-\omega(x)) :\psi^2:_a(x) dx} d\mu(\psi)_a,$$

one has

$$d\mu(\psi)_\omega = Z_\omega^{-1} e^{\frac{1}{2}\int (a-\omega(x)) :\psi^2:_a(x) dx} d\mu(\psi)_a,$$

and this is the Gaussian measure with mean zero and covariance

$$C = (-\Delta + \omega(x))^{-1}.$$

LEMMA 4.5. — *The vacuum energy densities defined in* (2.6) *satisfy the following bounds:*

i) $$\alpha_{\infty,2}^{+,0,-} \geq 0,$$

and

ii) $$\alpha_{\infty,1}^{+,0,-} \geq \begin{cases} -E_0 \\ -E_{+,0,-} + c_W^{+,0,-} - C\lambda^{1/2}, \end{cases}$$

for some C independent of λ.

Proof. — Since the half-Neumann and free boundary condition pressures are equal [GRS2], we consider the half-Neumann vacuum energy densities. We will place a superscript N on the Gaussian measures to denote the presence of (zero) Neumann boundary conditions on $\partial\Lambda$. By (2.3) and (2.6),

$$\alpha^0_{\Lambda,2} = \frac{1}{|\Lambda|} \ln \int e^{-:P_2(\phi)_\Lambda: + \frac{m_0^2}{2} :(\phi-\xi_0)^2_\Lambda: + c^0_W|\Lambda|} d\mu(\phi-\xi_0)^N_{m_0^2}$$

$$= \frac{1}{|\Lambda|} \ln \left[\int e^{-:P_2(\phi)_\Lambda: + \frac{m_0^2}{2} :(\phi-\xi_0)^2_\Lambda: + \left(1-\frac{m_0^2}{2}\right) :(\phi-\xi_0)^2_\Lambda:} d\mu(\phi-\xi_0)^N_2 \times Z_0^{-1}(\Lambda)\right] + c^0_W,$$

where

$$Z_0(\Lambda) = \int e^{\left(1-\frac{m_0^2}{2}\right) :(\phi-\xi_0)^2_\Lambda:} d\mu(\phi-\xi_0)^N_2.$$

Recalling the definition of c^0_W (2.4), one notes that

$$\lim_{\Lambda\uparrow R^2} \left[-\frac{1}{|\Lambda|} \ln Z_0(\Lambda) + c^0_W \right] = 0.$$

Thus, we see that

$$(4.14) \qquad \alpha^0_{\infty,2} = \lim_{\Lambda\uparrow R^2} \frac{1}{|\Lambda|} \ln \int e^{-:P_2(\phi)_\Lambda: + :(\phi-\xi_0)^2_\Lambda:} d\mu(\phi-\xi_0)^N_2$$

$$= \lim_{\Lambda\uparrow R^2} \frac{1}{|\Lambda|} \ln \int e^{-:P_2(\phi)_\Lambda: + :(\phi-\xi_0)^2_\Lambda:} d\mu(\phi-\xi_0)_2,$$

by [GRS2].

But

$$(4.15) \qquad \frac{1}{|\Lambda|} \ln \int e^{-:P_2(\phi)_\Lambda: + :(\phi-\xi_0)^2_\Lambda:} d\mu(\phi-\xi_0)_2$$

$$\geq \frac{1}{|\Lambda|} \ln e^{\int -:P_2(\psi_0+\xi_0)_\Lambda: + :\psi^2_{0\Lambda}: d\mu(\psi_0)_2}$$

$$= \frac{1}{|\Lambda|} \ln e^{(E_c - E_0)|\Lambda|} = E_c - E_0,$$

where we have used Jensen's inequality in the second line and have recalled that the Wick ordering is always with respect to mass^2 2.

Define now g_+ as g_{h_1} (see above) with

$$h_1(y) = \begin{cases} \xi_0 & , \quad y \in R^2 \setminus (\Lambda \cup N(\partial\Lambda)) \\ \xi_+ & , \quad y \in \Lambda \cup N(\partial\Lambda) \end{cases}.$$

Then, using (4.12), one has

$$\frac{1}{|\Lambda|}\ln\int e^{-:P_2(\phi)_\Lambda:+:(\phi-\xi_0)^2_\Lambda:}d\mu(\phi-\xi_0)_2$$
$$=\frac{1}{|\Lambda|}\ln\int e^{-:P_2(\phi+g_+)_\Lambda:+:(\phi+g_+-\xi_0)^2_\Lambda:}d\mu(\phi+g_+-\xi_0)_2$$
$$=\frac{1}{|\Lambda|}\ln\Big\{\int e^{-:P_2(\phi+g_+)_\Lambda:+:\phi^2_\Lambda:+2[\phi(g_+-\xi_0)]_\Lambda+(g_+-\xi_0)^2_\Lambda}$$
$$\times e^{-\frac{1}{2}\langle g_+-\xi_0,(-\Delta+2)(g_+-\xi_0)\rangle-\langle\phi,(-\Delta+2)(g_+-\xi_0)\rangle}d\mu(\phi)_2\Big\}$$
$$=\frac{1}{|\Lambda|}\ln\Big\{\int e^{-:P_2(\phi+\xi_+)_\Lambda:+:\phi^2_\Lambda:-\frac{1}{2}\int_{R^2\setminus\Lambda}[(g_+-\xi_0)(-\Delta+2)(g_+-\xi_0)]}$$
$$\times e^{\int_{R^2\setminus\Lambda}[\phi(-\Delta+2)(g_+-\xi_0)]}d\mu(\phi)_2\Big\}$$

(since $g_+\upharpoonright\Lambda=\xi_+$, by definition)

$$\geq\frac{1}{|\Lambda|}\ln$$
$$\Big\{e^{\int -:P_2(\phi+\xi_+)_\Lambda:+:\phi^2_\Lambda:-\frac{1}{2}[(g_+-\xi_0)(-\Delta+2)(g_+-\xi_0)]_{R^2\setminus\Lambda}+[\phi(-\Delta+2)(g_+-\xi_0)]_{R^2\setminus\Lambda}d\mu(\phi)_2}\Big\}$$

(by Jensen's inequality)

$$=\frac{1}{|\Lambda|}\ln e^{(E_c-E_+)|\Lambda|+C|\partial\Lambda|}=E_c-E_++C\frac{|\partial\Lambda|}{|\Lambda|},$$

where we have used the fact that for all $x\in R^2\setminus\Lambda$ such that dist $(x,\partial\Lambda)\geq 2L$, $g_+=\xi_0$. Since the same can be done for a correspondingly defined g_-, one may conclude that $\alpha^0_{\infty,2}\geq E_c-E_{+,-}$, which, in conjunction with the equations (4.14) and (4.15), yields

$$\alpha^0_{\infty,2}\geq\max\{E_c-E_0,E_c-E_+,E_c-E_-\}=0.$$

The argument of (4.14) and (4.15) applied to $\alpha^0_{\infty,1}$ yields $\alpha^0_{\infty,1}\geq -E_0$. In addition,

$$\alpha^+_{\Lambda,1}=\frac{1}{|\Lambda|}\ln\int e^{-:P_1(\phi)_\Lambda:+\frac{m_+^2}{2}:(\phi-\xi_+)^2_\Lambda:+c^+_W|\Lambda|}d\mu(\phi-\xi_+)_{m_+^2},$$

from which, writing $\psi_+=\phi-\xi_+$ and re-expressing $P_1(\phi)$ in terms of $\sum_{j=0}^{6}c_j^+{\psi_+}^j$ as in (1.3), we obtain

$$\alpha^+_{\Lambda,1}=\frac{1}{|\Lambda|}\ln\int e^{-:\sum_{j=3}^{6}c_j^+{\psi_+}^j{}_\Lambda:+(c^+_W-E_+)|\Lambda|}d\mu(\psi_+)_{m_+^2},$$

where $| c_j^+ | = 0(\lambda^{1/2})$, $j \geq 3$. Therefore, re-Wick ordering,

$$\alpha_{\Lambda,1}^+ \geq \frac{1}{|\Lambda|} \ln \int e^{-:\sum_{j=1}^{6} c_j' \psi_+{}^j{}_\Lambda :_{m_+^2} - C\lambda^{1/2}|\Lambda| + (c_W^+ - E_+)|\Lambda|} d\mu(\psi_+)_{m_+^2}$$

for some C. Use of Jensen's inequality again gives us

$$\alpha_{\infty,1}^+ \geq - C\lambda^{1/2} + c_W^+ - E_+ .$$

Clearly, the same argument can be used to obtain

$$\alpha_{\infty,1}^{0,-} \geq - C\lambda^{1/2} + c_W^{0,-} - E_{0,-} .$$

The rest of the lemma follows from the next proposition.

LEMMA 4.6. — $\alpha_{\infty,i}^+ = \alpha_{\infty,i}^0 = \alpha_{\infty,i}^-$ for all λ, σ, h and $i = 1, 2$.

Proof. — Using (2.4) and the argument in the previous lemma, it is easy to see that

$$\alpha_{\infty,i}^{+,-} = \lim_{\Lambda \uparrow R^2} \frac{1}{|\Lambda|} \ln \int e^{-:P_i(\phi)_\Lambda: + :(\phi - \xi_{+,-})_\Lambda^2:} d\mu(\phi - \xi_{+,-})_2^N .$$

Furthermore, since $\xi_+ - \xi_0$ is in the domain of $-\Delta_{\partial\Lambda}^N$ (in fact, $-\Delta_{\partial\Lambda}^N(\xi_+ - \xi_0) = 0$, and writing g_0 as g_{h_1} with

$$h_1(y) = \begin{cases} \xi_0 & , \quad y \in \Lambda \cup N(\partial\Lambda) \\ \xi_+ & , \quad y \in R^2 \backslash (\Lambda \cup N(\partial\Lambda)) \end{cases} ,$$

we have

$$(4.16) \quad \frac{1}{|\Lambda|} \ln \int e^{-:P_i(\phi)_\Lambda: + :(\phi - \xi_+)_\Lambda^2:} d\mu(\phi - \xi_+)_2^N$$

$$= \frac{1}{|\Lambda|} \ln \int e^{-:P_i(\phi)_\Lambda: + :(\phi - g_0 + g_0 - \xi_+)_\Lambda^2:} d\mu(\phi - g_0 + g_0 - \xi_+)_2^N$$

$$= \frac{1}{|\Lambda|} \ln \int e^{-:P_i(\phi)_\Lambda: + :(\phi - \xi_0)_\Lambda^2: + (\xi_+ - \xi_0)_\Lambda^2 + 2[(\phi - \xi_0)(\xi_0 - \xi_+)]_\Lambda}$$

$$\times e^{-\frac{1}{2}\langle \xi_+ - g_0, (-\Delta_{\partial\Lambda}^N + 2)(\xi_+ - g_0)\rangle - \langle \phi - g_0, (-\Delta_{\partial\Lambda}^N + 2)(g_0 - \xi_+)\rangle} d\mu(\phi - g_0)_2^N$$

by (4.12). But with Neumann boundary conditions on $\partial\Lambda$, the Gaussian measure and the integral factor across $\partial\Lambda$. Since

$$\int e^{-\frac{1}{2}\int_{R^2\backslash\Lambda}(\xi_+ - g_0)(-\Delta_{\partial\Lambda}^N + 2)(\xi_+ - g_0) - \int_{R^2\backslash\Lambda}(\phi - \xi_0)(-\Delta_{\partial\Lambda}^N + 2)(g_0 - \xi_+)} d\mu(\phi - g_0)_2^N = 1,$$

(4.16) is equal to

$$\frac{1}{|\Lambda|}\ln\int e^{-:P_i(\phi)_\Lambda:+:(\phi-\xi_0)^2_\Lambda:}d\mu(\phi-\xi_0)^{\mathrm{N}}_2.$$

Thus, taking the limit $\Lambda\nearrow \mathrm{R}^2$ and using (4.14), $\alpha^0_{\infty,i}=\alpha^+_{\infty,i}$. A similar argument for $\alpha^-_{\infty,i}$ completes the proof.

5. THE PHASE DIAGRAM

Using the vacuum energy bounds proven in the last chapter, we shall verify the phase diagram of figure 3. In the first section of this chapter we shall establish some further results that will be necessary.

5.1. Technical preliminaries.

We recall the chessboard estimate [FS]. If F_α is a measurable function of the fields with support in the lattice square Δ_α, then

$$(5.1.1)\qquad \left|\left\langle\prod_{\alpha\in\mathrm{N}}\mathrm{F}_\alpha\right\rangle^{+,0,-}\right|\leq e^{\sum\limits_{\alpha\in\mathrm{N}}(\alpha^{+,0,-}_\infty(\mathrm{F}_\alpha)-\alpha^{+,0,-}_\infty)|\Delta_\alpha|},$$

where N is some index set and

$$(5.1.2)\qquad \alpha^{+,0,-}_\infty(\mathrm{F}_\alpha)=\lim_{\Lambda\nearrow\mathrm{R}^2}\frac{1}{|\Lambda|}\ln\int\prod_{\Delta_\beta\in\Lambda}(\mathrm{F}_\alpha)_\beta d\phi^{+,0,-}_\Lambda.$$

$(\mathrm{F}_\alpha)_\beta$ is the function with support in Δ_β obtained by a series of reflections in lattice lines and translations of the function F_α (see [FS]).

Further, for a given unit lattice square Δ, we define

$$(5.1.3)\qquad \mathrm{F}^{\sigma(\Delta)}(\Delta)=\prod_{i=1}^n c_i:(\phi-\xi_{\sigma(\Delta)})^{m_i}:(\Delta),$$

where $\{c_i\}^n_{i=1}$ is a set of given coefficients. We denote the total degree of F by $\mathrm{N}(\mathrm{F}(\Delta))\equiv\Sigma^n_{i=1}m_i$. We wish to show

PROPOSITION 5.1.1. — Let $\{w_j\}^\nu_{j=1}$ be a collection of localized functions such that $w_j\in \mathrm{L}_q(\Delta_j)$ for some $q>1$. Then for any collection

$$\{\mathrm{F}^{\sigma_{1,j}}(w_j)\chi_{\sigma_{2,j}}(\Delta_j)\}^\nu_{j=1},$$

there exist constants K, $c > 0$ such that for all small enough λ and σ, $h \in T$, if $K(N) = K^N N\,!$, one has the following estimate:

$$\left| \left\langle \prod_{j=1}^{\nu} F^{\sigma_{1,j}}(w_j)\chi_{\sigma_{2,j}}(\Delta_j) \right\rangle^{+,0,-} \right|$$

$$\leq \prod_{\{j \mid \sigma_{1,j}=\sigma_{2,j}\}} \left[\left(\prod_{i=1}^{n_j} | c_{i,j} | \right) K(N(F_j)) \| w_j \|_p e^{c(E_c - E_{\sigma_{2,j}})} \right]$$

$$\times \prod_{\{j \mid \sigma_{1,j} \neq \sigma_{2,j}\}} \left[\lambda^{-N(F_j)/2} \left(\prod_{i=1}^{n_j} | c_{i,j} | \right) K(N(F_j)) \| w_j \|_p e^{c(E_c - E_{\sigma_{2,j}})} \right]$$

for any $p > 1$.

Proof. — By the chessboard inequality, e. g., for the choice of the + state,

$$(5.1.4) \qquad \left| \left\langle \prod_{j=1}^{\nu} F^{\sigma_{1,j}}(w_j)\chi_{\sigma_{2,j}}(\Delta_j) \right\rangle^{+} \right| \leq e^{\sum_{j=1}^{\nu} (\alpha_\infty^+(F^{\sigma_{2,j}}\chi_{\sigma_{2,j}}) - \alpha_\infty)|\Delta_j|},$$

where we have used lemma 4.6. For arbitrary Λ,

$$(5.1.5) \qquad \alpha_\Lambda^+(F^{\sigma_{1,j}}\chi_{\sigma_{2,j}}) = \frac{1}{|\Lambda|} \ln \int \prod_{\Delta \subset \Lambda} (F^{\sigma_{1,j}}(w_j))_\Delta \chi_{\sigma_{2,j}}(\Delta) d\phi_\Lambda^+ .$$

We note that the same spin characteristic function $\chi_{\sigma_{2,j}}(\Delta)$ is multiplied throughout Λ.

If $\sigma_{2,j} = +$, we estimate (5.1.5) through Hölder's inequality:

$$(5.1.6) \qquad \frac{1}{|\Lambda|} \frac{1}{p'} \ln \int \prod_{\Delta \subset \Lambda} (F^{\sigma_{1,j}}(w_j))_\Delta^{p'} d\mu(\phi - \xi_+)_{m_+^2}$$

$$+ \frac{1}{|\Lambda|} \frac{1}{p} \ln \int e^{-p\left[:P_2(\phi)_\Lambda: - \frac{m_+^2}{2} :(\phi - \xi_+)_\Lambda^2: \right]} \prod_{\Delta \subset \Lambda} \chi_{\sigma_{2,j}}(\Delta) d\mu(\phi - \xi_+)_{m_+^2} .$$

By proposition 4.4, the second term is estimated by

$$(5.1.7) \qquad \frac{1}{p} \ln a\lambda^{1/2} + E_c - E_+,$$

if we choose $p \simeq 1 + 10^{-6}$ (and require that p' is even). The first term of (5.1.6) can be estimated, using the checkerboard estimate [GRS1, 2], by

$$\frac{1}{|\Lambda|} \frac{1}{p'q} \left(\sum_{\Delta \subset \Lambda} \ln \int (F^{\sigma_{1,j}}(w_j))_\Delta^{p'q} d\mu(\phi - \xi_+)_{m_+^2} \right) .$$

If $\sigma_{1,j} = +$, this is bounded by

$$\ln\left[\left(\prod_{i=1}^{n_j} | c_{i,j} |\right) \mathrm{K}(\mathrm{N}(\mathrm{F}_j)) \| w_j \|_p\right], \tag{5.1.8}$$

$p > 1$, using a standard argument on Gaussian integrals [DG]. This bound is uniform in Λ and the interaction parameters. If, however, $\sigma_{1,j} = -$ or 0, because $| \xi_{-,0} - \xi_+ | = 0(\lambda^{-1/2})$, the same argument on Gaussian integrals leads to the bound

$$\ln\left[\left(\prod_{i=1}^{n_j} | c_{i,j} |\right) \mathrm{K}(\mathrm{N}(\mathrm{F}_j)) \lambda^{-\mathrm{N}(\mathrm{F}_j)/2} \| w_j \|_p\right]. \tag{5.1.9}$$

This bound is uniform in Λ and σ, $h \in \mathrm{T}$.

If $\sigma_{2,j} = -$ or 0, we must shift the mean of the Gaussian measure (and that of the second term of the interaction exponent) in order to employ the uniform bound of proposition 4.4. Let us consider the case $\sigma_{2,j} = -$; $\sigma_{2,j} = 0$ is treated similarly.

Defining $g(x) = g_{h_1}(x)$, with h_1 as specified after (4. 13), we can rewrite (5.1.5) as

$$\frac{1}{|\Lambda|} \ln \int \prod_{\Delta \subset \Lambda} (\mathrm{F}^{\sigma_{1,j}}(w_j))_\Delta \chi_{\sigma_{2,j}}(\Delta) e^{-:\mathrm{P}_2(\phi)_\Lambda: + \frac{m_+^2}{2} :(\phi - g)_\Lambda^2:} \tag{5.1.10}$$

$$\times e^{m_+^2 \int_\Lambda (\phi - g)(g - \xi_+) + \frac{m_+^2}{2} \int_\Lambda (g - \xi_+)^2} d\mu(\phi - \xi_+)_{m_+^2}$$

$$= \frac{1}{|\Lambda|} \ln \int \prod_{\Delta \subset \Lambda} (\mathrm{F}^{\sigma_{1,j}}(w_j))_\Delta \chi_{\sigma_{2,j}}(\Delta) e^{-:\mathrm{P}_2(\phi)_\Lambda: + \frac{m_+^2}{2} :(\phi - \xi_-)_\Lambda^2:}$$

$$\times e^{m_+^2 \int_{\mathrm{R}^2 \setminus \Lambda} (\phi - g)(g - \xi_+) + \frac{m_+^2}{2} \int_{\mathrm{R}^2 \setminus \Lambda} (g - \xi_+)^2} d\mu(\phi - g)_{m_+^2}$$

(using (4.12) and the fact that $g \restriction \Lambda = \xi_-$)

$$= \frac{1}{|\Lambda|} \ln \left[\int \prod_{\Delta \subset \Lambda} (\mathrm{F}^{\sigma_{1,j}}(w_j))_\Delta \chi_{\sigma_{2,j}}(\Delta) e^{-:\mathrm{P}_2(\phi)_\Lambda: + \frac{m_-^2}{2} :(\phi - \xi_-)_\Lambda^2:} \right.$$

$$\left. \times e^{m_+^2 \int_{\mathrm{R}^2 \setminus \Lambda} (\phi - g)(g - \xi_+) + \frac{m_+^2}{2} \int_{\mathrm{R}^2 \setminus \Lambda} (g - \xi_+)^2} d\mu(\phi - g)_\omega \mathrm{Z}_{+-}(\Lambda) \right],$$

where

$$\omega = m_+^2 - (m_+^2 - m_-^2)\chi_\Lambda(x)$$

and

$$\mathrm{Z}_{+-}(\Lambda) = \int e^{\frac{1}{2}(m_+^2 - m_-^2):\psi_\Lambda^2:} d\mu(\psi)_{m_+^2}.$$

Recalling (2.2), one has from [GRS2] that

$$\lim_{\Lambda \nearrow R^2} \frac{1}{|\Lambda|} \ln Z_{+-}(\Lambda) = 0(1), \tag{5.1.11}$$

uniformly for $\lambda \downarrow 0$ and σ, $h \in T$ (which is also true of the analogously defined $Z_{+0}(\Lambda)$). We thus concentrate on

$$\begin{aligned} &\frac{1}{|\Lambda|} \ln \int \prod_{\Delta \subset \Lambda} (F^{\sigma_1, j}(w_j))_\Delta \chi_{\sigma_2, j}(\Delta) e^{-:P_2(\phi)_\Lambda: + \frac{m_-^2}{2} :(\phi - \xi_-)_\Lambda^2:} \\ &\qquad \times e^{m_+^2 \int_{R^2 \setminus \Lambda} (\phi - g)(g - \xi_+) + \frac{m_+^2}{2} \int_{R^2 \setminus \Lambda} (g - \xi_+)^2} d\mu(\phi - g)_\omega \\ &\leq \frac{1}{|\Lambda|} \frac{1}{p'} \ln \int \prod_{\Delta \subset \Lambda} (F^{\sigma_1, j}(w_j))_\Lambda^{p'} e^{p' m_+^2 \int_{R^2 \setminus \Lambda} \left[(\phi - g)(g - \xi_+) + \frac{1}{2}(g - \xi_+)^2\right]} d\mu(\phi - g) \\ &+ \frac{1}{|\Lambda|} \frac{1}{p} \ln \int e^{-p[:P_2(\phi)_\Lambda: - \frac{m_-^2}{2} :(\phi - \xi_-)_\Lambda^2:]} \prod_{\Delta \subset \Lambda} \chi_{\sigma_2, j}^{P}(\Delta) d\mu(\phi - g)_\omega . \end{aligned} \tag{5.1.12}$$

With the earlier indicated choice of p, the second term is estimated by using proposition 4.4, since $\omega \upharpoonright \Lambda = m_-^2$ and $\omega(x) > 0$, for all x, so that a minor modification of lemma 4.3 (see e. g. [GJS4]) suffices to yield the bound:

$$\frac{1}{p} \ln a\lambda^{1/2} + E_c - E_- . \tag{5.1.13}$$

(It is easy to see that because $g \upharpoonright \Lambda = \xi_-$ and g is continuous, it is indeed valid to make use of proposition 4.4). Hölder's inequality applied again to the first term on the right-hand side of (5.1.12) yields the bound

$$\begin{aligned} &\frac{1}{|\Lambda|} \frac{1}{p'q} \ln \int \prod_{\Delta \subset \Lambda} (F^{\sigma_1, j}(w_j))_\Lambda^{p'q} d\mu(\phi - g) \\ &\qquad + \frac{1}{|\Lambda|} \frac{1}{p'q'} \ln \int e^{p'q' m_+^2 \int_{R^2 \setminus \Lambda} \left[(\phi - g)(g - \xi_+) + \frac{1}{2}(g - \xi_+)^2\right]} d\mu(\phi - g). \end{aligned}$$

Because, in $R^2 \setminus \Lambda$, g differs from ξ_+ only in a strip along $\partial\Lambda$, one observes that the second term is dominated by

$$\frac{1}{|\Lambda|} LC \,|\, \partial\Lambda \,|,$$

where C depends on λ. And the first term is estimated, using the arguments utilized previously, by

$$\ln \left[\left(\prod_{i=1}^{n_j} |c_{i,j}| \right) K(N(F_j)) \, \| w_j \|_p \right],$$

if $\sigma_{1,j} = -$, or by

$$\ln\left[\left(\prod_{i=1}^{n_j} | c_{i,j} |\right) K(N(F_j))\lambda^{-N(F_j)/2} \| w_j \|_p\right],$$

if $\sigma_{1,j} = +$ or 0. These estimates, with the analogous bounds for the choice $\sigma_{2,j} = 0$, (5.1.7)-(5.1.9) and lemma 4.5, yield the proposition.

LEMMA 5.1.2. — There exists a $b > 0$, such that for all sufficiently small λ, all σ, $h \in T$, and any set Y composed of unit lattice squares Δ, the following bound obtains:

$$\left\langle \prod_{\Delta \subset Y} \chi_{\sigma(\Delta),p}(\Delta) \right\rangle \leq e^{-b\lambda^{-1/2}|Y|},$$

where the expectation is in either the $+$, 0 or $-$ state and the choice of the spin configuration function $\sigma(.)$ is arbitrary.

Proof. — Follows directly from the chessboard estimates, proposition 4.4, lemma 4.5, and arguments in the previous proof.

LEMMA 5.1.3. — There exist strictly positive constants K, c, such that for all sufficiently small λ, all σ, $h \in T$ and any collection

$$\{ F^{\sigma_{1,j}}(\Delta_j) \}_{j=1}^{\nu}$$

of functions of the form (5.1.3) one has the following estimate:

$$\left| \left\langle \prod_{j=1}^{\nu} F^{\sigma_{1,j}}(\Delta_j)\chi_{\sigma_{2,j},p}(\Delta_j) \right\rangle \right| \leq \prod_{j=1}^{\nu}\left[\left(\prod_{i=1}^{n_j} | c_{i,j} |\right) K(N(F_j))\lambda^{-N(F_j)/2} e^{-c\lambda^{-1/2}}\right].$$

The expectation is in either the $+$, 0 or $-$ state.

Proof. — This follows readily from the argument of proposition 5.1.1, lemma 4.5 and proposition 4.4 *ii*).

We are now in a position to prove the essential estimate (3.9).

PROPOSITION 5.1.4. — There exists a $c > 0$ such that for all Δ_α, Δ_β, all small enough λ and σ, $h \in T$,

$$\langle \chi_{\sigma(\Delta_\alpha)}(\Delta_\alpha)\chi_{\sigma(\Delta_\beta)}(\Delta_\beta) \rangle^{+,0,-} \leq e^{-c\lambda^{-1/2}},$$

whenever $\sigma(\Delta_\alpha) \neq \sigma(\Delta_\beta)$.

Proof. — Using $1 = \chi_+(\Delta) + \chi_0(\Delta) + \chi_-(\Delta)$ at every $\Delta \subset \Lambda_0$, where Λ_0 is a large square containing Δ_α and Δ_β, we have, e. g. for the $+$ state,

$$\langle \chi_{\sigma_\alpha}(\Delta_\alpha)\chi_{\sigma_\beta}(\Delta_\beta) \rangle^+ = \sum_{\sigma(.)} \Big\langle \prod_{\Delta \subset \Lambda_0} \chi_{\sigma(\Delta)}(\Delta) \Big\rangle^+ , \tag{5.1.14}$$

where the sum $\Sigma_{\sigma(.)}$ is the sum over configurations $\sigma(.)$ such that $\sigma(\Delta_\alpha) = \sigma_\alpha$ and $\sigma(\Delta_\beta) = \sigma_\beta$. Following [GJS3, Fr2], we estimate (5.1.14) by

$$\sum_\gamma \sum_{\sigma(\Delta') \neq \sigma_\alpha} \Big\langle \prod_{(\Delta,\Delta') \in N(\gamma)} \chi_{\sigma_\alpha}(\Delta)\chi_{\sigma(\Delta')}(\Delta') \Big\rangle^+ ,$$

where $N(\gamma)$ is the set of nearest neighbor pairs of unit lattice squares bordering on a certain minimal connected contour γ, consisting of unit lattice lines, separating Δ_α and Δ_β (see [Fr2] for further details). Following [GJS3, Fr2] this proposition will be proven once one establishes that

$$\Big\langle \prod_{(\Delta,\Delta') \in N(\gamma)} \chi_{\sigma(\Delta)}(\Delta)\chi_{\sigma(\Delta')}(\Delta') \Big\rangle \leq e^{-\delta\lambda^{-1/2}|\gamma|}, \tag{5.1.15}$$

for some $\delta > 0$, where $N(\gamma)$ is a given set of $|\gamma|$ neighboring pairs of unit lattice squares and $\sigma(\Delta) \neq \sigma(\Delta')$. We note that, in fact, $|\gamma| \geq 4$. We may assume that all pairs in $N(\gamma)$ are mutually disjoint (separating them with Hölder's inequality if they are not) and that $\sigma(\Delta') \neq 0$. Then we recall $\chi_0 = \chi_{0,s} + \chi_{0,p}$ (see (4.5) and (4.6)), so that

$$\begin{aligned} &\Big\langle \prod_{(\Delta,\Delta') \in N(\gamma)} \chi_{\sigma(\Delta)}(\Delta)\chi_{\sigma(\Delta')}(\Delta') \Big\rangle \\ &\leq \sum_{\gamma'} \Big\langle \prod_{(\Delta,\Delta') \in N(\gamma')} \chi_{0,p}(\Delta) \Big\rangle^{1/2} \Big\langle \prod_{(\Delta,\Delta') \in N(\gamma \backslash \gamma')} \chi_{0,s \text{ or } -\sigma(\Delta')}(\Delta)\chi_{\sigma(\Delta')}(\Delta') \Big\rangle^{1/2} , \end{aligned} \tag{5.1.16}$$

where $\Sigma_{\gamma'}$ runs over the subsets of γ such that the pairs $(\Delta, \Delta') \in N(\gamma')$ satisfy $\sigma(\Delta) = 0$. But

$$\chi_{0,s \text{ or } -\sigma(\Delta')}(\Delta)\chi_{\sigma(\Delta')}(\Delta') \leq e^{\sigma(\Delta')(\phi(\Delta')-\phi(\Delta)) - \sigma(\Delta')\zeta_{\sigma(\Delta')} + \sigma(\Delta')\xi_0 + \lambda^{1/4}\xi_+}, \tag{5.1.17}$$

and if we choose functions $h^i_{\Delta,\Delta'}$ as in [Fr2] such that

$$\sigma(\Delta')(\phi(\Delta') - \phi(\Delta)) = \sum_{i=0}^{1} \phi(\partial_i h^i_{\Delta,\Delta'}),$$

we have, using (5.1.17) and the Gaussian domination bound [FSS, Fr2], i. e.,

$$\Big\langle e^{\sum_{i=0}^{1} \phi(\partial_i h^i)} \Big\rangle \leq e^{\sum_{i=0}^{1} \| h^i \|_2^2} , \tag{5.1.18}$$

that

$$\Big\langle \prod_{(\Delta,\Delta')\in N(\gamma\backslash\gamma')} \chi_{0,s\ or\ -\sigma(\Delta')}(\Delta)\chi_{\sigma(\Delta')}(\Delta') \Big\rangle \leq e^{-\delta'\lambda^{-1/2}|\gamma\backslash\gamma'|},$$

$\delta' > 0$. Thus, with lemma 5.1.2, (5.1.16) yields (5.1.15) and the proposition.

Next we shall prove (3.1)-(3.3) in the more general formulation that accomodates the large external fields that are permitted in T.

THEOREM 5.1.5. — There is a finite constant K such that for all sufficiently small λ, all σ, $h \in$ T, and every Δ_α, Δ_β:

(5.1.19 *a*) when $h \geq 0$,

$$|\lambda^2\langle(:\phi^2:(\Delta_\alpha) - \xi_0^2)(\xi_+^2 - :\phi^2:(\Delta_\beta))\rangle^{+,0,-}| \leq K\lambda^{1/2};$$

(5.1.19 *b*) when $h \leq 0$,

$$|\lambda^2\langle(:\phi^2:(\Delta_\alpha) - \xi_0^2)(\xi_-^2 - :\phi^2:(\Delta_\beta))\rangle^{+,0,-}| \leq K\lambda^{1/2}.$$

For $0 < \sigma_0 \leq 10^{-1}$,

$$\lim_{\lambda \searrow 0} \lambda\langle:\phi^2:(\Delta)\rangle_{\sigma_0}^{+,0,-} = 0, \tag{5.1.20}$$

$$\lim_{\lambda \searrow 0} \lambda\langle:\phi^2:(\Delta)\rangle_{-\sigma_0}^{+,0,-} = \omega_+^2(h=0), \tag{5.1.21}$$

where the subscripts indicate the sign of σ_0 in the interaction polynomial and $\omega_{+,0,-}^2 = \lambda\xi_{+,0,-}^2$. Furthermore, there exists a $C > 0$ such that

$$|\langle:\phi^2:(\Delta)\rangle^{+,0,-} - \langle\phi(\Delta)^2\rangle^{+,0,-}| \leq C. \tag{5.1.22}$$

Proof. — We have adapted and extended arguments of [Ga] to prove this proposition.

To prove (5.1.22), we note

$$\begin{aligned}|\langle:\phi^2:(\Delta) - \phi(\Delta)^2\rangle^{+,0,-}| \leq{}& \langle|:\phi^2:(\Delta) - \phi(\Delta)^2|\chi_0(\Delta)\rangle^{+,0,-}\\ &+ \langle|:\phi^2:(\Delta) - \phi(\Delta)^2|\chi_+(\Delta)\rangle^{+,0,-}\\ &+ \langle|:\phi^2:(\Delta) - \phi(\Delta)^2|\chi_-(\Delta)\rangle^{+,0,-} \leq 0(1),\end{aligned}$$

by proposition 5.1.1, since

$$:\phi^2:(\Delta) - \phi(\Delta)^2 = :\psi_{+,0,-}^2:(\Delta) - \psi_{+,0,-}(\Delta)^2.$$

We will now prove (5.1.19 *a*); (5.1.19 *b*) is shown similarly. We rewrite (5.1.19 *a*)):

$$\begin{aligned}&\lambda^2\langle(:\phi^2:(\Delta_\alpha) - \xi_0^2)(\xi_+^2 - :\phi^2:(\Delta_\beta))\rangle^{+,0,-} \\ &= \lambda^2\langle(:\phi^2:(\Delta_\alpha) - \xi_0^2)\chi_0(\Delta_\alpha)(\xi_+^2 - :\phi^2:(\Delta_\beta))\rangle^{+,0,-}\\ &+ \lambda^2\langle(:\phi^2:(\Delta_\alpha) - \xi_0^2)(\chi_+(\Delta_\alpha) + \chi_-(\Delta_\alpha))\\ &\qquad\qquad (\xi_+^2 - :\phi^2:(\Delta_\beta))(\chi_+(\Delta_\beta) + \chi_-(\Delta_\beta))\rangle^{+,0,-}\\ &+ \lambda^2\langle(:\phi^2:(\Delta_\alpha) - \xi_0^2)(\chi_+(\Delta_\alpha) + \chi_-(\Delta_\alpha))(\xi_+^2 - :\phi^2:(\Delta_\beta))\chi_0(\Delta_\beta)\rangle^{+,0,-}.\end{aligned} \tag{5.1.23}$$

The absolute value of the first term is estimated by

$$(5.1.24)\quad \lambda(\langle \chi_0(\Delta_\alpha)(:\phi^2:(\Delta_\alpha) - \xi_0^2)^2 \rangle^{+,0,-})^{1/2} \times (\langle \lambda^2(\xi_+^2 - :\phi^2:(\Delta_\beta))^2 \rangle^{+,0,-})^{1/2} \le \lambda 0(\lambda^{-1/2+\varepsilon}),$$

by proposition 5.1.1, using $:\phi^2:(\Delta_\alpha) - \xi_0^2 = :\psi_0^2:(\Delta_\alpha) + 2\xi_0\psi_0(\Delta_\alpha)$ in the first factor ($\xi_0 = 0(h)$) and inserting $1 = \chi_+(\Delta_\beta) + \chi_0(\Delta_\beta) + \chi_-(\Delta_\beta)$ in the second factor and estimating each term in the resulting sum by 0(1). The absolute value of the second term in (5.1.23) has the bound

$$(5.1.25)\quad \lambda^{1/2}(\lambda^2 \langle (:\phi^2:(\Delta_\alpha) - \xi_0^2)^2 \rangle^{+,0,-})^{1/2} \times (\lambda \langle (\chi_+(\Delta_\beta) + \chi_-(\Delta_\beta))(\xi_+^2 - :\phi^2:(\Delta_\beta))^2 \rangle^{+,0,-})^{1/2}.$$

The first factor is estimated by 0(1), using proposition 5.1.1, since one can again insert $1 = \chi_+(\Delta_\alpha) + \chi_0(\Delta_\alpha) + \chi_-(\Delta_\alpha)$ and estimate each term. A further application of proposition 5.1.1 yields a bound to the last factor in (5.1.25) that is 0(1) ,since

$$\begin{aligned}(5.1.26)\quad & \lambda \langle \chi_-(\Delta_\beta)(\xi_+^2 - :\phi^2:(\Delta_\beta))^2 \rangle^{+,0,-} \\ &= \lambda \langle \chi_-(\Delta_\beta)(\xi_+^2 - \xi_-^2 - 2\xi_-\psi_-(\Delta_\beta) - :\psi_-^2:(\Delta_\beta))^2 \rangle^{+,0,-} \\ &\le \lambda e^{(E_c - E_-)}[(\xi_+^2 - \xi_-^2)^2 + |\xi_+^2 - \xi_-^2|\,|\xi_-|\,0(1) + |\xi_-|\,0(1)],\end{aligned}$$

by proposition 5.1.1. However, for $\sigma, h \in T$,

$$\begin{aligned}\lambda e^{(E_c - E_-)}(\xi_+^2 - \xi_-^2)^2 &\le 0(1)\lambda e^{(E_c - E_-)}(h\xi_+(0) - h\xi_-(0))^2 \\ &= 0(1)\lambda e^{(E_c - E_-)}h^2\xi_+^2(0).\end{aligned}$$

Because $E_c - E_- \le 0$ for all σ, h, if $0 \le h \le 1$ and $|\sigma| \le 10^{-1}$, the above is bounded by 0(1). For $h \ge 1$, $E_c - E_- \le h\xi_-(0)/2 = -h\xi_+(0)/2$, and the above is again majorized by 0(1). As this is the worst term in (5.1.26), the assertion is confirmed.

Finally, the third term in (5.1.23) is bounded by

$$(\lambda^4 \langle (:\phi^2:(\Delta_\alpha) - \xi_0^2)^4 \rangle^{+,0,-})^{1/4}(\lambda^4 \langle (\xi_+^2 - :\phi^2:(\Delta_\beta))^4 \rangle^{+,0,-})^{1/4} \times (\langle (\chi_+(\Delta_\alpha) + \chi_-(\Delta_\alpha))\chi_0(\Delta_\beta) \rangle^{+,0,-})^{1/2} \le e^{-c\lambda^{-1/2}},$$

for some $c > 0$, by propositions 5.1.1 and 5.1.4. Therefore, (5.1.23) is bounded by $0(\lambda^{1/2+\varepsilon}) + \lambda^{1/2}0(1) + \exp\{-c\lambda^{-1/2}\} = 0(\lambda^{1/2})$, proving (5.1.19 *a*).

Turning our attention to (5.1.20), we see that

$$\begin{aligned}(5.1.27)\quad & \lambda\,|\langle :\phi^2:(\Delta) \rangle_{\sigma_0}^{+,0,-}| \\ &\le \lambda \langle |:\phi^2:(\Delta)|\,\chi_0(\Delta) \rangle_{\sigma_0}^{+,0,-} + \lambda \langle |:\phi^2:(\Delta)|\,(\chi_+(\Delta) + \chi_-(\Delta)) \rangle_{\sigma_0}^{+,0,-}.\end{aligned}$$

By proposition 5.1.1, the second term is bounded by

$$0(1)(e^{E_c(\sigma_0) - E_+(\sigma_0)} + e^{E_c(\sigma_0) - E_-(\sigma_0)}) \le e^{-c\sigma_0\lambda^{-1}},$$

for $c > 0$ and λ small enough (we have used the fact that

$$E_{+,-}(\sigma) = \lambda^{-1}(\sigma + 0(\sigma^2)) - h\xi_{+,-}(0) + 0(h^2)$$

for small σ (see (2.7)), that $h \in T$ entails $|h| \leq \lambda^{-1/2+\varepsilon}$, and that $E_c \leq 0$ for all values of σ and h). Proposition 5.1.1 bounds the first term of (5.1.27) by $\lambda 0(\lambda^{-1+2\varepsilon})$. Therefore, (5.1.20) is verified.

To prove (5.1.21) we note

$$(5.1.28)\quad \lambda \,|\langle :\phi^2:(\Delta)\rangle^{+,0,-}_{-\sigma_0} - \xi_+^2\,| \\ \leq \lambda^{1/2}\langle \lambda^{1/2}\,|\,\xi_+^2 - :\phi^2:(\Delta)\,|\,(\chi_+(\Delta) + \chi_-(\Delta))\rangle^{+,0,-}_{-\sigma_0} \\ + \langle \lambda\,|\,\xi_+^2 - :\phi^2:(\Delta)\,|\,\chi_0(\Delta)\rangle^{+,0,-}_{-\sigma_0} \\ \leq 0(1)\lambda^{1/2} + 0(1)e^{E_c(-\sigma_0)-E_0(-\sigma_0)} \leq 0(\lambda^{1/2}) + e^{-c\sigma_0\lambda^{-1}},$$

for $c > 0$, using proposition 5.1.1 (here we have used the fact that (e. g. when $h \geq 0$) $E_c\ (-\sigma_0) = \lambda^{-1}(-\sigma_0 + 0(\sigma_0^2)) - h\xi_+(0) + 0(h^2)$, when $h \in T$ and λ is small enough. In the first term, (5.1.26) was again employed). Finally, we observe that $\lambda\,|\,\xi_+^2 - \xi_+^2(0)\,| = 0(\lambda^\varepsilon)$.

This completes the proof of theorem 5.1.5.

5.2. **The triple point and the double points.**

The necessary prerequisites being established, we can now follow the outline of chapter III to confirm the validity of the phase diagram in figure 3. Recalling the well-known fact that the vacuum energy density (2.6) is a convex function of any parameter appearing linearly in the interaction density (2.1), we conclude that $\alpha_{\infty,1}(\lambda, \sigma, h)$ is convex in σ and h. Thus, it is a continuous function of these parameters and its derivatives with respect to h and σ exist at all except at most countably many values of σ and h. With this in mind, we state a theorem from [Ga] that is itself an extension of a theorem from [Gu2].

LEMMA 5.2.1. — If $\partial\alpha_\infty(\lambda, \sigma, h)/\partial\sigma$ exists, then

$$\langle :\phi^2:(\Delta_\alpha) :\phi^2:(\Delta_\beta)\rangle^{+,0,-}$$

clusters in mean.

Proof. — As in [Ga]. See, however, the appendix for technical remarks dealing with the possible lack of translation invariance in the states provided by the compactness construction of [GJ1].

Remarks. — 1) The theorem of [Gu2] states that if the derivative of $\alpha^{+,0,-}(\lambda, \sigma, h)$ with respect to h exists, then $\langle \phi(\Delta_\alpha)\phi(\Delta_\beta)\rangle^{+,0,-}$ clusters in mean.

2) Due to lemma 4.6, if the derivative of α_∞^+ with respect to σ or h exists,

then the corresponding derivative of $\alpha_\infty^{0,-}$ exists. Thus, lemma 5.2.1 or Guerra's theorem can be applied to the $+$, 0 and $-$ boundary condition states simultaneously (see, furthermore, Appendix 1 and proposition 5.2.5).

3) The arguments to follow make good use of the proof in [Ga] of the existence of the triple point.

Lemma 5.2.1 and (5.1.19) of theorem 5.1.5 entail that whenever $\partial\alpha_\infty/\partial\sigma$ exists, $\lambda\langle :\phi^2:(\Delta)\rangle^{+,0,-}$ cannot lie in $[\omega_0^2 + K\lambda^{1/2}, \omega_+^2 - K\lambda^{1/2}]$ (here we take $h \geq 0$; the argument for $h \leq 0$ is similar). And (5.1.20) and (5.1.21) imply that for $0 < \sigma_0 \leq 10^{-1}$ and small enough λ,

$$\lambda\langle :\phi^2:(\Delta)\rangle_{\sigma_0}^{+,0,-} \leq \omega_0^2 + K\lambda^{1/2} \tag{5.2.1}$$

and

$$\lambda\langle :\phi^2:(\Delta)\rangle_{-\sigma_0}^{+,0,-} \geq \omega_+^2 - K\lambda^{1/2}. \tag{5.2.2}$$

Therefore, if we define, for fixed h and fixed, small enough λ,

$$S_{\lambda,h}^{+,0,-} = \{\sigma \mid \lambda\langle :\phi^2:(\Delta)\rangle_h^{+,0,-} \leq \omega_0^2 + K\lambda^{1/2}\},$$

this set is nonempty, is bounded from below, and possesses an infimum:

$$\sigma_D^{+,0,-}(\lambda, h) = \inf S_{\lambda,h}^{+,0,-}. \tag{5.2.3}$$

As previously commented, the second Griffiths' inequality (see, e. g. [GRS1, Si2]) entails that $\langle :\phi^2:(\Delta)\rangle^{+,0,-}$ is monotone decreasing in σ (monotone increasing in h). Thus, the monotonicity of $\langle :\phi^2:(\Delta)\rangle$ in σ and the almost everywhere differentiability of $\alpha_\infty(\sigma)$ give

$$\lambda\langle :\phi^2:(\Delta)\rangle_{\sigma_D^{+,0,-}(\lambda,h)+0}^{+,0,-} \leq \omega_0^2 + K\lambda^{1/2} \tag{5.2.4}$$

and (when $h \geq 0$)

$$\lambda\langle :\phi^2:(\Delta)\rangle_{\sigma_D^{+,0,-}(\lambda,h)-0}^{+,0,-} \geq \omega_+^2 + K\lambda^{1/2}. \tag{5.2.5}$$

LEMMA 5.2.2. — $\sigma_D^+(\lambda, h) = \sigma_D^0(\lambda, h) = \sigma_D^-(\lambda, h)$ for all sufficiently small λ and all $|h| \leq \lambda^{-1/2+\varepsilon}$.

Proof. — We show in Appendix 1 that whenever $\partial\alpha_\infty^{+,0,-}/\partial\sigma$ exists,

$$-\frac{\partial}{\partial\sigma}\alpha_\infty^{+,0,-} = \langle :\phi^2:(\Delta)\rangle^{+,0,-},$$

and $\langle :\phi^2:(\Delta)\rangle^{+,0,-}$ is independent of Δ (i. e., translation invariant). But by lemma 4.6, this entails that at such values of σ (λ, h fixed),

$$\langle :\phi^2:(\Delta)\rangle^+ = \langle :\phi^2:(\Delta))^0 = \langle :\phi^2:(\Delta)\rangle^-.$$

Let us assume, e. g., $\sigma_D^+(\lambda, h) > \sigma_D^0(\lambda, h)$. Then there exists a $\sigma_0 \in (\sigma_D^0(\lambda, h), \sigma_D^+(\lambda, h))$ such that $\partial\alpha_\infty/\partial\sigma|_{\sigma_0}$ exists. Thus,

$$\langle :\phi^2:(\Delta)\rangle_{\sigma_0}^+ = \langle :\phi^2:(\Delta)\rangle_{\sigma_0}^0;$$

but since $\sigma_0 < \sigma_{\mathrm{D}}^{+}(\lambda, h)$, $\lambda \langle :\phi^2:(\Delta) \rangle_{\sigma_0}^{+} \geq \omega_+^2 - \mathrm{K}\lambda^{1/2}$, for small λ, which implies that

$$\lambda \langle :\phi^2:(\Delta) \rangle_{\sigma_0}^{0} \geq \omega_+^2 - \mathrm{K}\lambda^{1/2}. \tag{5.2.6}$$

However, since

$$\langle :\phi^2:(\Delta) \rangle_{\sigma_{\mathrm{D}}^0(\lambda,h)+0}^{0} \geq \langle :\phi^2:(\Delta) \rangle_{\sigma_0}^{0}$$

for $\sigma_0 > \sigma_{\mathrm{D}}^0(\lambda, h)$, we conclude from (5.2.6) that

$$\lambda \langle :\phi^2:(\Delta) \rangle_{\sigma_{\mathrm{D}}^0(\lambda,h)+0}^{0} \geq \omega_+^2 - \mathrm{K}\lambda^{1/2}$$

for small λ, which contradicts (5.2.4). Thus, $\sigma_{\mathrm{D}}^{+}(\lambda, h) \leq \sigma_{\mathrm{D}}^{0}(\lambda, h)$. A similar argument may be applied to yield $\sigma_{\mathrm{D}}^{+}(\lambda, h) \geq \sigma_{\mathrm{D}}^{0}(\lambda, h)$. The same argument yields the rest of the lemma.

Note. — We comment further that, due to the independence of the vacuum energy density from (at least) the classical boundary conditions [GRS2], this argument entails that the location of the double point $\sigma_{\mathrm{D}}(\lambda, h)$ is, for given λ and h, independent of boundary conditions (see Appendix 1). That is to say, independently of boundary conditions, the quantity $\langle :\phi^2:(\Delta) \rangle$ is discontinuous in σ at $\sigma_{\mathrm{D}}(\lambda, h)$ (the magnitude of the discontinuity is also an invariant). This completes the proof of theorem 2.1.

We shall now prove the estimates (3.8) (and, in passing, the rest of theorem 3.1) in $\mathrm{R}^{+,0,-} \backslash \partial \mathrm{R}^{+,0,-}$, i. e., in $\mathrm{R}^{+,0,-}$ excluding the phase transition lines themselves, at which we shall directly construct the +, 0, − state through the limits discussed in chapter III.

PROPOSITION 5.2.3. — There exists a $\mathrm{K} > 0$ such that for all small enough λ and σ, $h \in \mathrm{T}$,

i) if $\sigma > \sigma_{\mathrm{D}}(\lambda, h)$, $\langle \chi_0(\Delta) \rangle_\sigma^0 \geq e^{-\mathrm{K}\lambda^\varepsilon}$,

ii) if $\sigma < \sigma_{\mathrm{D}}(\lambda, h)$, $h > 0$, $\langle \chi_+(\Delta) \rangle_\sigma^+ \geq e^{-\mathrm{K}\lambda^\varepsilon}$,

iii) if $\sigma < \sigma_{\mathrm{D}}(\lambda, h)$, $h < 0$, $\langle \chi_-(\Delta) \rangle_\sigma^- \geq e^{-\mathrm{K}\lambda^\varepsilon}$.

Remark. — The ε appearing in the estimates is that which occurs in the definition of T.

Proof. — We demonstrate *i)* first. By (5.1.22) and (5.2.4), we have for all $\sigma > \sigma_{\mathrm{D}}(\lambda, h)$:

$$\lambda \langle \phi(\Delta)^2 \rangle^0 \leq \omega_0^2 + \mathrm{K}_1\lambda^{1/2}. \tag{5.2.7}$$

But

$$\begin{aligned} \lambda \langle \phi(\Delta)^2 \rangle^0 &= \lambda \langle \phi(\Delta)^2\chi_0(\Delta) \rangle^0 \\ &\quad + \lambda \langle \phi(\Delta)^2\chi_+(\Delta) \rangle^0 + \lambda \langle \phi(\Delta)^2\chi_-(\Delta) \rangle^0 \\ &\geq 0 + \lambda \langle \phi(\Delta)^2\chi_{+,s}(\Delta) \rangle^0 + \lambda \langle \phi(\Delta)^2\chi_{-,s}(\Delta) \rangle^0 \\ &\geq \lambda(1 - \lambda^{1/4})^2\xi_+^2 \langle \chi_{+,s}(\Delta) \rangle^0 + \lambda(\xi_- + \lambda^{1/4}\xi_+)^2 \langle \chi_{-,s}(\Delta) \rangle^0 \end{aligned} \tag{5.2.8}$$

(using $\chi_\pm(\Delta) \geq \chi_{\pm,s}(\Delta)$ and definitions (4.5)). Moreover, $\lambda(1-\lambda^{1/4})^2\xi_+^2 = 0(1)$ and $\lambda(\xi_- + \lambda^{1/4}\xi_+)^2 = \lambda(1 + 0(\lambda^{1/2}h) - \lambda^{1/4})^2\xi_+^2$, so that (5.2.7) and (5.2.8) imply

$$(5.2.9)\quad \langle \chi_+(\Delta) \rangle^0 + \langle \chi_-(\Delta) \rangle^0 = \langle \chi_{+,s}(\Delta) \rangle^0 + \langle \chi_{+,p}(\Delta) \rangle^0 + \langle \chi_{-,s}(\Delta) \rangle^0 + \langle \chi_{-,p}(\Delta) \rangle^0 \leq \langle \chi_{+,s}(\Delta) \rangle^0 + \langle \chi_{-,s}(\Delta) \rangle^0 + e^{-c\lambda^{-1/2}} = \langle \chi_{+,s}(\Delta) \rangle^0 + \frac{\lambda(\xi_- + \lambda^{1/4}\xi_+)^2}{\lambda(1-\lambda^{1/4})^2\xi_+^2} \langle \chi_{-,s}(\Delta) \rangle^0 + \left(1 - \frac{\lambda(\xi_- + \lambda^{1/4}\xi_+)^2}{\lambda(1-\lambda^{1/4})^2\xi_+^2}\right) \langle \chi_{-,s}(\Delta) \rangle^0 + e^{-c\lambda^{-1/2}} \leq K_2\omega_0^2 + K_3\lambda^{1/2} + 0(\lambda^{1/2}h) < \chi_{-,s}(\Delta) \rangle^0 + e^{-c\lambda^{-1/2}} \leq K_2\omega_0^2 + K_4\lambda^\varepsilon$$

($\chi_{-,s}(\Delta) \leq 1$ and $|h| \leq \lambda^{-1/2+\varepsilon}$)

$$\leq K_5\lambda^\varepsilon$$

since, for all σ, $h \in T$, $\omega_0^2 = 0(\lambda^{2\varepsilon})$. Therefore, (5.2.9) implies

$$\langle \chi_0(\Delta) \rangle^0 = 1 - \langle \chi_+(\Delta) \rangle^0 - \langle \chi_-(\Delta) \rangle^0 \geq 1 - K_5\lambda^\varepsilon \geq e^{-K_6\lambda^\varepsilon}.$$

That completes the proof of part *i*).

We will prove *ii*) explicitly; *iii*) follows in a similar manner.

$$(5.2.10)\quad \lambda\langle \phi(\Delta)^2 \rangle^+ = \lambda\langle \phi(\Delta)^2\chi_0(\Delta) \rangle^+ + \lambda\langle \phi(\Delta)^2\chi_+(\Delta) \rangle^+ + \lambda\langle \phi(\Delta)^2\chi_-(\Delta) \rangle^+ \leq \lambda\langle \phi(\Delta)^2\chi_{0,s}(\Delta) \rangle^+ + \lambda\langle \phi(\Delta)^2\chi_{+,s}(\Delta) \rangle^+ + \lambda\langle \phi(\Delta)^2\chi_{-,s}(\Delta) \rangle^+ + e^{-c\lambda^{-1/2}}$$

(by lemma 5.1.3)

$$\leq \lambda(\xi_0 + \lambda^{1/4}\xi_+)^2 \langle \chi_{0,s}(\Delta) \rangle^+ + \lambda(1 + \lambda^{1/4})^2\xi_+^2 \langle \chi_{+,s}(\Delta) \rangle^+ + \lambda(\xi_- - \lambda^{1/4}\xi_+)^2 \langle \chi_{-,s}(\Delta) \rangle^+ + e^{-c\lambda^{-1/2}}.$$

But

$$\langle \chi_{+,s}(\Delta) \rangle^+ \leq 1 - \langle \chi_{-,s}(\Delta) \rangle^+ - \langle \chi_{0,s}(\Delta) \rangle^+$$

entails with (5.2.10):

$$(5.2.11)\quad \lambda\langle \phi(\Delta)^2 \rangle^+ \leq \lambda(\xi_0 + \lambda^{1/4}\xi_+)^2 \langle \chi_{0,s}(\Delta) \rangle^+ + (1 + \lambda^{1/4})^2\omega_+^2 + 0(\lambda^{1/2}h)\langle \chi_{-,s}(\Delta) \rangle^+ - (1 + \lambda^{1/4})^2\omega_+^2 \langle \chi_{0,s}(\Delta) \rangle^+ + e^{-c\lambda^{-1/2}}.$$

But (5.2.5) and (5.1.22) imply that for $\sigma < \sigma_D(\lambda, h)$,

$$(5.2.12)\quad \lambda\langle \phi(\Delta)^2 \rangle^+ \geq \omega_+^2 - K\lambda^{1/2},$$

so that, by (5.2.11), we have

$$[-\lambda(\xi_0 + \lambda^{1/4}\xi_+)^2 + (1 + \lambda^{1/4})^2\omega_+^2]\langle \chi_{0,s}(\Delta) \rangle^+ \leq (1 + \lambda^{1/4})^2\omega_+^2 + 0(\lambda^{1/2}h) - \omega_+^2 + K_6\lambda^{1/2} \leq K_7\lambda^\varepsilon.$$

Therefore

$$(5.2.13)\quad \langle \chi_{0,s}(\Delta) \rangle^+ \leq \frac{K_7\lambda^\varepsilon}{[-\lambda(\xi_0 + \lambda^{1/4}\xi_+)^2 + (1 + \lambda^{1/4})^2\omega_+^2]} \leq K_8\lambda^\varepsilon,$$

for small λ, since $\lambda(\xi_0+\lambda^{1/4}\xi_+)^2=0(\lambda^{2\varepsilon})$ and $\omega_+^2=1+\sigma+0(\sigma^2)+0(\lambda^{1/2}h)$. Using (5.2.13) in the right-hand side of (5.2.10), we have

$$\lambda\langle \phi(\Delta)^2\rangle^+ \leq K_9\lambda^{3\varepsilon} + (1+\lambda^{1/4})^2\omega_+^2\langle \chi_{+,s}(\Delta)\rangle^+ + (1+\lambda^{1/4})^2\omega_+^2\langle \chi_{-,s}(\Delta)\rangle^+ + 0(\lambda^{1/2}h)\langle \chi_{-,s}(\Delta)\rangle^+ + e^{-c\lambda^{-1/2}}.$$

Thus, with (5.2.12),

$$(1+\lambda^{1/4})^2\omega_+^2\langle \chi_{+,s}(\Delta)\rangle^+ \geq \omega_+^2 - K_{10}\lambda^{\varepsilon} - (1+\lambda^{1/4})^2\omega_+^2\langle \chi_{-,s}(\Delta)\rangle^+,$$

so that

$$(5.2.14)\qquad \langle \chi_{+,s}(\Delta)\rangle^+ \geq \frac{\omega_+^2 - K_{10}\lambda^{\varepsilon}}{(1+\lambda^{1/4})^2\omega_+^2} - \langle \chi_{-,s}(\Delta)\rangle^+ \geq 1 - K_{11}\lambda^{\varepsilon} - \langle \chi_{-,s}(\Delta)\rangle^+.$$

We now remark that

$$(5.2.15)\quad \lambda\langle \phi(\Delta_\alpha)\phi(\Delta_\beta)\rangle^+ = \lambda\langle \phi(\Delta_\alpha)\chi_+(\Delta_\alpha)\phi(\Delta_\beta)\chi_+(\Delta_\beta)\rangle^+ + \lambda\langle \phi(\Delta_\alpha)\chi_-(\Delta_\alpha)\phi(\Delta_\beta)\chi_-(\Delta_\beta)\rangle^+ + \lambda\langle \phi(\Delta_\alpha)\chi_0(\Delta_\alpha)\phi(\Delta_\beta)\chi_0(\Delta_\beta)\rangle^+ + \lambda\sum_{\sigma(\Delta_\alpha)\neq\sigma(\Delta_\beta)}\langle \phi(\Delta_\alpha)\chi_{\sigma(\Delta_\alpha)}(\Delta_\alpha)\phi(\Delta_\beta)\chi_{\sigma(\Delta_\beta)}(\Delta_\beta)\rangle^+.$$

The third term in (5.2.15) may be estimated by

$$\lambda\langle \phi(\Delta)^2\chi_0(\Delta)\rangle^+ \leq \lambda\langle \phi(\Delta)^2\chi_{0,s}(\Delta)\rangle^+ + e^{-c\lambda^{-1/2}} \leq \lambda(\xi_0+\lambda^{1/4}\xi_+)^2\langle \chi_{0,s}(\Delta)\rangle^+ + e^{-c\lambda^{-1/2}} \leq K_{12}\lambda^{3\varepsilon}$$

(by (5.2.13)). We note that we can estimate the last terms in (5.2.15) by

$$\sum_{\sigma(\Delta_\alpha)\neq\sigma(\Delta_\beta)}(\lambda^2\langle \phi(\Delta)^4\rangle^+)^{1/2}\langle \chi_{\sigma(\Delta_\alpha)}(\Delta_\alpha)\chi_{\sigma(\Delta_\beta)}(\Delta_\beta)\rangle^{+\,1/2} \leq e^{-a\lambda^{-1/2}}, \quad a>0,$$

by propositions 5.1.1 and 5.1.4. Therefore,

$$(5.2.16)\quad \lambda\langle \phi(\Delta_\alpha)\phi(\Delta_\beta)\rangle^+ \geq \lambda\langle \phi(\Delta_\alpha)\chi_+(\Delta_\alpha)\phi(\Delta_\beta)\chi_+(\Delta_\beta)\rangle^+ + \lambda\langle \phi(\Delta_\alpha)\chi_-(\Delta_\alpha)\phi(\Delta_\beta)\chi_-(\Delta_\beta)\rangle^+ - K_{13}\lambda^{3\varepsilon}.$$

Because $\chi_{\pm,s}(\Delta_\beta)=1-\chi_0(\Delta_\beta)-\chi_{\mp}(\Delta_\beta)-\chi_{\pm,p}(\Delta_\beta)$, we have from (5.2.16):

$$\begin{aligned}(5.2.17)\quad \lambda\langle \phi(\Delta_\alpha)\phi(\Delta_\beta)\rangle^+ &\geq (1-\lambda^{1/4})^2\omega_+^2\langle \chi_{+,s}(\Delta_\alpha)\rangle^+ \\ &-(1-\lambda^{1/4})^2\omega_+^2\langle \chi_{+,s}(\Delta_\alpha)\chi_0(\Delta_\beta)\rangle^+ - (1-\lambda^{1/4})^2\omega_+^2\langle \chi_{+,s}(\Delta_\alpha)\chi_-(\Delta_\beta)\rangle^+ \\ &+(1-\lambda^{1/4}+0(\lambda^{1/2}h))^2\omega_+^2\langle \chi_{-,s}(\Delta_\alpha)\rangle^+ \\ &-(1-\lambda^{1/4}+0(\lambda^{1/2}h))^2\omega_+^2\langle \chi_{-,s}(\Delta_\alpha)\chi_0(\Delta_\beta)\rangle^+ \\ &-(1-\lambda^{1/4}+0(\lambda^{1/2}h))^2\omega_+^2\langle \chi_{-,s}(\Delta_\alpha)\chi_+(\Delta_\beta)\rangle^+ - K_{14}\lambda^{3\varepsilon} \\ &\geq (1-\lambda^{1/4})^2\omega_+^2\langle \chi_{+,s}(\Delta_\alpha)\rangle^+ + (1-\lambda^{1/4})^2\omega_+^2\langle \chi_{-,s}(\Delta)\rangle^+ - K_{15}\lambda^{\varepsilon}\end{aligned}$$

(by proposition 5.1.4)

$$\geq (1 - \lambda^{1/4})^2 \omega_+^2 (1 - K_{11}\lambda^\varepsilon) - K_{15}\lambda^\varepsilon \geq \omega_+^2 - K_{16}\lambda^\varepsilon,$$

using (5.2.14). By the theorem of Guerra, by choosing a sequence of positive h_n converging to h such that $\partial\alpha_\infty/\partial h$ exists at each h_n, (5.2.17) implies that for all $h > 0$ and $\sigma < \sigma_D(\lambda, h)$,

$$(5.2.18) \qquad \lambda^{1/2} \langle \phi(\Delta) \rangle^+ \geq \omega_+ - K_{17}\lambda^\varepsilon$$

(note that $\langle \phi(\Delta) \rangle^+$ is monotone increasing in h, by Griffiths' second inequality). This confirms theorem 3.1 *iii*) in the interior of R^+.

We see that for $h > 0$ and $\sigma < \sigma_D(\lambda, h)$,

$$\begin{aligned}(5.2.19) \quad \lambda^{1/2} \langle \phi(\Delta) \rangle^+ &\leq \lambda^{1/2} \langle \phi(\Delta)\chi_{+,s}(\Delta) \rangle^+ \\ &+ \lambda^{1/2} \langle \phi(\Delta)\chi_{0,s}(\Delta) \rangle^+ + \lambda^{1/2} \langle \phi(\Delta)\chi_{-,s}(\Delta) \rangle^+ + e^{-c\lambda^{-1/2}} \\ &\leq (1 + \lambda^{1/4})\omega_+ \langle \chi_{+,s}(\Delta) \rangle^+ + \lambda^{1/2}(\xi_0 + \lambda^{1/4}\xi_+) \langle \chi_{0,s}(\Delta) \rangle^+ \\ &- (1 + 0(\lambda^{1/2}h) - \lambda^{1/4})\omega_+ \langle \chi_{-,s}(\Delta) \rangle^+ + e^{-c\lambda^{-1/2}} \\ &\leq (1 + \lambda^{1/4})\omega_+ \langle \chi_{+,s}(\Delta) \rangle^+ - (1 - \lambda^{1/4})\omega_+ \langle \chi_{-,s}(\Delta) \rangle^+ + K_{18}\lambda^\varepsilon,\end{aligned}$$

where we have used (5.2.13) in the last inequality. Using

$$\langle \chi_{+,s}(\Delta) \rangle^+ \leq 1 - \langle \chi_{-,s}(\Delta) \rangle^+,$$

(5.2.19) yields

$$\begin{aligned}\lambda^{1/2} \langle \phi(\Delta) \rangle^+ \leq (1 + \lambda^{1/4})\omega_+ &- (1 + \lambda^{1/4})\omega_+ \langle \chi_{-,s}(\Delta) \rangle^+ \\ &- (1 - \lambda^{1/4})\omega_+ \langle \chi_{-,s}(\Delta) \rangle^+ + K_{18}\lambda^\varepsilon,\end{aligned}$$

so that by using (5.2.18)

$$(5.2.20) \quad \langle \chi_{-,s}(\Delta) \rangle^+ \leq \frac{(1 + \lambda^{1/4})\omega_+ - \omega_+ + K_{19}\lambda^\varepsilon}{(1 + \lambda^{1/4})\omega_+ + (1 - \lambda^{1/4})\omega_+} \leq K_{20}\lambda^\varepsilon.$$

Therefore, by (5.2.20) and (5.2.13)

$$\begin{aligned}(5.2.21) \quad \langle \chi_+(\Delta) \rangle^+ &= 1 - \langle \chi_-(\Delta) \rangle^+ - \langle \chi_0(\Delta) \rangle^+ \\ &\geq 1 - \langle \chi_{-,s}(\Delta) \rangle^+ - \langle \chi_{0,s}(\Delta) \rangle^+ - e^{-c\lambda^{-1/2}} \geq 1 - K_{21}\lambda^\varepsilon,\end{aligned}$$

for $h > 0$, $\sigma < \sigma_D(\lambda, h)$. This completes the proof of the proposition.

We now wish to define the $+$, 0, $-$ state at the phase transition lines. By the monotonicity of $\langle :\phi^2:(\Delta) \rangle$ and $\langle \phi(\Delta) \rangle$ in σ and h, it is clear that the slope (in parameter space) of the phase transition line $\sigma_D(\lambda, h)$, for fixed λ, is strictly positive in the $h > 0$ half-plane (and, by symmetry, $\sigma_D(\lambda, h) = \sigma_D(\lambda, -h)$), as previously remarked in [GJ2]. Thus, because $\alpha_\infty(\lambda, \sigma, h)$ at each given (λ, σ), is continuously differentiable in h at all but at most countably many values of h, it is clear that one can find a sequence $\{ (\sigma_D(\lambda, h), h_n) \}^{+,0,-}$ converging to $(\sigma_D(\lambda, h), h)$ (or for $\sigma_0 \leq \sigma_T(\lambda)$, $h = 0$, a sequence $\{ (\sigma_0, h_n) \}^{+,-}$ converging to $(\sigma_0, 0)$ such that

$$\{ (\sigma_D(\lambda, h), h_n) \}^{+,0,-} \subset R^{+,0,-}(\{ (\sigma_0, h_n) \}^{+,-} \subset R^{+,-})$$

and such that

$$\frac{\partial}{\partial h}\alpha_\infty(\lambda, \sigma, h)\Big|_{(\sigma_D(\lambda,h),h_n)(\text{resp. } (\sigma_0,h_n))}$$

exists for each n. The only exception—the 0 state at $(\sigma_T(\lambda), 0)$—is reached by a sequence $\{(\sigma_n, h_n)\} \subset R^0$ converging to $(\sigma_T(\lambda), 0)$, at each point of which the vacuum energy density is differentiable with respect to the external field. Such a sequence, again, exists. However, the « canonical » sequence $\{(\sigma_n, 0)\} \downarrow (\sigma_T(\lambda), 0)$ used by Gawedzki [Ga] is not necessarily such a sequence. The $+, 0, -$ state at the point $(\sigma_0, h_0) \in \partial R^{+,0,-}$ is defined by

$$\lim_{n\to\infty}\left\langle \prod_{i=1}^{\nu}\phi(f_i)\right\rangle^{+,0,-}_{(\sigma_n,h_n)^{+,0,-}} \equiv \left\langle \prod_{i=1}^{\nu}\phi(f_i)\right\rangle^{+,0,-}_{(\sigma_0,h_0)}.$$

In fact, in appendix 2, it is shown that the generating functionals

$$\{ Z(f_1)_{(\sigma_n,h_n)^{+,0,-}} \}$$

(see (2.8)) converge uniformly to a functional $Z^{+,0,-}(f_1)_{(\sigma_0,h_0)}$ analytic in $f_1 \in L_{1,6/5} \supset \mathscr{S}(R^2)$, which determines a unique measure $d\phi^{+,0,-}_{(\sigma_0,h_0)}$ on $\mathscr{S}'(R^2)$, whose generalized Schwinger functions exist and are continuous on $\Pi\mathscr{L}_{P,\Sigma} \supset \Pi\mathscr{S}(R^2)$.

Thus, the estimates in proposition 5.2.3 extend to the phase transition lines, and we have proven the estimates (3.8) and theorem 3.1. In preparation to prove theorem 3.2, we introduce the following objects:

$$\chi^Z_{+,0,-}(\Delta) = \chi_{+,0,-}(\Delta)\,|_{\sigma=0,h=0}$$

(the dependence on σ and h is in $\zeta_{+,-}$, see (4.2)). Then we have the following lemma:

LEMMA 5.2.4. — There exists a $c > 0$ such that for all small enough λ, all σ, $h \in T$, all Δ and any $\sigma(\Delta)$,

$$\alpha_\infty^{+,0,-}(|\,\chi_{\sigma(\Delta)}(\Delta) - \chi^Z_{\sigma(\Delta)}(\Delta)\,|) \leq -\,c\lambda^{1/2}.$$

Proof. — This follows trivially from proposition 4.4 since

$$|\,\chi_{\sigma(\Delta)}(\Delta) - \chi^Z_{\sigma(\Delta)}(\Delta)\,| \leq \chi_{+,p}(\Delta) + \chi_{0,p}(\Delta) + \chi_{-,p}(\Delta).$$

We shall also need the following slight generalization of the result of Fröhlich and Simon mentioned in chapter III.

PROPOSITION 5.2.5. — If the Schwinger functions of a state are continuous from the right (or from the left) in the external field, then the state satisfies the Osterwalder-Schrader axioms, including clustering, and is independent of the classical boundary conditions (free, Dirichlet, Neumann, periodic, half-Dirichlet, etc.).

Proof. — Implicit in the proof of theorems 4.1, 4.2 and 4.4 of [FS]. See also [Su1].

Remark. — If the vacuum energy density is differentiable in h, then it is continuously differentiable, and it follows that proposition 5.2.5 is applicable.

As we have seen in chapter III, propositions 5.1.4, 5.2.5 and 5.2.3 imply that (σ, $h \in \mathrm{T}$) :

(5.2.22 *a*) for $h > 0$ and $\sigma = \sigma_{\mathrm{D}}(\lambda, h)$, or $h = 0$ and $\sigma \leq \sigma_{\mathrm{T}}(\lambda)$,

$$\langle \chi_0(\Delta) + \chi_-(\Delta) \rangle^+ \leq e^{-c\lambda^{-1/2}},$$

(5.2.22 *b*) for $h < 0$ and $\sigma = \sigma_{\mathrm{D}}(\lambda, h)$, or $h = 0$ and $\sigma \leq \sigma_{\mathrm{T}}(\lambda)$,

$$\langle \chi_+(\Delta) + \chi_0(\Delta) \rangle^- \leq e^{-c\lambda^{-1/2}},$$

(5.2.22 *c*) for $\sigma = \sigma_{\mathrm{D}}(\lambda, h)$,

$$\langle \chi_+(\Delta) + \chi_-(\Delta) \rangle^0 \leq e^{-c\lambda^{-1/2}}.$$

Griffiths' second inequality implies that $\langle \chi_+^{\mathrm{Z}}(\Delta) + \chi_-^{\mathrm{Z}}(\Delta) \rangle^0$ is monotone decreasing in σ ($\chi_{+,0,-}^{\mathrm{Z}}(\Delta)$ was defined to be independent of σ and h), and the FKG inequalities entail that $\langle \chi_\mp(\Delta) + \chi_0(\Delta) \rangle^\pm$ is monotone decreasing in h. And because lemma 5.2.4 and the chessboard estimate tell us that throughout T,

$$|\langle \chi_{+,0,-}(\Delta) \rangle - \langle \chi_{+,0,-}^{\mathrm{Z}}(\Delta) \rangle| \leq e^{-c\lambda^{-1/\lambda}},$$

(5.2.22) implies theorem 3.2.

Because $1 = \langle \chi_+(\Delta) \rangle + \langle \chi_0(\Delta) \rangle + \langle \chi_-(\Delta) \rangle$, an immediate corollary is:

COROLLARY 5.2.6. — There exists a $\mathrm{K} > 0$ such that for all small enough λ, all σ, $h \in \mathrm{T}$ and all Δ,

i) for $\sigma \geq \sigma_{\mathrm{D}}(\lambda, h)$, $\langle \chi_0(\Delta) \rangle_\sigma^0 \geq e^{-\mathrm{K}e^{-c\lambda^{-1/2}}}$,

ii) for $h \geq 0$, $\sigma \leq \sigma_{\mathrm{D}}(\lambda, h)$, $\langle \chi_+(\Delta) \rangle_\sigma^+ \geq e^{-\mathrm{K}e^{-c\lambda^{-1/2}}}$,

iii) for $h \leq 0$, $\sigma \leq \sigma_{\mathrm{D}}(\lambda, h)$, $\langle \chi_-(\Delta) \rangle_\sigma^- \geq e^{-\mathrm{K}e^{-c\lambda^{-1/2}}}$.

Because we have defined the limit states (with the exception of the 0 state at ($\sigma_{\mathrm{T}}(\lambda)$, 0)) to be right (or left) continuous in the external field, we may apply proposition 5.2.5 to yield the balance of theorem 2.2. Since the 0 state at ($\sigma_{\mathrm{T}}(\lambda)$, 0) is a limit of states satisfying the hypothesis of proposition 5.2.5, it satisfies all the Osterwalder-Schrader axioms (excepting possibly clustering and the linear growth condition) and is independent of the classical boundary conditions. The linear growth condition is proven in chapter VII, but we have no argument at present to show that the 0 state at ($\sigma_{\mathrm{T}}(\lambda)$, 0) is pure.

Although the $+$, 0, $-$ state at the phase transition lines is independent of the classical boundary conditions, it is, nevertheless, in principle dependent on the choice of sequence used to define it. However, by the second Griffiths' inequality, the Schwinger functions of the states are monotone increasing in h ($h \geq 0$) and monotone decreasing in σ. Let us consider, for example, the $+$ state at $\sigma_D(\lambda, h)$, $h \geq 0$ (or $\sigma \leq \sigma_T(\lambda)$, $h = 0$). By the monotonicity in σ and h,

$$\left\langle \prod_i \phi(f_i) \right\rangle^+_{\sigma_D(\lambda,h),h} = \lim_{(\sigma_n,h_n)\to(\sigma_D(\lambda,h),h)} \left\langle \prod_i \phi(f_i) \right\rangle, \quad f_i \geq 0,$$

for every sequence $(\sigma_n, h_n) \to (\sigma_D(\lambda, h), h)$ that is (eventually) contained in the second quadrant of parameter space, with $(\sigma_D(\lambda, h), h)$ regarded as the origin. In this quadrant, the Schwinger functions are jointly monotone decreasing in σ and h to the limit. Similarly, for $h \leq 0$ the $-$ state at $\sigma_D(\lambda, h)$ (or $\sigma \leq \sigma_T(\lambda)$, $h = 0$) is independent of the choice of sequence lying in the third quadrant of the axes with origin at $(\sigma_D(\lambda, h), h)$. When $h > 0$ ($h < 0$), the 0 state at $\sigma_D(\lambda, h)$ is independent of the choice of any sequence lying in the fourth quadrant (first quadrant). Thus, the 0 state at $(\sigma_D(\lambda, h), h \neq 0)$ coincides with the « canonical » limit (see [GJ2]) $(\sigma_n, h \neq 0) \downarrow (\sigma_D(\lambda, h)$ $h \neq 0)$. Only the 0 state at $(\sigma_T(\lambda), 0)$ depends *a priori* on the choice of the sequence $\{ (\sigma_n, h_n) \} \subset R^0$ that converges to it, without the benefit of large regions of equivalence.

6. THE POSITIONS OF THE PHASE TRANSITION LINES IN PARAMETER SPACE

The aim of this chapter is to prove theorem 2.5, as outlined in chapter III. As a first step, we note

LEMMA 6.1. — If we define $\alpha_\infty^{+,0,-}(\chi_{+,0,-})$ as in (5.1.2), the following inequalities hold:

i) $\alpha_\infty^{+,0,-}(\chi_{+,0,-}) \leq \alpha_\infty$;

moreover, there exists a $K > 0$ such that if $\sigma, h \in R^{+,0,-} \cap T$,

ii) $\alpha_\infty^{+,0,-}(\chi_{+,0,-}) - \alpha_\infty \geq - Ke^{-c\lambda^{-1/2}}$,

Proof. — *i)* is trivial, since $\chi_{+,0,-}(\Delta) \leq 1$. By the chessboard estimate

$$\langle \chi_+(\Delta) \rangle^+ \leq e^{\alpha_\infty^+(\chi_+)-\alpha_\infty}.$$

But for $\sigma, h \in R^+$, we have from corollary 5.2.6 that

$$e^{-Ke^{-c\lambda^{-1/2}}} \leq e^{\alpha_\infty^+(\chi_+)-\alpha_\infty}.$$

The 0, $-$ case in $R^{0,-}$ is proven similarly.

LEMMA 6.2. — There exists a $K > 0$ such that for all small enough λ and all σ, $h \in T$,

$$\alpha_{\infty,1}^{+,0,-}(\chi_{+,0,-}) \leq - E_{+,0,-} + c_W^{+,0,-} + K\lambda^{1/2}.$$

Proof.

$$\alpha_{\infty,1}^{+}(\chi_{+}) = \lim_{\Lambda \uparrow R^2} \frac{1}{|\Lambda|} \ln \int \prod_{\Delta \subset \Lambda} \chi_{+}(\Delta) e^{-:P_1(\phi)_\Lambda: + \frac{m_+^2}{2} :(\phi - \xi_+)_\Lambda^2: + c_W^+ |\Lambda|} d\mu(\phi - \xi_+)$$

$$= \lim_{\Lambda \uparrow R^2} \frac{1}{|\Lambda|} \ln \int \prod_{\Delta \subset \Lambda} \chi_{+}(\Delta) e^{-\left[:P_2(\phi)_\Lambda: - \frac{m_+^2}{2} :(\phi - \xi_+)_\Lambda^2: + (E_c - E_+)|\Lambda|\right]} d\mu(\phi - \xi_+) + c_W^+ - E_+$$

$$\leq - E_+ + c_W^+ + K\lambda^{1/2},$$

by proposition 4.4. The 0, — case is proven similarly.

Proof of theorem 2.5. — If σ, $h \in R^+ \cap T$, lemmas 6.1 and 6.2 imply

$$- E_+ + c_W^+ + K_1\lambda^{1/2} \geq \alpha_{\infty,1}^{+}(\chi_+) \geq \alpha_{\infty,1} - K_2 e^{-c\lambda^{-1/2}}$$
$$\geq - E_0 + c_W^0 - K_3\lambda^{1/2} - K_2 e^{-c\lambda^{-1/2}}$$

(using lemma 4.5 in the last inequality), which entails that

$$- E_+ + (c_W^+ - c_W^0) + K_4\lambda^{1/2} + E_0 \geq 0. \tag{6.1}$$

If we write $E_+(h = 0) \equiv \lambda^{-1}(\sigma + g_+(\sigma))$, then

$$E_+ = \lambda^{-1}(\sigma + g_+(\sigma)) - h\xi_+(o) - \frac{h^2}{2c_2} + f_+(\lambda, \sigma, h), \tag{6.2}$$

where $f_+(\lambda, \sigma, h) = 0(h^3\lambda^{1/2})$ (see (2.7)). $f_+(\lambda, \sigma, h)$ can, in fact, be calculated to arbitrary accuracy in h (see [Su1]). But (6.1) and (6.2) imply

$$\lambda(c_W^+ - c_W^0)|_{\sigma_D(\lambda,h)} - g_+(\sigma_D(\lambda, h)) + \lambda h\xi_+(o) + \frac{\lambda h^2}{2c_2} - \lambda f_+(\lambda, \sigma_D(\lambda, h), h) + \lambda E_0 + 0(\lambda^{3/2}) \geq \sigma_D(\lambda, h). \tag{6.3}$$

When σ, $h \in R^0 \cap T$ and $h \geq 0$, lemmas 6,1, 6.2 and 4.5 imply

$$- E_0 + c_W^0 + K_1\lambda^{1/2} \geq \alpha_{\infty,1}^{0}(\chi_0) \geq \alpha_{\infty,1} - K_1 e^{-c\lambda^{-1/2}}$$
$$\geq - E_+ + c_W^+ - K_3\lambda^{1/2} - K_2 e^{-c\lambda^{-1/2}},$$

so that, again,

$$\sigma_D(\lambda, h) \geq \lambda(c_W^+ - c_W^0)|_{\sigma_D(\lambda,h)} - g_+(\sigma_D(\lambda, h)) + \lambda h\xi_+(o) + \frac{\lambda h^2}{2c_2} - \lambda f_+(\lambda, \sigma_D(\lambda, h), h) + \lambda E_0 + 0(\lambda^{3/2}). \tag{6.4}$$

(6.3) and (6.4) give, with the same calculation for $h \leq 0$:

$$\sigma_D(\lambda, h) = \lambda(c_W^{\operatorname{sgn}(h)} - c_W^0)\,|_{\sigma_D(\lambda,h)} - g_+(\sigma_D(\lambda, h)) + \lambda\,|\,h\,|\,\xi_+(o) + \frac{\lambda h^2}{2c_2} - \lambda f_+(\lambda, \sigma_D(\lambda, h), |\,h\,|) + \lambda E_0 + 0(\lambda^{3/2}).$$

One sees at once that (because $g_+(\sigma) = -\frac{\sigma^2}{4} + 0(\sigma^3)$) whenever $|\,h\,| \leq \lambda^{-1/2+\varepsilon}$, we have

$$\begin{aligned}\sigma_D(\lambda, h) &= \lambda(c_W^{\operatorname{sgn}(h)} - c_W^0)\,|_{\sigma_D(\lambda,h)} + \lambda\,|\,h\,|\,\xi_+(o) + 0(\lambda h^2) + 0(\lambda^{3/2}) \\ &= \lambda\left(\frac{1}{\pi}\ln 4 - \frac{3}{4\pi}\right) + \lambda\,|\,h\,|\,\xi_+(o) + 0(\lambda h^2) + 0(\lambda^{3/2}),\end{aligned}$$

where we have used (2.2) and (2.4).

7. PERTURBATION THEORY IS ASYMPTOTIC

We have shown in [Su2], in the specific example of ϕ_2^4 deep in the two-phase region, that with the counterparts of theorem 3.2 and proposition 5.1.1 one can prove that the expansions generated by perturbation theory about the appropriate classical field value (the appropriate minimum of the polynomial) for the generalized Schwinger functions are asymptotic. Thus, we shall be somewhat telegraphic in this presentation of the proof of theorem 2.3 and shall rely heavily on [Su2]. Readers wishing to see all the details worked out specifically in the context of this model are referred to [Su1].

As previously mentioned in chapter III, we shall use the following result of [GJ1].

THEOREM 7.1. — The following formula is valid:

$$\int :(\phi - \xi_{+,0,-})^j:(h) R(\psi_{+,0,-}) d\phi^{+,0,-} = \int d\phi^{+,0,-} \int A^{+,0,-}(x)\left[\frac{\delta R}{\delta\psi_{+,0,-}(x)} - R(\psi_{+,0,-}):P'(\psi_{+,0,-}):(x)\right]dx,$$

where

$$A^{+,0,-}(x) = j\int dy C^{+,0,-}(x-y)h(y):(\phi - \xi_{+,0,-})^{j-1}:(y),$$

$$R(\psi_{+,0,-}) = \prod_{i=1}^{n} :\psi_{+,0,-}^{m_i}:(h_i),$$

$$C^{+,0,-} = (-\Delta + m_{+,0,-}^2)^{-1}.$$

We shall outline the proof of theorem 2.3 for the + state in $R^+ \cap T$. The argument is similar for the other cases.

The object of study is a typical generalized Schwinger function:

$$\left\langle \prod_{i=1}^{n} :(\phi - \xi_+)^{m_i}:(f_i) \right\rangle^+ , \tag{7.1}$$

where we shall require that suppt. $f_i \subset \Delta_i$, Δ_i a unit lattice square, $i = 1, \ldots, n$. The general case of unrestricted support is recovered as a sum over $\Delta_1, \ldots, \Delta_n$ of such localized monomials. Repeated integration by parts, applied to all the linear factors of the original product of Wick monomials and to the linear factors of the subsequent derivatives of the interaction polynomial $: P(\psi_+) :$ ((1.4)) brought into the integrand and continued until each term in the resultant sum either is a constant on path space or contains at least $r + 1$ (derivatives of the) polynomials $: P(\psi_+) :$, yields the following expansion for (7.1):

$$\sum_{i=1}^{r} \alpha_i^+(\sigma, h)\lambda^{i/2} = \sum_k \langle R_k(\psi_+) \rangle^+ , \tag{7.2}$$

where a typical term in the finite sum over $\langle R_K(\psi_+) \rangle^+$ is of the form

$$\left\langle \int \left(\prod_{\mu=1}^{M} : P^{(\alpha_\mu)}(\psi_+) : (y_\mu) \right) w(y) dy \right\rangle^+ , \tag{7.3}$$

and $M \geq r + 1$, $P^{(\alpha_\mu)}$ is the α_μ-th derivative of P, and

$$w(y) = \int v(x, y) \prod_{i=1}^{n} \chi_{\Delta_i}(x_i) f_i(x_i) dx.$$

$\chi_{\Delta_i}(x_i)$ denotes the characteristic function of the unit square Δ_i and $v(x,y)$ is a product of $N \geq M$ factors $C(x_i - x_j)$, $C(x_i - y_j)$, $C(y_i - y_j)$, where $C = (-\Delta + m_+^2)^{-1}$. It is easy to see that the constants $\alpha_i^+(\sigma, h)$ are exactly those given by perturbation theory about the minimum ξ_+.

Each term in the sum $\Sigma_K \langle R_K(\psi_+) \rangle^+$ can be represented graphically, with the lines of the graph due to the free covariances and the vertices provided by the derivatives of the original Wick monomials and of the interaction polynomial. The basic point is to estimate the vertices uniformly as $\lambda \downarrow 0$ and to control the integration over vertex positions by the exponential decay of the free covariances. Here, we shall give short shrift to the latter point and concentrate on the former.

For each term (7.3) we shall assume that not only every linear factor of the original product of Wick monomials has been integrated out, but that no two vertices with derivatives of the interaction polynomial are contracted to each other, unless one or both of them have been completely integrated out, i. e., unless one or both are constants on path space. For any factor that does not satisfy this assumption, one continues integrating by parts the nonconforming vertices until each term in the resulting sum fulfills the requirement. This merely increases M and the number of terms.

Then, by performing a localization sum

$$1 = \sum_{j \in Z^2} \chi_{\Delta_j}(.)$$

at each vertex, one obtains for (7.3):

$$0(\lambda^{M_1/2}) \sum_J \left\langle \left(\prod_{\mu=1}^{M_2} : P^{(\alpha_\mu)}(\psi_+) : (y_\mu) \chi_{\Delta_{j_\mu}}(y_\mu) \right) w_J(y) dy \right\rangle^+,$$

where $M = M_1 + M_2$, $0(\lambda^{M_1/2})$ contains the interaction coefficients of the M_1 completely integrated interaction vertices (recall lemma 4.2) and

$$w_J(y) = \int v(x, y) \prod_{i=1}^{n} \chi_{\Delta_i}(x_i) f_i(x_i) \prod_{k=1}^{M_1} \chi_{\Delta_{j_k}}(x_k) dx.$$

Here we have subsumed by x also the position variables (x_k) of the completely integrated vertices. Each $J = \{ j_v \}_{v=1}^{N} = \{ (j_{v,1}, j_{v,2}) \}_{v=1}^{N}$ $(j_v \in Z^4)$ denotes a choice of unit lattice square localizations for the M vertices (and thus the N covariances in $v(x, y)$). Next, at each square Δ_{j_μ} one performs the spin configuration expansion $1 = \chi_+(\cdot) + \chi_0(\cdot) + \chi_-(\cdot)$, so that (7.3) becomes

$$0(\lambda^{M_1/2}) \sum_J \sum_{\sigma(.)} \left\langle \int \left(\prod_{\mu=1}^{M_2} : P^{(\alpha_\mu)}(\psi_+) : (y_\mu) \chi_{\Delta_{j_\mu}}(y_\mu) \chi_{\sigma(\Delta_{j_\mu})}(\Delta_{j_\mu}) \right) w_J(y) dy \right\rangle^+,$$

where the sum $\Sigma_{\sigma(.)}$ is over the possible choices of $\sigma(\Delta_{j_\mu})$, $\mu = 1, \ldots, M_2$.

Finally, because there are no covariances in $v(x, y)$ (in the expression for $w_J(y)$) that join the vertices that are not completely integrated to each other (by the assumption described above), (7.3) is of the form:

$$(7.4) \qquad 0(\lambda^{M_1/2}) \sum_J \sum_{\sigma(.)} \left\langle \prod_{\mu=1}^{M_2} : P^{(\alpha_\mu)}(\psi_+) : (f_{j_\mu}) \chi_{\sigma(\Delta_{j_\mu})}(\Delta_{j\mu}) \right\rangle^+,$$

where $f_{j_\mu}(y_{j_\mu}) = \chi_{\Delta_{j_\mu}}(y_\mu) w_J(y_{j_\mu})$ and $w_J(y_{j_\mu})$ is the corresponding constituent of $w_J(y)$ (see [Su2]).

Each term in (7.4) is now in a form to which we may apply proposition 5.1.1. A given term with at least one $\sigma(\Delta_{j_\mu})$ differing from $+$ is estimated by Hölder's inequality:

$$(7.5) \qquad \left| \left\langle \prod_{\mu=1}^{M_2} : P^{(\alpha_\mu)}(\psi_+) : (f_{j_\mu}) \chi_{\sigma(\Delta_{j_\mu})}(\Delta_{j_\mu}) \right\rangle^+ \right|$$
$$\leq \left\langle \left(\prod_{\mu=1}^{M_2} : P^{(\alpha_\mu)}(\psi_+) : (f_{j_\mu}) \right)^2 \prod_{\mu=1}^{M_2} \chi_{\sigma(\Delta_{j_\mu})}(\Delta_{j_\mu}) \right\rangle^{+1/2} \langle \chi_{\sigma_0}(\Delta_0) \rangle^{+1/2},$$

where $\sigma_0 = 0$ or $-$. Application of proposition 5.1.1, lemma 4.2 and theorem 3.2 yields the following worst-case bound for (7.5):

$$K_1(5M_2)\lambda^{-M_2/2}e^{-c\lambda^{-1/2}}\prod_{\mu=1}^{M_2} \| f_{j_\mu} \|_2.$$

Proposition 5.1.1 and lemma 4.2 yield the following bound for the single term in the sum over spin configurations for which $\sigma(\Delta_{j_\mu}) = +, \mu = 1, \ldots, M_2$:

$$K_1(5M_2)\lambda^{M_2/2}\prod_{\mu=1}^{M_2} \| f_{j_\mu} \|_2.$$

Thus, (7.4) is estimated by

$$0(\lambda^{M_1/2})K_1(5M_2)3^{M_2}[\lambda^{M_2/2} + e^{-c\lambda^{-1/2}}\lambda^{-M_2/2}]\left\{\sum_{J}\prod_{\mu=1}^{M_2} \| f_{j_\mu} \|_2\right\}.$$

The factor 3^{M_2} is the number of terms in the sum $\Sigma_{\sigma(.)}$. By the exponential decay of the free covariances, the sum in the last factor has the bound [DG, GJS1, Su2]

$$\sum_{J}\prod_{\mu=1}^{M_2} \| f_{j_\mu} \|_2 \leq K_2^N.$$

And because $N \leq \Sigma_{i=1}^n m_i + 5M$, the total degree of the Wick monomials that have been brought into the integrand, we have the bound

$$\left| \sum_k \langle R_k(\psi_+) \rangle^+ \right| \leq 0(\lambda^{M/2}),$$

where $0(\lambda^{M/2})$ depends on M and N(A). Thus, theorem 2.3 is proven.

Similarly, a localization and spin configuration sum, in conjunction with proposition 5.1.1 and theorem 3.2, entail that

$$\left| \left\langle \prod_{i=1}^n (\phi - \xi_0)(f_i) \right\rangle^0 \right| \leq n\,!\, K^n \prod_{i=1}^n | f_i |_p$$

where $| f_i |_p \equiv \Sigma_{\Delta \subset R^2} \| f_i \chi_\Delta \|_p$, $p > 1$, with K uniform in λ $(\lambda \downarrow 0)$ and in σ, $h \in R^0 \cap T$. Because it is easy to see that $| f_i |_p \leq K | f_i |_{\mathcal{S}}$, for $| \,.\, |_{\mathcal{S}}$ a suitable Schwartz space seminorm (see [Su1]), one can directly verify the Osterwalder-Schrader linear growth condition, thus filling in the last gap in the proof of theorem 2.2.

ACKNOWLEDGMENTS

The author would like to acknowledge useful criticism by Prof. Konrad Osterwalder. In addition, the author would like to thank Prof. Osterwalder and the Mathematisches Seminar, Eidgenössische Technische Hochschule, Zürich, for their hospitality while most of this work was being done.

APPENDIX 1

BOUNDARY CONDITIONS AND EUCLIDEAN INVARIANCE OF $\langle :\phi^2:(\Delta)\rangle$

In this appendix we treat a technical problem that arises from the fact that the states that are obtained through the infinite volume limit of the normalized finite volume interacting measures in (2.3) are not known to be Euclidean invariant (their construction through a compactness argument does not yield total Euclidean invariance, only time translation invariance). However, at a few crucial points in section V.2 the Euclidean invariance of a few objects is tacitly employed. In order not to interrupt the main flow of the ideas of that section, we place the proof of the necessary properties, along with the technical results needed to show that $\sigma_D(\lambda, h)$ is independent of the (classical) boundary conditions, in the format of this appendix.

PROPOSITION A.1.1. — If $\partial\alpha_\infty(\lambda, \sigma, h)/\partial\sigma$ exists,

$$\langle :\phi^2:(x)\rangle^{\text{Periodic}} = \langle :\phi^2:(x)\rangle^{\text{H.D.}} = \langle :\phi^2:(x)\rangle^{\text{Free}}$$

for every x.

Proof. — We note that the chessboard estimates (valid for free and half-Dirichlet [FS] and periodic [Fr2] boundary conditions) imply for every $\chi > 0$,

$$\langle e^{\mp\chi:\phi^2:(\Lambda)}\rangle \leq e^{|\Lambda|(\alpha_\infty(\lambda,\sigma\pm\chi,h)-\alpha_\infty(\lambda,\sigma,h))}.$$

Therefore, if one subtracts by 1, divides by $\chi\,|\,\Lambda\,|$ and lets $\chi \to 0$, one obtains

$$\mp\frac{1}{|\Lambda|}\langle :\phi^2:(\Lambda)\rangle \leq \pm D_\sigma^{\pm}\alpha_\infty(\lambda, \sigma, h),$$

where $D_\sigma^{\pm}$ is the derivative from the right (resp., left) with respect to σ. When the derivative of the vacuum energy density with respect to σ exists, one has, thus,

$$-\frac{1}{|\Lambda|}\langle :\phi^2:(\Lambda)\rangle = \frac{\partial\alpha_\infty(\lambda, \sigma, h)}{\partial\sigma}$$

Because the free, periodic and half-Dirichlet pressures are equal [GRS2], one has at such points

$$\frac{1}{|\Lambda|}\langle :\phi^2:(\Lambda)\rangle^{\text{Periodic}} = \frac{1}{|\Lambda|}\langle :\phi^2:(\Lambda)\rangle^{\text{H.D.}} = \frac{1}{|\Lambda|}\langle :\phi^2:(\Lambda)\rangle^{\text{Free}}$$

for every Λ. The Euclidean invariance of the half-Dirichlet state [GRS1] implies the proposition.

Note. — Because the state $\langle\,.\,\rangle^{+,0,-}$ has free boundary conditions, lemma 4.6 and proposition A.1.1 yield the Euclidean invariance of $\langle :\phi^2:(x)\rangle^{+,0,-}$ (wherever $\partial\alpha_\infty(\lambda, \sigma, h)/\partial\sigma$ exists), which was used in the proof of lemma 5.2.1. Furthermore, the proof of proposition A.1.1 entails that

$$-\frac{\partial}{\partial\sigma}\alpha_\infty^{+,0,-} = \langle :\phi^2:(\Delta)\rangle^{+,0,-},$$

which was used in the proof of lemma 5.2.2. Because the validity of this argument is limited to those states with boundary conditions for which the chessboard estimate is known, we will need to introduce some further ideas in order to justify the note following lemma 5.2.2.

Let G be the set of functions on R^n (n arbitrary) that are sums of functions of the form

$$\mathcal{F}(f_1, \ldots, f_n) = \prod_{i=1}^{n} F(\phi(f_i)),$$

for $\phi \in \mathcal{S}'(R^2)$, $f_i \geq 0$ and

$$F(\phi(f)) = \theta(\phi(f))g(\phi(f)),$$

where g is a positive, increasing, polynomially bounded function on $(0, \infty)$ and $\theta(\phi(f)) = 1$ or sign $\phi(f)$. If $d\nu_1$ and $d\nu_2$ are probability measures on $\mathcal{S}'(R^2)$, we say $\nu_1 \leq \nu_2$(GKS) if and only if

$$\int \mathcal{F}(f_1, \ldots, f_n)d\nu_1 \leq \int \mathcal{F}(f_1, \ldots, f_n)d\nu_2$$

for any $\mathcal{F} \in G$, any n and any positive $f_1, \ldots, f_n \in \mathcal{S}(R^2)$.

We next recall a construction from [FS]. Let $P(x)$ be a semibounded polynomial and $Q_\pm(x) = P(x) \mp h_\infty x$, where h_∞ is large enough that the infinite volume probability measure $d\nu_{Q_\pm}$ corresponding to $Q_\pm$ can be constructed via Spencer's [Sp] large external field cluster expansion. Then we define the following measures for $|h| \leq h_\infty$:

$$d\nu_{P-h\phi,\pm} = \lim_{\Lambda \nearrow R^2} \frac{e^{(h \mp h_\infty)\phi_\Lambda} d\nu_{Q_\pm}}{\int e^{(h\mp h_\infty)\phi_\Lambda} d\nu_{Q_\pm}}.$$

These measures exist and obey all of the Osterwalder-Schrader axioms (including clustering) and are independent of the choice of h_∞ [FS]. Furthermore, $\alpha_\infty(P)_\pm = \alpha_\infty(P)$ and the chessboard estimate is known to be valid for these states. Thus, letting the P above be that in (2.1), wherever the vacuum energy density is differentiable in σ,

$$\langle : \phi^2 : (\Delta) \rangle_\pm = \langle : \phi^2 : (\Delta) \rangle. \quad \text{(A.1.1)}$$

We prove the following.

PROPOSITION A.1.2. — If $\alpha_\infty(\lambda, \sigma, h)$ is differentiable in σ, then $\langle : \phi^2 : (\Delta) \rangle$ is independent of the classical boundary conditions.

Proof. — The program for proving this proposition is the same as the proof of theorem 5.2 of [FS]. Given some boundary conditions y_0, we will find another set of boundary conditions $y_\pm$ such that

$$\pm\, d\nu_{\Lambda',\Lambda',y_0} \leq \pm\, d\nu_{\Lambda,\Lambda',y_\pm}(\text{GKS}), \quad \text{(A.1.2)}$$

where $d\nu_{\Lambda,\Lambda',y_\pm}$ has interaction P in Λ, $P \mp h_\infty\phi$ in Λ'/Λ and $y_\pm$ boundary conditions on $\partial\Lambda'$, and that for fixed Λ,

$$\lim_{\Lambda' \nearrow R^2} (d\nu_{\Lambda,\Lambda',y_\pm} - d\nu_{\Lambda,\Lambda',\text{Free}}) = 0. \quad \text{(A.1.3)}$$

(A.1.3) implies that $\langle\, . \,\rangle_\pm = \lim_{\Lambda \nearrow R^2} (\lim_{\Lambda' \nearrow R^2} d\nu_{\Lambda,\Lambda',y_\pm})$ and (A.1.2) gives us, thus,

$$\pm \langle\, . \,\rangle_{y_0} \leq \pm \langle\, . \,\rangle_\pm \text{ (GKS)}.$$

Therefore,

$$\langle :\phi^2:(\Delta)\rangle_- \leq \langle :\phi^2:(\Delta)\rangle_{y_0} \leq \langle :\phi^2:(\Delta)\rangle_+,$$

and (A.1.1) yields $\langle :\phi^2:(\Delta)\rangle_{y_0} = \langle :\phi^2:(\Delta)\rangle_\pm$. (A.1.2) and (A.1.3) are proven in [FS] for y_0 equal to the classical boundary conditions. We should point out that although (A.1.2) is proven in [FS] in the « FKG » sense for general polynomials, because the interaction polynomial $P_1(\phi)$ is even (excepting the external field), the « GKS » sense of the inequality (A.1.2) is maintained [GRS1]. It should, furthermore, be mentioned that, in order to maintain the validity of the GKS inequalities, only even boundary conditions can be admitted, which includes all of the classical boundary conditions, including the half-Neumann, half-periodic, etc., boundary conditions treated by Fröhlich and Simon's multiplicative B. C.-perturbation (the latter point follows because, if one re-Wick-orders our polynomial from y_1 B. C.-ordering to y_2 B. C.-ordering, one finds

$$:P_1(\phi):_{y_1} = :P_1(\phi):_{y_2} + :F(\phi):_{y_2},$$

where $F(\phi)$ is an even polynomial in ϕ. Thus, the GKS inequalities are retained under a perturbation due to $F(\phi)$).

APPENDIX 2

EXISTENCE OF THE LIMIT STATES

In this appendix we shall outline the proof of the existence of the limit states (their measures, generating functionals, and generalized Schwinger functions) at $\partial R^{+,0,-}$, as required in section V.2. We will discuss explicitly the proof for the 0 state at $(\sigma_T(\lambda), 0)$; the arguments for the other limits are similar. We have from [GJ1] that

$$Z(f_1) = \int e^{(\phi-\xi_0)(f_1)} d\phi^0$$

is bounded and analytic in $f_1 \in L_{1,6/5}$. In particular, this is true at each point (σ_n, h_n) of the sequence $\{(\sigma_n, h_n)\} \to (\sigma_T(\lambda), 0)$ chosen to define the 0 state at $(\sigma_T(\lambda), 0)$. Moreover, it is known [FS] that whenever the vacuum energy density is differentiable in h, the state constructed by the compactness argument incorporated in the proof of the existence theorem we have quoted from [GJ1] in Chapter II coincides with the state generated by the Fröhlich-Simon large external field boundary conditions construction. But for the latter state it is known that $Z(f_1)(f_1 \geq 0)$ is monotone increasing in h, and it is easy to see it is also monotone decreasing in σ. Therefore $\{ Z(f_1)_{(\sigma_n, h_n)} \}$ is a uniformly bounded family of analytic functionals (bounded, in fact, by $Z(f_1)_{(\sigma_T(\lambda), \sup h_n)}$), which, by Vitali's theorem converges uniformly on $L_{1,6/5}$ (possibly through a subsequence) to an analytic limit $Z^0(f_1)$.

Moreover, one sees that the limit determines a unique measure on $\mathcal{S}'(R^2)$. The measure is obtained from Minlos' theorem [Mi] once it is remarked that the uniform convergence $Z(f_1)_{(\sigma_n, h_n)} \to Z^0(f_1)$ entails that $J^0(f_1) \equiv Z^0(if_1)$ satisfies

i) $J^0(0) = 1$,
ii) J^0 is continuous on $\mathcal{S}(R^2)$,
iii) J^0 is of positive type.

i)-*iii*) follow from the corresponding properties of the $Z(if_1)_{(\sigma_n, h_n)}$. The measure $d\phi^0$ then generated is the unique measure for which

$$J^0(f_1) = e^{-i\langle \xi_0, f_1 \rangle} \int e^{i\phi(f_1)} d\phi^0.$$

To establish the existence of the generalized Schwinger functions of the limit states, we note that since we have already been assured of the validity of theorem 3.2 in $R^{+,0,-} \backslash \partial R^{+,0,-}$, we may employ theorem 3.2 *i*) and proposition 5.1.1 to conclude, as in Chapter VII (no integration by parts is necessary), that for fixed j and N,

$$\text{(A.2.1)} \qquad |\mathcal{S}_j^{(N)}(f_j)| \equiv \left| \int \prod_{\nu=1}^{N} :(\phi - \xi_0)^j : (f_{j_\nu}) d\phi^0 \right| \leq (jN)!\, K^N |f_j|_p^N,$$

where K is a constant uniform in $\{ (\sigma_n, h_n) \}$ and $|\ .\ |_p$ is given by

$$|f_j|_p = \sum_{\Delta \subset R^2} \| f_j \chi_\Delta \|_n \quad , \quad p > 1.$$

Denoting the Banach space defined with this norm by $\mathscr{L}_{p,\Sigma}$, one notes that

$$L_{1,6/6-} \supset \mathscr{L}_{j,\Sigma} \supset \mathscr{S}(R^2).$$

(A.2.1) entails that the family $\{ \mathscr{S}_j^{(N)}.(f_j)_{(\sigma_n, h_n)} \}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous on $\mathscr{L}_{j,\Sigma}$. Thus, it converges (possibly through a subsequence) to a limit $\mathscr{S}_j^{(N)}(f_j)^0$ continuous on $\mathscr{L}_{j,\Sigma}$. Because there are only countably many $\mathscr{S}_j^{(N)}$, one can find a subsequence so that all $\mathscr{S}_j^{(N)}(f_j)^0$ exist.

We comment that, once the existence of the generalized Schwinger functions has been established, as above, one can copy the argument of [GJ1] to prove that one can integrate by parts in the limit states, i. e., theorem 7.1 is valid for the limit states. The argument proceeds through the sequence of states at (σ_n, h_n) (for which theorem 7.1 holds) instead of through a sequence of finite volume states as the volume grows to infinity.

Note added in proof: We remark that because the lattice used in the proof of proposition 4.4 must coincide with the lattice used to define $\delta\phi$, the proof given here does not suffice for a unit lattice. A (more lengthy) proof of the vacuum energy bounds valid for a unit lattice is given explicitly in [Su1]. But in any case, one can just as well have started with a lattice composed of squares of area 10^{-6}. No results or arguments in the paper change; one must simply carry the normalization factor $|\Delta|^{-1}$ with every $F(\phi)(\Delta)$.

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(Manuscrit reçu le 8 novembre 1979)